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Set-functions and factorization

By

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1. Introduction. Let $\mathscr{A}$ be an algebra of subsets of some set Ω. Let us say that a set-function $\phi: \mathscr{A} \to \mathbb{R}$ is *monotone* if it satisfies $\phi(\emptyset) = 0$ and $\phi(A) \leqq \phi(B)$ whenever $A \subset B$. We say ϕ is normalized if $\phi(\Omega) = 1$. A monotone set-function ϕ is a *submeasure* if

$$\phi(A \cup B) \leqq \phi(A) + \phi(B)$$

whenever $A, B \in \mathscr{A}$ are disjoint, and ϕ is a *supermeasure* if

$$\phi(A \cup B) \geqq \phi(A) + \phi(B)$$

whenever $A, B \in \mathscr{A}$ are disjoint. If ϕ is both a submeasure and supermeasure it is a (finitely additive) measure.

If ϕ and ψ are two monotone set-functions on $\mathscr{A}$ we shall say that ϕ is *ψ-continuous* if $\lim_{n \to \infty} \phi(A_n) = 0$ whenever $\lim_{n \to \infty} \psi(A_n) = 0$. If ϕ is ψ-continuous and ψ is ϕ-continuous then ϕ and ψ are *equivalent.* A monotone set-function ϕ is called *exhaustive* if $\lim_{n \to \infty} \phi(A_n) = 0$ whenever (A_n) is a disjoint sequence in $\mathscr{A}$. The classical (unsolved) Maharam problem ([1], [5], [6] and [15]) asks whether every exhaustive submeasure is equivalent to a measure. A submeasure ϕ is called *pathological* if whenever λ is a measure satisfying $0 \leqq \lambda \leqq \phi$ then $\lambda = 0$. The Maharam problem has a positive answer if and only if there is no normalized exhaustive pathological submeasure.

While the Maharam problem remains unanswered, it is known (see e.g. [1] or [15]) that there are non-trivial pathological submeasures. In the other direction it is shown in [6] that if ϕ is a non-trivial *uniformly exhaustive* submeasure then ϕ cannot be pathological. ϕ is uniformly exhaustive if given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $\{A_1, \ldots, A_N\}$ are disjoint sets in $\mathscr{A}$ then

$$\min_{1 \leqq i \leqq N} \phi(A_i) < \varepsilon.$$

Let us say that a monotone set-function ϕ satisfies an *upper p-estimate* where $0 < p < \infty$ if ϕ^p is a submeasure, and a *lower p-estimate* if ϕ^p is a supermeasure. If ϕ is a normalized submeasure which satisfies a lower p-estimate for some $1 < p < \infty$ then ϕ is uniformly exhaustive and hence by results of [6] there is a non-trivial measure λ with $0 \leqq \lambda \leqq \phi$. In Section 2 we prove this by a direct argument which yields a quantitative

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estimate that λ can be chosen so that

$$\lambda(\Omega) \geqq 2(2^p - 1)^{-1/p} - 1.$$

Notice that the expression on the right tends to one as $p \to 1$ so this result can be regarded for p close to 1 as a perturbation result. The dual result for supermeasures (Theorem 2.2) is that if a normalized supermeasure ϕ satisfies an upper p-estimate where $0 < p < 1$ then there is a ϕ-continuous measure λ with $\lambda \geqq \phi$ and

$$\lambda(\Omega) \leqq 2(2^p - 1)^{-1/p} - 1.$$

While we believe these results with their relatively simple proofs have interest in their own right, one of our motivations for considering them was to use them in the study of some questions concerning quasi-Banach lattices, or function spaces.

It is well-known ([7]) that a Banach lattice X with a (crude) upper p-estimate is r-convex for every $0 < r < p$ (for the definitions, see Section 3). This result does not hold for arbitrary quasi-Banach lattices [2]; a quasi-Banach lattice need not be r-convex for any $r < \infty$. However, it is shown in [2] that if X has a crude lower q-estimate for some $q < \infty$ then the result is true. We provide first a simple proof of this fact, only depending on the arguments of Section 2. We then investigate this result further, motivated by the fact that if X satisfies a strict upper p-estimate (i.e. with constant one) and a strict lower p-estimate then X is p-convex (and in fact isometric to an $L_p(\mu)$-space.) We thus try to estimate the constant of r-convexity $M^{(r)}(X)$ when $0 < r < p$ and X has a strict upper p-estimate and a strict lower q-estimate where p, q are close. We find that an estimate of the form

$$\log M^{(r)}(X) \leqq c\theta(1 + |\log\theta|)$$

where $c = c(r, p)$ and $\theta = q/p - 1$. We show by example that such an estimate is best possible. Let us remark that in the case $r = 1 < p < q$ the constant $M^{(1)}(X)$ measures the distance (in the Banach-Mazur sense) of the space X from a Banach lattice.

Finally in Section 4 we apply these results to give extensions of some factorization theorems of Pisier [13] to the non-locally convex setting. Pisier showed the existence of a constant $B = B(p)$ so that if X is a Banach space and $T: C(\Omega) \to X$ is bounded satisfying for a suitable constant C and all disjointly supported functions $f_1, \ldots, f_n \in C(\Omega)$

$$\left(\sum_{k=1}^{n} \|Tf_k\|^p\right)^{1/p} \leqq C \max_{1 \leqq k \leqq n} \|f_k\|$$

then there is as probability measure μ on Ω so that for $f \in C(\Omega)$

$$\|Tf\| \leqq BC \|f\|_{L_{p,1}(\mu)}$$

where $L_{p,1}(\mu)$ denotes the Lorentz space $L_{p,1}$ with respect to μ.

Pisier's approach in [13] uses duality and so cannot be used in the case when X is a quasi-Banach space. Nevertheless the result can be extended and we prove that if $0 < r < 1$ there is a constant $B = B(r, p)$ so that if X is r-normable then there exists a probability measure μ so that for all $f \in C(\Omega)$,

$$\|Tf\| \leqq BC \|f\|_{L_{p,r}(\mu)}.$$

We apply these results to show if X is a quasi-Banach space of cotype two then any operator $T: C(\Omega) \to X$ is 2-absolutely summing and so factorizes through a Hilbert space. We conclude by presenting a dual result and make a general conjecture that if X and Y are quasi-Banach spaces such that X^* and Y have cotype two and $T: X \to Y$ is an approximately linear operator then T factorizes through a Hilbert space.

2. Submeasures and supermeasures. Let us define for $0 < p < \infty$,

$$K_p = \frac{2}{(2^p - 1)^{1/p}} - 1.$$

Notice that for p close to 1 we have $K_p \sim 1 - 4(p-1)\log 2$ while for p large we have $K_p \sim p^{-1} 2^{-p}$.

We now state our main result on submeasures with a lower estimate (see Section 1 for the definitions).

Theorem 2.1. *Let $\mathscr{A}$ be an algebra of subsets of Ω. Suppose that ϕ is a normalized submeasure on $\mathscr{A}$, which satisfies a lower p-estimate where $1 < p < \infty$. Then there is a measure λ on $\mathscr{A}$ with $0 \leqq \lambda \leqq \phi$ and $\lambda(\Omega) \geqq K_p$.*

Proof. By an elementary compactness argument we need only prove the result for the case when Ω is finite and $\mathscr{A} = 2^{\Omega}$. We fix such an Ω.

