A Characterization of Real $\mathcal{C}(K)$-Spaces

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1. INTRODUCTION. The aim of this note is to give a simple characterization of real Banach algebras that are isometrically isomorphic to (real) spaces $\mathcal{C}(K)$ of continuous functions on compact Hausdorff spaces K. The development of this criterion was influenced by the problem of introducing students to $\mathcal{C}(K)$-spaces in a course on Banach space theory that emphasizes real scalars over complex scalars (see [**1**]). One needs to know that certain spaces such as ℓ_∞ and $L_\infty(0, 1)$ are $\mathcal{C}(K)$-spaces in disguise. The standard derivation of such facts requires the Gelfand-Naimark theorem for commutative C^*-algebras. Our approach allows us to avoid complex analysis and the general methods of Banach algebras that depend heavily on the use of complex scalars. Although we invoke a few ideas from Banach algebra theory, the proof is sufficiently direct that it is accessible to the student of functional analysis who knows only the definition of a real Banach algebra. We show, in fact, that a real Banach algebra $\mathcal{A}$ with identity is a $\mathcal{C}(K)$-space if it satisfies one additional condition:

Theorem 1.1. *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity e such that $\|e\| = 1$. Then $\mathcal{A}$ is isometrically isomorphic to the algebra $\mathcal{C}(K)$ for some compact Hausdorff space K if and only if*

$$\|a^2 - b^2\| \le \|a^2 + b^2\| \quad (a, b \in \mathcal{A}). \tag{1.1}$$

Inequality (1.1) thus gives a very simple and easily verified criterion for checking whether a real Banach algebra is a $\mathcal{C}(K)$-space. We refer to [**3**] for a study of characterizations of the Banach algebra $\mathcal{C}(K)$. The following theorem is due to Arens [**2**]; however, the proof given by Arens is rather different and involves complexification of the algebra:

Theorem 1.2 (Arens). *Let $\mathcal{A}$ be a commutative real Banach algebra with an identity e such that $\|e\| = 1$. Then $\mathcal{A}$ is isometrically isomorphic to the algebra $\mathcal{C}(K)$ for some compact Hausdorff space K if and only if*

$$\|a\|^2 \le \|a^2 + b^2\| \quad (a, b \in \mathcal{A}). \tag{1.2}$$

It is easy to verify that every $\mathcal{C}(K)$-space satisfies (1.1) and (1.2), and therefore we only need to worry about the other direction of both theorems.

We note that Theorem 1.2 follows directly from Theorem 1.1. Indeed, if we assume (1.2) and if a and b belong to $\mathcal{A}$, then

$$\begin{aligned}\|a^2 - b^2\|^2 &\le \|(a^2 - b^2)^2 + 4a^2b^2\| \\ &= \|(a^2 + b^2)^2\| \\ &\le \|a^2 + b^2\|^2.\end{aligned}$$

Thus (1.1) is an immediate consequence of (1.2).

2. PRELIMINARIES. Unless otherwise specified, all Banach spaces are real. We use $\mathcal{C}(K)$ to denote the space of continuous real-valued functions on a compact Hausdorff space K equipped with the norm $\|f\|_{\mathcal{C}(K)} = \max_{s\in K} |f(s)|$. For reference we state the Stone-Weierstrass theorem.

Theorem 2.1 (Stone-Weierstrass). *Suppose that K is a compact Hausdorff topological space and that $\mathcal{A}$ is a closed subalgebra of $\mathcal{C}(K)$ containing the constant functions. If $\mathcal{A}$ separates the points of K (i.e., for any s and t in K with $s \neq t$ there exists f in $\mathcal{A}$ such that $f(s) \neq f(t)$), then $\mathcal{A} = \mathcal{C}(K)$.*

Suppose that $\mathcal{A}$ is a commutative real Banach algebra with identity e such that $\|e\| = 1$. The *state space* of $\mathcal{A}$ is the set

$$\mathcal{S} = \{\varphi \in \mathcal{A}^* : \|\varphi\| = \varphi(e) = 1\},$$

where $\mathcal{A}^*$ denotes the dual space of $\mathcal{A}$. An element of $\mathcal{S}$ is called a *state*. The set of states is nonempty by the Hahn-Banach theorem.

We recall that the weak* topology on $\mathcal{A}^*$ is the topology of pointwise convergence. Then $\mathcal{S}$ is compact in the weak* topology. This follows easily since, by Alaoglu's theorem, the closed unit ball $B_{\mathcal{A}^*}$ of $\mathcal{A}^*$ is weak* compact, and the relations in the definition of $\mathcal{S}$ define a weak* closed subset of $B_{\mathcal{A}^*}$. Now we just take into account the fact that the weak* topology is Hausdorff, so a weak* closed subset of a weak* compact set is also weak* compact.

