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N. J. KALTON (*)

The Existence of Primitives for Continuous Functions in a Quasi-Banach Space (**).

Abstract. – *We show that if X is a quasi-Banach space with trivial dual then every continuous function $f\colon [0, 1] \to X$ has a primitive, answering a question of M.M. Popov.*

Let X be a quasi-Banach space and let $f\colon [0, 1] \to X$ be a continuous function. We say that f has a primitive if there is a differentiable function $F\colon [0, 1] \to X$ so that $F'(t) = f(t)$ for $0 \leqslant t \leqslant 1$. M. M. Popov has asked where every continuous function $f\colon [0, 1] \to L_p$ where $0 < p < 1$ has a primitive; more generally, he asks the same question for any space with trivial dual [4]. We show here that the answer to this question is positive. We remark that by an old result of Mazur and Orlicz [3], [6], every continuous f is Riemann-integrable if and only if X is a Banach space.

Let us suppose for convenience that X is p-normed where $0 < p < 1$, and let $I = [0, 1]$. Let $C(I; X)$ be the usual quasi-Banach space of continuous functions $f\colon I \to X$ with the quasi-norm $\|f\|_\infty = \max_{0 \leqslant t \leqslant 1} \|f(t)\|$. We

(*) Department of Mathematics, University of Missouri Columbia, MO 65211, U.S.A. E-mail address: mathnjk@mizzou1.missouri.edu.

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also introduce the space $C^1(I:X)$ of all functions $f \in C(I; X)$ which are differentiable at each t and such that the function $g: I^2 \to X$ is continuous where $g(t, t) = f'(t)$ for $0 \leqslant t \leqslant 1$ and

$$g(s, t) = \frac{f(s) - f(t)}{s - t}$$

when $s \neq t$. It is easily verified that $C^1(I; X)$ is a quasi-Banach space under the quasi-norm

$$\|f\|_{C^1} = \|f(0)\| + \sup_{0 \leqslant s < t \leqslant 1} \frac{\|f(t) - f(s)\|}{t - s}.$$

Let $C_0^0(I; X)$ be the closed subspace of $C^1(I; X)$ of all f such that $f(0) = 0$. We consider the map $D: C_0^1(I; X) \to C(I; X)$ given by $Df = f'$. The following result is proved in [1].

THEOREM 1. *If X has trivial dual then for every $x \in X$ there exists $f \in C_0^1(I; X)$ such that $Df = 0$ and $f(1) = x$.*

From this we deduce the answer to the question of Popov.

THEOREM 2. *If X has trivial dual then the map $D: C_0^1(I; X) \to \to C(I; X)$ is surjective. In particular every continuous $f: I \to X$ has a primitive.*

PROOF. From Theorem 1 and the Open Mapping Theorem we deduce the existence of a constant $M \geqslant 1$ so that if $x \in X$ there exists $f \in C_0^1(I; X)$ so that $Df = 0$, $f(1) = x$ and $\|f\|_{C^1} \leqslant M\|x\|$.

Now suppose $g \in C(I; X)$ with $\|g\|_\infty < 1$. For any $\varepsilon > 0$ we show the existence of $f \in C_0^1(I; X)$ with $\|Df - g\|_\infty < \varepsilon$ and $\|f\|_{C^1} < 4^{1/p}M$. Once this is achieved the Theorem follows again from a well-known variant of the Open Mapping Theorem.

Since g is uniformly continuous, there is a piecewise linear function h so that $\|g - h\|_\infty < \varepsilon$ and $\|h\|_\infty < 1$. Since h has finite-dimensional range there exists $H \in C_0^1(I; X)$ with $DH = h$. Now let n be a natural number, and let $x_{kn} = H(k/n) - H((k-1)/n)$. For $k = 1, 2, \ldots, n$ define $f_{k,n} \in C_0^1(I; X)$ so that $Df = 0$, $\|f_{k,n}\|_{C_0^1} \leqslant M\|x_{kn}\|$ and $f_{k,n}(1) = x_{kn}$. Then we define $F_n \in C_0^1(I; X)$ by

$$F_n(t) = H(t) - H\left(\frac{k-1}{n}\right) - f_{kn}(nt - k + 1)$$

for $(k-1)/n \leqslant t \leqslant k/n$. Clearly $DF_n = DH = h$. It remains to estimate $\|F_n\|_{C_0^1}$.