Let γ be the greatest constant such that, whenever ϕ is a normalized submeasure on Ω satisfying a lower p-estimate then there is a measure λ with $0 \leqq \lambda \leqq \phi$ and $\phi(\Omega) \geqq \gamma$. It follows from a simple compactness argument that there is a normalized submeasure ϕ satisfying a lower p-estimate for which this constant is attained; that is if λ is a measure with $0 \leqq \lambda \leqq \phi$ then $\lambda(\Omega) \leqq \gamma$. We choose this ϕ and then pick an optimal measure λ with $0 \leqq \lambda \leqq \phi$ and $\lambda(\Omega) = \gamma$.

Let $\delta = (2^p - 1)^{-1/p}$. Let E be a maximal subset of Ω such that $\lambda(E) \geqq \delta\phi(E)$ and let $F = \Omega \backslash E$. Suppose $A \subset F$; then

$$\lambda(A) + \lambda(E) = \lambda(A \cup E) \leqq \delta\phi(A \cup E) \leqq \delta(\phi(A) + \phi(E)),$$

and so

$$\lambda(A) \leqq \delta\phi(A). \tag{1}$$

Let q be the conjugate index of p, i.e. $p^{-1} + q^{-1} = 1$. Let ν be any measure on $\mathscr{A}$ so that $0 \leqq \nu \leqq \phi$. Suppose $c_1, c_2 \geqq 0$ are such that $c_1^q + c_2^q = 1$. Consider the measure

$$\mu(A) = c_1\lambda(A \cap E) + c_2\nu(A \cap F).$$

Then for any A

$$\begin{aligned} \mu(A) &\leqq (c_1^q + c_2^q)^{1/q}(\lambda(A \cap E)^p + \nu(A \cap F)^p)^{1/p} \\ &\leqq (\phi(A \cap E)^p + \phi(A \cap F)^p)^{1/p} \leqq \phi(A). \end{aligned}$$

Hence $\mu(\Omega) \leqq \gamma$ which translates as

$$c_1\lambda(E) + c_2\nu(F) \leqq \gamma.$$

Taking the supremum over all c_1, c_2, we have:

$$\lambda(E)^p + \nu(F)^p \leqq \gamma^p. \tag{2}$$

Now take $\nu(A) = \delta^{-1}\lambda(A \cap F)$. It follows from (1) that $\nu \leqq \phi$ and hence from (2),

$$\lambda(E)^p + \delta^{-p}\lambda(F)^p \leqq \gamma^p.$$

If we set $t = \lambda(E)/\gamma$ then

$$t^p + (2^p - 1)(1-t)^p \leqq 1$$

and it follows by calculus that $t \geqq \frac{1}{2}$. Hence $\lambda(E) \geqq \gamma/2$.

Now, consider the submeasure $\psi(A) = \phi(A \cap F)$. By hypothesis on γ there exists a positive measure ν on Ω such that $0 \leqq \nu \leqq \psi$ and $\nu(\Omega) \geqq \gamma\psi(\Omega)$. Thus for all A we have $0 \leqq \nu(A) \leqq \phi(A \cap F)$, $\nu(E) = 0$ and $\nu(F) = \nu(\Omega) \geqq \gamma\phi(F)$. Returning to equation (2) we have:

$$\lambda(E)^p + \gamma^p \phi(F)^p \leqq \gamma^p.$$

However $\phi(F) \geqq 1 - \phi(E) \geqq 1 - \delta^{-1}\lambda(E)$. Thus, recalling that $t = \lambda(E)/\gamma$

$$t^p + (1 - \delta^{-1}\gamma t)^p \leqq 1,$$

which simplifies to

$$\gamma \geqq \delta\left(\frac{1-(1-t^p)^{1/p}}{t}\right).$$

Since $t \geqq \frac{1}{2}$ it follows again by calculus arguments that the right-hand side is minimized when $t = \frac{1}{2}$ and then

$$\gamma \geqq 2\delta(1-(1-2^{-p})^{1/p}) = K_p$$

and this completes the proof. □

In almost the same manner, we can prove the dual statement for supermeasures.

Theorem 2.2. *Let $\mathscr{A}$ be an algebra of subsets of a set Ω. Suppose $0 < p < 1$ and that ϕ is a normalized supermeasure on Ω which satisfies an upper p-estimate. Then there is a ϕ-continuous measure λ on $\mathscr{A}$ such that $\lambda \geqq \phi$, $\lambda(\Omega) \leqq K_p$.*

Proof. We first prove the existence of some measure λ with $\lambda \geqq \phi$ and $\lambda(\Omega) \leqq K_p$ without requiring continuity. As in the preceding proof it will suffice to consider the case when Ω is finite and $\mathscr{A} = 2^\Omega$. In this case there is a least constant $\gamma < \infty$ with the property that if ϕ is a normalized supermeasure on Ω then there is a measure $\lambda \geqq \phi$ with $\lambda(\Omega) \leqq \gamma$. We again may choose an extremal ϕ and associated extremal λ for which $\lambda(\Omega) = \gamma$.

Define $\delta = (2^p - 1)^{-1/p} > 1$ we now let E be a maximal subset so that $\lambda(E) \leqq \delta\phi(E)$ and defining $F = \Omega \backslash E$ we obtain in this case that if $A \subset F$ then $\lambda(A) \geqq \delta\phi(A)$.

In this case let q be defined by $\frac{1}{q} = \frac{1}{p} - 1$. Let ν be any measure on $\mathscr{A}$ such that $\nu(A) \geqq \phi(A)$ whenever $A \subset F$. Suppose $c_1, c_2 > 0$ satisfy $c_1^{-q} + c_2^{-q} = 1$. Consider the measure

$$\mu(A) = c_1\lambda(A \cap E) + c_2\nu(A \cap F).$$

Then for any A,

$$\begin{aligned}\phi(A) &\leqq (\phi(A \cap E)^p + \phi(A \cap F)^p)^{1/p}\\ &\leqq (\lambda(A \cap E)^p + \nu(A \cap F)^p)^{1/p}\\ &\leqq (c_1\lambda(A \cap E) + c_2\nu(A \cap F))(c_1^{-q} + c_2^{-q}) = \mu(A).\end{aligned}$$

Hence $\mu(\Omega) \geqq \gamma$ and so

$$c_1\lambda(E) + c_2\nu(F) \geqq \gamma.$$

Minimizing over c_1, c_2 yields

$$\lambda(E)^p + \nu(F)^p \geqq \gamma^p. \tag{3}$$

In particular if we let $v(A) = \delta^{-1}\lambda(A \cap F)$ and set $t = \lambda(E)/\gamma$ we obtain

$$t^p + (2^p - 1)(1 - t)^p \geqq 1.$$

Since in this case $p < 1$ we are led to the conclusion that $t \geqq \frac{1}{2}$.

Next we consider the supermeasure $\psi(A) = \phi(A \cap F)$ and deduce the existence of a measure $v \geqq \psi$ with $v(\Omega) \leqq \gamma\psi(\Omega) = \gamma\phi(F)$. In this case (3) gives that

$$\lambda(E)^p + \gamma^p \phi(F)^p \geqq \gamma^p.$$

Now as $\phi(F) \leqq 1 - \phi(E) \leqq 1 - \delta^{-1}\lambda(E)$ we have:

$$t^p + (1 - \delta^{-1}\gamma t)^p \geqq 1$$

and this again leads by simple calculus to the fact that $\gamma \leqq K_p$. This then completes the proof if we do not require continuity of λ.

