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Traces of compact operators and the noncommutative residue

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Abstract

We extend the noncommutative residue of M. Wodzicki on compactly supported classical pseudo-differential operators of order $-d$ and generalise A. Connes' trace theorem, which states that the residue can be calculated using a singular trace on compact operators. Contrary to the role of the noncommutative residue for the classical pseudo-differential operators, a corollary is that the pseudo-differential operators of order $-d$ do not have a 'unique' trace; pseudo-differential operators can be non-measurable in Connes' sense. Other corollaries are given clarifying the role of Dixmier traces in noncommutative geometry, including the definitive statement of Connes' original theorem.

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1. Introduction

A. Connes proved, [8, Theorem 1], that

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\mathrm{Res}_W(P)$$

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where P is a classical pseudo-differential operator of order $-d$ on a d-dimensional closed Riemannian manifold, Tr_ω is a Dixmier trace (a trace on the compact operators with singular values $O(n^{-1})$ which is not an extension of the canonical trace), [13], and Res_W is the noncommutative residue of M. Wodzicki, [44].

Connes' trace theorem, as it is known, has become the cornerstone of noncommutative integration in noncommutative geometry, [9]. Applications of Dixmier traces as the substitute noncommutative residue and integral in non-classical spaces range from fractals, [31,22], to foliations [3], to spaces of noncommuting co-ordinates, [10,21], and applications in string theory and Yang–Mills, [12,14,40,8], Einstein–Hilbert actions and particle physics' standard model, [11,6,30].

Connes' trace theorem, though, is not complete. There are other traces, besides Dixmier traces, on the ideal of compact operators whose singular values are $O(n^{-1})$. Wodzicki showed that the noncommutative residue is essentially the unique trace on classical pseudo-differential operators of order $-d$, so it should be expected that every suitably normalised trace computes the noncommutative residue. Also, all pseudo-differential operators have a notion of principal symbol and Connes' trace theorem opens the question of whether the principal symbol of non-classical operators can be used to compute their Dixmier trace.

We generalise Connes' trace theorem. We introduce an extension of the noncommutative residue that relies only on the principal symbol of a pseudo-differential operator, and we show that the extension calculates the Dixmier trace of the operator. The following definition and theorem apply to a much wider class of Hilbert–Schmidt operators, called Laplacian modulated operators, that we develop in the text. Here, in the introduction, we mention only pseudo-differential operators.

A pseudo-differential operator $P : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ is compactly based if Pu has compact support for all $u \in C^\infty(\mathbb{R}^d)$. Equivalently the (total) symbol of P has compact support in the first variable.

Definition 1.1 (*Extension of the Noncommutative Residue*). Let $P : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ be a compactly based pseudo-differential operator of order $-d$ with symbol p. The linear map

$$P \mapsto \mathrm{Res}(P) := \left[\left\{ \frac{d}{\log(1+n)} \int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p(x,\xi) d\xi \, dx \right\}_{n=1}^{\infty} \right]$$

we call the *residue* of P, where $[\cdot]$ denotes the equivalence class in ℓ_∞ / c_0.

Here ℓ_∞ denotes the space of bounded complex-valued sequences, and c_0 denotes the subspace of vanishing at infinity convergent sequences. Alternatively, any sequence $\mathrm{Res}_n(P)$, $n \in \mathbb{N}$, such that

$$\int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p(x,\xi) d\xi \, dx = \frac{1}{d} \mathrm{Res}_n(P) \log n + o(\log n)$$

defines the residue $\mathrm{Res}(P) = [\mathrm{Res}_n(P)] \in \ell_\infty / c_0$. We identify the equivalence classes of constant sequences in ℓ_∞ / c_0 with scalars. In the case that $\mathrm{Res}(P)$ is the class of a constant sequence, then we say that $\mathrm{Res}(P)$ is a scalar and identify it with the limit of the constant sequence. Note that a dilation invariant state $\omega \in \ell_\infty^*$ vanishes on c_0. Hence

$$\omega([c_n]) := \omega(\{c_n\}_{n=1}^\infty), \quad \{c_n\}_{n=1}^\infty \in \ell_\infty$$

is well-defined as a linear functional on ℓ_∞ / c_0.

Theorem 1.2 (*Trace Theorem*)**.** *Let* $P : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ *be a compactly based pseudo-differential operator of order* $-d$ *with residue* $\mathrm{Res}(P)$*. Then (the extension)* $P : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ *is a compact operator with singular values* $O(n^{-1})$ *and:*

(i)
$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\omega(\mathrm{Res}(P))$$
for a Dixmier trace Tr_ω*;*

(ii)
$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\mathrm{Res}(P)$$
for every Dixmier trace Tr_ω *iff*
$$\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} p(x,\xi)d\xi\, dx = \frac{1}{d}\mathrm{Res}(P)\ \log n + o(\log n)$$
for a scalar $\mathrm{Res}(P)$*;*

(iii)
$$\tau(P) = \frac{\tau\circ \mathrm{diag}\left(\left\{\frac{1}{n}\right\}_{n=1}^\infty\right)}{d(2\pi)^d}\mathrm{Res}(P)$$
for every trace τ *on the compact operators with singular values* $O(n^{-1})$ *iff*
$$\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} p(x,\xi)d\xi\, dx = \frac{1}{d}\mathrm{Res}(P)\ \log n + O(1)$$
for a scalar $\mathrm{Res}(P)$*.*

Here diag is the diagonal operator with respect to an arbitrary orthonormal basis of $L_2(\mathbb{R}^d)$. This theorem is Theorem 6.32 in Section 6.3 of the text. There is a version for closed manifolds, Theorem 7.6 in Section 7.3.

The result
$$\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} p(x,\xi)dxd\xi = \frac{1}{d}\mathrm{Res}_W(P)\log n + O(1) \tag{1.1}$$
for a classical pseudo-differential operator P demonstrates that the residue in Definition 1.1 is an extension of the noncommutative residue and, from Theorem 1.2(iii) we obtain the following. Henceforth, we denote by $\mathcal{L}_{1,\infty}$ the ideal of compact operators with singular values $O(n^{-1})$.

Theorem 1.3 (*Connes' Trace Theorem*)**.** *Let* $P : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ *be a classical compactly based pseudo-differential operator of order* $-d$ *with noncommutative residue* $\mathrm{Res}_W(P)$*. Then (the extension)* $P \in \mathcal{L}_{1,\infty}$ *and*
$$\tau(P) = \frac{1}{d(2\pi)^d}\mathrm{Res}_W(P)$$
for every trace τ *on* $\mathcal{L}_{1,\infty}$ *with* $\tau(\mathrm{diag}\{k^{-1}\}_{k=1}^\infty) = 1$*.*

This result is Corollary 6.35 in the text. We show the same result for manifolds, Corollary 7.22.

In the text we construct a pseudo-differential operator Q whose residue $\mathrm{Res}(Q)$ is not scalar. Using Theorem 1.2(ii) we obtain the following.

Theorem 1.4 (*Pseudo-Differential Operators do not have Unique Trace*)**.** *There exists a compactly based pseudo-differential operator $Q : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ of order $-d$ such that the value $\mathrm{Tr}_\omega(Q)$ depends on the Dixmier trace Tr_ω.*

The operator Q is nothing extravagant, one needs only to interrupt the homogeneity of the principal symbol; see Corollary 6.34 in the text. There is a similar example on closed manifolds, Corollary 7.23. In summary the pseudo-differential operators of order $-d$ form quite good examples for the theory of singular traces. Some operators, including classical ones, have the same value for every trace. Others have distinct trace even for the smaller set of Dixmier traces. Theorem 1.4 shows that the qualifier classical cannot be omitted from the statement of Theorem 1.3.

The Laplacian modulated operators we introduce are a wide enough class to admit the operators $M_f(1-\Delta)^{-d/2}$ where $f \in L_2(\mathbb{R}^d)$ (almost) has compact support, $M_f u(x) = f(x)u(x)$, $u \in C_c^\infty(\mathbb{R}^d)$, and Δ is the Laplacian on $\mathbb{R}^d$. Using Theorem 6.32(iii) (the version of Theorem 1.2(iii) for Laplacian modulated operators) we prove the next result as Corollary 6.38 in the text.

Theorem 1.5 (*Integration of Square Integrable Functions*)**.** *If $f \in L_2(\mathbb{R}^d)$ has compact support then $M_f(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$ such that*

$$\tau(M_f(1-\Delta)^{-d/2}) = \frac{\mathrm{Vol}\,\mathbb{S}^{d-1}}{d(2\pi)^d}\int_{\mathbb{R}^d} f(x)dx$$

for every trace τ on $\mathcal{L}_{1,\infty}$ with $\tau(\mathrm{diag}\{k^{-1}\}_{k=1}^\infty) = 1$.

The same statement can be made for closed manifolds, omitting of course the requirement for compact support of f, and with the Laplace–Beltrami operator in place of the ordinary Laplacian, Corollary 7.24.

Finally, through results on modulated operators, specifically Theorem 5.2, we obtain the following spectral formula for the noncommutative residue on a closed manifold, Corollary 7.19. The eigenvalue part of this formula was observed by T. Fack, [17], and proven in [2, Corollary 2.14] (i.e. the log divergence of the series of eigenvalues listed with multiplicity and ordered so that their absolute value is decreasing is equal to the noncommutative residue).

Theorem 1.6 (*Spectral Formula of the Noncommutative Residue*)**.** *Let P be a classical pseudo-differential operator of order $-d$ on a closed d-dimensional manifold (X, g). Then*

$$d^{-1}(2\pi)^{-d}\,\mathrm{Res}_W(P) = \lim_n \frac{1}{\log n}\sum_{j=1}^n (Pe_j, e_j) = \lim_n \frac{1}{\log n}\sum_{j=1}^n \lambda_j(P)$$

where $\{\lambda_j(P)\}_{j=1}^\infty$ are the non-zero eigenvalues of P with multiplicity in any order so that $|\lambda_j(P)|$ is decreasing, $(\cdot,\cdot)$ is the inner product on $L^2(X, g)$, and $(e_j)_{j=1}^\infty$ is an orthonormal basis of eigenvectors of the Hodge-Laplacian $-\Delta_g$ (the negative of the Laplace–Beltrami operator) such that $-\Delta_g e_j = \lambda_j e_j$, $\lambda_1 \le \lambda_2 \le \cdots$ are increasing.

Theorems 1.2–1.6 are the main results of the paper.

Our proof of the trace theorem, Theorem 1.2, uses commutator subspaces and it is very different to the original proof of Connes' theorem. We finish the Introduction by explaining our proof of Connes' theorem in its plainest form.

Wodzicki initiated the study of the noncommutative residue in [44]. The noncommutative residue $\mathrm{Res}_W(P)$ vanishes if and only if a classical pseudo-differential operator P is a finite sum of commutators. Paired with the study of commutator subspaces of ideals, [36], this result initiated an extensive work characterising commutator spaces for arbitrary two sided ideals of compact operators, [15]; see also the survey, [43]. Our colleague Nigel Kalton, whose sudden passing was a tremendous loss to ourselves personally and to mathematics in general, contributed fundamentally to this area, through, of course, [27,28,16,19,29].

The commutator subspace, put simply, is the kernel of all traces on a two-sided ideal of compact linear operators of a Hilbert space $\mathcal{H}$ to itself. If one could show that a compact operator T belongs to the ideal $\mathcal{L}_{1,\infty}$ (operators whose singular values are $O(n^{-1})$) and that it satisfies

$$T - c\ \mathrm{diag}\left\{\frac{1}{k}\right\}_{k=1}^{\infty} \in \mathrm{Com}\,\mathcal{L}_{1,\infty} \tag{1.2}$$

for a constant c (here $\mathrm{Com}\,\mathcal{L}_{1,\infty}$ denotes the commutator subspace, i.e. the linear span of elements $AB - BA$, $A \in \mathcal{L}_{1,\infty}$, B is a bounded linear operator of $\mathcal{H}$ to itself, and diag is the diagonal operator in some chosen basis), then

$$\tau(T) = c$$

for a constant c for every trace τ with $\tau(\mathrm{diag}\{k^{-1}\}_{k=1}^{\infty}) = 1$. This is the type of formula Connes' original theorem suggests. Our first result, Theorem 3.3, concerns differences in the commutator subspace, i.e. (1.2), and it states that

$$T - S \in \mathrm{Com}\,\mathcal{L}_{1,\infty} \Leftrightarrow \sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \lambda_j(S) = O(1)$$

by using the fundamental results of [15,27,16]. Here $\{\lambda_j(T)\}_{j=1}^{\infty}$ are the non-zero eigenvalues of T, with multiplicity, in any order so that $|\lambda_j(T)|$ is decreasing, with the same for S. If there are no non-zero eigenvalues the sum is zero. Actually, all our initial results involve general ideals but, to stay on message, we specialise to $\mathcal{L}_{1,\infty}$ in the introduction. Then our goal, (1.2), has the explicit spectral form

$$\sum_{j=1}^{n} \lambda_j(T) - c\ \log n = O(1). \tag{1.3}$$

The crucial step therefore is the following theorem on sums of eigenvalues of pseudo-differential operators. As far as we know the theorem is new. Results about eigenvalues are known, of course, for positive elliptic operators on closed manifolds. The following result is for all operators of order $-d$.

Theorem 1.7. *Let $P : C_c^{\infty}(\mathbb{R}^d) \to C_c^{\infty}(\mathbb{R}^d)$ be a compactly based pseudo-differential operator of order $-d$ and with symbol p. Then*

$$\sum_{j=1}^{n} \lambda_j(P) - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p(x,\xi)\,dx\,d\xi = O(1)$$

where $\{\lambda_j(P)\}_{j=1}^{\infty}$ are the non-zero eigenvalues of P, with multiplicity, in any order so that $|\lambda_j(P)|$ is decreasing.

The theorem is Theorem 6.23 in the text, which is shown for the so-called Laplacian modulated operators, and we have stated here the special case for compactly supported pseudo-differential operators. Given Theorem 1.7 the proof of Theorem 1.2 follows, as indicated in Section 6.3.

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space with inner product complex linear in the first variable and let $\mathcal{B}(\mathcal{H})$ (respectively, $\mathcal{K}(\mathcal{H})$) denote the bounded (respectively, compact) linear operators on $\mathcal{H}$. If $(e_n)_{n=1}^{\infty}$ is a fixed orthonormal basis of $\mathcal{H}$ and $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers define the operator

$$\operatorname{diag}\{a_n\}_{n=1}^{\infty} := \sum_{n=1}^{\infty} a_n e_n e_n^*,$$

where $e_n^*(h) := (h, e_n)$, $h \in \mathcal{H}$, and $(\cdot,\cdot)$ denotes the inner product. The Calkin space $\operatorname{diag}(\mathcal{I})$ associated to a two-sided ideal $\mathcal{I}$ of compact operators is the sequence space

$$\operatorname{diag}(\mathcal{I}) := \{\{a_n\}_{n=1}^{\infty} | \operatorname{diag}\{a_n\}_{n=1}^{\infty} \in \mathcal{I}\}.$$

The Calkin space is independent of the choice of orthonormal basis and an operator $T \in \mathcal{I}$ if and only if the sequence $\{s_n(T)\}_{n=1}^{\infty}$ of its singular values belongs to $\operatorname{diag}(\mathcal{I})$, [5], [42, Section 2].

The non-zero eigenvalues of a compact operator T form either a sequence converging to 0 or a finite set. In the former case we define an *eigenvalue sequence* for T as the sequence of eigenvalues $\{\lambda_n(T)\}_{n=1}^{\infty}$, each repeated according to algebraic multiplicity, and arranged in an order (not necessarily unique) such that $\{|\lambda_n(T)|\}_{n=1}^{\infty}$ is decreasing (see, [42, p. 7]). In the latter case we construct a similar finite sequence $\{\lambda_n(T)\}_{n=1}^{N}$ of non-zero eigenvalues and then set $\lambda_n(T) = 0$ for $n > N$. If T is quasinilpotent then we take the zero sequences as the eigenvalue sequence of T. The appearance of eigenvalues will always imply that they are ordered as to form an eigenvalue sequence. For a normal compact operator T, $|\lambda_n(T)| = \lambda_n(|T|) = s_n(T)$, $n \in \mathbb{N}$, for any eigenvalue sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$. This implies that $\{\lambda_n(T)\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ for a normal operator $T \in \mathcal{I}$. The following well-known lemma will be useful so we provide the proof for completeness.

Lemma 2.1. *Suppose* $\operatorname{diag}(\mathcal{I})$ *is a Calkin space and* $\nu \in \operatorname{diag}(\mathcal{I})$ *is a positive sequence. If* $a := \{a_n\}_{n=1}^{\infty}$ *is a complex-valued sequence such that* $|a_n| \le \nu_n$ *for all* $n \in \mathbb{N}$*, then* $a \in \operatorname{diag}(\mathcal{I})$.

Proof. Set, for $n \in \mathbb{N}$, $b_n := \frac{a_n}{\nu_n}$ if $\nu_n \neq 0$ and $b_n := 0$ if $\nu_n = 0$. Then $b := \{b_n\}_{n=1}^{\infty} \in \ell_\infty$. Hence $\operatorname{diag} b \in \mathcal{B}(\mathcal{H})$ and $\operatorname{diag} a = \operatorname{diag}(b \cdot \nu) = \operatorname{diag} b \cdot \operatorname{diag} \nu \in \mathcal{I}$ since $\mathcal{I}$ is an ideal. □

Corollary 2.2. *Suppose* $T \in \mathcal{I}$ *is normal. Then* $\{\lambda_n(T)\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ *where* $\{\lambda_n(T)\}_{n=1}^{\infty}$ *is an eigenvalue sequence of* T.

Proof. Using the spectral theorem for normal operators, $|\lambda_n(T)| = s_n(T)$, $n \in \mathbb{N}$. Hence $|\lambda_n(T)| \le \nu_n$ where $\nu_n := s_n(T) \in \operatorname{diag}(\mathcal{I})$ is positive. By Lemma 2.1 $\{\lambda_n(T)\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$. □

The statement that $\{\lambda_n(T)\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ for every $T \in \mathcal{I}$ is false in general. Geometrically stable ideals were introduced by Kalton, [28]. A two-sided ideal $\mathcal{I}$ is called *geometrically stable* if given any decreasing nonnegative sequence $\{s_n\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ we have $\{(s_1 s_2 \dots s_n)^{1/n}\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$. It is a theorem of Kalton and Dykema that if $\mathcal{I}$ is geometrically stable then $\{\lambda_n(T)\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ for all $T \in \mathcal{I}$, [16, Theorem 1.3]. An ideal $\mathcal{I}$ is called Banach (respectively, quasi-Banach) if there is a norm (respectively, quasi-norm) $\|\cdot\|_{\mathcal{I}}$ on $\mathcal{I}$ such that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is complete and we have $\|ATB\|_{\mathcal{I}} \le \|A\|_{\mathcal{B}(\mathcal{H})} \|T\|_{\mathcal{I}} \|B\|_{\mathcal{B}(\mathcal{H})}$, $A, B \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{I}$. Equivalently, $\operatorname{diag}(\mathcal{I})$ is a Banach (respectively, quasi-Banach) symmetric sequence space; see e.g. [42,32,29]. Every quasi-Banach ideal is geometrically stable, [28]. An example of a non-geometrically stable ideal is given in [16].