We use $\mathcal{A}_+$ to denote the norm-closure of the set of squares in $\mathcal{A}$, that is, $\mathcal{A}_+ = \overline{\{a^2 : a \in \mathcal{A}\}}$. If $\mathcal{A} = \mathcal{C}(K)$, then $\mathcal{A}_+$ is simply the positive cone, $\{f : f \geq 0\}$; however, for a general Banach algebra, $\mathcal{A}_+$ may not even be closed under addition.

The following lemma is quite trivial:

Lemma 2.2. *The following statements are true for a commutative real Banach algebra $\mathcal{A}$:*

(i) *If $x, y \in \mathcal{A}_+$, then $xy \in \mathcal{A}_+$.*
(ii) *If $x \in \mathcal{A}_+$ and $\lambda \geq 0$, then $\lambda x \in \mathcal{A}_+$.*

Part (i) of the next proposition is the well-known "Square Root Lemma" from Banach algebra theory (see, for example, [**6**, Theorem 3.4.5, p. 361]).

Proposition 2.3. *A commutative real Banach algebra $\mathcal{A}$ with an identity e of norm 1 has the following properties:*

(i) *If $x \in \mathcal{A}$ is such that $\|x\| \leq 1$, then $e + x \in \mathcal{A}_+$.*
(ii) *$\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_+$.*

Proof. Let x in $\mathcal{A}$ have $\|x\| < 1$. By writing $(1+t)^{1/2}$ in its binomial series, valid for scalars t with $|t| < 1$, we see that the series $\sum_{n=0}^{\infty} \binom{1/2}{n} x^n$ is absolutely convergent, therefore convergent to some y in $\mathcal{A}$. By expanding $(1+t)^{1/2}(1+t)^{1/2}$ for a real variable t when $|t| < 1$ it becomes clear that

$$\sum_{m+n=k} \binom{1/2}{m}\binom{1/2}{n} = \begin{cases} 1 & \text{if } k = 0 \text{ or } 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

We deduce that $y^2 = e + x$. Since $\mathcal{A}_+$ is closed, we see that $e + x \in \mathcal{A}_+$ if $\|x\| \le 1$. This establishes (i).

When $\|x\| \le 1$ we can write

$$x = \tfrac{1}{2}(e + x) - \tfrac{1}{2}(e - x).$$

Part (ii) now follows (with the use of Lemma 2.2). ■

3. PROOF OF THE MAIN THEOREM. We now turn to the proof of Theorem 1.1. Let us first note two simple deductions from (1.1). The hypothesis gives

$$\|x - y\| \le \|x + y\| \quad (x, y \in \mathcal{A}_+). \tag{3.1}$$

So, if x and y belong to $\mathcal{A}_+$ we also have

$$\|x\| \le \tfrac{1}{2}\big(\|x - y\| + \|x + y\|\big) \le \|x + y\|. \tag{3.2}$$

Before completing the proof, we prove two preparatory lemmas.

Lemma 3.1. *Suppose that $\mathcal{A}$ satisfies* (1.1). *Then $\varphi(x) \ge 0$ whenever $\varphi \in \mathcal{S}$ and $x \in \mathcal{A}_+$.*

Proof. Take $x \in \mathcal{A}_+$ with $\|x\| = 1$. By Proposition 2.3, $e - x \in \mathcal{A}_+$ and, by (3.2),

$$\|e - x\| \le \|(e - x) + x\| = 1.$$

Hence for φ in $\mathcal{S}$ we have

$$1 = \|\varphi\| \ge \varphi(e - x) = 1 - \varphi(x),$$

whence $\varphi(x) \ge 0$. ■

We recall that a point x in a nonempty convex subset S of a vector space is an *extreme point of S* if whenever $x = \lambda x_1 + (1 - \lambda)x_2$ with x_1, x_2 in S and $0 < \lambda < 1$, then $x = x_1 = x_2$. We use $\partial_e S$ to denote the set of extreme points of S. We will use the Krein-Milman theorem, which in our context states that if S is a weak* compact convex subset of the dual of a Banach space, then $\partial_e S$ is nonempty and S is the weak* closure of the convex hull of $\partial_e S$.