Now suppose ϕ is a normalized supermeasure on $\mathscr{A}$ satisfying an upper p-estimate. Let λ be a minimal measure subject to the conditions $\lambda \geqq \phi$ and $\lambda(\Omega) \leqq K_p$. (It follows from an argument based on Zorn's Lemma that such a minimal measure exists). Suppose $\lim_{n \to \infty} \phi(F_n) = 0$. Consider the measures $\lambda_n(A) = \lambda(A \cap E_n)$ where $E_n = \Omega \backslash F_n$. Let $\mathscr{U}$ be any free ultrafilter on the natural numbers and define $\lambda_{\mathscr{U}}(A) = \lim_{\mathscr{U}} \lambda_n(A)$. Clearly $\lambda_{\mathscr{U}} \leqq \lambda$. Now for any A

$$\lambda_n(A) = \lambda(A \cap E_n) \geqq \phi(A \cap E_n) \geqq (\phi(A)^p - \phi(F_n))^{1/p}.$$

Hence $\lambda_{\mathscr{U}} = \lambda$ by minimality. Thus $\lim_{\mathscr{U}} \lambda(F_n) = 0$ for every such ultrafilter and this means $\lim_{n \to \infty} \lambda(F_n) = 0$. □

The following corollary is proved for more general uniformly exhaustive submeasures in [6].

Corollary 2.3. *Let $\mathscr{A}$ be an algebra of subsets of Ω and let ϕ be a submeasure on Ω such that for some constant $c > 0$ and some $q < \infty$, we have:*

$$\phi(A_1 \cup \dots \cup A_n) \geqq c(\phi^q(A_1) + \dots + \phi^q(A_n))^{1/q}$$

whenever $A_1, \dots, A_n$ are disjoint. Then there is a measure μ on $\mathscr{A}$ such that μ and ϕ are equivalent.

Proof. Define ψ by

$$\psi(A) = \sup\left(\sum_{k=1}^{n} \phi^q(A_k)\right)$$

where the supremum is computed over all n and all disjoint $(A_1, \dots, A_n)$ so that $A = \bigcup_{k=1}^{n} A_k$. It is not difficult to show that ψ is a supermeasure satisfying a $1/q$-upper estimate and clearly $c^q\psi \leqq \phi^q \leqq \psi$. By Theorem 2.2 we can pick a measure $\mu \geqq \psi$ which is equivalent to ψ and hence to ϕ. □

3. Convexity in lattices. Let Ω be a compact Hausdorff space and suppose $\mathscr{B}(\Omega)$ denotes the σ-algebra of Borel subsets of Ω. Let $B(\Omega)$ denote the space of all real-valued Borel functions on Ω. An admissible extended-value quasinorm on $B(\Omega)$ is a map $f \to \|f\|_X$, $(B(\Omega) \to [0, \infty])$ such that:

(a) $\|f\|_X \leqq \|g\|_X$ for all $f, g \in B(\Omega)$ with $|f| \leqq |g|$ pointwise.

(b) $\|\alpha f\|_X = |\alpha| \, \|f\|_X$ for $f \in B(\Omega)$, $\alpha \in \mathbb{R}$

(c) There is a constant C so that if $f, g \geqq 0$ have disjoint supports then $\|f+g\|_X \leqq C(\|f\|_X + \|g\|_X)$.
(d) There exists a strictly positive u with $0 < \|u\|_X < \infty$.
(e) If $f_n \geqq 0$ and $f_n \uparrow f$ pointwise, then $\|f_n\|_X \to \|f\|_X$.

The space $X = \{f : \|f\|_X < \infty\}$ is then a quasi-Banach function space on Ω equipped with the quasi-norm $\|f\|_X$ (more precisely one identifies functions f, g such that $\|f - g\|_X = 0$). We say that X is order-continuous if, in addition, we have:

(f) If $f_n \downarrow 0$ pointwise and $\|f_1\|_X < \infty$ then $\|f_n\|_X \downarrow 0$.

Conversely if X is a quasi-Banach lattice which contains no copy of c_0 and has a weak order-unit then standard representation theorems can be applied to represent X as an order-continuous quasi-Banach function space on some compact Hausdorff space Ω in the above sense. More precisely, if u is a weak order-unit then there is a compact Hausdorff space Ω and a lattice embedding $L: C(\Omega) \to X$ so that $L[0, \chi_\Omega] = [0, u]$. Since X contains no copy of c_0 we can use a result of Thomas [16] to represent L is the form

$$Lf = \int_\Omega f \, d\Phi$$

where Φ is regular X-valued Borel measure on Ω. This formula then extends L to all bounded Borel functions. We now define the quasi-Banach function space Y by

$$\|f\|_Y = \sup_n \|L(\min(|f|, n\chi_\Omega)\|_X$$

and it may be verified by standard techniques that L extends to a lattice isomorphism of Y onto X (which is an isometry if we assume that the quasi-norm on X is continuous).

For an arbitrary quasi-Banach function space X and $0 < p < \infty$ we define the p-convexity constant $M^{(p)}(X)$ to be the least constant (possibly infinite) such that for $f_1, \ldots, f_n \in X$

$$\left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_X \leqq M^{(p)} \left(\sum_{i=1}^n \|f_i\|_X^p \right)^{1/p}$$

and we let the p-concavity constant $M_{(p)}(X)$ be the least constant such that

$$M_{(p)} \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_X \geqq \left(\sum_{i=1}^n \|f_i\|_X^p \right)^{1/p}.$$

We also let $M^{(0)}(X)$ be the least constant such that

$$\| \, |f_1 \cdots f_n|^{1/n} \|_X \leqq M^{(0)} \left(\prod_{i=1}^n \|f_i\|_X \right)^{1/n}.$$

X is called p-convex if $M^{(p)}(X) < \infty$ and p-concave if $M_p(X) < \infty$; we will say that X is geometrically convex if $M^{(0)}(X) < \infty$. In [2] X is called L-convex if it is p-convex for some $p > 0$; it follows from [2] and [4] that X is L-convex if and only if it is geometrically convex.

Let us now turn to upper and lower estimates. We say X satisfies a crude upper p-estimate with constant a if for any disjoint $f_1, \dots, f_n$ we have

$$\|f_1 + \cdots + f_n\|_X \leqq a \left(\sum_{i=1}^{n} \|f_i\|_X^p \right)^{1/p}$$

and we say that X satisfies an upper p-estimate if $a = 1$. We say that X satisfies a crude lower q-estimate with constant b if for any disjoint $f_1, \dots, f_n$ we have

$$b \|f_1 + \cdots + f_n\|_X \geqq \left(\sum_{i=1}^{n} \|f_i\|_X^p \right)^{1/p};$$

and X satisfies a lower q-estimate if $b = 1$.

