SCHAUDER DECOMPOSITIONS AND COMPLETENESS

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A sequence $(E_n)_{n=1}^{\infty}$ of non-trivial subspaces of a topological vector space E is said to be a Schauder decomposition of E if there exists a sequence $(Q_n)_{n=1}^{\infty}$ of continuous orthogonal projections, such that $Q_n(E) = E_n$ for each n and, for each $x \in E$, $x = \sum_{n=1}^{\infty} Q_n x$. If, in addition, the projections $P_n = \sum_{i=1}^{n} Q_i$ are equicontinuous, then $(E_n)_{n=1}^{\infty}$ is said to be an equi-Schauder decomposition of E. It is obvious that a Schauder basis is equivalent to a Schauder decomposition in which each subspace is one-dimensional, and that it is equi-Schauder if and only if the corresponding decomposition is equi-Schauder. For more information on Schauder decompositions see, for example [**2** and **3**].

In this paper, it will be shown that if E is locally convex and possesses an equi-Schauder decomposition, the properties of sequential completeness, quasi-completeness or completeness of E may be related very simply to the properties of the decomposition; and that if E possesses an equi-Schauder basis, these three types of completeness are equivalent.

If $(E_n)_{n=1}^{\infty}$ is a Schauder decomposition of E, the sequences $(Q_n)_{n=1}^{\infty}$ and $(P_n)_{n=1}^{\infty}$ will always denote the corresponding sequences of projections as defined above.

LEMMA. *Let $\langle E, F\rangle$ be a separated dual pair of vector spaces and let τ be an $\langle E, F\rangle$ polar topology on E. Suppose $(E_n)_{n=1}^{\infty}$ is an equi-Schauder decomposition for (E, τ) and let $(x_\alpha)_{\alpha \in A}$ be a τ-Cauchy net on E such that for each n $(Q_n x_\alpha)_{\alpha \in A}$ converges. Then*:

(i) *$(\lim_\alpha P_n x_\alpha)_{n=1}^{\infty}$ is a τ-Cauchy sequence.*

(ii) *If $w\text{-}\lim_n \lim_\alpha P_n x_\alpha$ exists (where w-lim denotes the limit with respect to the weak topology $\sigma(E, F)$), then $\lim_\alpha x_\alpha$ exists and $\lim_\alpha x_\alpha = w\text{-}\lim_n \lim_\alpha P_n x_\alpha$.*

(i) Let U be any τ-neighbourhood of 0, and let V be a closed absolutely convex τ-neighbourhood of 0 such that $V+V+V \subset U$. Then since $(P_n)_{n=1}^{\infty}$ is τ-equicontinuous at 0, there exists a τ-neighbourhood W of 0 such that $P_n(W) \subset V$ for all n. Since $(x_\alpha)_{\alpha \in A}$ is a τ-Cauchy net, there exists $\beta \in A$ such that whenever $\gamma \geqslant \beta$, $x_\gamma - x_\beta \in W$, and so for all n $P_n(x_\gamma - x_\beta) \in V$.

For each n $(P_n x_\alpha)_{\alpha \in A}$ is a convergent net, for $P_n x_\alpha = \sum_{i=1}^{n} Q_i x_\alpha$; let $\lim_\alpha P_n x_\alpha = y_n$. Then since V is closed, $y_n - P_n(x_\beta) \in V$ for all n. The sequence $\big(P_n(x_\beta)\big)_{n=1}^{\infty}$ is convergent and so there exists k such that whenever $m, n \geqslant k$, $P_m(x_\beta) - P_n(x_\beta) \in V$.

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Thus whenever $m, n \geqslant k$

$$y_m - y_n = \big(y_m - P_m(x_\beta)\big) + \big(P_m(x_\beta) - P_n(x_\beta)\big) + \big(P_n(x_\beta) - y_n\big) \in V + V + V \subset U.$$

(ii) Suppose $w\text{-}\lim_n y_n = y$, and let U be any τ-neighbourhood of 0; then there exists a subset G of F whose polar G^0 is a τ-neighbourhood contained in U. Then since by part (i) (y_n) is τ-Cauchy, there exists k such that whenever $m, n \geqslant k$, $y_m - y_n \in G^0$, i.e.

$$\sup_{g \in G} |g(y_m - y_n)| \leqslant 1,$$

and so

$$\sup_{g \in G} |g(y - y_n)| \leqslant 1,$$

i.e. $y - y_n \in G^0 \subset U$, whenever $n \geqslant k$. Thus

$$\lim_n y_n = y.$$

Choosing V as in part (i), there exists m such that, whenever $n \geqslant m$, $y - y_n \in V$; and choosing β as in part (i), whenever $\alpha \geqslant \beta$, $y_n - P_n(x_\alpha) \in V$ for each n. For fixed $\alpha \geqslant \beta$, there exists $l \geqslant m$ such that $x_\alpha - P_l(x_\alpha) \in V$; and so we have

$$y - x_\alpha = (y - y_l) + (y_l - P_l x_\alpha) + (P_l x_\alpha - x_\alpha) \in V + V + V \subset U,$$

and so $\tau\text{-}\lim_\alpha x_\alpha = y$.

Naturally in part (ii) of the lemma, the existence of $w\text{-}\lim_n \lim_\alpha P_n x_\alpha$ may be replaced by the existence of $\lim_n \lim_\alpha P_n x_\alpha$, since this is a stronger condition; and it is in this form that the lemma is applied to derive the main criteria for completeness. First we define a Schauder decomposition (E_n) to be *complete* if whenever $\left(\sum_{n=1}^{\infty} x_n\right)_{m=1}^{\infty}$ is a Cauchy sequence with $x_n \in E_n$ then $\sum_{n=1}^{\infty} x_n$ converges; and a Schauder basis (x_n) is complete if the corresponding decomposition is complete.

THEOREM. *Let (E_n) be an equi-Schauder decomposition for the locally convex space E. Then E is complete (resp. quasi-complete; resp. sequentially complete) if and only if:*

(i) *Each E_n is complete (resp. quasi-complete; resp. sequentially complete), and*

(ii) *$(E_n)_{n=1}^{\infty}$ is a complete decomposition.*

If E is complete (resp. quasi-complete; resp. sequentially complete) then both conditions follow at once since each E_n is closed. Conversely if $(x_\alpha)_{\alpha \in A}$ is a net (resp. bounded net; resp. sequence) then by the application of the lemma to the dual pair $\langle E, E' \rangle$ the result follows.

The most important implications of this result appear to be those for bases; for we have

COROLLARY 1. *Let E be a locally convex space with an equi-Schauder basis (x_n). Then the following are equivalent*:

(i) *E is complete,*

(ii) *E is sequentially complete,*

(iii) *(x_n) is a complete basis.*

COROLLARY 2. *A sequentially complete barrelled space with a basis is complete.*

For each $x \in E$ $(P_n x)_{n=1}^{\infty}$ is bounded, and so, since E is barrelled $(P_n)_{n=1}^{\infty}$ is equicontinuous, i.e. the basis is equi-Schauder and we can apply Corollary 1.

Corollary 2 raises the following question: does there exist a separable sequentially complete barrelled space which is not complete? If there did, it could not have a basis, and so there would be a contradiction to the existence of a basis for every separable barrelled space. Komura [**1**] has shown that there does indeed exist a barrelled space which is sequentially complete (in fact quasi-complete) but not complete, but his example is probably non-separable. It is known from an example of Singer [**4**] that not every separable locally convex space has a Schauder basis, but his example is not barrelled.

References

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