Arch. Math., Vol. 50, 538–549 (1988)

0003-889X/88/5006-0538 $ 3.90/0

Harmonic functions in non-locally convex spaces

By

N. J. KALTON *)

1. Introduction. Let Ω be an open subset of $\mathbb{C}$ and let X be a complex quasi-Banach space. A function $f: \Omega \to X$ is called *analytic* if given $z_0 \in \Omega$ there exists $\delta > 0$ so that if $|z - z_0| < \delta$, $f(z)$ can be expanded in a power series

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n$$

where $x_n \in X$. If $\Omega = \Delta$ the open unit disk it follows from results of Turpin [7] that f can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \qquad |z| < 1.$$

The general properties of analytic functions were studied in [4]. Unfortunately, in the general non-locally convex setting there exist examples of analytic functions on Δ which extend continuously to $\bar{\Delta} = \Delta \cup \mathbb{T}$ and vanish on $\mathbb{T}$. However, in certain spaces X, such as L_p when $0 < p < 1$, a form of the maximum modulus principle holds i.e. for some C

$$\| f(z) \| \leqq C \max_{|w|=1} \| f(w) \|$$

whenever $f: \bar{\Delta} \to X$ is continuous and analytic on Δ. Such spaces are characterized in [5] and are termed A-convex.

At the other extreme one may ask at what rate one can have $f(z) \to 0$ as $|z| \to 1$. In [3] it is shown that if X is p-normable and

$$\| f(z) \| = o(1 - |z|)^{\frac{1}{p} - 1}$$

then $f \equiv 0$, while easy examples show that one can have f non-trivial and

$$\| f(z) \| = O(1 - |z|)^{\frac{1}{p} - 1}.$$

Let us say a map $h: \Delta \to X$ is *harmonic* if $h(z) = f(z) + g(\bar{z})$ where f, g are both analytic. One may ask similar questions for harmonic functions. It is, in fact, trivial to show that a maximum modulus principle for harmonic functions will imply X is locally convex, so we shall concentrate on the rate of decay of h.

*) Research supported by NSF grant DMS-8301099.

Our main results show that if X is p-normable an harmonic function can still decay no faster than $(1-|z|)^{1/p-1}$. Precisely we show that if $\beta > \frac{1}{p} - 1$ and

$$\|h(z)\| = O(1-|z|)^{\beta}$$

then $h \equiv 0$, while for $\frac{1}{2} < p < 1$ we prove that if

$$\|h(z)\| = o(1-|z|)^{\frac{1}{p}-1}$$

then $h \equiv 0$. In all probability this stronger conclusion holds for $p \leqq \frac{1}{2}$.

For $\frac{1}{2} < p < 1$ the maximal rate of decay can be achieved in L_p (which contrasts very strongly with the analytic case since L_p is A-convex). The example is simply the Poisson kernel. However $p = \frac{1}{2}$ is a natural barrier since we also show that if X is *A-convex* and $h\colon \Delta \to X$ is harmonic with

$$\|h(z)\| = O(1-|z|)$$

then $h \equiv 0$. Thus the maximal rate of decay can be achieved in an A-convex p-normable space only when $p \geqq \frac{1}{2}$.

We also relate our ideas to representing operators on L_p and give an atomic decomposition of L_p for $\frac{1}{2} < p < 1$ suggested by our theorems.

For some further results on harmonic functions see [6]. We may add, that as the referee has pointed out, there is an extensive literature on analytic and harmonic vector-valued functions, using techniques of factorization through locally convex spaces and tensor products. (See e.g. [1], [2], [9].) These techniques are readily applicable to problems on open sets, but seem to give little information on boundary behaviour questions.

2. Basic notation. Throughout this paper we deal only with complex vector spaces, although our results on harmonic functions can easily be lifted to the real case by complexification. A p-Banach space $(0 < p \leqq 1)$ is a complete metrizable topological vector space whose topology is given by a quasi-norm $x \to \|x\|$ such that

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\,\|x\|,\ \alpha \in \mathbb{C},\ x \in X$;
(iii) $\|x_1 + x_2\|^p \leqq \|x_1\|^p + \|x_2\|^p,\ x_1, x_2 \in X$.