Lemma 3.1. *Suppose $0 < p < q < \infty$. If X is a quasi-Banach function space satisfying a crude upper p-estimate with constant a and a crude lower q-estimate with constant b then there is an equivalent function space quasinorm $\| \ \|_Y$ satisfying an upper p and a lower q-estimate with*

$$\|f\|_X \leqq \|f\|_Y \leqq ab \|f\|_X .$$

Proof. First we define

$$\|f\|_W = \inf \left(\sum_{i=1}^{n} \|f \chi_{A_k}\|_X^p \right)^{1/p}$$

where the infimum is taken over all possible Borel partitions $\{A_1, \dots, A_n\}$ of Ω. It is clear that $\|f\|_W \leqq \|f\|_X \leqq a \|f\|_W$ and it can be verified that W satisfies an upper p-estimate and a crude lower q-estimate with constant b. Next we define

$$\|f\|_V = \sup \left(\sum_{i=1}^{n} \|f \chi_{A_k}\|_W^q \right)^{1/q}$$

and finally set $\|f\|_Y = a \|f\|_V$. We omit the details. $\square$

We now give a simple proof of the result proved in [2] that any quasi-Banach function space which satisfies a lower estimate is L-convex. We recall first that if μ is any Borel measure on Ω then $L_{p,\infty}(\mu)$ is the space of all Borel functions such that

$$\|f\|_{L_{p,\infty}(\mu)} = \sup_{t>0} t (\mu\{|f| > t\})^{1/p} < \infty .$$

Theorem 3.2. *Let X be a p-normable quasi-Banach function space which satisfies a crude lower q-estimate. Then*:

(i) *X is r-convex for $0 < r < p$*
(ii) *There is a measure μ on Ω such that $\|f\|_X = 0$ if and only if $f = 0$ $\mu - a.e.$*

Proof. We may assume by Lemma 3.1 that X has an upper p-estimate and a lower q-estimate. Now suppose $f_1, \dots, f_n \in X_+$ and $\left(\sum_{i=1}^{n} f_i^r \right)^{1/r} = f$. Consider the submeasure $\phi(A) = \|f \chi_A\|_X^p$ for $A \in \mathscr{B}(\Omega)$. This has a lower q/p-estimate and hence there is a Borel measure μ with $\mu(\Omega) \geqq K_{q/p}\, \phi(\Omega)$

and such that $\mu(A) \leqq \|f\chi_A\|^p$ for any Borel set A. Thus for any $g \in B(\Omega)$ (note that μ is supported on the set where f is finite)

$$\|gf^{-1}\|_{L_{p,\infty}(\mu)} \leqq \|g\|_X.$$

Now the space $L_{p,\infty}(\mu)$ is r-convex with $M^{(r)}(L_{p,\infty}(\mu)) \leqq C = C(p,r)$. Thus

$$\|f\|_X = \phi(\Omega)^{1/p} \leqq K_{q/p}^{-1/p} \|\chi_\Omega\|_{L_{p,\infty}(\mu)}$$

$$\leqq C K_{q/p}^{-1/p} \left(\sum_{i=1}^n \|f_i f^{-1}\|^r_{L_{p,\infty}(\mu)} \right)^{1/r} \leqq C' \left(\sum_{i=1}^n \|f_i\|^r_X \right)^{1/r}$$

where $C' = C'(p,q,r)$. For (ii) let u be a strictly positive function with $0 < \|u\|_X < \infty$ and define $\phi(A) = \|u\chi_A\|^p$; then by Corollary 2.3 there is a measure μ equivalent to ϕ and the conclusion follows quickly. □

Now suppose μ is any (finite) Borel measure on Ω. We define the Lorentz space $L_{p,q}$ for $0 < p, q < \infty$ by

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \frac{q}{p} t^{q/p-1} f^*(t)^q \, dt \right)^{1/q}.$$

Here f^* is the decreasing rearrangement of $|f|$ i.e. $f^*(t) = \inf_{\mu(E) \leqq t} \sup_{\omega \notin E} |f(\omega)|$. It can easily seen by integration by parts that

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty q t^{q-1} \mu(|f| > t)^{q/p} \, dt \right)^{1/q}.$$

It is then clear that if $p \leqq q$ then $L_{p,q}$ satisfies an upper p and a lower q-estimate. If $p > q$ then $L_{p,q}$ has an upper q and a lower p-estimate.

Suppose $p < q$. We define the functional

$$\|f\|_{A_{p,q}} = \sup \left(\sum_{i=1}^n \left(\inf_{\omega \in A_i} |f(\omega)| \right)^q \mu(A_i)^{q/p} \right)^{1/q} \tag{4}$$

where the supremum is taken over all Borel partitions $\{A_1, \ldots, A_n\}$ of Ω.

Proposition 3.3. *Suppose* $0 < p < q$. *Then*:

(i) *The* $A(p,q)$*-quasi-norm is the smallest admissible quasi-norm which satisfies a lower* q*-estimate and such that* $\|\chi_A\| \geqq \mu(A)^{1/p}$ *for any Borel set.*

(ii) *If* $f \in B(\Omega)$ *and* f^* *is the decreasing rearrangement of* $|f|$ *on* $[0, \infty)$ *then*

$$\|f\|_{A_{p,q}} = \sup_{\mathcal{T}} \left(\sum_{j=1}^n f^*(\tau_j)^q (\tau_j - \tau_{j-1})^{q/p} \right)^{1/q} \tag{5}$$

where $\mathcal{T} = \{\tau_0 = 0 < \tau_1 < \ldots < \tau_n\}$ *runs through all possible finite subsets of* $\mathbb{R}$.

(iii) *If* $f \in B(\Omega)$ *then*

$$\|f\|_{A_{p,q}} \leqq \|f\|_{L_{p,q}} \leqq ((1+\theta)^{2(1+\theta)} \theta^{-\theta})^{1/q} \|f\|_{A_{p,q}}$$

where $1 + \theta = \dfrac{q}{p}$.

Proof. (i) is clear from the definition.

(ii) Suppose $f \in B(\Omega)$ and let $\{A_1, \dots, A_n\}$ be any Borel partition of Ω. Suppose that $1 \leqq j, k \leqq n$. Suppose $\inf_{A_j} |f| \leqq \inf_{A_k} |f|$. Then it is easy to verify that if we let $A'_k = \{\omega \in A_j \cup A_k : |f(\omega)| \geqq \inf_{A_k} |f|\}$ and let $A'_j = (A_j \cup A_k) \setminus A'_k$ then the partition obtained by replacing A_j, A_k by A'_j, A'_k increases the right-hand side of (4). In particular it follows that (5) defines the $\Lambda_{p,q}$ quasinorm when μ is nonatomic. Further if f^* is constant on an interval $[\alpha, \beta]$ it suffices to consider $\mathcal{T}$ where no τ_j lies in (α, β) and this yields the conclusion for general μ.

(iii) The first inequality in (iii) is immediate from (i) since $L_{p,q}$ satisfies a lower q-estimate. For the right-hand inequality we observe that if $h = 1 + \theta = q/p$:

$$\begin{aligned} \|f\|_{L_{p,q}} &= \left(\int_0^\infty \frac{q}{p} t^{q/p-1} (f^*(t))^q \, dt \right)^{1/q} \\ &\leqq \left(\sum_{n=-\infty}^{\infty} \frac{q}{p} (h^{n+1} - h^n) h^{(n+1)(q/p-1)} f^*(h^n)^q \right)^{1/q} \\ &= \left(\sum_{n=-\infty}^{\infty} h(h-1) h^{q/p-1} h^{q(n+1)/p} f^*(h^{n+1})^q \right)^{1/q} \\ &\leqq h^{\frac{2}{p}} (h-1)^{\left(\frac{1}{q} - \frac{1}{p}\right)} \|f\|_{\Lambda_{p,q}}. \end{aligned}$$

The result then follows. □

Under the hypothesis $p > q$ we define $\Lambda_{p,q}$ by

$$\|f\|_{\Lambda_{p,q}} = \inf \left(\sum_{i=1}^{n} \left(\sup_{\omega \in A_i} |f(\omega)|^q \right) \mu(A_i)^{q/p} \right)^{1/q}$$

where the infimum is again computed over all Borel partitions of Ω. Proposition 3.4 now has an analogue whose proof is very similar and we omit most of the details.