Lemma 3.2. *Suppose that $\mathcal{A}$ satisfies* (1.1). *Let K be the set of all multiplicative states of $\mathcal{A}$* (*i.e., $K = \{\varphi \in \mathcal{S} : \varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathcal{A}\}$*). *Then K is a compact Hausdorff space in the weak* topology of $\mathcal{A}^*$ which contains the set $\partial_e \mathcal{S}$ of extreme points of $\mathcal{S}$* (*and, in particular, K is nonempty*).

Proof. It is trivial to show that K is a closed subset of the closed unit ball of $\mathcal{A}^*$. This ensures that K is compact for the weak* topology.

Since $\mathcal{S}$ is convex and compact in the weak* topology of $\mathcal{A}^*$, the Krein-Milman theorem guarantees that $\partial_e \mathcal{S}$ is nonempty. Suppose that φ lies in $\partial_e \mathcal{S}$. We claim that φ is in K. Since $\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_+$, it suffices to show that $\varphi(xy) = \varphi(x)\varphi(y)$ whenever $x \in \mathcal{A}_+$ and $y \in \mathcal{A}$.

Consider x in $\mathcal{A}_+$ with $\|x\| \le 1$ and y in $\mathcal{A}$ with $\|y\| \le 1$. By Proposition 2.3, $e \pm y \in \mathcal{A}_+$. Therefore, by Lemma 3.1 and Lemma 2.2 (i),

$$\varphi(x(e \pm y)) \ge 0,$$

which implies that

$$|\varphi(xy)| \le \varphi(x). \tag{3.3}$$

Similarly, $e - x \in \mathcal{A}_+$ by Proposition 2.3, so

$$|\varphi((e-x)y)| \le 1 - \varphi(x). \tag{3.4}$$

Notice that both inequalities (3.3) and (3.4) hold, in fact, for arbitrary y in $\mathcal{A}$.

If $\varphi(x) = 0$, inequality (3.3) yields $\varphi(xy) = \varphi(x)\varphi(y)$. Similarly, if $\varphi(x) = 1$, using (3.4) it is immediate that $\varphi(xy) = \varphi(x)\varphi(y)$.

If $0 < \varphi(x) < 1$, we define ψ_1 and ψ_2 on $\mathcal{A}$ by

$$\psi_1(y) = \varphi(x)^{-1}\varphi(xy)$$

and

$$\psi_2(y) = (1 - \varphi(x))^{-1}\varphi((e-x)y).$$

Using (3.3) and (3.4) we see that ψ_1 and ψ_2 are states. Now we can write

$$\varphi = \varphi(x)\psi_1 + (1 - \varphi(x))\psi_2.$$

By the fact that φ is an extreme point of $\mathcal{S}$ we must have $\psi_1 = \varphi$ and, therefore,

$$\varphi(xy) = \varphi(x)\varphi(y) \qquad (x \in \mathcal{A}_+,\ y \in \mathcal{A}). \qquad \blacksquare$$

Conclusion of the Proof of Theorem 1.1. Suppose that $\mathcal{A}$ satisfies condition (1.1). Let $J : \mathcal{A} \to \mathcal{C}(K)$ be the natural map given by

$$Jx(\varphi) = \varphi(x).$$

Clearly, J is an algebra homomorphism, $J(e) = 1$, and $\|J\| = 1$. In order to prove that J is an isometry we need to establish the following:

Claim. *Suppose that x in $\mathcal{A}$ is such that $\|Jx\|_{\mathcal{C}(K)} \le 1$. Then for each $\epsilon > 0$ there exists a $t_\varepsilon > 0$ for which*

$$\|e - t_\varepsilon(1+\epsilon)e - t_\varepsilon x\| < 1.$$

If the claim fails to be true, there is an x in $\mathcal{A}$ with $\|Jx\|_{\mathcal{C}(K)} \le 1$ so that for some $\epsilon > 0$ we have

$$\|e - t(1+\epsilon)e - tx\| \ge 1 \qquad (t \ge 0).$$

By the Hahn-Banach theorem (invoked to separate $\{e - t(1+\epsilon)e - tx :\ t \ge 0\}$ from the open unit ball) we can find a linear functional φ with $\|\varphi\| = 1$ and

$$\varphi(e - t(1+\epsilon)e - tx) \ge 1 \qquad (t \ge 0).$$

In particular, φ lies in $\mathcal{S}$ and $\varphi((1+\epsilon)e + x) \le 0$. Hence $|\varphi(x)| \ge 1 + \epsilon$. But now, using the the Krein-Milman theorem and Lemma 3.2, we deduce that there exists φ' in K with $|\varphi'(x)| \ge 1 + \epsilon$. Thus $\|Jx\|_{\mathcal{C}(K)} > 1$, a contradiction.