A quasi-Banach space is a p-Banach space for some p, $0 < p \leqq 1$.

We recall [4] that if $\sigma > 0$ and X is a p-Banach space $A_\sigma(X)$ denotes the space of all analytic functions $f\colon \Delta \to X$ so that if $\nu = [\sigma]$,

$$\|f\|_\sigma = \sum_{k=0}^{\nu} \|f^{(k)}(0)\| + \sup_{|z|<1} (1-|z|^2)^{\nu+1-\sigma} \|f^{(\nu+1)}(z)\| < \infty.$$

We set $V_\sigma(X)$ to be the set of all analytic f so that

$$\sup_{|z|<1} (1-|z|^2)^{-\sigma} \|f(z)\| < \infty.$$

Then $V_\sigma(X) \subset A_\sigma(X)$ (see [4]) and $V_\sigma(X) = \{0\}$ if $\sigma > \frac{1}{p} - 1$.

We also need the class $E_\sigma(X)$ of all continuous maps $f\colon \mathbb{C} \to X$ so that $f|\Delta$ and $f\left(\frac{1}{z}\right)$ $(z \in \Delta)$ are both in $A_\sigma(X)$ and further $\lim\limits_{|z|\to\infty} f(z) = 0$.

For $0 < p < 1$ let $L_p(\mathbb{T})$ denote the space of all measurable functions $f: \mathbb{T} \to \mathbb{C}$ so that

$$\| f \|_p = \left\{ \frac{1}{2\pi} \int | f(e^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.$$

If $\sigma = \frac{1}{p} - 1$ and X is a p-Banach space, the space of operators $\mathscr{L}(L_p, X)$ can be identified with $E_\sigma(X)$ ([4]) by the identification $T \leftrightarrow f$ where

$$T((1 - wz)^{-1}) = f(z).$$

Here $w \to (1 - wz)^{-1} \in L_p(\mathbb{T})$.

If $I \subset \mathbb{R}$ is a bounded closed interval then a function $f: I \to X$ is in $C_\sigma(I; X)$ if there is a constant $\gamma \geqq 0$ so that for every subinterval $J \subset I$ there is a polynomial $g: \mathbb{R} \to X$ of degree at most $v = [\sigma]$, with

$$\| f(t) - g(t) \| \leqq \gamma \, |J|^\sigma, \qquad t \in J.$$

The least such constant is denoted $\gamma_\sigma(f)$ and we quasi-norm $C_\sigma(I; X)$ by

$$\| f \|_\sigma = \gamma_\sigma(f) + \max_{t \in I} \| f(t) \|.$$

If X is p-normable and $\sigma > \frac{1}{p} - 1$ then $C_\sigma(I; X)$ admits an integration theory due to Turpin and Waelbroeck (cf. [4], [7], [8]). See the full discussion in [4]. If $\mu \in M(I)$ is a regular Borel measure supported on I then we can define

$$L_\mu(f) = \int_I f \, d\mu$$

and further

$$\| L_\mu(f) \| \leqq C \, \| f \|_\sigma \, \| \mu \|$$

where $C = C(\sigma, p, I)$ is independent of f, μ.

If $\mathbb{T}$ is the unit circle we transport these ideas to $\mathbb{T}$ by setting $C_\sigma(\mathbb{T}, X)$ to be those functions $f: \mathbb{T} \to X$ so that $\tilde{f} \in C_\sigma([-2\pi, 2\pi], X)$ where $\tilde{f}(t) = f(e^{it})$. We set $\| f \|_\sigma = \| \tilde{f} \|_\sigma$.

If $f \in A_\sigma(X)$ where $\sigma > 0$ then f extends continuously to $\bar{\Delta}$ and its boundary values belong to C_σ. If $\sigma > \frac{1}{p} - 1$ then the Turpin-Waelbroeck integral can be used to recapture f from its boundary values (see [2]).