If $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals we denote by $\mathcal{I}_1\mathcal{I}_2$ the ideal generated by all products AB, BA for $A \in \mathcal{I}_1$ and $B \in \mathcal{I}_2$. If $A, B \in \mathcal{B}(\mathcal{H})$ we let $[A, B] = AB - BA$. We define $[\mathcal{I}_1, \mathcal{I}_2]$ to be the linear span of all $[A, B]$ for $A \in \mathcal{I}_1$ and $B \in \mathcal{I}_2$. It is a theorem that $[\mathcal{I}_1, \mathcal{I}_2] = [\mathcal{I}_1\mathcal{I}_2, \mathcal{B}(\mathcal{H})]$, [15, Theorem 5.10]. The space $\operatorname{Com}\mathcal{I} := [\mathcal{I}, \mathcal{B}(\mathcal{H})] \subset \mathcal{I}$ is called the commutator subspace of an ideal $\mathcal{I}$.

3. A theorem on the commutator subspace

There is a fundamental description of the normal operators $T \in \operatorname{Com}\mathcal{I}$ given by Dykema, Figiel, Weiss and Wodzicki, [15]; see also [28, Theorem 3.1].

Theorem 3.1. *Suppose $\mathcal{I}$ is a two-sided ideal in $\mathcal{K}(\mathcal{H})$ and $T \in \mathcal{I}$ is normal. Then the following statements are equivalent:*

(i) $T \in \operatorname{Com}\mathcal{I}$;

(ii) *for any eigenvalue sequence* $\{\lambda_n(T)\}_{n=1}^{\infty}$,

$$\left\{\frac{1}{n}\sum_{j=1}^{n}\lambda_j(T)\right\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I}); \tag{3.1}$$

(iii) *for any eigenvalue sequence* $\{\lambda_n(T)\}_{n=1}^{\infty}$,

$$\frac{1}{n}\left|\sum_{j=1}^{n}\lambda_j(T)\right| \le \mu_n \tag{3.2}$$

for a positive decreasing sequence $\mu = \{\mu_n\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$.

We would like to observe the following refinement of Theorem 3.1.

Theorem 3.2. *Suppose $\mathcal{I}$ is a two-sided ideal in $\mathcal{K}(\mathcal{H})$ and $T, S \in \mathcal{I}$ are normal. Then the following statements are equivalent:*

(i) $T - S \in \operatorname{Com}\mathcal{I}$;

(ii) *for any eigenvalue sequences* $\{\lambda_j(T)\}_{j=1}^{\infty}$ *of* T *and* $\{\lambda_j(S)\}_{j=1}^{\infty}$ *of* S,

$$\left\{\frac{1}{n}\left(\sum_{j=1}^{n}\lambda_j(T) - \sum_{j=1}^{n}\lambda_j(S)\right)\right\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I}); \tag{3.3}$$

(iii) *for any eigenvalue sequences* $\{\lambda_j(T)\}_{j=1}^{\infty}$ *of* T *and* $\{\lambda_j(S)\}_{j=1}^{\infty}$ *of* S,

$$\frac{1}{n}\left|\sum_{j=1}^{n}\lambda_j(T) - \sum_{j=1}^{n}\lambda_j(S)\right| \le \mu_n \tag{3.4}$$

for a positive decreasing sequence $\mu = \{\mu_n\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$.

Proof. Observe that the normal operator

$$V = \begin{pmatrix} T & 0 \\ 0 & -S \end{pmatrix} = \begin{pmatrix} T - S & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}$$

belongs to $\operatorname{Com}\mathcal{I}$ if and only if $T - S \in \operatorname{Com}\mathcal{I}$. Indeed, it is straightforward to see that the eigenvector sequence of the operator $\begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}$ satisfies (3.1) and, since $S \in \mathcal{I}$, we have $\begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix} \in \operatorname{Com}\mathcal{I}$ by Theorem 3.1.

(iii) ⇒ (ii) Let $a_n := n^{-1} \sum_{j=1}^{n} (\lambda_j(T) - \lambda_j(S))$. By Lemma 2.1 $\{a_n\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$.

(ii) ⇒ (i) We have

$$\sum_{j=1}^{n} \lambda_j(V) = \sum_{j=1}^{r} \lambda_j(T) - \sum_{j=1}^{s} \lambda_j(S)$$

where $r + s = n$ and $|\lambda_{r+1}(T)|, |\lambda_{s+1}(S)| \le |\lambda_{n+1}(V)|$. Hence

$$\left| \sum_{j=1}^{n} \lambda_j(V) - \sum_{j=1}^{n} \lambda_j(T) + \sum_{j=1}^{n} \lambda_j(S) \right| \le n|\lambda_n(V)|. \tag{3.5}$$

Since $\nu := \{|\lambda_n(V)|\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ is positive and decreasing

$$\frac{1}{n} \sum_{j=1}^{n} \lambda_j(V) - \frac{1}{n} \left(\sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \lambda_j(S) \right) \in \operatorname{diag}(\mathcal{I})$$

by Lemma 2.1. Hence $\frac{1}{n} \sum_{j=1}^{n} \lambda_j(V) \in \operatorname{diag}(\mathcal{I})$ if $\frac{1}{n}(\sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \lambda_j(S)) \in \operatorname{diag}(\mathcal{I})$. It follows from Theorem 3.1 that $V \in \operatorname{Com}\mathcal{I}$.

(i) ⇒ (iii) We note from Eq. (3.5) that

$$\frac{1}{n} \left| \sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \lambda_j(S) \right| \le |\lambda_n(V)| + \frac{1}{n} \left| \sum_{j=1}^{n} \lambda_j(V) \right|.$$

The sequence $\nu := \{|\lambda_n(V)|\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ is positive and decreasing and, since $V \in \operatorname{Com}\mathcal{I}$, there exists a decreasing sequence ν' such that $\frac{1}{n} \left| \sum_{j=1}^{n} \lambda_j(V) \right| \le \nu'_n$. Now (iii) follows by setting $\mu = \nu + \nu'$. □

In [28], which used results in [15] although it appeared chronologically earlier, it was shown that Theorem 3.1 can be extended to non-normal operators under the hypothesis that $\mathcal{I}$ is geometrically stable. Theorem 3.2 can be extended similarly.

Theorem 3.3. *Suppose $\mathcal{I}$ is a geometrically stable ideal in $\mathcal{K}(\mathcal{H})$ and $T, S \in \mathcal{I}$. Then the following statements are equivalent:*

(i) $T - S \in \operatorname{Com}\mathcal{I}$;

(ii) *for any eigenvalue sequences $\{\lambda_j(T)\}_{j=1}^{\infty}$ of T and $\{\lambda_j(S)\}_{j=1}^{\infty}$ of S,*

$$\left\{ \frac{1}{n} \left(\sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \lambda_j(S) \right) \right\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I}); \tag{3.6}$$

(iii) *for any eigenvalue sequences $\{\lambda_j(T)\}_{j=1}^{\infty}$ of T and $\{\lambda_j(S)\}_{j=1}^{\infty}$ of S,*

$$\frac{1}{n} \left| \sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \lambda_j(S) \right| \le \mu_n \tag{3.7}$$

for a positive decreasing sequence $\mu = \{\mu_n\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$.

Proof. Let $T \in \mathcal{I}$. From [16, Corollary 2.5] $T = N + Q$ where $Q \in \mathcal{I}$ is quasinilpotent and $N \in \mathcal{I}$ is normal with eigenvalues and multiplicities the same as T. From [28, Theorem 3.3] we know $Q \in \operatorname{Com}\mathcal{I}$. Hence $T = N_T + Q_T$ and $S = N_S + Q_S$ where $Q_S, Q_T \in \operatorname{Com}\mathcal{I}$ are quasinilpotent and N_T, N_S are normal with eigenvalues and multiplicities the same as T and S, respectively. Since $T - S \in \operatorname{Com}\mathcal{I}$ if and only if $N_T - N_S \in \operatorname{Com}\mathcal{I}$ the results follow from Theorem 3.2. $\square$

We recall that $\operatorname{diag}\{\lambda_n(T)\}_{n=1}^{\infty} \in \mathcal{I}$ when $\mathcal{I}$ is geometrically stable. Theorem 3.3 therefore has the following immediate corollary.

Corollary 3.4. *Let $\mathcal{I}$ be a geometrically stable ideal in $\mathcal{K}(\mathcal{H})$ and $T \in \mathcal{I}$. Then $T - \operatorname{diag}\{\lambda_n(T)\}_{n=1}^{\infty} \in \operatorname{Com}\mathcal{I}$.*

Proof. Set $S := \operatorname{diag}\{\lambda_n(T)\}_{n=1}^{\infty} \in \mathcal{I}$. Then $\lambda_n(S) = \lambda_n(T)$, $n \in \mathbb{N}$, and $T - S \in \operatorname{Com}\mathcal{I}$ by Theorem 3.3. $\square$

4. Applications to traces

Suppose $\mathcal{I}$ is a two-sided ideal of compact operators. A trace $\tau : \mathcal{I} \to \mathbb{C}$ is a linear functional that vanishes on the commutator subspace, i.e. it satisfies the condition

$$\tau([A, B]) = 0, \quad A \in \mathcal{I},\ B \in \mathcal{B}(\mathcal{H}).$$

Note that we make no assumptions about continuity or positivity of the linear functional. The value

$$\tau(\operatorname{diag}\{a_n\}_{n=1}^{\infty}), \quad \{a_n\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$$

is independent of the choice of orthonormal basis. Therefore any trace $\tau : \mathcal{I} \to \mathbb{C}$ induces a linear functional $\tau \circ \operatorname{diag}$ (defined by the above value) on the Calkin space $\operatorname{diag}(\mathcal{I})$.

Corollary 4.1. *There are non-trivial traces on $\mathcal{I}$ if and only if $\operatorname{Com}\mathcal{I} \neq \mathcal{I}$, which occurs if and only if $\{\frac{1}{n}\sum_{j=1}^{n} s_j\}_{n=1}^{\infty} \notin \operatorname{diag}(\mathcal{I})$ for some positive sequence $\{s_n\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$.*

The proof is evident by considering the quotient vector space $\mathcal{I}/\operatorname{Com}\mathcal{I}$ and applying Theorem 3.1, so we omit it. The condition in Corollary 4.1 implies that traces on two-sided ideals other than the ideal of nuclear operators exist (e.g. the quasi-Banach ideal $\mathcal{L}_{1,\infty}$ such that $\operatorname{diag}(\mathcal{L}_{1,\infty}) = \ell_{1,\infty}$), [26]; see also [15, Section 5] for other examples of ideals that do and do not support non-trivial traces. In [16] it was shown that every trace on a geometrically stable ideal is determined by its associated functional applied to an eigenvalue sequence, which is an extension of the Lidskii theorem.

Corollary 4.2 (*Lidskii Theorem*)**.** *Let $\mathcal{I}$ be a geometrically stable ideal in $\mathcal{K}(\mathcal{H})$. Suppose $T \in \mathcal{I}$. Then*

$$\tau(T) = \tau \circ \operatorname{diag}(\{\lambda_n(T)\}_{n=1}^{\infty}) \tag{4.1}$$

for every trace $\tau : \mathcal{I} \to \mathbb{C}$ and any eigenvalue sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$ of T.

The proof, given Corollary 3.4, is trivial and therefore omitted. For ideals that are not geometrically stable ideals, it is possible that for some $T \in \mathcal{I}$ we have that $\{\lambda_n(T)\}_{n=1}^{\infty} \notin \operatorname{diag}(\mathcal{I})$. The Lidskii formulation as above will not be possible in this case. A general characterisation of traces on non-geometrically stable ideals requires an explicit formula for products $T = AS \in \mathcal{I}$ where A and S do not commute. Such a formula is also of interest when studying linear functionals on the bounded operators of the form $A \mapsto \tau(AS)$, $A \in \mathcal{B}(\mathcal{H})$, where $S \in \mathcal{I}$ is Hermitian and $\tau : \mathcal{I} \to \mathbb{C}$ is a trace (e.g. in A. Connes' noncommutative geometry, [9, Section 4]). We now characterise traces of products.

We introduce some terminology for systems of eigenvectors that are ordered to correspond with eigenvalue sequences. If T is a compact operator of infinite rank, we define an orthonormal sequence $(e_n)_{n=1}^{\infty}$ to be an *eigenvector sequence* for T if $Te_n = \lambda_n(T)e_n$ for all $n \in \mathbb{N}$ where $\{\lambda_n\}_{n=1}^{\infty}$ is an eigenvalue sequence. If T is Hermitian, an eigenvector sequence exists and there is an eigenvector sequence which forms a complete orthonormal system.

We will also need the following lemmas (see for example [15]).

Lemma 4.3. *Let $\mathcal{I}$ be a two-sided ideal in $\mathcal{K}(\mathcal{H})$. Suppose $D = \mathrm{diag}\{\alpha_n\}_{n=1}^{\infty}$ where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $|\alpha_n| \le \mu_n$ where $\{\mu_n\}_{n=1}^{\infty} \in \mathrm{diag}(\mathcal{I})$ is decreasing. Then*

$$\left|\sum_{j=1}^{n} \alpha_j - \sum_{j=1}^{n} \lambda_j(D)\right| \le 2n\mu_n.$$

Proof. We have $\lambda_j(D) = \alpha_{m_j}$ where $m_1, m_2, \ldots$ are distinct. Thus

$$\sum_{j=1}^{n} \lambda_j(D) = \sum_{j\in\mathbb{A}} \alpha_j$$

where $\mathbb{A} = \{m_1, \ldots, m_n\}$. If $k \in \mathbb{A} \setminus \{1, 2, \ldots, n\}$ we have $|\alpha_k| \le \mu_k \le \mu_n$. On the other hand, since

$$\mathbb{A} = \left\{m \in \mathbb{N} : \ \alpha_m = \lambda_j(D), \ \text{ for some } j \le n\right\},$$

if $k \in \{1, 2, \ldots, n\} \setminus \mathbb{A}$ we have $|\alpha_k| \le |\lambda_n(D)| \le \mu_n$. Hence

$$\left|\sum_{j=1}^{n} \lambda_j(D) - \sum_{j=1}^{n} \alpha_j\right| \le 2n\mu_n. \quad \square$$

Lemma 4.4. *Suppose $S \in \mathcal{K}(\mathcal{H})$ is Hermitian and $(e_j)_{j=1}^{\infty}$ is an eigenvector sequence for S. Suppose A is Hermitian and $H := \frac{1}{2}(AS + SA)$. Then we have*

$$\left|\sum_{j=1}^{n} \lambda_j(H) - \sum_{j=1}^{n} (ASe_j, e_j)\right| \le ns_{n+1}(H) + ns_{n+1}(S)\left(\frac{1}{n}\sum_{j=1}^{n} s_j(A)\right) \tag{4.2}$$

if $A \in \mathcal{K}(\mathcal{H})$ is compact, and

$$\left|\sum_{j=1}^{n} \lambda_j(H) - \sum_{j=1}^{n} (ASe_j, e_j)\right| \le ns_{n+1}(H) + ns_{n+1}(S)\|A\| \tag{4.3}$$

if $A \in \mathcal{B}(\mathcal{H})$ is bounded but not compact.

Proof. Let $(f_n)_{n=1}^{\infty}$ be an eigenvector sequence for H. Let P_n and Q_n be the orthogonal projections of $\mathcal{H}$ on $[e_1, \ldots, e_n]$ and $[f_1, \ldots, f_n]$ respectively and let R_n be the orthogonal projection on the linear span $[e_1, \ldots, e_n, f_1, \ldots, f_n]$.

If A is compact define $\beta_n := \frac{1}{n}\sum_{j=1}^n s_j(A)$. Otherwise set $\beta_n := \|A\|$. Since $\operatorname{rank}(R_n - P_n) \leq n$ we have

$$|\mathrm{Tr}(AS(R_n - P_n))| \leq n s_{n+1}(S)\beta_n.$$

Similarly

$$|\mathrm{Tr}(SA(R_n - P_n))| = |\mathrm{Tr}(A(R_n - P_n)S)| \leq n s_{n+1}(S)\beta_n,$$

and hence

$$|\mathrm{Tr}(H(R_n - P_n))| \leq n s_{n+1}(S)\beta_n.$$

Similarly

$$|\mathrm{Tr}(H(R_n - Q_n))| \leq n s_{n+1}(H).$$

Hence

$$\begin{aligned}|\mathrm{Tr}(H(P_n - Q_n))| &\leq |\mathrm{Tr}(H(P_n - R_n))| + |\mathrm{Tr}(H(R_n - Q_n))| \\ &\leq n s_{n+1}(H) + n s_{n+1}(S)\beta_n. \quad \square\end{aligned}$$

Theorem 4.5. *Let $\mathcal{I}_1$ be a two-sided ideal in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{I}_1 = \mathrm{Com}\,\mathcal{I}_1$ and $\mathcal{I}_2$ be a two-sided ideal in $\mathcal{K}(\mathcal{H})$. Let $\mathcal{I} = \mathcal{I}_1\mathcal{I}_2$. Suppose $S \in \mathcal{I}_2$ is Hermitian and that $(e_n)_{n=1}^\infty$ is an eigenvector sequence for S. Suppose $A \in \mathcal{I}_1$ is such that, for some decreasing positive sequence $\{\mu_n\}_{n=1}^\infty \in \mathrm{diag}(\mathcal{I})$, we have $|(ASe_n, e_n)| \leq \mu_n$, $n \in \mathbb{N}$. Then AS, $\mathrm{diag}\{(ASe_n, e_n)\}_{n=1}^\infty \in \mathcal{I}$,*

$$AS - \mathrm{diag}\{(ASe_n, e_n)\}_{n=1}^\infty \in \mathrm{Com}\,\mathcal{I}$$

and, hence, for every trace $\tau : \mathcal{I} \to \mathbb{C}$ we have

$$\tau(AS) = \tau \circ \mathrm{diag}(\{(ASe_n, e_n)\}_{n=1}^\infty) = \tau \circ \mathrm{diag}(\{(Ae_n, e_n)\lambda_n(S)\}_{n=1}^\infty).$$

Proof. Our assumptions imply that $AS \in \mathcal{I}$. Also by the assumption that $|(ASe_n, e_n)| \leq \mu_n$ for $\mu \in \mathrm{diag}(\mathcal{I})$, it follows from Lemma 2.1 that $\mathrm{diag}\{(ASe_n, e_n)\}_{n=1}^\infty \in \mathcal{I}$.

First let us assume that A is Hermitian. Set $D := \mathrm{diag}\{(ASe_n, e_n)\}_{n=1}^\infty$ and $\alpha_n := (ASe_n, e_n)$, $n \in \mathbb{N}$. By assumption $|\alpha_n| \leq \mu_n$ where $\mu \in \mathrm{diag}(\mathcal{I})$ is positive and decreasing and, by applying Lemma 4.3 to the sequence $\alpha_n := (ASe_n, e_n)$, we have

$$\left|\sum_{j=1}^n \lambda_j(D) - \sum_{j=1}^n (ASe_j, e_j)\right| \leq 2n\mu_n. \tag{4.4}$$

From Lemma 4.4 we have that

$$\left|\sum_{j=1}^n \lambda_j(H) - \sum_{j=1}^n (ASe_j, e_j)\right| \leq n s_{n+1}(H) + n s_{n+1}(S)\beta_n \tag{4.5}$$

where $H = \frac{1}{2}(AS + SA)$ and $\beta_n := \frac{1}{n}\sum_{j=1}^n s_j(A)$, $n \in \mathbb{N}$, if A is compact, or $\beta_n := \|A\|$, $n \in \mathbb{N}$, if A is bounded but not compact.