Let

$$\eta(\varepsilon) = \sup_{|t-s| \leq \varepsilon} \frac{\|H(t) - H(s)\|}{|t-s|}.$$

It is easy to see that $\lim_{\varepsilon \to 0} \eta(\varepsilon) = \|h\|_\infty < 1$. Now suppose $(k-1)/n \leq s < t \leq k/n$ for some $1 \leq k \leq n$. Then

$$\|F_n(t) - F_n(s)\| \leq \left(\eta\left(\frac{1}{n}\right)^p + n^p \|f_{kn}\|_{C^1}^p\right)^{1/p} (t-s) \leq$$

$$\leq \left(\eta\left(\frac{1}{n}\right)^p + M^p n^p \|x_{kn}\|^p\right)^{1/p} (t-s) \leq (M^p+1)^{1/p} \eta\left(\frac{1}{n}\right)(t-s).$$

Since $F_n(k/n) = 0$ for $0 \leq k \leq n$ we obtain that for any $0 \leq s < t \leq 1$,

$$\|F_n(t) - F_n(s)\| \leq 2^{1/p}(M^p+1)^{1/p} \eta\left(\frac{1}{n}\right) \min\left(t-s, \frac{1}{n}\right).$$

By taking n large enough we have $\|F_n\|_{C_0^1} < 4^{1/p} M$. Thus the theorem follows. ∎

We close with a few remarks on the general problem of classifying those quasi-Banach spaces X so that the map $D: C_0^1(I; X) \to C(I; X)$ is surjective; let us say that such a space is a D-space. The following facts are clear:

PROPOSITION 3. (1) *Any quotient of a D-space is a D-space.*

(2) *If X and Y are D-spaces then $X \oplus Y$ is a D-space.*

PROOF. (1) Let E be a closed subspace of X and let $\pi: X \to X/E$ be the quotient map. Let $\pi: C(I: X) \to C(I; X/E)$ be the induced map $\tilde{\pi} f = f \circ \pi$. We start with the observation that $\tilde{\pi}$ is surjective. If $g \in C(I; X/E)$ with $\|g\|_\infty < 1$ then we can find $f \in C(I; X)$ with $\|f\|_\infty < 2^{1/p-1}$ and $\|\tilde{\pi} f - g\|_\infty < 1$. To do this suppose N is an integer and let f_N be a function which is linear on each interval $[(k-1)/N, k/N]$ for $1 \leq k \leq N$ and such that $\pi f_N(k/N) = g(k/N)$ with $\|f_N(k/N)\| < 1$ for $0 \leq k \leq N$. For large enough N we have $\|g - \tilde{\pi} f_N\|_\infty < 1$ and our claim is substantiated.

Now if X is a D-space and $g \in C(I; X/E)$ then there exists $f \in C(I; X)$ with $\tilde{\pi} f = g$. Let $F \in C_0^1(I; X)$ with $DF = f$. Then if $G = \tilde{\pi} F$ we have $DG = g$.

(2) Is trivial. ∎

In [1] the notion of the core is defined: if X is a quasi-Banach space then core X is the maximal subspace with trivial dual.

THEOREM 4. *If core $X = \{0\}$ then X is a D-space if and only if X is a Banach space (i.e. is locally convex).*

PROOF. Suppose core $X = \{0\}$ and X is a D-space. Suppose $DF = 0$ where $F \in C_0^1(I; X)$. Let Y be the closed subspace generated by $\{F(s): 0 \leqslant s \leqslant 1\}$. We show $Y = \{0\}$; if not there exists a nontrivial continuous linear functional y^* on Y. Then $D(y^* \circ F) = 0$ so that $y^*(F(s)) = 0$ for $0 \leqslant s \leqslant 1$. But then $y^* = 0$ on Y. We conclude that $Y = \{0\}$ and so $F = 0$. Hence D is one-one and surjective and by the Closed Graph Theorem D is an isomorphism.