Throughout this paper we use the convention that C denotes a constant which may vary from line to line and may depend on σ, p, q, v etc. but is independent of f, g, x etc.

3. Harmonic functions. Let X be a complex p-Banach space. A function $h: \Delta \to X$ is called *harmonic* if we can write

$$h(z) = f_1(z) + f_2(\bar{z})$$

where f_1, f_2 are analytic. If h is harmonic then we can expand h in a series expansion

$$(*) \qquad h(re^{i\theta}) = \sum_{n \in \mathbb{Z}} x_n \, r^{|n|} \, e^{in\theta}$$

where $\sum \| x_n \|^p \, r^{|n| p} < \infty$ whenever $0 \leqq r < 1$. Conversely if h has such a series expansion then h is harmonic.

It is clear that if h is harmonic then h is infinitely differentiable and that the coefficients x_n in the series expansion (*) can be computed from the partial derivatives of h. Thus we shall set

$$d_n(h) = x_n, \quad n \in \mathbb{Z}.$$

Let $H(X)$ denote the space of all bounded harmonic functions $h: \Delta \to X$ with the sup norm

$$\| h \| = \sup_{z \in \Delta} \| h(z) \|.$$

Then we first investigate the continuity of the maps $d_n: H(X) \to X$ for $n \in \mathbb{Z}$.

For $h \in H(X)$, $\nu \geqq 1$ and $0 < r < 1$ we write

$$M_\nu(r; h) = \sup_{|z| \leqq r} \| f^{(\nu)}(z) \| + \sup_{|z| \leqq r} \| g^{(\nu)}(z) \|$$

where

$$f(z) = \sum_{n \geqq 0} d_n(h) z^n, \quad g(z) = \sum_{n > 0} d_{-n}(h) z^n.$$

Let

$$M_\nu(h) = \sup_{r < 1} M_\nu(r; h)$$

so that M_ν can be $+\infty$.

Now suppose σ is chosen with $\sigma > \frac{1}{p} - 1$ and that $\nu = [\sigma]$.

Lemma 3.1. *There exists $C = C(p, \sigma)$ and $0 < \beta < 1$ where $\beta = \beta(p, \sigma)$ so that if $h \in H(X)$ then for $n \in \mathbb{Z}$*

$$\| d_n(h) \| \leqq C(\| h \|^{1-\beta} M_{\nu+1}(h)^\beta + \| h \|).$$

Proof. It suffices to consider the case $M_{\nu+1} < \infty$. We define

$$f_0(z) = \sum_{n=\nu+1}^{\infty} d_n(h) z^n, \quad g_0(z) = \sum_{n=\nu+1}^{\infty} d_{-n}(h) z^n.$$

Then $f_0, g_0 \in A_\sigma(X)$ and

$$\| f_0 \|_\sigma \leqq C M_{\nu+1}, \quad \| g_0 \|_\sigma \leqq C M_{\nu+1}.$$

In particular f_0 and g_0 extend continuously to $\Delta \cup \mathbb{T}$. Hence if we put

$$f(z) = \sum_{n=0}^{\infty} d_n(h) z^n, \quad g(z) = \sum_{n=1}^{\infty} d_{-n}(h) z^n$$

then f and g belong to $A_\sigma(X)$ and extend continuously to $\Delta \cup \mathbb{T}$. Thus h extends continuously to $\Delta \cup \mathbb{T}$.

Note also that

$$h(e^{i\theta}) = f(e^{i\theta}) + g(e^{-i\theta})$$

for $0 \leqq \theta \leqq 2\pi$ and that $h \in C_\sigma(\mathbb{T}, X)$. Further by Theorem 6.4 of [2] we can compute $d_n(h)$ ($n \in \mathbb{Z}$) as Turpin-Waelbroeck integrals:

$$d_n(h) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})\, e^{-in\theta}\, d\theta,$$

for $n \in \mathbb{Z}$.