Suppose $A \in \mathcal{I}_1$ where $\mathcal{I}_1$ is an ideal of compact operators such that $\mathrm{Com}\,\mathcal{I}_1 = \mathcal{I}_1$. Then $|A| \in \mathrm{Com}\,\mathcal{I}_1$ and, from the equivalent conditions in Theorem 3.1 there exists a decreasing sequence $\mu' \in \mathrm{diag}(\mathcal{I}_1)$ such that $\frac{1}{n}\sum_{j=1}^n s_j(A) \leq \mu'_n$, $n \in \mathbb{N}$. Set $\nu := \{2\mu_n + s_{n+1}(H) + \mu'_n s_{n+1}(S)\}_{n=1}^\infty$ which is positive and decreasing. By assumption $\{\mu_n\}_{n=1}^\infty \in \mathrm{diag}(\mathcal{I})$. By the fact that $\mathcal{I}$ is a two-sided ideal of compact operators then $H = \frac{1}{2}(AS + SA) \in \mathcal{I}$ and $\{s_{n+1}(H)\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$. Finally $\mu'_n s_{n+1}(S) \in \operatorname{diag}(\mathcal{I}_1) \cdot \operatorname{diag}(\mathcal{I}_2) \subset \operatorname{diag}(\mathcal{I})$. Hence $\nu \in \operatorname{diag}(\mathcal{I})$. Then, using (4.4) and (4.5),

$$\left|\sum_{j=1}^{n} \lambda_j(D) - \sum_{j=1}^{n} \lambda_j(H)\right| \leq n\nu_n. \tag{4.6}$$

Now suppose $A \in \mathcal{I}_1 = \mathcal{B}(\mathcal{H}) = \operatorname{Com}\mathcal{B}(\mathcal{H})$, [36,23]. Then $\mathcal{I} = \mathcal{I}_2$ and, in this case, we define the sequence $\nu := \{2\mu_n + s_{n+1}(H) + \|A\| s_{n+1}(S)\}_{n=1}^{\infty} \in \operatorname{diag}(\mathcal{I})$ which is positive and decreasing. Therefore, in this case, (4.6) still holds for this new choice of decreasing positive sequence $\nu \in \operatorname{diag}(\mathcal{I})$.

With (4.6) satisfied for the cases A compact or A bounded but not compact, $D - H \in \operatorname{Com}\mathcal{I}$ by an application of Theorem 3.2. Since $H - AS = \frac{1}{2}[S, A] \in [\mathcal{I}_1, \mathcal{I}_2] = [\mathcal{B}(\mathcal{H}), \mathcal{I}] = \operatorname{Com}\mathcal{I}$ (see the preliminaries), we obtain $D - AS \in \operatorname{Com}\mathcal{I}$ and the result of the theorem when A and S are Hermitian.

The general case follows easily by splitting A into real and imaginary parts. □

Corollary 4.6. *Let $\mathcal{I}$ be a two-sided ideal in $\mathcal{K}(\mathcal{H})$. Suppose $S \in \mathcal{I}$ is Hermitian and $(e_n)_{n=1}^{\infty}$ is an eigenvector sequence for S. Then*

$$AS - \operatorname{diag}\{(ASe_n, e_n)\}_{n=1}^{\infty} \in \operatorname{Com}\mathcal{I}$$

for every $A \in \mathcal{B}(\mathcal{H})$ and hence for every trace $\tau : \mathcal{I} \to \mathbb{C}$ we have

$$\tau(AS) = \tau \circ \operatorname{diag}(\{(ASe_n, e_n)\}_{n=1}^{\infty}) = \tau \circ \operatorname{diag}(\{(Ae_n, e_n)\lambda_n(S)\}_{n=1}^{\infty}).$$

Proof. Set $\mathcal{I}_1 := \mathcal{B}(\mathcal{H}) = \operatorname{Com}\mathcal{B}(\mathcal{H})$ and $\mathcal{I}_2 := \mathcal{I}$. As

$$|(ASe_n, e_n)| \leq \|A\| |\lambda_n(S)| \in \operatorname{diag}(\mathcal{I})$$

and $\|A\| |\lambda_n(S)|$ is a positive decreasing sequence, we obtain the result from Theorem 4.5. □

We can now identify the form of every trace on any ideal of compact operators.

Corollary 4.7. *Let $\mathcal{I}$ be a two-sided ideal in $\mathcal{K}(\mathcal{H})$. Suppose $T \in \mathcal{I}$. Then*

$$\tau(T) = \tau \circ \operatorname{diag}(\{s_n(T)(f_n, e_n)\}_{n=1}^{\infty})$$

for every trace $\tau : \mathcal{I} \to \mathbb{C}$, where

$$T = \sum_{n=1}^{\infty} s_n(T) f_n e_n^*$$

is a canonical decomposition of T ($\{s_n(T)\}_{n=1}^{\infty}$ is the sequence of singular values of T, $(e_n)_{n=1}^{\infty}$ an orthonormal basis such that $|T|e_n = s_n(T)e_n$, $(f_n)_{n=1}^{\infty}$ an orthonormal system such that $Te_n = s_n(T)f_n$, and $e_n^*(\cdot) := (\cdot, e_n)$).

Proof. Let $T = U|T|$ be the polar decomposition of the compact operator T into the positive operator $|T|$ and the partial isometry U. The eigenvalue sequence $\{\lambda_n(|T|)\}_{n=1}^{\infty}$ defines the singular values $\{s_n(T)\}_{n=1}^{\infty}$ of T. Let $(e_n)_{n=1}^{\infty}$ be any orthonormal system such that $|T|e_n = s_n(T)e_n$ (an eigenvector sequence of $|T|$). Since U is bounded and $|T| \in \mathcal{I}$ is positive we apply Corollary 4.6 (with $A = U$ and $S = |T|$) and obtain

$$\tau(T) = \tau(U|T|) = \tau \circ \operatorname{diag}(\{(Ue_n, e_n)s_n(T)\}_{n=1}^{\infty}).$$

We set $f_n := Ue_n$, $n \in \mathbb{N}$. If $(e_n)_{n=1}^{\infty}$ forms a complete system, then we have the decomposition, [42, Theorem 1.4], $|T| = \sum_{n=1}^{\infty} s_n(T) e_n e_n^*$. It follows that $T = U|T| = \sum_{n=1}^{\infty} s_n(T) f_n e_n^*$. $\square$

5. Modulated operators and the weak-ℓ_1 space

The traces of interest in Connes' trace theorem (and in Connes' noncommutative geometry in general) are traces on the ideal $\mathcal{L}_{1,\infty}$ associated to the weak-ℓ_1 space $\ell_{1,\infty}$, i.e. $\mathrm{diag}(\mathcal{L}_{1,\infty}) = \ell_{1,\infty}$. It is indicated below that the ideal $\mathcal{L}_{1,\infty}$ is geometrically stable (it is a quasi-Banach ideal), so the Lidskii formulation applies to all its traces.

We were led in our investigation, following results like Corollary 4.6, to ask to what degree the Fredholm formulation applies to traces on $\mathcal{L}_{1,\infty}$. The Fredholm formulation of the canonical trace on trace class operators (usually taken as the definition of the canonical trace) is

$$\mathrm{Tr}(T) = \sum (Te_n, e_n), \quad T \in \mathcal{L}_1$$

where the usual sum $\sum : \ell_1 \to \mathbb{C}$ can be understood as the functional $\mathrm{Tr} \circ \mathrm{diag}$. In the Fredholm formulation $(e_n)_{n=1}^{\infty}$ is any orthonormal basis and the same basis can be used for all trace class operators $T \in \mathcal{L}_1$, which is quite distinct to the statement of Corollary 4.6. We know that the Fredholm formulation is false for traces on $\mathcal{L}_{1,\infty}$ (but we do not offer any proof of this fact here[2]), i.e. if $\tau : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is a non-zero trace there does not exist any basis $(e_n)_{n=1}^{\infty}$ such that $\tau(T) = \tau \circ \mathrm{diag}(\{(Te_n, e_n)\}_{n=1}^{\infty})$ for all $T \in \mathcal{L}_{1,\infty}$.

We considered whether there were restricted Fredholm formulations, which may hold for some subspace of $\mathcal{L}_{1,\infty}$ instead of the whole ideal. To this end we introduce new left ideals of the Hilbert–Schmidt operators. We have used the term modulated operators, see Definition 5.1, for the elements of the left ideals and the precise statement of a 'restricted Fredholm formulation' is Theorem 5.2. It is the aim of this section to prove Theorem 5.2.

Our purpose for introducing modulated operators is to study operators on manifolds modulated by the Laplacian, where this definition is made precise in Section 6. Compactly supported pseudo-differential operators of order $-d$ on $\mathbb{R}^d$ offer examples of these so-called Laplacian modulated operators, and this will be our avenue to proving extensions and variants of Connes' trace theorem.

Notation. *Henceforth we use big O, theta Θ, and little o notation, meaning $f(s) = O(g(s))$ if $|f(s)| \le C|g(s)|$ for a constant $C > 0$ for all $s \in \mathbb{N}$ or $s \in \mathbb{R}$, $f(s) = \Theta(g(s))$ if $c|g(s)| \le |f(s)| \le C|g(s)|$ for constants $C > c > 0$ for all $s \in \mathbb{N}$ or $s \in \mathbb{R}$, and $f(s) = o(g(s))$ if $|f(s)||g(s)|^{-1} \to 0$ as $s \to \infty$, respectively.*

Let $\mathcal{L}_2$ denote the Hilbert–Schmidt operators on the Hilbert space $\mathcal{H}$.

Definition 5.1. Suppose $V : \mathcal{H} \to \mathcal{H}$ is a positive bounded operator. An operator $T : \mathcal{H} \to \mathcal{H}$ is *V-modulated* if

$$\|T(1 + tV)^{-1}\|_{\mathcal{L}_2} = O(t^{-1/2}). \tag{5.1}$$

We denote by $\mathrm{mod}(V)$ the set of V-modulated operators.

It follows from the definition that $\mathrm{mod}(V)$ is a subset of Hilbert–Schmidt operators $\mathcal{L}_2$ and that it forms a left ideal of $\mathcal{B}(\mathcal{H})$ (see Proposition 5.4 below).

[2] Private communication by D. Zanin.

The main result of this section is the following theorem.

Theorem 5.2. *Suppose T is V-modulated where $0 < V \in \mathcal{L}_{1,\infty}$, and $(e_n)_{n=1}^\infty$ is an orthonormal basis such that $Ve_n = s_n(V)e_n$, $n \in \mathbb{N}$. Then*
(i) $T, \operatorname{diag}\{(Te_n, e_n)\}_{n=1}^\infty \in \mathcal{L}_{1,\infty}$;
(ii) $T - \operatorname{diag}\{(Te_n, e_n)\}_{n=1}^\infty \in \operatorname{Com}\mathcal{L}_{1,\infty}$;
(iii) *every eigenvalue sequence $\{\lambda_n(T)\}_{n=1}^\infty$ of T satisfies*

$$\sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} (Te_j, e_j) = O(1)$$

where $O(1)$ denotes a bounded sequence.

Remark 5.3 (*Fredholm Formula*)**.** Evidently from Theorem 5.2, if T is V-modulated then

$$\tau(T) = \tau \circ \operatorname{diag}(\{(Te_n, e_n)\}_{n=1}^\infty)$$

for every trace $\tau : \mathcal{L}_{1,\infty} \to \mathbb{C}$.

The proof of Theorem 5.2 is provided in a section below.

5.1. Properties of modulated operators

We establish the basic properties of V-modulated operators. This will simplify the proof of Theorem 5.2 and results in later sections.

Notation. *The symbols $\doteq$, $\dot{\leq}$, $\dot{\geq}$, may also be used to denote equality or inequality up to a constant. Where it is necessary to indicate that the constant depends on parameters $\theta_1, \theta_2, \ldots$ we write $\doteq_{\theta_1,\theta_2,\ldots}$, $\dot{\leq}_{\theta_1,\theta_2,\ldots}$, $\dot{\geq}_{\theta_1,\theta_2,\ldots}$. The constants may not be the same in successive uses of the symbol. We introduce this notation to improve text where the value of constants has no relevance to statements or proofs.*

Proposition 5.4. *Suppose $V : \mathcal{H} \to \mathcal{H}$ is positive and bounded. The set of V-modulated operators,* $\operatorname{mod}(V)$*, is a subset of $\mathcal{L}_2$ that forms a left ideal of $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose $T \in \operatorname{mod}(V)$. Then

$$\|T\|_{\mathcal{L}_2} = \|T(1+V)^{-1}(1+V)\|_{\mathcal{L}_2} \leq \|T(1+V)^{-1}\|_{\mathcal{L}_2}\|1+V\|$$

and T is Hilbert–Schmidt. Suppose $A_1, A_2 \in \mathcal{B}(\mathcal{H})$ and $T_1, T_2 \in \operatorname{mod}(V)$. We have, for $t \geq 1$,

$$\|(A_1T_1 + A_2T_2)(1+tV)^{-1}\|_{\mathcal{L}_2} \leq \|A_1\|\|T_1(1+tV)^{-1}\|_{\mathcal{L}_2} + \|A_2\|\|T_2(1+tV)^{-1}\|_{\mathcal{L}_2}$$

so

$$\|(A_1T_1 + A_2T_2)(1+tV)^{-1}\|_{\mathcal{L}_2} = O(t^{-1/2}).$$

Hence $\operatorname{mod}(V)$ forms a left ideal of $\mathcal{B}(\mathcal{H})$. □

Proposition 5.4 establishes $\operatorname{mod}(V)$ as a left ideal of $\mathcal{B}(\mathcal{H})$. The conditions by which bounded operators act on the right of $\operatorname{mod}(V)$ are more subtle.

If X is a set, let χ_E denote the indicator function of a subset $E \subset X$.

Lemma 5.5. *Suppose $V : \mathcal{H} \to \mathcal{H}$ is a positive bounded operator with $\|V\| \leq 1$. Then $T \in \operatorname{mod}(V)$ iff*

$$\|T\chi_{[0,2^{-n}]}(V)\|_{\mathcal{L}_2} = O(2^{-n/2}). \tag{5.2}$$

Proof. Let f, g be real-valued bounded Borel functions such that $|f| \le |g|$ and $Tg(V) \in \mathcal{L}_2$. Then

$$|f(V)T^*| = \sqrt{T|f|^2(V)T^*} \le \sqrt{T|g|^2(V)T^*} = |g(V)T^*|.$$

Hence $\|Tf(V)\|_{\mathcal{L}_2} \le \|Tg(V)\|_{\mathcal{L}_2}$. Let $f(x) = \chi_{[0,2^{-n}]}(x)$, $x \ge 0$, and $g(x) = 2(1+2^n x)^{-1}$, $x \ge 0$. Then $|f| \le |g|$ and

$$\|T\chi_{[0,2^{-n}]}(V)\|_{\mathcal{L}_2} \dot{\le} \|T(1+2^n V)^{-1}\|_{\mathcal{L}_2} \dot{\le} 2^{-n/2}.$$

Hence (5.1) implies (5.2).

Conversely, we note that

$$(\chi_{[0,2^{-(j-1)}]} - \chi_{[0,2^{-j}]})(x)(1+tx)^{-1} \le (1+t2^{-j})^{-1}\chi_{[0,2^{-(j-1)}]}(x), \quad x \ge 0.$$

Then if (5.2) holds and $2^{k-1} \le t < 2^k$ where $k \in \mathbb{N}$, we have

$$\begin{aligned}\|T(1+tV)^{-1}\|_{\mathcal{L}_2} &\le \sum_{j=1}^{k} \|T(\chi_{[0,2^{-(j-1)}]} - \chi_{[0,2^{-j}]})(V)(1+tV)^{-1}\|_{\mathcal{L}_2} \\ &\quad + \|T\chi_{[0,2^{-k}]}(V)(1+tV)^{-1}\|_{\mathcal{L}_2} \\ &\dot{\le} \sum_{j=1}^{k}(1+t2^{-j})^{-1}\|T\chi_{[0,2^{-(j-1)}]}(V)\|_{\mathcal{L}_2} + 2^{-k/2} \\ &\dot{\le} t^{-1}\sum_{j=1}^{k} 2^j 2^{-(j-1)/2} + 2^{-k/2} \\ &\dot{\le} t^{-1/2},\end{aligned}$$

and the condition for being modulated is satisfied. □

Proposition 5.6. *Suppose $V_1 : \mathcal{H} \to \mathcal{H}$ and $V_2 : \mathcal{H}' \to \mathcal{H}'$ are bounded positive operators. Let $B : \mathcal{H} \to \mathcal{H}'$ be a bounded operator and let $A : \mathcal{H}' \to \mathcal{H}$ be a bounded operator such that for some $a > 1/2$ we have*

$$\|V_1^a Ax\|_{\mathcal{H}} \dot{\le} \|V_2^a x\|_{\mathcal{H}'}, \quad x \in \mathcal{H}'.$$

If $T \in \mathrm{mod}(V_1)$ then $BTA \in \mathrm{mod}(V_2)$.

Remark 5.7. In particular, if $\mathcal{H} = \mathcal{H}'$, $V = V_1 = V_2$, $B = 1_{\mathcal{H}}$, $\|VAx\| \dot{\le} \|Vx\|$ for all $x \in \mathcal{H}$, then $TA \in \mathrm{mod}(V)$ if $T \in \mathrm{mod}(V)$.

Proof. We may suppose that $\|V_1\|_{\mathcal{B}(\mathcal{H})}, \|V_2\|_{\mathcal{B}(\mathcal{H}')} \le 1$. Let $P_n = \chi_{[0,2^{-n}]}(V_1)$ and $Q_n = \chi_{[0,2^{-n}]}(V_2)$, $n = \mathbb{Z}_+$. If $j \le k$ we have

$$\|(I-P_j)AQ_k x\|_{\mathcal{H}} \le 2^{ja}\|V_1^a AQ_k x\|_{\mathcal{H}} \le 2^{ja}\|V_2^a Q_k x\|_{\mathcal{H}'} \le 2^{(j-k)a}\|x\|_{\mathcal{H}'}, \quad x \in \mathcal{H}'.$$

Thus

$$\|(I-P_j)AQ_k\|_{\mathcal{B}(\mathcal{H}',\mathcal{H})} \le 2^{(j-k)a}.$$

If T is V_1-modulated then we have by Lemma 5.5 that

$$\|TP_j\|_{\mathcal{L}_2(\mathcal{H})} \dot{\le} 2^{-j/2}, \quad j \in \mathbb{N}.$$

Thus

$$\begin{aligned}
\|BTAQ_k\|_{\mathcal{L}_2(\mathcal{H}')} \le\ & \|B\|_{\mathcal{B}(\mathcal{H},\mathcal{H}')}\|TP_kAQ_k\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H}')} \\
& + \|B\|_{\mathcal{B}(\mathcal{H},\mathcal{H}')}\sum_{j=1}^{k}\|T(P_{j-1}-P_j)AQ_k\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H}')} \\
\dot{\le}\ & \|TP_k\|_{\mathcal{L}_2(\mathcal{H})}\|AQ_k\|_{\mathcal{B}(\mathcal{H}',\mathcal{H})} \\
& + \sum_{j=1}^{k}\|T(P_{j-1}-P_j)\|_{\mathcal{L}_2(\mathcal{H})}\|(P_{j-1}-P_j)AQ_k\|_{\mathcal{B}(\mathcal{H}',\mathcal{H})} \\
\dot{\le}\ & \|TP_k\|_{\mathcal{L}_2(\mathcal{H})} + \sum_{j=1}^{k}\|TP_{j-1}\|_{\mathcal{L}_2(\mathcal{H})}\|(1-P_{j-1})AQ_k\|_{\mathcal{B}(\mathcal{H}',\mathcal{H})} \\
& + \sum_{j=1}^{k}\|TP_{j-1}\|_{\mathcal{L}_2(\mathcal{H})}\|(1-P_j)AQ_k\|_{\mathcal{B}(\mathcal{H}',\mathcal{H})} \\
\dot{\le}\ & 2^{-k/2} + (1+2^{-a})2^{1/2}2^{-k/2}\sum_{j=1}^{k}2^{(a-1/2)(j-k)} \\
\dot{\le}\ & 2^{-k/2}.
\end{aligned}$$

Hence BTA is V_2-modulated by Lemma 5.5. □

5.2. Proof of Theorem 5.2

To prove Theorem 5.2, we will need the following lemmas.