Combining the claim with Proposition 2.3 (i) we see that $\|Jx\|_{\mathcal{C}(K)} \le 1$ implies that $(1+\epsilon)e + x \in \mathcal{A}_+$ for all $\epsilon > 0$, so $e + x \in \mathcal{A}_+$. Applying the same reasoning to $-x$ we have $e - x \in \mathcal{A}_+$. Hence, by (3.1), we obtain

$$\|x\| = \tfrac{1}{2}\|(e+x) - (e-x)\| \le \tfrac{1}{2}\|(e+x) + (e-x)\| = 1.$$

Thus J is an isometry. Finally, J maps $\mathcal{A}$ onto $\mathcal{C}(K)$ by the Stone-Weierstrass theorem. ∎

4. CONCLUDING REMARKS. Condition (1.1) may appear to be innocuous to the reader, but there are well-known commutative real algebras with identity where (1.1) fails. We illustrate this with a few examples suggested by the referee:

(a) Any complex Banach space $\mathcal{C}_{\mathbb{C}}(K)$ of continuous functions on a compact Hausdorff space K is in particular a commutative real Banach algebra. One readily sees that condition (1.1) fails by taking, for instance, a to be the constant function 1 and b the constant function i.

(b) The real algebra $\mathcal{C}^{(1)}[0, 1]$ of continuously differentiable real-valued functions on $[0, 1]$ with the norm

$$\|f\| = \max_{0\le t\le 1} |f(t)| + \max_{0\le t\le 1} |f'(t)|$$

is a commutative Banach algebra with unit that seems similar to $\mathcal{C}[0, 1]$ but fails to obey (1.1). Take, for instance, $a = e^x$ and $b = e^{-x}$.

(c) Let $\ell_1(\mathbb{Z}_+)$ be the space of all formal power series $\sum_{n=0}^{\infty} a_n t^n$ (with real coefficients) with $(a_n)_{n=0}^{\infty} \in \ell_1$ and with the norm

$$\left\| \sum_{n=0}^{\infty} a_n t^n \right\| = \sum_{n=0}^{\infty} |a_n|.$$

To see that condition (1.1) fails in $\ell_1(\mathbb{Z}_+)$ take, for instance, $a = 1 - 2t^2$ and $b = 2t + t^2$.

We now observe that our proof of Theorem 1.1 required the full force of hypothesis (1.1) only at the very last step. Prior to that we used only the weaker hypothesis

$$\|a^2\| \le \|a^2 + b^2\| \qquad (a, b \in \mathcal{A}). \tag{4.1}$$

Condition (4.1) implies (3.2), which was used in Lemmas 3.1 and 3.2. However, this hypothesis allows us to deduce only that $\mathcal{A}$ is 2-isomorphic to $\mathcal{C}(K)$, i.e.,

$$\frac{1}{2}\|x\| \le \|Jx\|_{\mathcal{C}(K)} \le \|x\| \quad (x \in \mathcal{A}),$$

so that $\|J\| = 1$ and $\|J^{-1}\| \le 2$. That this is best possible is clear from the norm on $\mathcal{C}(K)$ given by

$$|||f||| = \|f_+\|_{\mathcal{C}(K)} + \|f_-\|_{\mathcal{C}(K)},$$

where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Under this norm $\mathcal{C}(K)$ is a commutative real Banach algebra satisfying inequality (4.1) but not inequality (1.1).

We also note that the Gelfand-Naimark representation of (*complex*) commutative C^*-algebras (see [**5**] and [**4**, p. 242]) can be obtained from Theorem 1.1. To this end, we recall that a map $a \mapsto a^*$ is called an *involution* on the complex algebra $\mathcal{A}$ if it enjoys the following properties:

(i) $(a^*)^* = a$,

(ii) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^* \quad (\alpha, \beta \in \mathbb{C})$,

(iii) $(ab)^* = b^* a^*$.

In the Banach algebra $\mathcal{C}_{\mathbb{C}}(K)$ of complex-valued continuous functions on a compact Hausdorff space K we have a natural involution, namely, complex conjugation: $f^*(s) = \overline{f(s)}$ for s in K.

If $a^* = a$, then a is called *self-adjoint*. Every x in $\mathcal{A}$ can be written in a unique way as $x = a + ib$, where a and b are self-adjoint ($a = (x + x^*)/2$, $b = (x - x^*)/2i$).