If we set

$$\phi(\theta) = f_0(e^{i\theta}) + g_0(e^{-i\theta})$$

then $\phi \in C_\sigma([0, 2\pi], X)$ and in fact

$$\|\phi\|_{C_\sigma} \leqq C(\|f_0\|_\sigma + \|g_0\|_\sigma) \leqq CM_{\nu+1}.$$

Now ([2]) there is a sequence of C^∞-functions $u_m\colon [0, 2\pi] \to X$ ($m \geqq 2(\nu + 1)$) with rank $u_m \leqq m$ so that

$$\|\phi(\theta) - u_m(\theta)\| \leqq C\, m^{-\sigma} M_{\nu+1}.$$

Let

$$v_m(\theta) = u_m(\theta) + \sum_{-\nu}^{\nu} d_n(h)\, e^{in\theta}.$$

Then rank $v_m \leqq 2m$ and

$$\|h(e^{i\theta}) - v_m(\theta)\| \leqq C\, m^{-\sigma} M_{\nu+1}.$$

Now by [2] Equation 3.7 we have

$$\left\| d_n(h) - \frac{1}{2\pi} \int_0^{2\pi} v_m(\theta)\, e^{-in\theta}\, d\theta \right\| \leqq C\, m^{\frac{1}{p} - \sigma - 1}\, M_{\nu+1}.$$

However

$$\|v_m(\theta)\| \leqq C(\|h\| + m^{-\sigma} M_{\nu+1})$$

and as rank $v_m \leqq 2m$

$$\left\| \frac{1}{2\pi} \int_0^{2\pi} v_m(\theta)\, e^{-in\theta}\, d\theta \right\| \leqq C\, m^{\frac{1}{p} - 1} (\|h\| + m^{-\sigma} M_{\nu+1}).$$

Combining

$$\|d_n(h)\| \leqq C\, m^{\frac{1}{p} - 1} (\|h\| + m^{-\sigma} M_{\nu+1})$$

for $m \geqq 2(\nu + 1)$. By adjusting the constant we may suppose

$$\|d_n(h)\| \leqq C\, t^{\frac{1}{p} - 1} (\|h\| + t^{-\sigma} M_{\nu+1})$$

for all $t \geqq 1$. If $M_{\nu+1} > \|h\|$ set $t^\sigma = M_{\nu+1}/\|h\|$, and conclude

$$\|d_n(h)\| \leqq CM_{\nu+1}^\beta\, \|h\|^{1-\beta}$$

where $\beta = \frac{1}{\sigma}(\frac{1}{p} - 1)$. If $M_{\nu+1} \leqq \|h\|$,

$$\|d_n(h)\| \leqq C\, \|h\|.$$

The conclusion now follows.

Theorem 3.2. *For each $n \in \mathbb{Z}$, there exists a constant $C = C(p, \sigma, n)$ so that if $h \in H(X)$*

$$\| d_n(h) \| \leqq C \| h \|.$$

Proof. Suppose $\| h \| = 1$ and,

$$h(z) = f(z) + g(\bar{z})$$

where f, g are analytic. If $z_0 \in \Delta$ set

$$h_0(z) = h(z_0 + (1 - r) z)$$

where $|z_0| = r$. Note that

$$M_{\nu+1}(h_0) \leqq (1 - r)^{\nu+1} M_{\nu+1}(h).$$

Thus

$$\| d_n(h_0) \| \leqq C((1 - r)^{\beta(\nu+1)} M_{\nu+1}^{\beta} + 1)$$

by Lemma 3.1. In particular taking $n = \nu + 1$

$$\| f^{(\nu+1)}(z_0) \| \leqq C(1 - r)^{-(\nu+1)}[(1 - r)^{\beta(\nu+1)} M_{\nu+1}^{\beta} + 1]$$

and a similar inequality holds for $g^{(\nu+1)}$.