Lemma 5.8. *Let E be an n-dimensional Hilbert space and suppose $A : E \to E$ is a linear map. Then there is an orthonormal basis $(f_j)_{j=1}^n$ of E so that*

$$(Af_j, f_j) = \frac{1}{n}\mathrm{Tr}(A), \quad j \in \mathbb{N}.$$

Proof. This follows from the Hausdorff–Toeplitz theorem on the convexity of the numerical range $W(A)$. Suppose $(f_j)_{j=1}^k$ is an orthonormal sequence of maximal cardinality such that $(Af_j, f_j) = \frac{1}{n}\mathrm{Tr}(A)$ for $j = 1, 2 \dots, k$. Assume $k < n$ and let F be the orthogonal complement of $[f_j]_{j=1}^k$ (here $k = 0$ is permitted and then $F = E$). Let P be the orthogonal projection onto F and consider $PA : F \to F$. Then $\mathrm{Tr}(PA) = (1 - k/n)\mathrm{Tr}(A)$ and by the convexity of $W(PA)$ we can find $f_{k+1} \in F$ with $(Af_{k+1}, f_{k+1}) = (n-k)^{-1}\mathrm{Tr}(PA) = \frac{1}{n}\mathrm{Tr}(A)$, giving a contradiction. □

If $a := \{a_n\}_{n=1}^\infty$ is a sequence of complex numbers let a^* denote the sequence of absolute values $|a_n|$, $n \in \mathbb{N}$, arranged to be decreasing. The weak-ℓ_p spaces, $p \ge 1$, are defined by

$$\ell_{p,\infty} := \{\{a_n\}_{n=1}^\infty | a^* = O(n^{-1/p})\}.$$

Let $\mathcal{L}_{p,\infty}$, $p \ge 1$, denote the two-sided ideal of compact operators $T : \mathcal{H} \to \mathcal{H}$ such that $s_n(T) = O(n^{-1/p})$ (i.e. $\mathrm{diag}(\mathcal{L}_{p,\infty}) = \ell_{p,\infty}$), with quasi-norm

$$\|T\|_{\mathcal{L}_{p,\infty}} := \sup_n n^{1/p} s_n(T).$$

Here, as always, $\{s_n(T)\}_{n=1}^{\infty}$ denotes the singular values of T. The ideal $\mathcal{L}_{p,\infty}$ is a quasi-Banach (hence geometrically stable) ideal.

Lemma 5.9. *If $p, q \geq 1$ such that $p^{-1} + q^{-1} = 1$ then $\mathcal{L}_{1,\infty} = \mathcal{L}_{p,\infty}\mathcal{L}_{q,\infty}$.*

Proof. Suppose $A \in \mathcal{L}_{p,\infty}$ and $B \in \mathcal{L}_{q,\infty}$. Using an inequality of Fan, [18],

$$s_{2n}(AB) \leq s_{n+1}(A)s_n(B) \leq s_n(A)s_n(B) = O(n^{-1/p})O(n^{-1/q}) = O(n^{-1}).$$

Similarly $s_{2n}(BA) = O(n^{-1})$. Hence, $AB, BA \in \mathcal{L}_{1,\infty}$, and $\mathcal{L}_{p,\infty}\mathcal{L}_{q,\infty} \subset \mathcal{L}_{1,\infty}$.

However $\mathrm{diag}\{n^{-1}\}_{n=1}^{\infty} = \mathrm{diag}\{n^{-1/p}\}_{n=1}^{\infty}\mathrm{diag}\{n^{-1/q}\}_{n=1}^{\infty}$. So $\mathrm{diag}\{n^{-1}\}_{n=1}^{\infty} \in \mathcal{L}_{p,\infty}\mathcal{L}_{q,\infty}$. Since $\mathcal{L}_{1,\infty}$ is the smallest two-sided ideal that contains $\mathrm{diag}\{n^{-1}\}_{n=1}^{\infty}$ then $\mathcal{L}_{1,\infty} \subset \mathcal{L}_{p,\infty}\mathcal{L}_{q,\infty}$.

By the two inclusions $\mathcal{L}_{1,\infty} = \mathcal{L}_{p,\infty}\mathcal{L}_{q,\infty}$. □

Lemma 5.10. *Suppose $0 < p < 2$ and that $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Suppose $(v_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$ such that*

$$\sum_{j=n+1}^{\infty} \|v_j\|^2 = O(n^{1-\frac{2}{p}}). \tag{5.3}$$

Then

$$Tx = \sum_{j=1}^{\infty}(x, e_j)v_j$$

defines an operator $T \in \mathcal{L}_{p,\infty}$.

Proof. Observe that $\sum_{n=1}^{\infty} \|v_n\|^2 < \infty$ so that $T : \mathcal{H} \to \mathcal{H}$ is bounded and $T \in \mathcal{L}_2$. For $n = \mathbb{Z}_+$, let

$$T_n x := \sum_{j=n+1}^{\infty} (x, e_j)v_j.$$

Each T_n is also in $\mathcal{L}_2$. Recalling that (see [25, Theorem 7.1])

$$\sum_{j=n+1}^{\infty} s_j^2(T) = \min\{\|T - K\|_{\mathcal{L}_2}^2 \mid \mathrm{rank}(K) \leq n\},$$

we have

$$\begin{aligned} ns_{2n}(T)^2 \leq & \sum_{j=n}^{2n} s_j(T)^2 \leq \sum_{j=1}^{\infty} s_j(T_n)^2 \\ = & \|T_n\|_{\mathcal{L}_2}^2 = \sum_{j=1}^{\infty} \|T_n e_j\|^2 = \sum_{j=n+1}^{\infty} \|v_j\|^2 \\ \dot{\leq} & \ n^{1-2/p}. \end{aligned}$$

Hence $s_{2n}(T) \dot{\leq} n^{-1/p}$. □

Lemma 5.11. *Suppose $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator and $(f_n)_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ such that*

$$\sum_{j=n+1}^{\infty} \|Tf_j\|^2 = O(n^{-1})$$

and

$$|(Tf_n, f_n)| = O(n^{-1}).$$

Then T, $\mathrm{diag}\{(Tf_n, f_n)\}_{n=1}^{\infty} \in \mathcal{L}_{1,\infty}$,

$$T - \mathrm{diag}\{(Tf_n, f_n)\}_{n=1}^{\infty} \in \mathrm{Com}\,\mathcal{L}_{1,\infty}$$

and

$$\sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} (Tf_j, f_j) = O(1).$$

Proof. Suppose $1 < p < 2$ is fixed and $2 < q < \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Set

$$Sx := \sum_{j=1}^{\infty} j^{1/p-1}(x, f_j)f_j, \quad x \in \mathcal{H}.$$

Clearly S is Hermitian, $S \in \mathcal{L}_{q,\infty}$ (S has singular values $n^{-1/q}$) and $Sf_n = n^{-1/q} f_n$ (so $(f_n)_{n=1}^{\infty}$ is an eigenvector sequence for S). Now set

$$Ax := \sum_{j=1}^{\infty} j^{1-1/p}(x, f_j)Tf_j, \quad x \in \mathcal{H}.$$

We show that $A \in \mathcal{L}_{p,\infty}$. Set $v_j = j^{1-1/p}Tf_j$. Notice that

$$\begin{aligned}
\sum_{j=n+1}^{\infty} j^{2-2/p}\|Tf_j\|^2 &= \sum_{k=0}^{\infty} \sum_{j=2^k n+1}^{2^{k+1}n} j^{2-2/p}\|Tf_j\|^2 \\
&\dot{\leq} \sum_{k=0}^{\infty} (2^{k+1})^{2-2/p} n^{2-2/p} 2^{-k} n^{-1} \\
&\dot{\leq}_p n^{1-2/p} \sum_{k=0}^{\infty} 2^{k(1-2/p)} \\
&\dot{\leq}_p n^{1-2/p}.
\end{aligned}$$

Then we have

$$\sum_{j=n+1}^{\infty} \|v_j\|^2 = \sum_{j=n+1}^{\infty} j^{2-2/p}\|Tf_j\|^2 \dot{\leq}_p n^{1-2/p}. \tag{5.4}$$

Hence, by an application of Lemma 5.10, $A \in \mathcal{L}_{p,\infty}$.

By construction $AS = T$ and by assumption $|(ASf_n, f_n)| = |(Tf_n, f_n)| = O(n^{-1})$. Thus Theorem 4.5 can be applied to $A \in \mathcal{L}_{p,\infty} = \mathrm{Com}\,\mathcal{L}_{p,\infty}$ (this last equality follows easily from Corollary 4.1) and $S \in \mathcal{L}_{q,\infty}$, i.e. we use $\mathcal{I}_1 = \mathcal{L}_{p,\infty}$ and $\mathcal{I}_2 = \mathcal{L}_{q,\infty}$, noting from Lemma 5.9 that $\mathcal{I} = \mathcal{L}_{1,\infty} = \mathcal{L}_{p,\infty}\mathcal{L}_{q,\infty}$. Hence, from Theorem 4.5, $T = AS \in \mathcal{L}_{1,\infty}$, $D := \operatorname{diag}\{(Tf_n, f_n)\}_{n=1}^{\infty} = \operatorname{diag}\{(ASf_n, f_n)\}_{n=1}^{\infty} \in \mathcal{L}_{1,\infty}$, and

$$T - D = AS - \operatorname{diag}\{(ASf_j, f_j)\}_{j=1}^{\infty} \in \operatorname{Com}\mathcal{L}_{1,\infty}.$$

By Theorem 3.3

$$\sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \lambda_j(D) = O(1).$$

By Lemma 4.3

$$\sum_{j=1}^{n} \lambda_j(D) - \sum_{j=1}^{n} (Tf_j, f_j) = O(1)$$

and the results are shown. $\square$

Proof of Theorem 5.2. Let $T \in \operatorname{mod}(V)$ where $0 < V \in \mathcal{L}_{1,\infty}$ and $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ such that $Ve_n = s_n(V)e_n$. Since $s_n(V) \dot{\leq} n^{-1}$ we have that

$$\left(\sum_{j=n+1}^{\infty} \|Te_j\|^2\right)^{1/2} \leq (1 + ns_n(V))\|T(1+nV)^{-1}\|_{\mathcal{L}_2} \dot{\leq} n^{-1/2}. \tag{5.5}$$

Then we have

$$\sum_{j=n+1}^{\infty} \|Te_j\|^2 \dot{\leq} n^{-1}.$$

Let $D := \operatorname{diag}\{(Te_n, e_n)\}_{n=1}^{\infty}$ be the specific diagonalisation with respect to the basis $(e_n)_{n=1}^{\infty}$, i.e. $D = \sum_{n=1}^{\infty}(Te_n, e_n)e_n e_n^*$. Note that if $D' := \operatorname{diag}\{(Te_n, e_n)\}_{n=1}^{\infty}$ is the diagonalisation according to any arbitrary orthonormal basis $(h_n)_{n=1}^{\infty}$, i.e. $D' = \sum_{n=1}^{\infty}(Te_n, e_n)h_n h_n^*$, then there exists a unitary U with $h_n = Ue_n$, $n \in \mathbb{N}$, and thus $D' = UDU^*$. Since $\|De_j\| \leq \|Te_j\|$ then we also have

$$\sum_{j=n+1}^{\infty} \|De_j\|^2 \dot{\leq} n^{-1}.$$

Thus

$$\sum_{j=n+1}^{\infty} \|(T-D)e_j\|^2 \dot{\leq} 2n^{-1}.$$

(i) By Lemma 5.10, $T, D \in \mathcal{L}_{1,\infty}$ (set $v_j = Te_j$ and $v_j = De_j$ respectively). It follows that $D' \in \mathcal{L}_{1,\infty}$ for any diagonalisation D' since $D' = UDU^*$ for a unitary U and $\mathcal{L}_{1,\infty}$ is a two-sided ideal.

(ii) Notice by design that $(Te_j, e_j) = (De_j, e_j)$, $j \in \mathbb{N}$. Thus $T - D$ satisfies Lemma 5.11 (where $((T-D)e_j, e_j) = 0$, $j \in \mathbb{N}$) and $T - D \in \operatorname{Com}\mathcal{L}_{1,\infty}$. If $D' := \operatorname{diag}\{(Te_n, e_n)\}_{n=1}^{\infty}$ is an arbitrary diagonalisation then $D' = UDU^*$ for a unitary U and clearly $D' - D \in \operatorname{Com}\mathcal{L}_{1,\infty}$. Hence $T - D' \in \operatorname{Com}\mathcal{L}_{1,\infty}$.

(iii) Construct a new basis $(f_n)_{n=1}^\infty$ of $\mathcal{H}$ using Lemma 5.8 such that $(f_j)_{j=2^{k-1}}^{2^k-1}$ is a basis of $[e_j]_{j=2^{k-1}}^{2^k-1}$, $k = \mathbb{Z}_+$. Let P_k denote the projection onto $[e_j]_{j=2^{k-1}}^{2^k-1}$. If $2^{k-1} \le n \le 2^k - 1$ then

$$\sum_{j=n+1}^\infty \|Tf_j\|^2 \le \sum_{j=2^{k-1}}^\infty \|Te_j\|^2 \dot{\lesssim} n^{-1}. \tag{5.6}$$

Similarly

$$\|TP_k\|_{\mathcal{L}_2}^2 = \sum_{j=2^{k-1}}^{2^k-1} \|Te_j\|^2 \dot{\lesssim} 2^{-k}.$$

Thus

$$\|P_kTP_k\|_{\mathcal{L}_1} \le \|TP_k\|_{\mathcal{L}_2}\|P_k\|_{\mathcal{L}_2} \dot{\lesssim} 2^{-k/2}2^{k/2} \dot{\lesssim} 1.$$

Hence, if $2^{k-1} \le n \le 2^k - 1$, then

$$|(Tf_n, f_n)| \le 2^{-k}\|P_kTP_k\|_{\mathcal{L}_1} \dot{\lesssim} n^{-1}, \quad n \in \mathbb{N}. \tag{5.7}$$

Using (5.6) and (5.7), from Lemma 5.11 we obtain

$$\left|\sum_{j=1}^n \lambda_j(T) - \sum_{j=1}^n (Tf_j, f_j)\right| \dot{\lesssim} 1, \quad n \in \mathbb{N}. \tag{5.8}$$

Now, if $2^{k-1} \le n \le 2^k - 1$, then

$$\begin{aligned}\left|\sum_{j=1}^n (Te_j, e_j) - \sum_{j=1}^n (Tf_j, f_j)\right| &= \left|\sum_{j=2^{k-1}}^n (Te_j, e_j) - \sum_{j=2^{k-1}}^n (Tf_j, f_j)\right| \\ &\le 2\|P_kTP_k\|_{\mathcal{L}_1} \dot{\lesssim} 1.\end{aligned} \tag{5.9}$$

Hence (iii) is shown from (5.8) and (5.9). $\square$

5.3. Corollaries of interest

Before we specialise to operators modulated by the Laplacian we note some results of interest.

Corollary 5.12. *Let $\{T_i\}_{i=1}^N$ be a finite collection of V-modulated operators where $0 < V \in \mathcal{L}_{1,\infty}$. Then*

$$\sum_{j=1}^n \lambda_j\left(\sum_{i=1}^N T_i\right) - \sum_{i=1}^N \sum_{j=1}^n \lambda_j(T_i) = O(1). \tag{5.10}$$

Proof. Let $T_0 = \sum_{i=1}^N T_i \in \mathrm{mod}(V)$. Choose an eigenvector sequence $(e_n)_{n=1}^\infty$ of V with $Ve_n = s_n(V)e_n$, $n \in \mathbb{N}$. Then, by Theorem 5.2, for $i = 0, \ldots, N$, there are constants C_i such that

$$\left|\sum_{j=1}^n \lambda_j(T_i) - \sum_{j=1}^n (T_ie_j, e_j)\right| < C_i, \quad 0 \le i \le N.$$

Hence

$$\left|\sum_{j=1}^{n}\left(\lambda_j(T_0)-\sum_{i=1}^{N}\lambda_j(T_i)\right)\right| < C + \left|\sum_{j=1}^{n}\left((T_0e_j,e_j)-\sum_{i=1}^{N}(T_ie_j,e_j)\right)\right| = C$$

where $C = C_0 + \cdots + C_N$. $\square$

In the proof of Theorem 5.2 we noted that a V-modulated operator, $T \in \operatorname{mod}(V)$, satisfied the condition

$$\sum_{k=n+1}^{\infty} \|Te_k\|^2 = O(n^{-1})$$

for an eigenvector sequence $(e_n)_{n=1}^{\infty}$ of $0 < V \in \mathcal{L}_{1,\infty}$. We show a converse statement.

Proposition 5.13. *Suppose $0 < V \in \mathcal{L}_{1,\infty}$ is such that the singular values of V satisfy $s_n(V) = \Theta(n^{-1})$ and $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ such that $Ve_n = s_n(V)e_n$, $n \in \mathbb{N}$. Then $T \in \operatorname{mod}(V)$ iff*

$$\sum_{k=n+1}^{\infty} \|Te_k\|^2 = O(n^{-1}). \tag{5.11}$$

Proof. The only if statement is contained in the proof of Theorem 5.2. We show the if statement. Without loss $\|V\| \le 1$.

Clearly T is bounded and Hilbert–Schmidt. Let $c > 0$ be such that $s_n(V) \ge cn^{-1}$. Then

$$s_{\lfloor ck \rfloor}(V) > \frac{c}{\lfloor ck \rfloor} > \frac{c}{ck} = k^{-1}.$$

Hence $N(k^{-1}) \ge \lfloor ck \rfloor \stackrel{.}{\ge} k$ where

$$N(\lambda) = \max\{k \in \mathbb{N} | s_k(V) > \lambda\}$$

and we have

$$N(k^{-1})^{-1} \stackrel{.}{\le} k^{-1}.$$

Now note

$$\|T\chi_{[0,k^{-1}]}(V)\|_2^2 = \sum_{j=N(k^{-1})+1}^{\infty} \|Te_j\|^2 \stackrel{.}{\le} N(k^{-1})^{-1} \stackrel{.}{\le} k^{-1}.$$

Thus $\|T\chi_{[0,2^{-n}]}(V)\|_2 \stackrel{.}{\le} 2^{-n/2}$, $n \in \mathbb{N}$. By Lemma 5.5, $T \in \operatorname{mod}(V)$. $\square$

6. Applications to pseudo-differential operators

We define in this section operators that are modulated with respect to the operator $(1-\Delta)^{-d/2}$ where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^d$, termed by us Laplacian modulated operators. We show that the Laplacian modulated operators include the class of pseudo-differential operators of order $-d$, and that the Laplacian modulated operators admit a residue map which extends the noncommutative residue. Finally we show, with the aid of the results established, that singular traces applied to Laplacian modulated operators calculate the residue.