Proposition 3.4. *Suppose $0 < q < p$. Then*:

(i) *The $\Lambda(p, q)$-quasinorm is the largest admissible quasi-norm which satisfies an upper q-estimate and such that $\|\chi_A\| \leqq \mu(A)^{1/p}$, for any Borel set A.*

(ii) *If $f \in B(\Omega)$ then*

$$\|f\|_{\Lambda_{p,q}} = \inf_{\mathcal{T}} \left(\sum_{j=1}^{n} f^*(\tau_{j-1})^q (\tau_j - \tau_{j-1})^{q/p} \right)^{1/q}$$

where $\mathcal{T} = \{\tau_0 = 0 < \tau_1 < \dots < \tau_n = \mu(\Omega)\}$ runs through all possible finite subsets of $[0, \mu(\Omega)]$.

(iii) *If $f \in B(\Omega)$ then*

$$\|f\|_{L_{p,q}} \geqq \|f\|_{\Lambda_{p,q}} \geqq ((1+\theta)^{-2-\theta} \theta^\theta)^{1/p} \|f\|_{L_{p,q}}$$

where $\theta = \dfrac{p}{q} - 1$.

Proof. We will only proof the second inequality in (iii). We define h = $1+\theta = p/q$.

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \frac{q}{p} t^{q/p-1} f^*(t)^q \, dt\right)^{1/q}$$
$$\geqq \left(\sum_{n=-\infty}^{\infty} \frac{q}{p}(h^{n+1}-h^n) h^{n(q/p-1)} f^*(h^{(n+1)})^q\right)^{1/q}$$
$$= \left(\sum_{n=-\infty}^{\infty} h^{-1}(h-1) h^{(n-1)q/p} f^*(h^n)^q\right)^{1/q}$$
$$\geqq (h-1)^{\left(\frac{1}{q}-\frac{1}{p}\right)} h^{-1/p-1/q} \|f\|_{A_{p,q}}.$$

The result then follows. □

We now immediately deduce the following:

Proposition 3.5. *Let X be quasi-Banach function space on Ω satisfying a crude upper p-estimate with constant a and a crude lower q-estimate with constant b. Then if $f \in X_+$ with $\|f\|_X = 1$:*
(i) *There is a probability measure μ on Ω such that $f > 0$ μ – a.e. and if $g \in X$,*

$$\|g f^{-1}\|_{A_{p,q}(\mu)} \leqq ab K_{q/p}^{-1/p} \|g\|_X.$$

(ii) *There is a probability measure λ on Ω such that $f > 0$ λ – a.e. and if $g \in X$*

$$\|g\|_X \leqq ab K_{p/q}^{1/q} \|g f^{-1}\|_{A_{q,p}}(\lambda).$$

Proof. We first introduce an equivalent quasinorm $\| \ \|_Y$ with an exact upper p and lower q-estimate as in Lemma 3.1 so that $\|g\|_X \leqq \|g\|_Y \leqq ab \, \|g\|_X$ for all g.

(i) As in Theorem 3.2 we consider the submeasure $\phi(A) = \|f\chi_A\|_Y^p$. There is a probability measure μ such that

$$0 \leqq \mu(A) \leqq K_{q/p}^{-1} \frac{\phi(A)}{\phi(\Omega)}$$

for all Borel sets A. Then for $g \in X$, and any Borel partition $\{A_1, \ldots, A_n\}$ of Ω,

$$\left(\sum_{i=1}^{n} \left(\inf_{A_i} |g f^{-1}|^q\right) \mu(A_i)^{q/p}\right)^{1/q}$$
$$\leqq K_{q/p}^{-1/p} \phi(\Omega)^{-1/p} \left(\sum_{i=1}^{n} \left(\inf_{A_i} |g f^{-1}|^q\right) \|f\chi_A\|_Y^q\right)^{1/q}$$
$$\leqq K_{q/p}^{-1/p} \|f\|_Y^{-1} \|g\|_Y \leqq K_{q/p}^{-1/p} \, ab \, \|g\|_X.$$

Thus (i) follows. The proof of (ii) is very similar. In this case we consider the supermeasure $\phi(A) = \|f\chi_A\|_Y^q$. There is a probability measure λ on Ω such that

$$0 \leqq K_{p/q}^{-1} \frac{\phi(A)}{\phi(\Omega)} \leqq \lambda$$

for all Borel sets A. Thus for $g \in X$, and any Borel partition $\{A_1, \ldots, A_n\}$ of Ω,

$$\begin{aligned}\|g\|_X \leqq \|g\|_Y &\leqq \left(\sum_{i=1}^{n}\left(\sup_{A_i}|gf^{-1}|^p\right)\|f\chi_{A_i}\|_Y^p\right)^{1/p}\\ &\leqq \left(\sum_{i=1}^{n}\left(\sup_{A_i}|gf^{-1}|^p\right)\phi(A_i)^{p/q}\right)^{1/p}\\ &\leqq K_{p/q}^{1/q}\|f\|_Y\left(\sum_{i=1}^{n}\left(\sup_{A_i}|gf^{-1}|^p\right)\lambda(A_i)^{p/q}\right)^{1/p}\end{aligned}$$

and the result follows. □

Theorem 3.6. *Suppose $0 < r < p < \infty$. Then there is a constant $c = c(r, p)$ such that if X is a quasi-Banach function space satisfying an upper p-estimate and a lower q-estimate where $q/p = 1 + \theta < 2$ then $\log M^{(r)}(X) \leqq \theta\left(c + \frac{1}{p}|\log\theta|\right)$.*

Proof. We use Proposition 3.5. Suppose $f_1, \ldots, f_n$ are nonnegative functions in X with $\|f\|_X = 1$ where $f = \left(\sum_{i=1}^{n} f_i^r\right)^{1/r}$. Then there is a probability measure μ on Ω with $f > 0$ a.e. and such that if $g \in X$

$$\|gf^{-1}\|_{\Lambda_{p,q}} \leqq K_{q/p}^{-1/p}\|g\|_X.$$

Notice that $K_{q/p}^{-1} \leqq e^{c_1\theta}$ for some c_1. We also have $q/p \leqq e^{\theta}$ and $(q-r)/(p-r) \leqq c_2\theta$ where c_2 depends only on p, r. Let $w_i = f_i f^{-1}$.