Thus

$$M_{\nu+1}(r, h) \leqq C(1 - r)^{-(\nu+1)}[(1 - r)^{\beta(\nu+1)} M_{\nu+1}^{\beta} + 1].$$

Now if $0 < r < R < 1$ we can utilize this to conclude

$$M_{\nu+1}(r, h) \leqq C\left(1 - \frac{r}{R}\right)^{-(\nu+1)} \left(\left(1 - \frac{r}{R}\right)^{\beta(\nu+1)} M_{\nu+1}(R, h)^{\beta} + 1\right).$$

Let us suppose $M_{\nu+1}(\frac{1}{2}, h) \geqq 1$. Then for $\frac{1}{2} \leqq r < R < 1$,

$$M_{\nu+1}(r) \leqq 2C\left(1 - \frac{r}{R}\right)^{-(\nu+1)} M_{\nu+1}(R)^{\beta}.$$

Let $r_n = \dfrac{3}{4} - \dfrac{1}{4^n}$ for $n \geqq 1$. Then

$$1 - \frac{r_n}{r_{n+1}} \geqq \frac{1}{4^n}$$

and so if

$$M_n = M_{\nu+1}(r_n, h), \quad M_n \leqq (2C)\, 4^{n(\nu+1)} M_{n+1}^{\beta}.$$

Let $A_n = \beta^n \log M_n$. Then

$$A_n \leqq \beta^n \log(2C) + n\beta^n(\nu + 1) \log 4 + A_{n+1}$$

so that

$$A_{n+1} - A_1 \geqq -\log(2C) \sum_{n=1}^{\infty} \beta^n - \left(\sum_{n=1}^{\infty} n\beta^n\right)(\nu + 1) \log 4.$$

Since $A_n \to 0$ we conclude $\beta \log M_1 \leqq C$ where $C = C(p, \sigma)$ so that

$$M_1 \leqq C \quad \text{or} \quad M_{\nu+1}(\tfrac{1}{2}, h) \leqq C.$$

Now considering $h_0(z) = h(\frac{1}{2}z)$ we have $M_{\nu+1}(h_0) \leqq C$ and so

$$\| d_n(h_0) \| \leqq C$$

so that

$$\| d_n(h) \| \leqq C\, 2^{|n|}.$$

Corollary 3.3. *If h is harmonic on Δ and is written in the form $h(z) = f(z) + g(\bar{z})$ where f and g are analytic then for $n \geqq 1$*

$$\| f^{(n)}(z_0) \| \leqq C(p, \sigma, n)\, r^{-n} \max_{|z - z_0| \leqq r} \| h(z) \|$$

$$\| g^{(n)}(z_0) \| \leqq C(p, \sigma, n)\, r^{-n} \max_{|z - z_0| \leqq r} \| h(z) \|$$

for every r with $0 < r < 1 - |z_0|$.

4. The failure of the maximum modulus principle. Let us consider now an example. Let $P(z, w)$ be the Poisson kernel

$$P(z, w) = \frac{1 - |z|^2}{|w - z|^2}, \qquad z \in \Delta,\ w \in \mathbb{T}.$$

Let $P(z) = P(z, \cdot)$. Then the map $z \to P(z)$ $(\Delta \to L_p(\mathbb{T}))$ is harmonic. In fact

$$P(z) = \sum_{n \geqq 0} \bar{w}^n z^n + \sum_{n < 0} \bar{w}^n \bar{z}^n$$

If $0 < p < 1$, P extends continuously to $\Delta \cup \mathbb{T}$ if we put $P(z) = 0$ for $z \in \mathbb{T}$.

Thus the maximum modulus principle fails for harmonic functions in L_p when $0 < p < 1$. It is not difficult to establish that if X is a p-Banach space satisfying an estimate of the form

$$\| h(0) \| \leqq C \max_{w \in \mathbb{T}} \| h(w) \|$$

for every harmonic function, then X is locally convex. In fact if h has finite rank

$$h(0) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})\, d\theta$$

and hence one can deduce without difficulty

$$\left\| \frac{x_1 + \cdots + x_n}{n} \right\| \leqq C \max_{i \leqq n} \| x_i \|$$

for any $x_1, \ldots, x_n \in X$.