Definition 6.1. Suppose $d \in \mathbb{N}$ and that $T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is a bounded operator. We will say that T is *Laplacian modulated* if T is $(1-\Delta)^{-d/2}$-modulated.

From Proposition 5.4 a Laplacian modulated operator is Hilbert–Schmidt. We recall every Hilbert–Schmidt operator on $L_2(\mathbb{R}^d)$ has a unique symbol in the following sense.

Lemma 6.2. *A bounded operator $T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is Hilbert–Schmidt iff there exists a unique function $p_T \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

$$(Tf)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p_T(x,\xi) \hat{f}(\xi) d\xi, \quad f \in L_2(\mathbb{R}^d). \tag{6.1}$$

Further, $\|T\|_{\mathcal{L}_2} = (2\pi)^{d/2} \|p_T\|_{L_2}$ and if $\{\phi_n\}_{n=1}^\infty \subset C_c^\infty(\mathbb{R}^d)$ is such that $\phi_n \nearrow 1$ pointwise, then

$$p_T(x,\xi) = \lim_n e_{-\xi}(x)(T\phi_n e_\xi)(x), \quad x, \xi \text{ a.e.},$$

where $e_\xi(x) = e^{i\langle x,\xi\rangle}$.

Proof. It follows from Plancherel's theorem that T is Hilbert–Schmidt iff it can be represented in the form (6.1) for some unique $p_T \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ and that $\|T\|_{\mathcal{L}_2} = (2\pi)^{d/2} \|p_T\|_{L_2}$. We have

$$(T\phi_n e_\xi)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\eta\rangle} p_T(x,\eta) \hat{\phi}_n(\eta - \xi) d\eta, \quad x \text{ a.e.}$$

If $\phi_n \nearrow 1$ pointwise then $\hat{\phi}_n \to (2\pi)^d \delta$ in the sense of distributions, where δ is the Dirac distribution. If we set $g_x(\eta) := e^{i\langle x,\eta\rangle} p_T(x,\eta)$, then $g_x \in L_2(\mathbb{R}^d)$ is a tempered distribution. Hence $g_x \star \delta = g_x$ and

$$\lim_n e_{-\xi}(x)(T\phi_n e_\xi)(x) = e_{-\xi}(x)(g_x \star \delta)(\xi) = p_T(x,\xi) \quad x, \xi \text{ a.e.} \quad \square$$

The function p_T in (6.1) is called the *symbol* of the Hilbert–Schmidt operator T. Being Laplacian modulated places the following condition on the symbol.

Proposition 6.3. *Suppose $d \in \mathbb{N}$ and that $T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is a bounded operator. Then T is Laplacian modulated iff T can be represented in the form*

$$(Tf)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p_T(x,\xi) \hat{f}(\xi) d\xi \tag{6.2}$$

where $p_T \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ is such that

$$\left(\int_{\mathbb{R}^d} \int_{|\xi| \geq t} |p_T(x,\xi)|^2 d\xi \, dx \right)^{1/2} = O(t^{-d/2}), \quad t \geq 1. \tag{6.3}$$

Proof. If T is Laplacian modulated or if it satisfies (6.2), then T is Hilbert–Schmidt. So we are reduced to showing that a Hilbert–Schmidt operator T is Laplacian modulated iff its symbol p_T satisfies (6.3).

If Q_t, $t > 0$, is the Fourier projection

$$(Q_t f)(\xi) = \int_{|\eta| \geq t} \hat{f}(\eta) e^{i\langle \xi,\eta\rangle} \, d\eta$$

then the Hilbert–Schmidt operator TQ_t has the form

$$(TQ_t f)(\xi) = \int_{|\xi|\geq t} e^{i\langle x,\xi\rangle} p_T(x,\xi)\hat{f}(\xi)d\xi.$$

By Lemma 6.2

$$\|TQ_t\|_{\mathcal{L}_2} = \left(\int_{\mathbb{R}^d}\int_{|\xi|\geq t} |p_T(x,\xi)|^2\, dxd\xi\right)^{1/2}.$$

Define also

$$(P_t f)(\xi) = \int_{(1+|\eta|^2)^{-d/2}\leq t} \hat{f}(\eta)e^{i\langle \xi,\eta\rangle}\, d\eta,$$

or, via Fourier transform,

$$P_t = \chi_{[0,t]}\left((1-\Delta)^{-d/2}\right).$$

Note that, for $t \geq 1$,

$$|\eta| \geq t \Rightarrow (1+|\eta|^2)^{-d/2} \leq t^{-d}$$

and that

$$|\eta| \geq 2^{-1/2}t \Leftarrow (1+|\eta|^2)^{-d/2} \leq t^{-d}.$$

Hence

$$Q_t \leq P_{t^{-d}} \leq Q_{2^{-1/2}t}.$$

Note also that $P_t \leq P_{t_1}$ and $Q_t \leq Q_{t_1}$ if $t_1 \leq t$. Fix $t \geq 1$ and $n \in \mathbb{Z}$ such that $2^{n-1} \leq t < 2^n$.

We now prove the if statement. Suppose T is Laplacian modulated. Then, by Lemma 5.5,

$$\|TQ_t\|_{\mathcal{L}_2} \leq \left\|TP_{t^{-d}}\right\|_{\mathcal{L}_2} \leq \left\|TP_{2^{-dn}}\right\|_{\mathcal{L}_2} \mathrel{\dot{\lesssim}} 2^{-dn/2} \mathrel{\dot{\lesssim}} t^{-d/2}.$$

Hence (6.3) is satisfied.

We prove the only if statement. Let T satisfy (6.3). Then

$$\|TP_{2^{-n}}\|_{\mathcal{L}_2} \leq \left\|TQ_{2^{-1/2}2^{n/d}}\right\|_{\mathcal{L}_2} \mathrel{\dot{\lesssim}} 2^{d/4}2^{-n/2} \mathrel{\dot{\lesssim}} 2^{-n/2}.$$

By Lemma 5.5, T is Laplacian modulated. $\square$

For $p \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ define

$$\|p\|_{\text{mod}} := \|p\|_{L_2} + \sup_{t\geq 1} t^{d/2}\left(\int_{\mathbb{R}^d}\int_{|\xi|\geq t} |p(x,\xi)|^2 d\xi dx\right)^{1/2}. \tag{6.4}$$

Define

$$S^{\text{mod}} := \{p \in L_2(\mathbb{R}^d \times \mathbb{R}^d) \mid \|p\|_{\text{mod}} < \infty\}. \tag{6.5}$$

Proposition 6.3 says that each Laplacian modulated operator T is associated uniquely to $p_T \in S^{\text{mod}}$ and vice-versa. We can call S^{mod} the symbols of Laplacian modulated operators.

If $\phi \in C_c^\infty(\mathbb{R}^d)$ define the multiplication operator $(M_\phi f)(x) = \phi(x)f(x),\ f \in L_2(\mathbb{R}^d)$.

Definition 6.4. We will say a bounded operator $T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is *compactly based* if $M_\phi T = T$ for some $\phi \in C_c^\infty(\mathbb{R}^d)$, and *compactly supported* if $M_\phi T M_\phi = T$.

We omit proving the easily verified statements that a Hilbert–Schmidt operator T is compactly based if and only if $p_T(x, \xi)$ is (almost everywhere) compactly supported in the x-variable, and is compactly supported if and only if the kernel of T is compactly supported.

Example 6.5 (*Pseudo-Differential Operators*). We recall that $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$, $\xi \in \mathbb{R}^d$, and a multi-index of order $|\beta|$ is $\beta = (\beta_1, \dots, \beta_d) \in (\mathbb{N} \cup \{0\})^d$ such that $|\beta| := \sum_{i=1}^d \beta_i$. If $p \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that, for each multi-index α, β,

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \,\dot{\le}_{\alpha,\beta}\, \langle \xi \rangle^{m-|\beta|} \tag{6.6}$$

we say that p belongs to the symbol class $S^m := S^m(\mathbb{R}^d \times \mathbb{R}^d)$, $m \in \mathbb{R}$, (in general terminology we have just defined the uniform symbols of Hörmander type (1,0); see e.g. [24] and [37, Chapter 2]). If $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz functions (the smooth functions of rapid decrease), an operator $P : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ associated to a symbol $p \in S^m$,

$$(Pu)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} p(x, \xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d) \tag{6.7}$$

is called a pseudo-differential operator of order m.

If $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, are the Sobolev–Hilbert spaces consisting of those $f \in L_2(\mathbb{R}^d)$ with

$$\|f\|_s := \|(1 - \Delta)^{s/2} f\|_{L_2} < \infty$$

and P is a pseudo-differential operator of order m, then P has an extension to a continuous linear operator

$$P : H^s(\mathbb{R}^d) \to H^{s-m}(\mathbb{R}^d), \quad s \in \mathbb{R}; \tag{6.8}$$

see, e.g. [37, Theorem 2.6.11]. If P is of order 0 this implies that P has a bounded extension $P : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$.

The compactly based Laplacian modulated operators extend the compactly based pseudo-differential operators of order $-d$.

Proposition 6.6. *If P is a compactly based (respectively, compactly supported) pseudo-differential operator of order $-d$ then the bounded extension of P is a compactly based (respectively, compactly supported) Laplacian modulated operator. Also, the symbol of P is equal to (provides the L_2-equivalence class) the symbol of the bounded extension of P as a Laplacian modulated operator.*

Proof. Let P have symbol $p \in S^{-d}$ that is compactly based in the first variable. Then

$$\int_{\mathbb{R}^d} \int_{|\xi| \ge t} |p(x, \xi)|^2 d\xi dx \,\dot{\le}\, \int_{|\xi| \ge t} \langle \xi \rangle^{-2d} d\xi \,\dot{\le}\, \langle t \rangle^{-d}. \tag{6.9}$$

Hence $p \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ and, if P_0 is the extension of P then P_0 is Hilbert–Schmidt. Let p_0 be the symbol of P_0 as a Hilbert–Schmidt operator. Let $\phi \in C_c^\infty(\mathbb{R}^d)$, and $e_\xi(x) := e^{i\langle x, \xi \rangle}$, $\xi \in \mathbb{R}^d$. Since

$$(P - P_0)\phi e_\xi = 0$$

we have, by Lemma 6.2,

$$p(x,\xi) - p_0(x,\xi) = \lim_n e_{-\xi}(x)(P - P_0)\phi_n e_\xi(x) = 0, \quad x, \xi \text{ a.e.}$$

where $\{\phi_n\}_{n=1}^\infty \subset C_c^\infty(\mathbb{R}^d)$ is such that $\phi_n \nearrow 1$ pointwise. Then (6.9) implies that the symbol of P_0 satisfies (6.3), hence P_0 is Laplacian modulated. $\square$

The Laplacian modulated operators form a bimodule for sufficiently regular operators.

Lemma 6.7. *Let T be Laplacian modulated and $R, S : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ be bounded such that $S : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ is bounded for some $s < -d/2$. Then RTS is Laplacian modulated.*

Proof. The Laplacian modulated operators form a left ideal so RT is Laplacian modulated. By Proposition 5.6, the result is shown if $\|(1-\Delta)^{-da/2}Su\|_{L_2} \dot{\leq} \|(1-\Delta)^{-da/2}u\|_{L_2}$ for all $u \in C_c^\infty(\mathbb{R}^d)$ where $a > 1/2$. However, this is the same statement as $\|Su\|_s \dot{\leq} \|u\|_s$ for $s < -d/2$. $\square$

Remark 6.8. From (6.8), $R, S : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ for any $s \in \mathbb{R}$ for all zero order pseudo-differential operators R and S. Hence the Laplacian modulated operators form a bimodule for the pseudo-differential operators of order 0.

The next example confirms that the Laplacian modulated operators are a wider class than the pseudo-differential operators.

Example 6.9 (*Square-Integrable Functions*)**.** For $f, g \in L_2(\mathbb{R}^d)$ set

$$M_f : L_\infty(\mathbb{R}^d) \to L_2(\mathbb{R}^d), \qquad (M_f h)(x) := f(x)h(x), \quad x \text{ a.e.}$$

and

$$T_g : L_2(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d),$$
$$(T_g h)(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} g(\xi)\hat{h}(\xi)d\xi = (\hat{g} \star h)(x), \quad x \text{ a.e.}$$

Define a subspace $L_{\mathrm{mod}}(\mathbb{R}^d)$ of $L_2(\mathbb{R}^d)$ by

$$L_{\mathrm{mod}}(\mathbb{R}^d) := \left\{ g \in L_2(\mathbb{R}^d) \,\middle|\, \left(\int_{|\xi|\geq t} |g(\xi)|^2 d\xi\right)^{1/2} = O(t^{-d/2}),\ t \geq 1 \right\}.$$

Remark 6.10. It is clear that the function $\langle\xi\rangle^{-d} = (1+|\xi|^2)^{-d/2}$, $\xi \in \mathbb{R}^d$, belongs to $L_{\mathrm{mod}}(\mathbb{R}^d)$.

Proposition 6.11. *If $f \in L_2(\mathbb{R}^d)$ and $g \in L_{\mathrm{mod}}(\mathbb{R}^d)$ then $M_f T_g$ is Laplacian modulated with symbol $f \otimes g \in S^{\mathrm{mod}}$. If f has compact support then $M_f T_g$ is compactly based.*

Proof. First note that $\|T_g h\|_{L_\infty} = \|\hat{g} \star h\|_{L_\infty} \leq \|g\|_{L_2}\|h\|_{L_2}$ by Young's inequality. Hence $T_g : L_2 \to L_\infty$ is continuous and linear. Also $\|M_f h\|_{L_2} \leq \|f\|_2\|h\|_{L_\infty}$, so $M_f : L_\infty \to L_2$ is continuous and linear. The composition $M_f T_g : L_2 \to L_2$ is continuous and linear (and everywhere defined). We have that

$$(M_f T_g h)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} f(x)g(\xi)\hat{h}(\xi)d\xi, \quad x \text{ a.e.}$$

is an integral operator with symbol $f \otimes g \in S^{\mathrm{mod}}$. Hence $M_f T_g$ is Laplacian modulated. If f has compact support, $\phi f = f$ for some $\phi \in C_c^\infty(\mathbb{R}^d)$, and hence $M_\phi M_f T_g = M_f T_g$. $\square$

6.1. Residues of Laplacian modulated operators

We define in this section the residue of a compactly based Laplacian modulated operator. We show that it is an extension of the noncommutative residue of classical pseudo-differential operators of order $-d$ defined by M. Wodzicki, [44].

We make some observations about the symbol of a compactly based operator.

Lemma 6.12. *Let T be a compactly based Laplacian modulated operator with symbol p_T. Then:*

$$\int_{\mathbb{R}^d}\int_{r\le|\xi|\le 2r} |p_T(x,\xi)| d\xi\, dx = O(1), \quad r \ge 1; \tag{6.10}$$

$$\int_{\mathbb{R}^d}\int_{|\xi|\le r} |p_T(x,\xi)| d\xi\, dx = O(\log(1+r)), \quad r \ge 1; \tag{6.11}$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_T(x,\xi)| \langle\xi\rangle^{-\theta} d\xi\, dx < \infty, \quad \theta > 0; \tag{6.12}$$

and, if A is a positive $d \times d$-matrix with spectrum contained in $[a,b]$, $b > a > 0$ are fixed values, then

$$\int_{\mathbb{R}^d}\int_{|\xi|\le r} p_T(x,\xi) d\xi\, dx - \int_{\mathbb{R}^d}\int_{|A\xi|\le r} p_T(x,\xi) d\xi\, dx = O(1), \quad r \ge 1. \tag{6.13}$$

Proof. We prove (6.10). Using (6.3), if $r \ge 1$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{r\le|\xi|\le 2r} |p_T(x,\xi)| d\xi dx \\ &\quad \dot{\le} \left(\int_{r\le|\xi|\le 2r} d\xi\right)^{1/2} \left(\int_{\mathbb{R}^d}\int_{|\xi|\ge r} |p_T(x,\xi)|^2 d\xi dx\right)^{1/2} \dot{\le}\ r^{d/2} r^{-d/2}.\end{aligned}$$

We prove (6.11). Fix $n \in \mathbb{N}$ such that $2^{n-1} \le r < 2^n$, then

$$\{\xi \in \mathbb{R}^d | 0 \le |\xi| \le r\} \subset [0,1] \cup \cup_{k=1}^n \{\xi \in \mathbb{R}^d | 2^{k-1} < |\xi| \le 2^k\}.$$

By (6.10) the integral of $|p_T(x,\xi)|$ over each individual set in the union on the right hand side of the previous display is controlled by some constant C. Then the integral over the initial set on the left hand side of the display is controlled by

$$C(n+2) \le C(3 + \log_2 r)\ \dot{\le}\ \log(1+r).$$

We prove (6.12). Consider

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_T(x,\xi)| (1+|\xi|^2)^{-\theta/2} d\xi dx \\ &\quad \dot{\le} \sum_{n=1}^\infty 2^{-n\theta} \int_{\mathbb{R}^d}\int_{2^{n-1}\le|\xi|\le 2^n} |p_T(x,\xi)| d\xi dx\ \dot{\le} \sum_{n=1}^\infty 2^{-n\theta} < \infty\end{aligned}$$

by (6.10).

We prove (6.13). It follows from (6.10) that

$$\left|\int_{\mathbb{R}^d}\int_{|A\xi|\le r} p_T(x,\xi)d\xi dx - \int_{\mathbb{R}^d}\int_{|\xi|\le b^{-1}r} p_T(x,\xi)d\xi dx\right|$$
$$\le \int_{\mathbb{R}^d}\int_{b^{-1}r\le|\xi|\le a^{-1}r} |p_T(x,\xi)|d\xi dx.$$

From (6.10)

$$\int_{\mathbb{R}^d}\int_{b^{-1}r\le|\xi|\le a^{-1}r} |p_T(x,\xi)|d\xi dx = O(1).$$

Also

$$\left|\int_{\mathbb{R}^d}\int_{|\xi|\le r} p_T(x,\xi)d\xi dx - \int_{\mathbb{R}^d}\int_{|\xi|\le b^{-1}r} p_T(x,\xi)d\xi dx\right|$$
$$\le \int_{\mathbb{R}^d}\int_{c_b r\le|\xi|\le c_b^{-1}r} |p_T(x,\xi)|d\xi dx$$

where $c_b = b$ if $b \le 1$ and $c_b = b^{-1}$ if $b > 1$. From (6.10) again

$$\int_{\mathbb{R}^d}\int_{c_b r\le|\xi|\le c_b^{-1}r} |p_T(x,\xi)|d\xi dx = O(1).$$

Formula (6.13) follows. □

We notice from (6.11) that, $n \in \mathbb{N}$,

$$\frac{d}{\log(1+n)}\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} p_T(x,\xi)d\xi\, dx = O(1)$$

(as $\log(1+n^{1/d}) \sim d^{-1}\log(1+n)$). If ℓ_∞ are the bounded sequences and c_0 denotes the closed subspace of sequences convergent to zero, let ℓ_∞/c_0 denote the quotient space.