We note first that for any $w \geqq 0$ in $B(\Omega)$ we have

$$\begin{aligned}\int w^r\,d\mu &= \int_0^1 w^*(t)^r\,dt\\ &\leqq \left(\frac{q}{p}\right)^{-r/q}\|w\|_{L_{p,q}}^r\left(\int_0^1 t^{-\frac{r(q-p)}{p(q-r)}}\,dt\right)^{1-r/q}\\ &\leqq \frac{p}{q}\left(\frac{q-r}{p-r}\right)^{1-r/q}\|w\|_{L_{p,q}}^r \leqq e^{c_3\theta}\|w\|_{L_{p,q}}^r\end{aligned}$$

where $c_3 = c_3(r, p)$. Thus

$$\int w^r\,d\mu \leqq \theta^{-r\theta/q}e^{c_4\theta}\|w\|_{\Lambda_{p,q}}^r$$

where $c_4 = c_4(r, p)$. Applying this to the w_i and summing we have

$$1 \leqq \theta^{-r\theta/p}e^{c_5\theta}\sum_{i=1}^{n}\|f_i\|_X^r,$$

with c_5 depending only on r, p. The result now follows. □

Example. We show that the estimate in the previous theorem is essentially best possible. For convenience we consider the case $p = 1$ and $q = 1 + \theta$. The conclusion of the theorem is that, for $0 < r < p$, $M^{(r)}(X) \leqq \exp(F_r(\theta))$ where

$$\lim_{\theta\to 0}\frac{F_r(\theta)}{\theta|\log\theta|} = 1.$$

Since $M^{(0)}(X) \leqq M^{(r)}(X)$ for all $r > 0$ a similar conclusion is attained in the case $r = 0$. We establish a converse by considering only the case $r = 0$.

For $\theta > 0$ we let $X = X_\theta = \Lambda_{1,q}[0, 1]$ and we let $\kappa(\theta) = M^{(0)}(X)$. We will set $\phi(\theta) = \exp(-|\log\theta|^{1/2})$ so that $\lim_{\theta\to 0} \phi(\theta) = 0$. We further set $\psi(\theta) = (2^{1/q} - 1)^{-2}$; then $\psi(\theta) = 1 + 4\theta\log 2 + O(\theta^2)$.

We will consider the function $f = f_\theta \in X$ defined by $f(t) = t^{-1}\chi_{[1-\phi, 1]}$. It follows easily from the definition of $M^{(0)}(X)$ that

$$\exp\left(-\frac{1}{\phi}\int_{1-\phi}^{1} \log t\, dt\right) \|\chi_{[1-\phi, 1]}\|_X \leqq \kappa \|f\|_X.$$

To obtain this one derives the integral version of geometric convexity and applies it to suitable rotations of f. Thus if $\beta(\theta) = 1 + (1 - \phi)\phi^{-1}\log(1 - \phi)$ then

$$e^\beta \phi \leqq \kappa \|f\|_X.$$

Turning to the estimation of $\|f\|_X$ we note that $\|f\|_X^q$ is the supremum of expressions of the form

$$\sum_{j=1}^{n} (\tau_j - \tau_{j-1})^q \tau_j^{-q}$$

where $1 - \phi = \tau_0 < \tau_1 < \ldots < \tau_n = 1$. Now if $\tau_j > \psi\tau_{j-1}$ it can be checked that this expression is increased by interpolating $(\tau_j\tau_{j-1})^{1/2}$ into the partition. We there fore may suppose that we consider only partitions where $\tau_j \leqq \psi\tau_{j-1}$. In this case we estimate:

$$\begin{aligned}(\tau_j - \tau_{j-1})^q \tau_j^{-q} &\leqq (\tau_j - \tau_{j-1})(\tau_j - \tau_{j-1})^\theta \tau_{j-1}^{-1+\theta}\\ &\leqq (\psi - 1)^\theta (\tau_j - \tau_{j-1})\tau_{j-1}^{-1}\end{aligned}$$

and after summing we get the estimate

$$\|f\|_X^q \leqq (\psi - 1)^\theta |\log(1 - \phi)|.$$

Thus

$$\kappa \geqq e^\beta \phi(\psi - 1)^{-\theta/q} |\log(1 - \phi)|^{-1/q}.$$

Now for small θ we can estimate $|\log(1 - \phi)| \leqq (1 + \phi)\phi$. Thus

$$\kappa^q \geqq e^{q\beta}\phi^\theta(\psi - 1)^{-\theta}(1 + \phi)^{-1}$$

so that

$$\liminf_{\theta\to 0} \frac{\log\kappa}{\theta|\log\theta|} \geqq \liminf_{\theta\to 0} \frac{(\log\phi - \log(\psi - 1))}{|\log\theta|} \geqq 1.$$

Thus we conclude from this calculation and the theorem that

$$\log\kappa(\theta) = -\theta\log\theta + o(\theta|\log\theta|)$$

as $\theta \to 0$. □

4. The factorization theorems of Pisier. We next show how the results of Proposition 3 quickly give extensions of some factorization theorems due to Pisier [13]. Our approach is valid for quasi-Banach spaces since it does not depend on any duality.

Theorem 4.1. *Suppose $0 < r \leqq p < q < \infty$. Suppose Ω is a compact Hausdorff space and that Y is an r-Banach space. Suppose $T: C(\Omega) \to Y$ is a bounded linear operator.*

Then the following conditions on T are equivalent:

(i) *There is a constant C_1 so that for any $f_1, \ldots, f_n$ in $C(\Omega)$ with disjoint support we have*

$$\left(\sum_{i=1}^{n} \|Tf_i\|^q\right)^{1/q} \leqq C_1 \max_{1 \leqq i \leqq n} \|f_i\|.$$

(ii) *There is a constant C_2 so that for any $f_1, \ldots, f_n$ we have*

$$\left(\sum_{i=1}^{n} \|Tf_i\|^q\right)^{1/q} \leqq C_2 \left\|\left(\sum_{i=1}^{n} |f_i|^p\right)^{1/q}\right\|.$$

(iii) *There is a constant C_3 and a probability measure μ on Ω such that for all $f \in C(\Omega)$,*

$$\|Tf\| \leqq C_3 \|f\|^{1-p/q} \left(\int |f|^p \, d\mu\right)^{1/q}.$$

(iv) *There is a constant C_4 and a probability measure μ on Ω such that* for *all $f \in C(\Omega)$,*

$$\|Tf\| \leqq C_4 \|f\|_{L_{q,r}(\mu)}.$$

Proof. Some implications are essentially trivial. Thus (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i); (iii) $\Rightarrow$ (ii) is easy and we omit it. It remains to show (i) $\Rightarrow$ (iv).

To do this we first notice that if f_n is a sequence of disjointly supported functions in $C(\Omega)$ then $Tf_n \to 0$. Thus the theorem of Thomas [16] already cited allows us to find a regular Y-valued Borel measure Φ on Ω so that

$$Tf = \int f \, d\Phi$$

and use this formula to extend T to the bounded Borel functions $B_\infty(\Omega)$. It is easy to verify the condition (i) remains in effect for disjointly supported bounded Borel functions.

Now we introduce a quasi-Banach function space Z by defining

$$\|f\|_Z = \sup\left(\sum_{i=1}^{n} \|Tg\|^q\right)^{1/q},$$

where the supremum is over disjoint $g_i \in B_\infty(\Omega)$ with $|g_i| \leqq |f|$. It is immediate that $\| \, \|_Z$ satisfies an upper r-estimate and a lower q-estimate. Also, we see that $\chi_\Omega \in Z$. Thus by Proposition 3.5 we can find a probability measure μ on Ω so that for some C, C_4

$$\|g\|_Z \leqq C \|g\|_{\Lambda_{q,r}(\mu)} \leqq C_4 \|g\|_{L_{q,r}(\mu)}. \quad \square$$

We now prove the companion factorization result for operators on L_p-spaces.