At the opposite extreme one may ask for the precise rate at which an harmonic function can decay at the boundary on Δ in a p-normable space. In [2] it is shown that if X is p-normable and $f\colon \Delta \to X$ is analytic and satisfies

$$\| f(z) \| = o(1 - |z|^2)^{\frac{1}{p} - 1}$$

as $|z| \to 1$ then $f \equiv 0$. The existence of an analytic f such that

$$\| f(z) \| \leqq C(1 - |z|^2)^{\frac{1}{p} - 1}$$

is equivalent to $\mathscr{L}(L_p/H_p, X) \neq \{0\}$.

For $\sigma > 0$ we let $VH_\sigma(x)$ be the space of all harmonic functions $h\colon \Delta \to X$ such that

$$\| h(z) \| \leqq C(1 - |z|^2)^\sigma.$$

Lemma 4.1. *If* $h \in VH_\sigma(X)$ *and*

$$h(z) = f(z) + g(\bar{z})$$

where f *and* g *are analytic then* $f, g \in A_\sigma(X)$ *and*

$$\sup (1 - |z|^2)^{1-\sigma} \| f'(z) \| < \infty$$
$$\sup (1 - |z|^2)^{1-\sigma} \| g'(z) \| < \infty.$$

Proof. If

$$\| h(z) \| \leqq B(1 - |z|^2)^\sigma$$

then by Corollary 3.3 if $n \geqq 1$

$$\| f^{(n)}(z) \| \leqq CB(1 - |z|^2)^{\sigma - n}$$
$$\| g^{(n)}(z) \| \leqq CB(1 - |z|^2)^{\sigma - n}.$$

If $\nu = [\sigma]$ take $n = \nu + 1$ and then $f, g \in A_\sigma(X)$. The second conclusion takes $n = 1$. We see immediately that:

Theorem 4.2. *If* $VH_\sigma(X) \neq \{0\}$ *and* $\sigma > 1$ *then* $V_{\sigma-1}(X) \neq \{0\}$.

[Here $V_\sigma(X)$ is the class of analytic functions $f\colon \Delta \to X$ so that $\| f(z) \| \leqq B(1 - |z|^2)^\sigma$].

Corollary 4.3. *Let* X *be an* A*-convex quasi-Banach space and suppose* $h\colon \Delta \to X$ *is harmonic and*

$$\| h(z) \| = o(1 - |z|).$$

Then $h \equiv 0$.

Proof. The argument of Lemma 4.1 here shows that $\| f'(z) \| \to 0$ as $|z| \to 1$ and $\| g'(z) \| \to 0$ as $|z| \to 1$. Since X is A-convex $f' \equiv 0$ and $g' \equiv 0$ so that h is constant.

Remark. The example $h(z) = P(z)$ in weak $L_{1/2} = L(\frac{1}{2}, \infty)$ shows one can have $\| h(z) \| = O(1 - |z|)$ in an A-convex space with $h \neq 0$.

Theorem 4.4. *Suppose X is p-normable and $\sigma = \frac{1}{p} - 1$. Suppose $h \in VH_\sigma(X)$. Then there exists $T \in \mathscr{L}(L_p, X)$ so that*

$$T(P(z)) = h(z).$$

Proof. Suppose

$$h(z) = f(z) + g(\bar{z})$$

where f, g are analytic and $f(0) = 0$. Then $f, g \in A_\sigma(X)$. Define $F: \mathbb{C} \to X$ by

$$\begin{aligned} F(z) &= g(z), & |z| \leqq 1 \\ &= -f\left(\frac{1}{z}\right), & |z| > 1. \end{aligned}$$

Note that if $|z_n| > 1$ and $z_n \to z_0$ where $|z_0| = 1$ then

$$F(z_n) = -f\left(\frac{1}{z_n}\right) = g\left(\frac{1}{\bar{z}_n}\right) - h\left(\frac{1}{z_n}\right)$$

so that $\lim\limits_{n\to\infty} F(z_n) = \lim\limits_{n\to\infty} g(\bar{z}_n^{-1}) = g(\bar{z}_0^{-1}) = F(z_0)$. Thus $F \in E_\sigma(X)$ and there exists ([4] Theorem 8.1) $T \in \mathscr{L}(L_p, X)$ so that

$$T((1 - wz)^{-1}) = F(z).$$

Now if $|z| < 1$

$$P(z, w) = (1 - w\bar{z})^{-1} - (1 - wz^{-1})^{-1}$$

so that

$$TP(z) = g(\bar{z}) + f(z) = h(z).$$

Corollary 4.5. *If X is p-normable and $\sigma > \frac{1}{p} - 1$ then $VH_\sigma(X) = \{0\}$.*