Definition 6.13. Let T be a compactly based Laplacian modulated operator with symbol p_T. The linear map

$$T \mapsto \mathrm{Res}(T) := \left[\frac{d}{\log(1+n)}\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} p_T(x,\xi)d\xi\, dx\right]$$

we call the *residue* of T, where $[\cdot]$ denotes the equivalence class in ℓ_∞/c_0.

Note that any sequence $\mathrm{Res}_n(T)$, $n \in \mathbb{N}$, such that

$$\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} p_T(x,\xi)d\xi\, dx = \frac{1}{d}\mathrm{Res}_n(T)\log n + o(\log n) \tag{6.14}$$

defines the residue $\mathrm{Res}(T) = [\mathrm{Res}_n(T)] \in \ell_\infty/c_0$.

We show that Res, applied to compactly based pseudo-differential operators, depends only on the principal symbol and extends the noncommutative residue.

Example 6.14 (*Noncommutative Residue*)**.** Let S^m_{base} be the symbols of the compactly based pseudo-differential operators of order m. An equivalence relation is defined on symbols

$p, q \in S^m_{\text{base}}$ by $p \sim q$ if $p - q \in S^{m-1}_{\text{base}}$. The *principal symbol* of a compactly based pseudo-differential operator P of order m with symbol $p \in S^m_{\text{base}}$ is the equivalence class $[p] \in S^m_{\text{base}}/S^{m-1}_{\text{base}}$.

Lemma 6.15. *Let P be a compactly based pseudo-differential operator of order $-d$. Then* $\text{Res}(P)$ *depends only on the principal symbol of P.*

Proof. By Proposition 6.6 (the extension) P is Laplacian modulated and $\text{Res}(P)$ is well defined. If $p(x, \xi) \in S^m_{\text{base}}$, $m < -d$, then $q(x, \xi) := p(x, \xi)\langle\xi\rangle^{\theta} \in S^{\text{mod}}$, $\theta = -d - m > 0$. Then

$$\int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p(x, \xi) d\xi \, dx = \int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} q(x, \xi)\langle\xi\rangle^{-\theta} d\xi \, dx = O(1)$$

by (6.12). It follows from (6.14) that the residue depends only on the equivalence class of a symbol $p \in S^{-d}_{\text{base}}$. $\square$

The asymptotic expansion of $p \in S^m_{\text{base}}$ means (for our purposes) a sequence $\{p_{m-j}\}_{j=0}^{\infty}$ such that $p_{m-j} \in S^{m-j}_{\text{base}}$ and $p - \sum_{j=0}^{n} p_{m-j} \in S^{m-n-1}_{\text{base}}$, $n \ge 0$. A pseudo-differential P of order m is *classical* if its symbol p has an asymptotic expansion $\{p_{m-j}\}_{j=0}^{\infty}$ where each p_{m-j} is a homogeneous function of order $m - j$ in ξ except in a neighbourhood of zero. The principal symbol of P is the leading term $p_m \in S^m$ in the asymptotic expansion. When $d > 1$ let ds denote the volume form of the $d - 1$-sphere

$$\mathbb{S}^{d-1} = \{\xi \in \mathbb{R}^d \,|\, |\xi| = 1\},$$

according to radial and spherical co-ordinates of $\mathbb{R}^d$, i.e. $d\xi = r^{d-1} dr ds$, $\xi \in \mathbb{R}^d \setminus \{0\}$, $r > 0$, $s \in \mathbb{S}^{d-1}$. When $d = 1$ let $S^{d-1} = \{-1, 1\}$ with counting measure ds.

We understand the scalars to be embedded in ℓ_∞/c_0 as the classes $[a_n]$ where $a_n \in c$, i.e. if $\lambda \in \mathbb{C}$ then $[a_n] = \lim_{n\to\infty} a_n = \lambda$.

Proposition 6.16 (*Extension of the Noncommutative Residue*)**.** *Let P be a compactly based classical pseudo-differential operator of order $-d$ with principal symbol p_{-d}. Then* $\text{Res}(P)$ *is the scalar*

$$\text{Res}(P) = \text{Res}_W(P) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} p_{-d}(x, s) ds \, dx$$

where Res_W *denotes the noncommutative residue.*

Proof. By the previous lemma we need only consider the principal symbol p_{-d} of P, which we assume without loss to be homogeneous for $|\xi| \ge 1$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p_{-d}(x, \xi) d\xi \, dx &= \int_{\mathbb{R}^d} \int_{1 \le |\xi| \le n^{1/d}} |\xi|^{-d} p_{-d}(x, \xi/|\xi|) d\xi \, dx + O(1) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{S}^d} p_{-d}(x, s) ds \, dx \int_1^{n^{1/d}} r^{-d} r^{d-1} dr + O(1) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{S}^d} p_{-d}(x, s) ds \, dx \; \log(n^{1/d}) + O(1) \\
&= \frac{1}{d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^d} p_{-d}(x, s) ds \, dx \; \log n + O(1).
\end{aligned}$$

The result follows from (6.14). $\square$

Thus, the residue (as in Definition 6.13) of a classical compactly based pseudo-differential operator of order $-d$ is a scalar and coincides with the noncommutative residue. The residue of an arbitrary pseudo-differential operator is not always a scalar.

Example 6.17 (*Non-Measurable Pseudo-Differential Operators*)**.** We construct a compactly supported pseudo-differential operator Q of order $-d$ whose residue is not a scalar. The following lemma will simplify the construction. The lemma is a standard result on pseudo-differential operators, but we will use it several times.

Lemma 6.18. *Suppose P is a pseudo-differential operator with symbol $p \in S^m$ and $\psi, \phi \in C_c^\infty(\mathbb{R}^d)$. The compactly supported operator $Q = M_\psi P M_\phi$ has symbol $q \in S^m_{\text{base}}$ such that $q \sim \psi p \phi$.*

Proof. The operator $Q := M_\psi P M_\phi$ is a pseudo-differential operator of order m, [41, Corollary 3.1]. Evidently it is compactly supported. Let q be the symbol of Q. From [41, Theorem 3.1]

$$q(x,\xi) \sim \sum_\alpha \frac{(-i)^\alpha}{\alpha!} (\partial_\xi^\alpha \partial_y^\alpha \psi(x) p(x,\xi)\phi(y))|_{y=x}$$

where the asymptotic sum runs over all multi-indices α. Since

$$\sum_{|\alpha|\geq 1} \frac{(-i)^\alpha}{\alpha!} (\partial_\xi^\alpha \partial_y^\alpha \psi(x) p(x,\xi)\phi(y))|_{y=x} \in S^{m-1}_{\text{base}}$$

(see [41, p. 3]) we obtain $q(x,\xi) - \psi(x)p(x,\xi)\phi(x) \in S^{m-1}_{\text{base}}$. $\square$

Proposition 6.19. *There is a compactly supported pseudo-differential operator Q of order $-d$ such that*

$$\operatorname{Res}(Q) = [\sin \log \log n^{1/d}].$$

Remark 6.20. Obviously $\operatorname{Res}(Q)$ is not a scalar.

Proof. Set

$$p'(\xi) = \frac{\sin \log \log |\xi| + \cos \log \log |\xi|}{|\xi|^d}, \quad \xi \in \mathbb{R}^d, |\xi| \geq e.$$

One confirms by calculation that p' satisfies

$$|\partial_\xi^\alpha p'(\xi)| \leq 2.3^{|\alpha|}(d+|\alpha|)!|\xi|^{-d-|\alpha|}, \quad |\xi| > e.$$

Let $g \in C_c^\infty(\mathbb{R}^d)$ be $g(\xi) = g_1(|\xi|)$ where g_1 is positive and increasing such that

$$g_1(\xi) = \begin{cases} 0 & \xi \leq 3 \\ 1 & \xi \geq 4 \end{cases}, \quad \xi \in \mathbb{R}^d.$$

Then $p := gp' \in S^{-d}$, and denote by P the pseudo-differential operator with symbol p.

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be such that

$$\int_{\mathbb{R}^d} |\phi(x)|^2 dx = (\operatorname{Vol} \mathbb{S}^{d-1})^{-1}.$$

If Q is the operator $M_{\overline{\phi}} P M_\phi$ of Lemma 6.18 with symbol $q \sim \overline{\phi} p \phi$ then, provided $n \geq 4^d$,

$$\begin{aligned}\int_{\mathbb{R}^d} \int_{|\xi| \leq n^{1/d}} q(x, \xi) d\xi \, dx &= \int_{\mathbb{R}^d} \int_{|\xi| \leq n^{1/d}} |\phi(x)|^2 p(x, \xi) d\xi \, dx + O(1) \\ &= \int_{\mathbb{R}^d} |\phi(x)|^2 dx \int_{4 \leq |\xi| \leq n^{1/d}} p'(\xi) d\xi + O(1) \\ &= \int_4^{n^{1/d}} (\sin \log \log r + \cos \log \log r) r^{-d} r^{d-1} dr + O(1) \\ &= \frac{1}{d} (\sin \log \log n^{1/d}) \log n + O(1).\end{aligned}$$

The result follows from (6.14). $\square$

The operator Q will be used in Corollary 6.34 to provide an example of a non-measurable pseudo-differential operator.

Example 6.21 (*Integration of Square-Integrable Functions*)**.** The residue can be used to calculate the integral of a compactly supported square integrable function.

Proposition 6.22. *If $f \in L_2(\mathbb{R}^d)$ has compact support and Δ is the Laplacian on $\mathbb{R}^d$ then* $\mathrm{Res}(M_f(1-\Delta)^{-d/2})$ *is the scalar*

$$\mathrm{Res}(M_f(1-\Delta)^{-d/2}) = \mathrm{Vol}\, \mathbb{S}^{d-1} \int_{\mathbb{R}^d} f(x) dx.$$

Proof. Since $(1-\Delta)^{-d/2} = T_g$ where $g(\xi) = \langle \xi \rangle^{-d} \in L_{\mathrm{mod}}(\mathbb{R}^d)$, $M_f(1-\Delta)^{-d/2}$ is a compactly based Laplacian modulated operator by Proposition 6.11. Then

$$\begin{aligned}\int_{\mathbb{R}^d} \int_{|\xi| \leq n^{1/d}} f(x) \langle \xi \rangle^{-d} d\xi \, dx &= \int_{\mathbb{R}^d} f(x) dx \int_{1 \leq |\xi| \leq n^{1/d}} |\xi|^{-d} d\xi + O(1) \\ &= \frac{1}{d} \mathrm{Vol}\, \mathbb{S}^{d-1} \int_{\mathbb{R}^d} f(x) dx \; \log n + O(1).\end{aligned}$$

The result follows from (6.14). $\square$

6.2. Eigenvalues of Laplacian modulated operators

We now come to our main technical theorem. This result is at the heart of Connes' trace theorem.

Theorem 6.23. *Suppose $T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is compactly supported and Laplacian modulated with symbol p_T. Then $T \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$ and*

$$\sum_{j=1}^n \lambda_j(T) - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{|\xi| \leq n^{1/d}} p_T(x, \xi) d\xi dx = O(1) \tag{6.15}$$

where $\{\lambda_j(T)\}_{j=1}^\infty$ is any eigenvalue sequence of T.

Remark 6.24. If T is only compactly based, then (6.15) still holds but it may not be true that $T \in \mathcal{L}_{1,\infty}$. See the proof below.

The following lemmas are required for the proof. Let $\mathcal{Q}_z$ denote the unit cube on $\mathbb{R}^d$ centred on $z \in \mathbb{R}^d$.

Lemma 6.25. *There exists* $0 < \phi \in C_c^\infty(\mathbb{R}^d)$ *such that:*
(i) $\phi(x) = 1$, $x \in \pi \mathcal{Q}_0$, $\phi(x) = 0$, $x \notin 2\pi \mathcal{Q}_0$*;*
(ii) $\{u_m\}_{m\in\mathbb{Z}^d}$ *form an orthonormal set in* $L_2(\mathbb{R}^d)$ *where*

$$u_m(x) = \frac{1}{(2\pi)^{d/2}}\phi(x)e^{i\langle x,m\rangle}, \quad m \in \mathbb{Z}^d;$$

(iii) *for each* $N \in \mathbb{N}$ $|\hat{\phi}(\xi)| \stackrel{\cdot}{\lesssim}_N \langle\xi\rangle^{-N}$.

Proof. (i) Let h be a non-negative C^∞-function on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} h(t)\,dt = 1,$$

and for some $\delta < \pi/2$ we have $\operatorname{supp} h = (-\delta, \delta)$. We then define $g = h \star \chi_{[-\pi,\pi]}$. Then

$$\hat{g}(\xi) = 2\pi \hat{h}(\xi)\operatorname{sinc}(\pi\xi).$$

Hence

$$\begin{aligned}
&\hat{g}(0) = 2\pi,\\
&\hat{g}(n) = 0, \quad n \in \mathbb{Z} \setminus \{0\},\\
&\operatorname{supp}(g) = (-\pi - \delta, \pi + \delta)
\end{aligned}$$

and

$$g(t) = 1, \quad -\pi + \delta < t < \pi - \delta.$$

We also have that $\sqrt{g} \in C_c^\infty(\mathbb{R})$. Let us define

$$\phi(x) := \prod_{j=1}^{d} \sqrt{g(x_j)}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Then

$$\widehat{|\phi|^2}(\xi) := \prod_{j=1}^{d} \hat{g}(\xi_j), \quad \xi = (\xi_1, \ldots \xi_d) \in \mathbb{R}^d$$

and

$$\begin{aligned}
&\widehat{|\phi|^2}(0) = (2\pi)^d,\\
&\widehat{|\phi|^2}(m) = 0, \quad m \in \mathbb{Z}^d \setminus \{0\},\\
&\operatorname{supp}(\phi) \subset [-2\pi, 2\pi]^d,\\
&\phi(x) = 1, \quad x \in [-\pi, \pi]^d.
\end{aligned}$$

(ii) Since

$$\int_{\mathbb{R}^d} \overline{u_{m_1}(x)}u_{m_2}(x)dx = \frac{\widehat{|\phi|^2}(m_1 - m_2)}{(2\pi)^d} \quad m_1, m_2 \in \mathbb{Z}^d$$

the family $(u_m)_{m\in\mathbb{Z}^d}$ is orthonormal in $L_2(\mathbb{R}^d)$.

(iii) Since ϕ is smooth and compactly supported the estimate now follows from standard results; see e.g. [20, Problem 8.16, p. 113]. □

Lemma 6.26. *For $n \in \mathbb{N}$ let $\phi_n = \phi(x/n)$. Then:*

(i) $\phi_n(x) = 1$, $x \in \pi n \mathcal{Q}_0$, $\phi_n(x) = 0$, $x \notin 2\pi n \mathcal{Q}_0$;
(ii) $\{u_{m,n}\}_{m\in\mathbb{Z}^d}$ *form an orthonormal set in* $L_2(\mathbb{R}^d)$ *where*

$$u_{m,n}(x) = \frac{1}{(2\pi n)^{d/2}}\phi_n(x)e^{i\langle x,m/n\rangle}, \quad m \in \mathbb{Z}^d.$$

Proof. (i) Note

$$\phi(x/n) := \prod_{j=1}^{d} \sqrt{g(x_j/n)}, \quad x = (x_1,\ldots,x_d) \in \mathbb{R}^d.$$

Then

$$\widehat{|\phi_n|^2}(\xi) := \prod_{j=1}^{d} n^d \hat{g}(n\xi_j) = n^d \widehat{|\phi|^2}(n\xi)$$

and

$$\widehat{|\phi_n|^2}(0) = (2\pi n)^d,$$
$$\widehat{|\phi_n|^2}(m/n) = 0, \quad m \in \mathbb{Z}^d \setminus \{0\}.$$

(ii) Since

$$\int_{\mathbb{R}^d} \overline{u_{m_1,n}} u_{m_2,n}(x)dx = \frac{\widehat{|\phi_n|^2}((m_1-m_2)/n)}{(2\pi n)^d} \quad m_1, m_2 \in \mathbb{Z}^d$$

the family $(u_{m,n})_{m\in\mathbb{Z}^d}$ is orthonormal in $L_2(\mathbb{R}^d)$. □

For $n \in \mathbb{N}$, let $\mathcal{H}_n$ denote the Hilbert space generated by $(u_{m,n})_{m\in\mathbb{R}^d}$. Let P_n be the projection such that $\mathcal{H}_n = P_n L_2(\mathbb{R}^d)$. Clearly $L_2(\pi n \mathcal{Q}_0) \subset \mathcal{H}_n$. Let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ be the positive compact operator defined by

$$V_n u_{m,n} = (1 + |m|^2)^{-d/2} u_{m,n}, \quad m \in \mathbb{Z}^d.$$

Then

$$T_n := P_n T P_n : \mathcal{H}_n \to \mathcal{H}_n$$

for any bounded operator T. We now show that if T is Laplacian modulated then T_n is V_n-modulated. Let σ_l denote the bi-dilation isometry of $L_2(\mathbb{R}^d, \mathbb{R}^d)$ to itself:

$$(\sigma_l p)(x,\xi) = p(lx, \xi/l), \quad l > 0.$$

Lemma 6.27. *Let T be Laplacian modulated, with symbol $p_T \in S^{\mathrm{mod}}$. Then*

$$\sum_{|m|\geq n^{1/d}} \|T u_{m,l}\|_{L_2}^2 \dot{\leq}_d \|\sigma_l p_T\|_{\mathrm{mod}}^2 \, n^{-1}, \quad n \geq 1.$$

Proof. Fix $l \in \mathbb{N}$. Denote p_T just by p and $u_{m,l}$ by u_m. Note that

$$\widehat{u_m}(\xi) = (2\pi l)^{-d/2}\hat{\phi}_l(\xi - m/l) = (2\pi l)^{-d/2} l^d \hat{\phi}(l\xi - m).$$

Then

$$\begin{aligned} Tu_m(x) &= (2\pi l)^{-d/2} l^d \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{\phi}(l\xi - m)d\xi \\ &= (2\pi l)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x,\xi/l\rangle} p(x,\xi/l)\hat{\phi}(\xi - m)d\xi \\ &= (2\pi l)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x/l,\xi\rangle} p(x,\xi/l)\hat{\phi}(\xi - m)d\xi \end{aligned}$$

and

$$|Tu_m(x)|^2 \leq (2\pi)^{-d} l^{-d} \left(\int_{\mathbb{R}^d} |p(x,\xi/l)||\hat{\phi}(\xi - m)|d\xi\right)^2.$$

By Lemma 6.26 $|\hat{\phi}(\xi)| \dot{\leq}_N \langle\xi\rangle^{-N}$ for any integer N. Hence

$$\begin{aligned} \|Tu_m\|_{L_2}^2 &\dot{\leq}_N (2\pi)^{-d} l^{-d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p(x,\xi/l)|\langle\xi - m\rangle^{-N} d\xi\right)^2 dx \\ &\dot{\leq}_N (2\pi)^{-d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p(lx,\xi/l)|\langle\xi - m\rangle^{-N} d\xi\right)^2 dx. \end{aligned} \tag{6.16}$$

Temporarily denote $p(lx,\xi/l)$ by $p_l(x,\xi)$.