Theorem 4.2. *Suppose $0 < r \leqq q < s < \infty$. Let (Ω, μ) be a σ-finite measure space. Let Y be an r-Banach space, and let $T: L_s(\mu) \to Y$ be a bounded linear operator. Then the following conditions are equivalent:*

(i) *There is a constant C_1 so that for any disjoint $f_1, \ldots, f_n$ in $L_s(\mu)$ we have:*

$$\left(\sum_{i=1}^{n} \|Tf_i\|^q\right)^{1/q} \leqq C_1 \left\|\sum_{i=1}^{n} |f_i|\right\|_{L_s(\mu)}.$$

(ii) *There is a constant C_2 and a probability measure λ on Ω so that for any $f \in L_s(\mu)$ and any Borel set E we have*

$$\| T(f\chi_E)\| \leqq C_2 \|f\|_{L_s(\mu)} \lambda(E)^{1/q-1/s}.$$

(iii) *There is a constant C_3 and a probability measure λ on Ω so that for any $f \in L_s(\mu)$ and $g \in B(\Omega)$ with $|g| \leqq 1$,*

$$\| T(fg)\| \leqq C_3 \|f\|_{L_s(\mu)} \|g\|_{L_{t,r}(\lambda)}$$

where $1/t = 1/q - 1/s$.

(iv) *There exists $w \in L_1(\mu)$ with $w \geqq 0$ and $\int w\,d\mu = 1$ and a constant C_4 such that for all $f \in L_s(\mu)$*

$$\| Tf \| \leqq C_4 \|f w^{-1/s}\|_{L_{q,r}(w\mu)}.$$

Proof. We omit the simple proof that (ii) $\Rightarrow$ (i). Also (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii) are obvious. We first consider (i) $\Rightarrow$ (iii). For this direction we define a quasi-Banach function space Z by

$$\|g\|_Z = \sup \left(\sum_{i=1}^{n} \| T(f_i g)\|^q \right)^{1/q},$$

where the supremum is over disjoint $f_i \in L_s(\mu)$ with $\left\| \sum_{i=1}^{n} |f_i| \right\|_s \leqq 1$ and $f_i g \in L_s(\mu)$. It is clear that Z satisfies an upper r-estimate. We show that Z satisfies a lower t-estimate, where $1/t = 1/q - 1/s$. Let us suppose that we have disjoint $g_1, \dots, g_m \in Z$ such that

$$\sum_{k=1}^{m} \|g_k\|_Z^t > 1,$$

and let $g = g_1 + \cdots + g_m$. Then there are f_{ik} with $f_{ik}(\omega) = 0$ whenever $g_k(\omega) = 0$, and such that

$$\sum_{k=1}^{m} \left(\sum_{i=1}^{n} \| T(f_{ik} g_k)\|^q \right)^{t/q} = 1$$

and

$$\left\| \sum_{i=1}^{n} |f_{ik}| \right\|_s \leqq 1.$$

We let

$$\alpha_k = \left(\sum_{i=1}^{n} \| T(f_{ik} g_k)\|^q \right)^{t/qs}$$

and $f'_{ik} = \alpha_k f_{ik}$. Then we have that

$$\sum_{k=1}^{m} \sum_{i=1}^{n} \| T(f'_{ik} g)\|^q \geqq 1,$$

and

$$\left\| \sum_{k=1}^{m} \sum_{i=1}^{n} |f'_{ik}| \right\|_s^s \leqq \sum_{k=1}^{m} \alpha_k^s \leqq 1,$$

that is, $\|g\|_Z \geqq 1$.

We also notice that $\chi_\Omega \in Z$. Thus by Propositions 3.4 and 3.5, there is a probability measure λ so that for some C_3 and all bounded g

$$\|g\|_Z \leqq C_3 \|g\|_{L_{t,r}(\lambda)}.$$

Now we show that (ii) $\Rightarrow$ (iv). We notice that λ is μ-continuous, and so by the Radon-Nikodym theorem, we can find $w = d\lambda/d\mu$. Then for any measurable set A we have:

$$\| T(\chi_A w^{1/s}) \| \leqq C_2 \, \| \chi_A w^{1/s} \|_{L_s(\mu)} \lambda(A)^{1/q - 1/s} = C_2 \, \lambda(A)^{1/q}.$$

Now define a function space W by

$$\| f \|_W = \sup_{|g| \leqq |f|, g \in B_\infty} \| T(g w^{1/s}) \|.$$

Then W has an upper r-estimate and

$$\| \chi_A \|_W \leqq C_2 \, \lambda(A)^{1/q}.$$

Hence by Proposition 3.4

$$\| f \|_W \leqq C_4 \, \| f \|_{L_{q,r}(\lambda)}$$

for some C_4. □

To conclude the paper let us observe that Theorem 4.1 can be used to extend other factorization results to non-locally convex spaces. Let us first recall that an operator $T: X \to Y$ where X is a Banach space and Y is a quasi-Banach space is called 2-absolutely summing if there is a constant C so that for $x_1, \ldots, x_n \in X$ we have

$$\left(\sum_{i=1}^{n} \| T x_i \|^2 \right)^{1/2} \leqq C \max_{\| x^* \| \leqq 1} \left(\sum_{i=1}^{n} | x^*(x_i) |^2 \right)^{1/2}.$$

A quasi-Banach space X is of cotype p if there is a constant C so that if $x_1, \ldots, x_n \in X$ then

$$\mathrm{Ave}_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leqq C \left(\sum_{i=1}^{n} \| x_i \|^p \right)^{1/p}.$$

For the next theorem for Banach spaces see [12], p. 62. In the following more general formulation, we understand from the referee that it has apparently been known for some time to Maurey and Pisier, with a somewhat different proof based on extrapolation.

Theorem 4.3. *Suppose Ω is a compact Hausdorff space and Y is a quasi-Banach space with cotype two. Suppose $T: C(\Omega) \to Y$ is a bounded operator. Then T is 2-absolutely summing and hence there is a probability measure μ on Ω and a constant C so that $\| Tf \| \leqq C \, \| f \|_{L_2(\mu)}$ for $f \in C(\Omega)$.*

Proof. We may assume that X is an r-Banach space where $0 < r < 1$. We first note that if $f_1, \ldots, f_n \in X$ then since X has cotype two,

$$\begin{aligned} \left(\sum_{i=1}^{n} \| T f_i \|^2 \right)^{1/2} &\leqq C_1 \, \mathrm{Ave}_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i T f_i \right\| \\ &\leqq C_1 \, \| T \| \, \mathrm{Ave}_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i f_i \right\| \\ &\leqq C_1 \, \| T \| \left\| \sum_{i=1}^{n} | f_i | \right\| \end{aligned}$$

where C_1 depends on the cotype two constant of X. Applying Theorem 4.1, we see that there is a probability measure ν on Ω and a constant C_2 so that $\| Tf \| \leqq C_2 \, \| f \|_{L_{2,r}(\nu)}$ for $f \in C(\Omega)$. In particular it follows that $\|Tf\| \leqq C_3 \|f\|_{L_4(\nu)}$. From this we conclude that if $f_1, \ldots, f_n \in C(\Omega)$, using Khintchine's inequality,

$$\begin{aligned}\left(\sum_{i=1}^n \|Tf_i\|^2\right)^{1/2} &\leqq C_1 \operatorname{Ave}_{\varepsilon_i = \pm 1} \left\|\sum_{i=1}^n \varepsilon_i T f_i\right\| \\ &\leqq C_4 \operatorname{Ave}_{\varepsilon_i = \pm 1} \left\|\sum_{i=1}^n \varepsilon_i f_i\right\|_{L_4(\nu)} \\ &\leqq C_5 \left\|\left(\sum_{i=1}^n |f_i|^2\right)^{1/2}\right\|_{L_4(\nu)} \\ &\leqq C_5 \left\|\left(\sum_{i=1}^n |f_i|^2\right)^{1/2}\right\|_{C(\Omega)}\end{aligned}$$

so that T is 2-absolutely summing. It now follows from the Grothendieck-Pietsch Factorization Theorem (which applies to non-locally convex X) that there is a probability measure μ on Ω satisfying the conclusions of the Theorem. □

Remarks. We conclude with some comments on Theorem 4.3. We remark first that the theorem, taking $X = L_1$, gives a circuitous proof of Grothendieck's inequality, which is equivalent to the assertion that every operator from $C(\Omega)$ to L_1 is 2-absolutely summing.