Proof. Let $\frac{1}{q} = \sigma + 1$. Then if $h \in VH_\sigma(\bar{X})$ there exists $T \in \mathscr{L}(L_q, X)$ so that

$$T(P(z)) = h(z).$$

However X is p-normable and $p > q$ so that $T = 0$.

Corollary 4.6. *If $\frac{1}{2} < p < 1$ and $\sigma = \frac{1}{p} - 1$ then $VH_\sigma(X) \neq \{0\}$ if and only if $\mathscr{L}(L_p, X) \neq \{0\}$.*

Proof. In this case $P(z) \in VH_\sigma(L_p)$.

Remark. For $p < \frac{1}{2}$ this theorem is false by Corollary 4.3.

We conclude this section by proving an atomic decomposition for L_p where $\frac{1}{2} < p \leqq 1$, which is suggested by Corollary 4.6.

For $0 < r < 1$ and $n \in \mathbb{N}$ set

$$A(r, n) = \{rw_n^k : k = 0, 1, 2, \ldots, n-1\}$$

where $w_n = \exp(2\pi i/n)$.

Theorem 4.7. *Suppose $\frac{1}{2} < p \leqq 1$. There exists $\delta = \delta(p) > 0$ with the following property. Let $r_k \to 1$ and $n_k \to \infty$ be chosen so that $\delta \leqq n_k(1 - r_k) \leqq M < \infty$. Let $A = \cup A(r_k, n_k)$ and let $\{z_m\}_{m=1}^{\infty}$ be an ordering of A. Then there exists a constant C so that if $f \in L_p(\mathbb{T})$*

$$f = \sum_{m=1}^{\infty} c_m P(z_m)(1 - |z_m|^2)^{1-\frac{1}{p}}$$

and $(\sum |c_m|^p)^{\frac{1}{p}} \leqq C \, \| f \|_p$.

Remark. Note that

$$\sup_m \| P(z_m)(1 - |z_m|^2)^{1-\frac{1}{p}} \|_p = B < \infty.$$

Proof. We shall suppose

$$2^{2/p} \frac{e^{-\delta}}{(1 - e^{-p\delta})^{1/p}} = \beta < 1.$$

Define $T: l_p \to L_p$ by

$$T(c_m) = \sum c_m P(z_m)(1 - |z_m|^2)^{1-\frac{1}{p}}.$$

Then T is bounded and $\| T \| = B$. We must show that T is an open mapping.

Suppose $f \in L_p$ is a trigonometric polynomial i.e.

$$f(w) = \sum_{k=-N}^{N} a_k w^k.$$

Suppose $A(r, n) \subset A$, with $n > 2N$ and $r = r_m$, $n = n_m$. We compute $g = g_m$ where

$$g = \frac{1}{n} \sum_{k=0}^{n-1} f(w_n^k) P(r w_n^k).$$

Then $g = T(d)$ where $d = d_m$ and

$$\| d \| = \frac{1}{n}(1 - r^2)^{\frac{1}{p}-1} \left(\sum_{k=0}^{n-1} |f(w_n^k)|^p \right)^{\frac{1}{p}}.$$

Now

$$g = \sum_{-N}^{N} r^{|k|} a_k w^k + \sum_{l=1}^{\infty} \sum_{-N}^{N} r^{k+ln} a_k w^{k+ln} + \sum_{l=1}^{\infty} \sum_{-N}^{N} r^{ln-k} a_k w^{k-ln}.$$

Let

$$F_1 = F_1(r, w) = \sum_{-N}^{N} r^{|k|} a_k w^k$$

$$F_2 = F_2(r, w) = \sum_{-N}^{N} r^k a_k w^k$$

$$F_3 = F_3(r, w) = \sum_{-N}^{N} r^{-k} a_k w^k.$$

Then

$$\| f - g_m \|^p \leqq \| f - F_1 \|^p + \frac{r^{np}}{1 - r^{np}} (\| F_2 \|^p + \| F_3 \|^p).$$