Let M be a constant so that $\|DF\|_\infty \leqslant 1$ implies $\|F\|_{C^1} \leqslant M$ for $F \in C_0^1(I; X)$. Let ϕ be any C^∞-real function on $\boldsymbol{R}$ with $\phi(t) = 0$ for $t \leqslant 0$ and $\phi(t) = 1$ for $t \geqslant 1$. Let $K = \max_{0 \leqslant t \leqslant 1} |\phi'(t)|$. For any N and any $x_1, \ldots, x_N \in X$ with $\max \|x_k\| \leqslant 1$, we define $F(t) = \sum_{k=1}^{N} \phi(Nt - k + 1)\, x_k$. Then $F \in C_0^1(I; X)$ and $\|DF\|_\infty \leqslant NK$. Hence $\|F(1)\| \leqslant NMK$, i.e.

$$\left\| \frac{1}{N}(x_1 + \ldots + x_N) \right\| \leqslant MK\,.$$

This implies X is locally convex. ∎

Combining Proposition 3 and Theorem 4 gives that if X is a D-space then $X/\text{core } X$ is a Banach space. It is, however, possible to construct an example to show that the converse to this statement is false, and there does not seem, therefore to be any nice classification of D-spaces in general.

To construct the example we observe the following theorem. First for any quasi-Banach space X let $a_N(X) = \sup\{\|x_1 + \ldots + x_N\|: \|x_i\| \leqslant 1\}$ (so that $a_N(X) \geqslant N$).

THEOREM 5. *Suppose X is a D-space; then for some constant C we have $a_N(X) \leqslant C a_N(\text{core } X)$.*

PROOF. Let $b_N = a_N(\text{core } X)$. Suppose $x_1, \ldots, x_N \in X$ with $\|x_i\| \leqslant 1$ and define as in Theorem 4, $F(t) = \sum_{k=1}^{N} \phi(Nt - k + 1)\, x_k$. Then $\|DF\|_\infty \leqslant NK$ and so by the Open Mapping Theorem, for some constant $M = M(X)$, there exists $G \in C_0^1(I; X)$ with $DG = DF$ and $\|G\|_{C^1} \leqslant MNK$. Then $\|G(k/N) - G((k-1)/N)\| \leqslant MK$ for $1 \leqslant k \leqslant N$.

Let $H(t) = F(t) - G(t)$. Since $DH = 0$ and $X/\text{core } X$ is a Banach space

H has range in core X. Now for $1 \leqslant k \leqslant N$, $H(k/N) - H((k-1)/N) = x_k - (G(k/N) - G((k-1)/N))$ so that $\|H(k/N) - H((k-1)/N)\| \leqslant (M^p K^p + 1)^{1/p}$. Hence if $C^p = M^p K^p + 1$, we have $\|H(1)\| \leqslant Cb_N$ or $\|x_1 + \ldots + x_n\| \leqslant Cb_n$.

To construct our example we start with the Ribe space Z([2],[5]) which is a space with a one-dimensional subspace L so that Z/L is isomorphic to l_1. A routine calculation shows $a_N(Z) \geqslant cN \log N$ for some $c > 0$. Then let Y be any quasi-Banach space with trivial dual so that $a_N(Y) = O(N \log N)$ (for example a Lorentz space $L(1, p)$ where $1 < p < \infty$). Let $j: L \to Y$ be an isometry and let X be the quotient of $Y \times Z$ by the subspace of all (jz, z) for $z \in L$. Then Z embeds into X so that $a_n(X) \geqslant cN \log N$ but core $X \sim Y$ so that X cannot be a D-space. However $X/\text{core } X$ is isomorphic to Z/L which is a Banach space.

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