We also note that there are, by now, many known non-locally convex spaces of cotype two. The most natural examples are the spaces L_p when $0 < p < 1$; in this case, Theorem 4.3 is known, and is a consequence of work of Maurey [8] (see [17], p. 271). However, more recently Pisier [14] has shown that the spaces L_p/H_p have cotype two when $p < 1$, and indeed essentially establishes Theorem 4.3 for this space. It also follows from work of Xu [18] that the Schatten ideals S_p have cotype two when $p < 1$, and for these examples Theorem 4.3 is apparently new.

We conclude by noting a dual result and then make a conjecture based on these observations. First let us recall that if X is a quasi-Banach space then its dual X^* defined in the usual way is a Banach space; here X^* need not separate the points of X and may indeed reduce to zero. We define the canonical map (not necessarily injective) $j: X \to X^{**}$. The Banach envelope of X is the closure $\hat{X}$ of $j(X)$.

We shall say that an operator $T: X \to Y$ is *strongly approximable* if T is in the smallest subspace $\mathscr{A}$ of $\mathscr{L}(X, Y)$ containing the finite-rank operators and closed under the pointwise convergence of uniformly bounded nets.

The following theorem is essentially known.

Theorem 4.4. *Suppose (Ω, μ) is a σ-finite measure space. Let X be a quasi-Banach space such that X^* has cotype $q < \infty$. Then, for $0 < p < 1$, there is a constant $C = C(p, X)$ so that if $T: X \to L_p(\mu)$ is a strongly approximable operator then there exists $w \geqq 0$ with $\int w^s \, d\mu \leqq C$, where $1/s = 1/p - 1$, and such that $\|w^{-1} T x\|_{L_1(\mu)} \leqq C \|T\| \|x\|$ for $x \in X$.*

Remark. If X is a Banach space, then this theorem is due to Mezrag [9], [10] with no approximability assumptions on T. If X is not locally convex this result is essentially proved in [3] and we here show how to obtain the actual statement from the equivalent Theorem 2.2 of [3]. Note also that for spaces X with trivial dual, the theorem holds vacuously since the only strongly approximable operators are identically zero.

Proof. It is shown in Theorem 2.2 of [3] that given $\varepsilon > 0$ there exists $C_0(\varepsilon)$ so that, for any probability measure ν, if $T: X \to L_p(\nu)$ is strongly approximable then there exists a Borel subset E of Ω so that $\nu(E) > 1 - \varepsilon$ and $\int_E |Tx| \, d\mu \leqq C_0 \|T\| \|x\|$ for $x \in X$. Let us fix $\varepsilon = 1/2$ and let $C_0 = C_0(1/2)$. Suppose $x_1, \ldots, x_n \in X$ and let $f = \sum_{i=1}^n |Tx_i|$. If $\|f\|_p > 0$, define $\nu = \|f\|_p^{-p} |f|^p \mu$.

Consider the operator $S: X \to L_p(\nu)$ defined by $Sx = |f|^{-1} Tx$ (set $Sx(\omega) = 0$ when $f(\omega) = 0$.) Then $\|S\| \leqq \|f\|_p^{-1} \|T\|$. Thus there is a Borel subset E of Ω with $\nu(E) > 1/2$ and so that $\int_E |Sx| \, d\nu \leqq C_0 \|f\|_p^{-1} \|T\| \|x\|$ for $x \in X$. Now

$$\frac{1}{2} \leqq \nu(E) = \int_E \sum_{i=1}^n |Sx_i| \, d\nu$$
$$\leqq C_0 \|f\|_p^{-1} \|T\| \sum_{i=1}^n \|x_i\| .$$

and so we obtain an inequality

$$\left(\int \left(\sum_{i=1}^n |Tx_i| \right)^p d\mu \right)^{1/p} \leqq 2 C_0 \|T\| \sum_{i=1}^n \|x_i\|,$$

where C_1 depends only on p, X. Now by the factorization results of Maurey [8] (see [17] p. 264) we obtain the desired conclusion. □

Theorem 4.5. *Let X be a quasi-Banach space such that X^* has cotype two. Suppose Ω is a compact Hausdorff space and μ is a σ-finite measure on Ω. Then for $0 < p < 1$, there is a constant $C = C(p, X)$ so that if $T: X \to L_p(\mu)$ is a strongly approximable bounded operator, then there exists $v \geqq 0$, with $\int v^t \, d\mu \leqq C$ where $1/t = 1/p - 1/2$, and such that $\|v^{-1} Tx\|_{L_2(\mu)} \leqq C \|T\| \|x\|$ for $x \in X$.*

Proof. By Theorem 4.4 we can find $w \geqq 0$ with $\int w^s \, d\mu \leqq C_0$ such that $\|w^{-1} Tx\|_{L_1} \leqq C_0 \|T\| \|x\|$. Now since L_1 is a Banach space we have $\|wTx\|_{L_1} \leqq C_0 \|T\| \|x\|_{\hat{X}}$ so that $wT = Sj$ where $S: \hat{X} \to L_1$ is bounded with $\|S\| \leqq C_0 \|T\|$. Since $\hat{X}^* = X^*$ has cotype two and L_1 has cotype two we can factor S through a Hilbert space [11] and then apply Maurey's factorization results to obtain $u \geqq 0$ with $\int u^2 \, d\mu \leqq C_1$ and such that $\|u^{-1} Sx\|_{L_2} \leqq C_1 \|T\| \|x\|$ for $x \in X$. Letting $v = uw$ completes the proof. □

Remarks. We discuss a question motivated by Theorems 4.3 and 4.5. An operator $T: X \to Y$ is called approximable if given any compact set $K \subset X$ and any $\varepsilon > 0$ there exists a finite-rank operator $F: X \to F$ with $\|Tx - Fx\| < \varepsilon$ for $x \in K$. Pisier has shown that if X, Y are Banach spaces such that X^* and Y have cotype two and $T: X \to Y$ is an approximable linear operator then T factorizes through a Hilbert space (see [11], [12]). Does the same result hold if we assume X, Y are quasi-Banach spaces? Theorem 4.3 and 4.5 provide evidence that this perhaps is true.

Note added 17 December, 1992. After the initial preparation of this note, the first author and Sik-Chung Tam showed that the conjecture in the last paragraph is true, at least for strongly approximable operators.

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