Thus

$$\limsup_{m \to \infty} \| f - g_m \|^p \leqq 2 \left(\limsup_{m \to \infty} \frac{r^{np}}{1 - r^{np}} \right) \| f \|^p \leqq 2 \frac{e^{-p\delta}}{1 - e^{-p\delta}} \| f \|^p$$

while

$$\| d_m \|^p = n^{1-p} (1 - r)^{1-p} (1 + r)^{1-p} \frac{1}{n} \sum_{k=0}^{n-1} | f(w_n^k) |^p$$

so that

$$\limsup_{m \to \infty} \| d_m \|^p \leqq 2^{1-p} M^{1-p} \| f \|^p.$$

By a density argument we conclude that if $f \in L_p$ there exists $d \in l_p$ with

$$\begin{aligned} \| d \| &< 2^{\frac{1}{p}} M^{\frac{1}{p} - 1} \| f \| \\ \| f - S d \| &< 2^{2/p} \frac{e^{-\delta}}{(1 - e^{-p\delta})^{1/p}} \| f \| \\ &= \beta \| f \|. \end{aligned}$$

This implies that T is onto since $\beta < 1$.

Corollary 4.8. *Suppose $\frac{1}{2} < p \leqq 1$. Let X be p-normable and suppose $h \in H(X)$ satisfies*

$$\| h(z) \| = o(1 - |z|)^{\frac{1}{p} - 1}.$$

Then $h \equiv 0$.

Proof. Suppose $p < 1$. By Theorem 4.7 there is a set $\{z_n\}$ so that the map $S: l_p \to L_p$ is onto where

$$S(e_n) = P(z_n)(1 - |z_n|^2)^{1 - \frac{1}{p}}.$$

Let $T: L_p \to X$ be the map given by Theorem 4.4. Then $T \circ S(e_n) \to 0$ and hence T is compact. Thus [3] $T = 0$ and $h = 0$.

For $p = 1$, this follows from the Maximum Modulus Principle for harmonic functions.

References

[1] K. D. BIERSTEDT und R. MEISE, Lokalkonvexe Unterräume in topologischen Vektorräumen und das Produkt. Manuscripta Math. **8**, 143–172 (1973).

[2] B. GRAMSCH und D. VOGT, Holomorphe Funktionen mit Werten in nicht lokalkonvexen Vektorräumen. J. Reine Angew. Math. **243**, 159–170 (1970).

[3] N. J. KALTON, Compact operators on symmetric function spaces. Bull. Acad. Polon. Sci. **36**, 815–816 (1978).

[4] N. J. KALTON, Analytic functions in non-locally convex spaces and applications. Studia Math. **83**, 275–303 (1986).

[5] N. J. KALTON, Plurisubharmonic functions on quasi-Banach spaces. Studia Math. **84**, 297–324 (1987).
[6] N. J. KALTON, Some applications of vector-valued analytic and harmonic functions. Probability and Banach spaces (J. Bastero, M. San Miguel, editors) LNM **1221**, 114–140, Berlin-Heidelberg-New York 1986.
[7] P. TURPIN, Convexités dans les espaces vectoriels topologiques. Diss. Math. **131**, Warsaw 1974.
[8] L. WAELBROECK, Topological vector spaces and algebras. LNM **230**, Berlin-Heidelberg-New York 1971.
[9] L. WAELBROECK, Galbs, tensor products and convex-holomorphic mappings. Toeplitz Centenniel, (editor I. Gohberg) Operator Theory **4**, 463–476, Basel 1982.

Eingegangen am 27. 1. 1987*)

Anschrift des Autors:

Nigel J. Kalton
College of Arts and Science
Dept. of Mathematics
University of Missouri-Columbia
Columbia, Missouri 65211, USA

*) Eine Neufassung ging am 27. 8. 1987 ein.