Now let $s \in \mathcal{Q}_m$. Then

$$|m - s| \leq \sqrt{\sum_{i=1}^{d} (1/2)^2} = d^{1/2}/2.$$

Hence

$$\langle m - s\rangle := 1 + |m - s| \leq 1 + d^{1/2}/2 \leq 2d^{1/2}.$$

Using Peetre's inequality,

$$\langle x\rangle^N \langle y\rangle^{-N} \dot{\leq}_N \langle x - y\rangle^N, \quad x, y \in \mathbb{R}^d,$$

we have

$$\langle\xi - s\rangle^N \langle\xi - m\rangle^{-N} \dot{\leq}_N \langle\xi - s - (\xi - m)\rangle^N = \langle m - s\rangle^N \dot{\leq}_{N,d} 1, \quad \xi \in \mathbb{R}^d, s \in \mathcal{Q}_m, m \in \mathbb{Z}^d.$$

Multiplying throughout by $\langle\xi - s\rangle^{-N}$ we obtain

$$\langle\xi - m\rangle^{-N} \dot{\leq}_{d,N} \langle\xi - s\rangle^{-N}, \quad s \in \mathcal{Q}_m. \tag{6.17}$$

Substituting (6.17) into (6.16) provides

$$\|Tu_m\|_{L_2}^2 \dot{\leq}_{d,N} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p_l(x,\xi)|\langle\xi - s\rangle^{-N} d\xi\right)^2 dx, \quad s \in \mathcal{Q}_m,$$

and hence

$$\|Tu_m\|_{L_2}^2 = \int_{\mathcal{Q}_m} \|Tu_m\|_{L_2}^2 ds \dot{\leq}_{d,N} \int_{\mathcal{Q}_m} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p_l(x,\xi)|\langle\xi - s\rangle^{-N} d\xi\right)^2 dx ds. \tag{6.18}$$

Now let $n > d^{d/2}$. Then $n^{1/d}/2 > d^{1/2}/2$ and $n^{1/d} - d^{1/2}/2 < n^{1/d}/2$. Hence

$$|m| \geq n^{1/d} \Longrightarrow |s| \geq n^{1/d}/2, \quad s \in \mathcal{Q}_m.$$

Using (6.18) we have

$$\begin{aligned}\sum_{|m|\geq n^{1/d}} \|Tu_m\|_{L_2}^2 &\dot{\leq}_{d,N} \sum_{|m|\geq n^{1/d}} \int_{\mathcal{Q}_m} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 dxds \\ &\dot{\leq}_{d,N} \int_{|s|\geq n^{1/d}/2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 dxds. \qquad (6.19)\end{aligned}$$

We split the integrand in (6.19) into two parts according to the condition $|\xi - s| \geq |s|/2$,

$$\begin{aligned}\left(\int_{\mathbb{R}^d} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 &\dot{\leq} \left(\int_{|\xi-s|\geq|s|/2} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 \\ &\quad + \left(\int_{|\xi-s|<|s|/2} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 \qquad (6.20)\end{aligned}$$

where the inequality is from $(a+b)^2 \leq 2(a^2+b^2)$, $a, b > 0$.

We consider the first term from (6.20). As $|\xi - s| \geq |s|/2$, then $\langle \xi - s\rangle^{-N} \leq 2^N \langle s\rangle^{-N}$. Then

$$\begin{aligned}&\left(\int_{|\xi-s|\geq|s|/2} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 \\ &\quad \leq \int_{|\xi-s|\geq|s|/2} |p_l(x,\xi)|^2 d\xi \int_{|\xi-s|\geq|s|/2} \langle \xi - s\rangle^{-2N} d\xi \\ &\quad \dot{\leq} \langle s\rangle^{-N} \int_{\mathbb{R}^d} |p_l(x,\xi)|^2 d\xi \int_{|\xi-s|\geq|s|/2} \langle \xi - s\rangle^{-N} d\xi \\ &\quad \leq \langle s\rangle^{-N} \int_{\mathbb{R}^d} |p_l(x,\xi)|^2 d\xi \int_{\mathbb{R}^d} \langle \xi\rangle^{-N} d\xi\end{aligned}$$

where we used the Holder inequality (assuming $N > d$). We now set $N = 2d$ and then

$$\begin{aligned}&\int_{|s|\geq n^{1/d}/2} \int_{\mathbb{R}^d} \left(\int_{|\xi-s|\geq|s|/2} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 dxds \\ &\quad \dot{\leq} \|p_l\|_{L_2}^2 \int_{|s|\geq n^{1/d}/2} \langle s\rangle^{-2d} ds \\ &\quad \dot{\leq} \|p\|_{L_2}^2 n^{-1}.\end{aligned}$$

Now we consider the second term from (6.20). We have

$$\begin{aligned}&\left(\int_{|\xi-s|<|s|/2} |p_l(x,\xi)| \langle \xi - s\rangle^{-N} d\xi \right)^2 \leq \left(\int_{|\xi|<|s|/2} |p_l(x,\xi+s)| \langle \xi\rangle^{-N} d\xi \right)^2 \\ &\quad \dot{\leq} \int_{|\xi|<|s|/2} |p_l(x,\xi+s)|^2 \langle \xi\rangle^{-N} d\xi \int_{\mathbb{R}^d} \langle \xi\rangle^{-N} d\xi \\ &\quad \dot{\leq} \int_{|\xi|<|s|/2} |p_l(x,\xi+s)|^2 \langle \xi\rangle^{-N} d\xi\end{aligned}$$

where we used the Holder inequality and assumed $N > d$. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_l(x,\xi+s)|^2\langle\xi\rangle^{-N} d\xi dx ds \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_l(x,s)|^2\langle\xi\rangle^{-N} d\xi dx ds \\
&\quad \dot{\leq} \|p_l\|_{L_2}^2.
\end{aligned}$$

Hence we can interchange the order of integration. As $|s|/2 \geq n^{1/d}/4$ and $|\xi| < |s|/2$, then $|\xi + s| > |s|/2 \geq n^{1/d}/4$. By Fubini's Theorem

$$\begin{aligned}
&\int_{|s|\geq n^{1/d}/2}\int_{|\xi|<|s|/2} |p_l(x,\xi+s)|^2\langle\xi\rangle^{-N} d\xi ds \\
&\quad \leq \int_{\mathbb{R}^d}\left(\int_{|\xi+s|\geq n^{1/d}/4} |p_l(x,\xi+s)|^2 ds\right)\langle\xi\rangle^{-N} d\xi \\
&\quad = \int_{\mathbb{R}^d}\left(\int_{|s|\geq n^{1/d}/4} |p_l(x,s)|^2 ds\right)\langle\xi\rangle^{-N} d\xi \\
&\quad = \int_{|s|\geq n^{1/d}/4} |p_l(x,s)|^2 ds \int_{\mathbb{R}^d}\langle\xi\rangle^{-N} d\xi.
\end{aligned}$$

By choosing $N > d$,

$$\begin{aligned}
&\int_{|s|\geq n^{1/d}/2}\int_{\mathbb{R}^d}\left(\int_{|\xi-s|<|s|/2} |p_l(x,\xi)|\langle\xi-s\rangle^{-N} d\xi\right)^2 dx ds \\
&\quad \dot{\leq} \int_{\mathbb{R}^d}\int_{|s|\geq n^{1/d}/4} |p_l(x,s)|^2 ds dx \\
&\quad \dot{\leq} \left(\sup_{n\geq\min\{4,d^{d/2}\}} (n^{1/d}/4)^{-d}\int_{\mathbb{R}^d}\int_{|s|\geq n^{1/d}/4} |p_l(x,s)|^2 ds dx\right) n^{-1} \\
&\quad \dot{\leq} \left(\sup_{t\geq 1} t^d \int_{\mathbb{R}^d}\int_{|s|\geq t} |p_l(x,s)|^2 ds dx\right) n^{-1}.
\end{aligned}$$

The inequality of the proposition is shown for $n \geq \min\{4, d^{d/2}\}$. It is trivial to adjust the statement by a constant so that it holds also for $1 \leq n < \min\{4, d^{d/2}\}$. $\square$

Fix $l \in \mathbb{N}$. Let $\{u_m\}_{m\in\mathbb{Z}^d}$ be the basis of $\mathcal{H}_l$. Let m_n, $n \in \mathbb{N}$, be the Cantor enumeration of $\mathbb{Z}^d$. Then $V_l u_{m_n} = (1 + |m_n|^2)^{-d/2} u_{m_n}$ is ordered so that $(1 + |m_n|^2)^{-d/2}$ are the singular values of V_l.

Lemma 6.28. *Let T be Laplacian modulated with symbol p_T and $l \in \mathbb{N}$. Then $T_l : \mathcal{H}_l \to \mathcal{H}_l$ is V_l-modulated.*

Proof. Let Λ_k be the hypercube centred on 0 of dimensions k^d. Then there exists an integer k such that $m_n \notin \Lambda_k$ but $m_n \in \Lambda_{k+1}$. By the Cantor enumeration $(k-1)^d < n \leq k^d$. So $n^{1/d} \leq k$. Hence, when $j > n$, m_j is not in the ball of radius $(k-1)/2$ centred on 0, which is smaller than the ball $n^{1/d}/4$ (ruling out the trivial case $n = 1$). In summary, when $j > n$, then

$$m_j \notin \{m \in \mathbb{Z}^d, |m| \leq n^{1/d}/4\}.$$

Hence

$$\sum_{j>n} \|T_l u_{m_j}\|_{L_2}^2 \dot{\leq} \sum_{|m|>n^{1/d}/4} \|T u_m\|_{L_2}^2 \dot{\leq} n^{-1}$$

by Lemma 6.27 and T_l is V_l-modulated by Proposition 5.13 since $V_l \in \mathcal{L}_{1,\infty}$ and $s_n(V_l) = (1+|m_n|^2)^{-d/2} = \Theta(n^{-1})$. $\square$

Set $v_n = u_{m_n}$, $n \in \mathbb{N}$, where m_n is the Cantor enumeration.

Remark 6.29. As T_l is V_l-modulated where $0 < V_l \in \mathcal{L}_{1,\infty}$ we can use Theorem 5.2 to conclude that, for fixed $l \in \mathbb{N}$:

(i) $T_l \in \mathcal{L}_{1,\infty}$; and
(ii)

$$\sum_{j=1}^{n} \lambda_j(T_l) - \sum_{j=1}^{n} (T_l v_j, v_j) = O(1)$$

for any eigenvalue sequence $\{\lambda_j(T_l)\}_{j=1}^{\infty}$ of T_l.

We need a final lemma.

Lemma 6.30. *If T is compactly based in $l\pi \mathcal{Q}_0$ with symbol p_T, then*

$$\sum_{j=1}^{n} (T v_j, v_j) - \frac{1}{(2\pi)^d} \int_{|\xi| \leq n^{1/d}} \int_{\mathbb{R}^d} p_T(x,\xi) d\xi\, dx = O(1).$$

Proof. It is clear that there are constants $0 < 2a < 1 < b/2 < \infty$ depending on d so that $(v_j)_{j=1}^n = (u_m)_{m \in \mathbb{A}_n}$ where $\{|m| \leq 2an^{1/d}\} \subset \mathbb{A}_n \subset \{|m| \leq bn^{1/d}/2\}$. Let

$$G_n(x,\xi) := \frac{1}{(2\pi)^d} \sum_{m \in \mathbb{A}_n} e^{i\langle x, \xi - m\rangle} \hat{\phi}(\xi - m).$$

Now

$$(T u_m, u_m) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \phi(x) e^{-i\langle x, m\rangle} \left(\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p_T(lx, \xi/l) \hat{\phi}(\xi - m) d\xi \right) dx.$$

We use the notation $p_l(x,\xi) = p_T(lx,\xi/l)$ again and note that p_l has support in a compact set K within $\pi \mathcal{Q}_0$. Here the double integral converges absolutely and we can apply Fubini's theorem to obtain that, since $\phi(x) = 1$ for $x \in K$,

$$\sum_{j=1}^{n} (T v_j, v_j) = \sum_{m \in \mathbb{A}_n} (T u_m, u_m) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_K G_n(x,\xi) p_l(x,\xi) dx\, d\xi. \tag{6.21}$$

To obtain a bound

$$\int_{\mathbb{R}^d} \int_K G_n(x,\xi) p_l(x,\xi) dx\, d\xi - \int_K \int_{|\xi| \leq n^{1/d}} p_l(x,\xi) d\xi\, dx = O(1)$$

we will compare $G_n(x,\xi)$ with the function $H_n(\xi) := \chi_{\{|\xi| \leq n^{1/d}\}}$. For fixed ξ, consider the smooth periodic function

$$\psi_\xi(x) = \sum_{m \in \mathbb{Z}^d} e^{-i\langle x,\xi\rangle} \phi(x + 2\pi m). \tag{6.22}$$

For every x, the Fourier series,

$$\psi_\xi(x) = \sum_{m\in\mathbb{Z}^d} e^{i\langle x,m\rangle}\hat{\psi}_\xi(m)$$

converges, where

$$\begin{aligned}\hat{\psi}_\xi(m) &= \frac{1}{(2\pi)^d}\int_{2\pi Q} e^{-i\langle x,m\rangle}\psi_\xi(x)dx \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle x,m+\xi\rangle}\phi(x)dx = \frac{1}{(2\pi)^d}\hat{\phi}(m+\xi).\end{aligned}$$

Hence

$$\psi_\xi(x) = \frac{1}{(2\pi)^d}\sum_{m\in\mathbb{Z}^d} e^{-i\langle x,m\rangle}\hat{\phi}(\xi-m). \tag{6.23}$$

Note from (6.22) that $e^{i\langle x,\xi\rangle}\psi_\xi(x) = 1$ for all $x \in \pi Q_0$. Using (6.23) gives the formula

$$1 = \frac{1}{(2\pi)^d}\sum_{m\in\mathbb{Z}^d} e^{i\langle x,\xi-m\rangle}\hat{\phi}(\xi-m) \tag{6.24}$$

for $x \in \pi Q_0$. Now suppose $|\xi| < an^{1/d}$. Then, using (6.24),

$$|H_n(\xi) - G_n(x,\xi)| = |1 - G_n(x,\xi)| \le \frac{1}{(2\pi)^d}\sum_{m\notin A_n}|\hat{\phi}(\xi-m)|.$$

If $m \notin \mathbb{A}_n$ then $|m| > 2an^{1/d}$. Hence $|\xi - m| > |\xi|$ and $|\xi - m| \ge |m|/2$. This implies

$$|\hat{\phi}(\xi-m)| \dot{\le}_N \langle \xi-m\rangle^{-N} \dot{\le}_N \langle m\rangle^{-N}.$$

We choose $N > d$. We obtain, when $|\xi| < an^{1/d}$,

$$|H_n(\xi) - G_n(x,\xi)| \dot{\le}_N \sum_{|m|\ge 2an^{1/d}}\langle m\rangle^{-N} \dot{\le}_N n^{1-N/d}. \tag{6.25}$$

We also have an estimate $|G_n(x,\xi)| \dot{\le} 1$ when $x \in K$. Hence, for $an^{1/d} \le |\xi| \le bn^{1/d}$,

$$|H_n(\xi) - G_n(x,\xi)| \le 1. \tag{6.26}$$

If $|\xi| > bn^{1/d}$ then $|m - \xi| \ge |m|$ when $m \in \mathbb{A}_n$. Hence $|\xi - m| \ge |\xi|/2$ and

$$|\hat{\phi}(\xi-m)| \dot{\le}_N \langle \xi-m\rangle^{1-d-N/d} \dot{\le}_N \langle\xi\rangle^{1-d-N/d}, \quad m \in \mathbb{A}_n$$

Hence, for $|\xi| > bn^{1/d}$,

$$|G_n(x,\xi)| \dot{\le}_N \langle\xi\rangle^{1-d-N/d}\sum_{|m|\le bn^{1/d}} 1 \dot{\le}_N \langle\xi\rangle^{1-d-N/d} n \dot{\le}_N \langle\xi\rangle^{1-N/d}.$$

Then, for $|\xi| > bn^{1/d}$,

$$|H_n(\xi) - G_n(x,\xi)| \dot{\le}_N |\xi|^{1-N/d}. \tag{6.27}$$

Combining (6.25)–(6.27) we have, when $x \in K$, that

$$|H_n(\xi) - G_n(x,\xi)| \dot{\le}_N \begin{cases} n^{1-N/d} & |\xi| < an^{1/d} \\ 1 & an^{1/d} \le |\xi| \le bn^{1/d} \\ |\xi|^{1-N/d} & |\xi| > bn^{1/d}. \end{cases}$$

With this result we can show

$$\int_K \int_{\mathbb{R}^d} |G_n(x,\xi) - H_n(\xi)||p_l(x,\xi)| d\xi\, dx = O(1) \tag{6.28}$$

by considering the regions $|\xi| < an^{1/d}$, $an^{1/d} \le |\xi| \le bn^{1/d}$, and $|\xi| > bn^{1/d}$, and a choice of $N \ge (1+\epsilon)d$, $\epsilon > 0$. Consider

$$\begin{aligned}
&\int_K \int_{|\xi|<an^{1/d}} |G_n(x,\xi) - H_n(\xi)||p_l(x,\xi)| d\xi\, dx \\
&\quad \dot{\leq}_N\ n^{1-N/d} \int_K \int_{|\xi|<an^{1/d}} |p_l(x,\xi)| d\xi\, dx \\
&\quad \dot{\leq}_N\ n^{1-N/d} \log(n) \\
&\quad \dot{\leq}_N\ n^{1+\epsilon-N/d}\ \dot{\leq}\ 1
\end{aligned}$$

by using (6.11). Similarly

$$\begin{aligned}
&\int_K \int_{|\xi|>bn^{1/d}} |G_n(x,\xi) - H_n(\xi)||p_l(x,\xi)| d\xi\, dx \\
&\quad \dot{\leq} \int_K \int_{|\xi|>bn^{1/d}} |\xi|^{1-N/d} |p_l(x,\xi)| d\xi\, dx\ \dot{\leq}\ 1
\end{aligned}$$

by applying (6.12) since $1 - N/d < 0$. Finally,

$$\begin{aligned}
&\int_K \int_{an^{1/d}\le|\xi|\le bn^{1/d}} |G_n(x,\xi) - H_n(\xi)||p_l(x,\xi)| d\xi\, dx \\
&\quad \dot{\leq} \int_K \int_{an^{1/d}\le|\xi|\le bn^{1/d}} |p_l(x,\xi)| d\xi\, dx\ \dot{\leq}\ 1
\end{aligned}$$

by applying (6.10). Hence (6.28) is shown. Combining (6.21) and (6.28) shows that

$$\sum_{j=1}^n (Tv_j, v_j) - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{|\xi|\le n^{1/d}} p_l(x,\xi) d\xi\, dx = O(1).$$

Applying (6.13) shows that

$$\int_{\mathbb{R}^d} \int_{|\xi|\le n^{1/d}} p_l(x,\xi) d\xi\, dx - \int_{\mathbb{R}^d} \int_{|\xi|\le n^{1/d}} p(x,\xi) d\xi\, dx = O(1)$$

since

$$\int_{\mathbb{R}^d} \int_{|\xi|\le n^{1/d}} p(lx,\xi/l) d\xi\, dx = \int_{\mathbb{R}^d} \int_{|l\xi|\le n^{1/d}} p(x,\xi) d\xi\, dx.$$

The result is shown. □

Taking into account Remark 6.29 and Lemma 6.30, we are now in a position to prove Theorem 6.23.