Theorem 4.1 (Gelfand-Naimark). *If $\mathcal{A}$ is a commutative complex Banach algebra with an identity e such that $\|e\| = 1$ and an involution $*$ such that*

$$\|a^* a\| = \|a\|^2 \qquad (a \in \mathcal{A}), \tag{4.2}$$

then $\mathcal{A}$ is isometrically $*$-algebra isomorphic to $\mathcal{C}_{\mathbb{C}}(K)$ for some compact Hausdorff space K.

Proof. Note first that (4.2) implies that the involution is isometric on $\mathcal{A}$:

$$\|a^*\|^2 = \|(a^{**})a^*\| = \|aa^*\| = \|a\|^2.$$

Let $\mathcal{A}_{\mathbb{R}}$ be the closed subalgebra of $\mathcal{A}$ comprising all self-adjoint elements of $\mathcal{A}$. Then $\mathcal{A}_{\mathbb{R}}$ is a real Banach algebra. We show that it satisfies condition (1.1). Hence for a and b in $\mathcal{A}_{\mathbb{R}}$ we have $\|(a+ib)^2\| = \|(a-ib)^2\|$, from which it follows that

$$\begin{aligned}
\|a^2 - b^2\| &= \tfrac{1}{2}\|(a+ib)^2 + (a-ib)^2\| \\
&\le \|(a+ib)^2\| \le \|(a+ib)\|^2 \\
&= \|(a+ib)(a-ib)\| = \|a^2 + b^2\|.
\end{aligned}$$

Accordingly, we can invoke Theorem 1.1 to produce an isometric isomorphism of real Banach algebras $J : \mathcal{A}_{\mathbb{R}} \to \mathcal{C}(K) = \mathcal{C}_{\mathbb{R}}(K)$ for some compact Hausdorff space K. The map J trivially extends to a $*$-algebra isomorphism $\tilde{J} : \mathcal{A} \to \mathcal{C}_{\mathbb{C}}(K)$. Finally, if $x = a + ib$ with a and b in $\mathcal{A}_{\mathbb{R}}$ we have

$$\|x\|^2 = \|a^2 + b^2\| = \|\tilde{J}(a^2 + b^2)\| = \|\tilde{J}x\|^2,$$

so $\tilde{J}$ is an isometry. ∎

ACKNOWLEDGMENTS The authors would like to thank Robert Burckel for his helpful comments. The second author was supported by NSF grant DMS-0244515.

REFERENCES

1. F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.

2. R. Arens, Representation of *-algebras, *Duke Math. J.* **14** (1947) 269–282.
3. R. B. Burckel, *Characterizations of $C(X)$ among Its Subalgebras*, Lecture Notes in Pure and Applied Mathematics, vol. 6. Marcel Dekker, New York, 1972.
4. J. B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1985.
5. I. Gelfand and M. Naimark, On the imbedding of normed rings into the ring of operators in Hilbert space, *Rec. Math. [Mat. Sbornik] N.S.* **12** (1943) 197–213.
6. T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras, Vol. I., Algebras and Banach Algebras*, Encyclopedia of Mathematics and its Applications, no. 49, Cambridge University Press, Cambridge, 1994.

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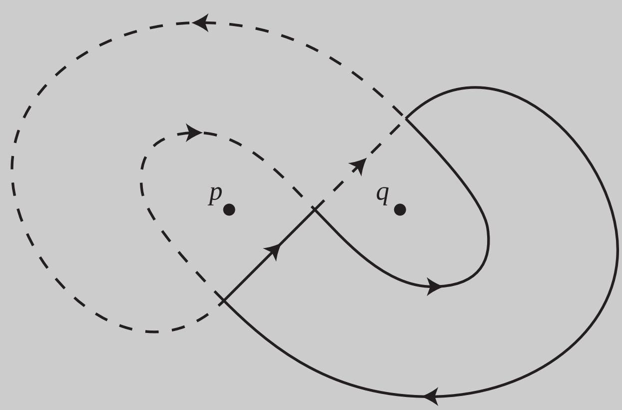
Oral Exam Solution



Let C_1 be the dashed contour and C_2 the solid contour. Then C_1 and C_2 are null-homotopic in $\mathbb{C} \setminus \{p, q\}$, so by the Cauchy integral theorem,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0 + 0 = 0.$$

The problem can also be solved by observing that the original contour C is homologous to zero in $\mathbb{C} \setminus \{p, q\}$ and then applying the homology version of Cauchy's theorem; see B. Palka, *An Introduction to Complex Function Theory*, Springer-Verlag, New York, 1991, Theorem 5.1, p. 188. The purpose of my question is to test if the candidate knows the homology version of Cauchy's theorem and, if not, to show them how to reduce the homology version to the homotopy version.

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