Proof of Theorem 6.23. Suppose T is compactly based and Laplacian modulated. Choose $l \in \mathbb{N}$ sufficiently large so that $0 < \phi \in C_c^\infty(\mathbb{R}^d)$, $\phi(x) = 1$ for $x \in 2\pi l Q_0$, and $M_\phi T = T$. Since $M_\phi T = T$ then $S = TM_\phi = M_\phi T M_\phi$ is compactly supported and Laplacian modulated (see Remark 6.8).

From Lemma 6.28 the operator $S = S_l$ is V_l-modulated (if T is compactly supported then, without loss, $T = S$ and $T \in \mathcal{L}_{1,\infty}$, which is the first part of the theorem). By Remark 6.29

$$\sum_{j=1}^{n} \lambda_j(S) - \sum_{j=1}^{n} (Sv_j, v_j) = O(1).$$

However, up to the irrelevant multiplicity of zero as an eigenvalue,

$$\lambda_j(S) = \lambda_j(TM_\phi) = \lambda_j(M_\phi T) = \lambda_j(T).$$

Also, since $\phi v_j = v_j$, then

$$(Sv_j, v_j) = (Tv_j, v_j).$$

So

$$\sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} (Tv_j, v_j) = O(1).$$

The result of the theorem now follows from Lemma 6.30. $\square$

6.3. *Traces of Laplacian modulated operators*

In this section we prove several versions of Connes' trace theorem.

Let $\mathcal{H}$ be a separable Hilbert space. In [13], J. Dixmier constructed a trace on the Banach ideal of compact operators

$$\mathcal{M}_{1,\infty} := \left\{ T \in \mathcal{K}(\mathcal{H}) \;\middle|\; \sup_{n \in \mathbb{N}} \frac{1}{\log(1+n)} \sum_{j=1}^{n} s_j(T) < \infty \right\}$$

by linear extension of the weight

$$\mathrm{Tr}_\omega(T) := \omega\left(\left\{ \frac{1}{\log(1+n)} \sum_{j=1}^{n} s_j(T) \right\}_{n=1}^{\infty} \right), \quad T > 0.$$

Here ω is a dilation invariant state on ℓ_∞. By the inclusion $\mathcal{L}_{1,\infty} \subset \mathcal{M}_{1,\infty}$ a Dixmier trace Tr_ω restricts to a trace on $\mathcal{L}_{1,\infty}$. All Dixmier traces are normalised, meaning that

$$\mathrm{Tr}_\omega\left(\mathrm{diag}\left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \right) = 1.$$

The commutator subspace has been used previously to study spectral forms of Dixmier traces, e.g. [17,2]. Despite the Lidskii theorem, Corollary 4.2, it is not evident that

$$\mathrm{Tr}_\omega(T) := \omega\left(\left\{ \frac{1}{\log(1+n)} \sum_{j=1}^{n} \lambda_j(T) \right\}_{n=1}^{\infty} \right)$$

for an eigenvalue sequence $\{\lambda_j(T)\}_{j=1}^{\infty}$ of a compact operator T. By a combination of results from [39] and [2] the result is true for the restriction to $\mathcal{L}_{1,\infty}$. A more comprehensive proof that the result is true for all $T \in \mathcal{M}_{1,\infty}$ can be found in the monograph [35]; see Theorem 7.3.1 in that reference.

Lemma 6.31. *Suppose $T \in \mathcal{L}_{1,\infty}(\mathcal{H})$. Then*

$$\mathrm{Tr}_\omega(T) = \omega\left(\left\{\frac{1}{\log(1+n)}\sum_{j=1}^{n}\lambda_j(T)\right\}_{n=1}^{\infty}\right) \tag{6.29}$$

for any eigenvalue sequence $\{\lambda_j(T)\}_{j=1}^{\infty}$ of T, and any dilation invariant state ω.

Proof. From [39, Remark 13] it follows that

$$\mathrm{Tr}_\omega(T) = \omega\left(\left\{\frac{1}{\log(1+n)}\sum_{\{j||\lambda_j|>1/n\}}\lambda_j(T)\right\}_{n=1}^{\infty}\right).$$

A similar result was obtained in [17] and [2], but not for all dilation invariant states. The argument of [2, Corollary 2.12] that, when $T \in \mathcal{L}_{1,\infty}$,

$$\sum_{\{j||\lambda_j|>1/n\}}\lambda_j(T) - \sum_{j=1}^{n}\lambda_j(T) = O(1)$$

provides the result. □

Connes' trace theorem, [8], states that a Dixmier trace applied to a compactly supported classical pseudo-differential operator P of order $-d$ yields the noncommutative residue up to a constant,

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\mathrm{Res}_W(P). \tag{6.30}$$

Connes' statement was given for closed manifolds, but it is equivalent to (6.30). We shall consider manifolds in our final section.

The main result of our paper is the generalisation of Connes' trace theorem below. We note that a dilation invariant state ω is a generalised limit, i.e. it vanishes on c_0. Hence

$$\omega([c_n]) := \omega(\{c_n\}_{n=1}^{\infty}), \quad \{c_n\}_{n=1}^{\infty} \in \ell_\infty$$

is well-defined as a linear functional on ℓ_∞/c_0.

Theorem 6.32 (*Trace Theorem*)**.** *Suppose T is compactly based and Laplacian modulated such that $T \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$. Then:*

(i)

$$\mathrm{Tr}_\omega(T) = \frac{1}{d(2\pi)^d}\omega(\mathrm{Res}(T))$$

where $\mathrm{Res}(T) \in \ell_\infty/c_0$ *is the residue of T;*

(ii)

$$\mathrm{Tr}_\omega(T) = \frac{1}{d(2\pi)^d}\mathrm{Res}(T)$$

for every Dixmier trace Tr_ω *iff* $\mathrm{Res}(T)$ *is scalar;*

(iii)

$$\tau(T) = \frac{\tau \circ \operatorname{diag}\left(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right)}{(2\pi)^d d} \operatorname{Res}(T)$$

for every trace $\tau : \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)) \to \mathbb{C}$ *iff*

$$\int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p_T(x, \xi) d\xi\, dx = \frac{1}{d} \operatorname{Res}(T) \log n + O(1) \tag{6.31}$$

for a scalar $\operatorname{Res}(T)$.

Proof. (i) By Theorem 6.23 and the formula (6.14) we have that

$$\frac{1}{\log(1+n)} \sum_{j=1}^{n} \lambda_j(T) = \frac{1}{d(2\pi)^d} \operatorname{Res}_n(T) + o(1) \tag{6.32}$$

for any eigenvalue sequence $\{\lambda_j(T)\}_{j=1}^{\infty}$ and any representative $\operatorname{Res}_n(T)$ of the equivalence class $\operatorname{Res}(T)$. By the condition that $T \in \mathcal{L}_{1,\infty}$ we apply Lemma 6.31 and obtain

$$\begin{aligned} \operatorname{Tr}_\omega(T) &:= \omega\left(\left\{\frac{1}{\log(1+n)} \sum_{j=1}^{n} \lambda_j(T)\right\}_{n=1}^{\infty}\right) \\ &= \omega\left(\frac{1}{d(2\pi)^d} \operatorname{Res}_n(T) + o(1)\right) = \frac{1}{d(2\pi)^d} \omega\left(\operatorname{Res}(T)\right) \end{aligned}$$

since ω vanishes on sequences convergent to zero.

(ii) As $T \in \mathcal{L}_{1,\infty}$, by [38, Theorem 20] $\operatorname{Tr}_\omega(T)$ is the same value for all Dixmier traces if and only if $\operatorname{Tr}_\omega(T) = \lim_{n\to\infty} \frac{1}{\log(1+n)} \sum_{j=1}^{n} \lambda_j(T)$ and the limit on the right exists. From (6.32) $\lim_{n\to\infty} \frac{1}{\log(1+n)} \sum_{j=1}^{n} \lambda_j(T)$ exists if and only if $\operatorname{Res}(T)$ is scalar and $\lim_{n\to\infty} \frac{1}{\log(1+n)} \sum_{j=1}^{n} \lambda_j(T) = (2\pi)^{-d} d^{-1} \operatorname{Res}(T)$.

(iii) Suppose T satisfies (6.31). Then, by Theorem 6.23,

$$\sum_{k=1}^{n} \lambda_k(T) - \frac{1}{(2\pi)^d d} \operatorname{Res}(T) \sum_{k=1}^{n} \frac{1}{k} = O(1).$$

By Theorem 3.3,

$$T - \operatorname{diag}\left\{\frac{1}{(2\pi)^d d} \operatorname{Res}(T) \frac{1}{k}\right\}_{k=1}^{\infty} \in \operatorname{Com} \mathcal{L}_{1,\infty}.$$

Conversely, suppose the previous display is given. Then, by Theorem 3.3, for some decreasing sequence $\nu \in \ell_{1,\infty}$,

$$\left| \sum_{k=1}^{n} \lambda_k(T) - \frac{1}{(2\pi)^d d} \operatorname{Res}(T) \sum_{k=1}^{n} \frac{1}{k} \right| \le n\nu_n \mathrel{\dot{\lesssim}} 1,$$

and the symbol p_T of T satisfies (6.31) by Theorem 6.23. $\square$

Theorem 6.32 allows the Dixmier trace of any compactly based pseudo-differential operator of order $-d$ to be computed from its symbol.

Corollary 6.33. *If P is a compactly based pseudo-differential operator of order $-d$, then $P \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$ and*

$$\operatorname{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\omega(\operatorname{Res}(P))$$

for any dilation invariant state ω. Here $\operatorname{Res}(P)$ *is given by Definition* 6.13.

Proof. If P is compactly based, there exists $0 < \phi \in C_c^\infty(\mathbb{R}^d)$ with $M_\phi P = P$. The operators P and $P' = M_\phi P M_\phi$ are Laplacian modulated by Proposition 6.6. Since P' is compactly supported then $P' \in \mathcal{L}_{1,\infty}$ by Theorem 6.23. Note that $P - P' = [M_\phi, P] \in G^{-\infty}$ is a smoothing operator belonging to the Shubin class, see [41, IV], and the extensions of $G^{-\infty}$ belong to the trace class operators $\mathcal{L}_1$ on $\mathbb{R}^d$, [41, Section 27]; see also [1]. Hence $P = P' + (P - P') \in \mathcal{L}_{1,\infty}$. Now apply Theorem 6.32. □

Not every pseudo-differential operator is measurable in Connes' sense, [9, Section 4].

Corollary 6.34 (*Non-Measurable Pseudo-Differential Operators*)**.** *There is a compactly supported pseudo-differential operator Q of order $-d$ such that the value* $\operatorname{Tr}_\omega(Q)$ *depends on the dilation invariant state ω.*[3]

Proof. Let Q be the operator from Proposition 6.19. Since $\operatorname{Res}(Q) = [\sin\log\log n^{1/d}]$ is not a scalar, $\operatorname{Tr}_\omega(Q)$ depends on the state ω by Theorem 6.32(ii). □

Contrary to the case of general pseudo-differential operators of order $-d$, the next corollary shows that the classical pseudo-differential operators of order $-d$ have unique trace.

Corollary 6.35 (*Connes' Trace Theorem*)**.** *Suppose P is a compactly based classical pseudo-differential operator of order $-d$ with noncommutative residue* $\operatorname{Res}_W(P)$*. Then $P \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$ and*

$$\tau(P) = \frac{1}{d(2\pi)^d}\operatorname{Res}_W(P),$$

for every trace τ on $\mathcal{L}_{1,\infty}$ such that $\tau(\operatorname{diag}\{n^{-1}\}_{n=1}^\infty) = 1$.

Proof. By Corollary 6.33, $P \in \mathcal{L}_{1,\infty}$. Proposition 6.16 and its proof shows both that $\operatorname{Res}(P) = \operatorname{Res}_W(P)$ and that (6.31) is satisfied. The result now follows from Theorem 6.32. □

Remark 6.36. Corollary 6.34 indicates that the qualifier classical cannot be omitted from the statement of Connes' trace theorem.

Remark 6.37. Corollary 6.35 is stronger than Connes' original theorem. To see that the set of Dixmier traces restricted to $\mathcal{L}_{1,\infty}$ is smaller than the set of arbitrary normalised traces on $\mathcal{L}_{1,\infty}$, consider that the positive part of the common kernel of Dixmier traces is exactly the positive part of the separable subspace $(\mathcal{L}_{1,\infty}^0)^+$, [34], while the positive part of the common kernel of arbitrary normalised traces is exactly $\mathcal{L}_1^+ = (\operatorname{Com}\mathcal{L}_{1,\infty})^+$, easily seen from Theorem 3.1 or see [26].

[3] Such operators are called *non-measurable*.

Corollary 6.38 (*Integration of Square Integrable Functions*). *Let $f \in L_2(\mathbb{R}^d)$ be compactly supported. Then $M_f(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$, and*

$$\tau(M_f(1-\Delta)^{-d/2}) = \frac{\operatorname{Vol}\mathbb{S}^{d-1}}{d(2\pi)^d}\int_{\mathbb{R}^d} f(x)dx,$$

for every trace τ on $\mathcal{L}_{1,\infty}$ such that $\tau(\operatorname{diag}\{n^{-1}\}_{n=1}^{\infty}) = 1$.

Proof. It follows from [4, Section 5.7] that $M_f(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$. On the other hand, from Proposition 6.11, $M_f(1-\Delta)^{-d/2}$ is compactly based and Laplacian modulated. From the proof of Proposition 6.22 $\operatorname{Res}(M_f(1-\Delta)^{-d/2})$ is scalar, equal to $\operatorname{Vol}\mathbb{S}^{d-1}\int_{\mathbb{R}^d} f(x)dx$, and (6.31) is satisfied. From Theorem 6.32,

$$\tau(M_f(1-\Delta)^{-d/2}) = \frac{\tau \circ \operatorname{diag}\left(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right)\operatorname{Vol}\mathbb{S}^{d-1}}{d(2\pi)^d}\int_{\mathbb{R}^d} f(x)dx$$

for every trace on $\mathcal{L}_{1,\infty}$. $\square$

We now transfer our notions and results to the setting of closed Riemannian manifolds.

7. Closed Riemannian manifolds

In this section we introduce the notion of a Hodge-Laplacian modulated operator.

Notation. *Henceforth X will always denote a d-dimensional closed Riemannian manifold (X, g) with metric g, and Δ_g denotes the Laplace–Beltrami operator with respect to g,* [7, *p. 3*].

Definition 7.1. A bounded operator $T : L_2(X) \to L_2(X)$ is *Hodge-Laplacian modulated* if it is $(1-\Delta_g)^{-d/2}$-modulated for some metric g.

This definition is independent of the choice of metric g.

Lemma 7.2. *Suppose T is a bounded operator $T : L_2(X) \to L_2(X)$. If T is $(1-\Delta_{g_1})^{-d/2}$-modulated then it is $(1-\Delta_{g_2})^{-d/2}$-modulated for any pair of metrics g_1 and g_2.*

Proof. The operator $(1-\Delta_{g_1})^{-d/2}(1-\Delta_{g_2})^{d/2}$ is a zero-order pseudo-differential operator on X; see [41, Section 4] for pseudo-differential operators on manifolds. Hence it has a bounded extension, [41, Section 6.4], and there exists a constant C such that

$$\|(1-\Delta_{g_1})^{-d/2}f\|_{L_2} \le C\|(1-\Delta_{g_2})^{-d/2}f\|_{L_2}$$

for $f \in L_2(X)$. By Proposition 5.6 any $(1-\Delta_{g_1})^{-d/2}$-modulated operator is $(1-\Delta_{g_2})^{-d/2}$-modulated. $\square$

The positive bounded operator $(1-\Delta_g)^{-d/2} : L_2(X) \to L_2(X)$ is a compact operator (alternatively Δ_g has compact resolvent, [7, p. 8]).

Thus there exists an orthonormal basis $(e_n)_{n=1}^{\infty}$ of eigenvectors

$$-\Delta_g e_n = s_n e_n, \quad n \in \mathbb{N},$$

ordered such that the eigenvalues $s_1 \le s_n \le \cdots$ are increasing.

Also, by Weyl's asymptotic formula, [7, p. 9],

$$s_n^{-d/2} \sim l_d\, n^{-1}$$

for a constant l_d. Therefore

$$(1-\Delta_g)^{-d/2} \in \mathcal{L}_{1,\infty}.$$

Due to these spectral properties we can invoke Theorem 5.2 and arrive directly at a trace theorem by making the following definition of the residue.

Lemma 7.3. *If T is a Hodge-Laplacian modulated operator and $(e_n)_{n=1}^{\infty}$ is the above eigenvector sequence of the Laplace–Beltrami operator, then*

$$\sum_{j=1}^{n}(Te_j, e_j) = O(\log(1+n)).$$

Proof. From Theorem 5.2(i) $T \in \mathcal{L}_{1,\infty}$. Then $\mu_n(T) \dot{\leq} n^{-1}$ and

$$\left|\sum_{j=1}^{n}\lambda_j(T)\right| \leq \sum_{j=1}^{n}\mu_j(T) \dot{\leq} \log(1+n).$$

By Theorem 5.2(iii)

$$\sum_{j=1}^{n}(Te_j, e_j) = \sum_{j=1}^{n}\lambda_j(T) + O(1)$$

and the result follows. □

Definition 7.4. If T is a Hodge-Laplacian modulated operator the class

$$\operatorname{Res}(T) := d(2\pi)^d \left[\left\{\frac{1}{\log(1+n)}\sum_{j=1}^{n}(Te_j, e_j)\right\}_{n=1}^{\infty}\right] \tag{7.1}$$

is called the *residue* of T, where $[\cdot]$ denotes an equivalence class in ℓ_∞/c_0, and $(e_n)_{n=1}^{\infty}$ is the above eigenvector sequence of the Laplace–Beltrami operator.

Remark 7.5. The residue is evidently linear and vanishes on trace class Hodge-Laplacian operators. The proof of Lemma 7.3 shows that the residue is independent of the metric, since the eigenvalue sequence of a compact operator $T \in \mathcal{K}(L_2(X))$ does not depend on the metric.

Theorem 7.6 (*Trace Theorem for Closed Manifolds*)**.** *Let T be Hodge-Laplacian modulated. Then $T \in \mathcal{L}_{1,\infty}(L_2(X))$ and:*

(i)

$$\operatorname{Tr}_\omega(T) = \frac{1}{d(2\pi)^d}\omega(\operatorname{Res}(T))$$

for any Dixmier trace Tr_ω *where* $\operatorname{Res}(T) \in \ell_\infty/c_0$ *is the residue of T;*

(ii)

$$\operatorname{Tr}_\omega(T) = \frac{1}{d(2\pi)^d}\operatorname{Res}(T)$$

for every Dixmier trace Tr_ω *iff* $\operatorname{Res}(T)$ *is scalar;*

(iii)

$$\tau(T) = \frac{\tau \circ \operatorname{diag}\left(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right)}{(2\pi)^d d} \operatorname{Res}(T)$$

for every trace $\tau : \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)) \to \mathbb{C}$ *iff*

$$\sum_{j=1}^{n} (Te_j, e_j) = \frac{1}{d(2\pi)^d} \operatorname{Res}(T) \log(1+n) + O(1) \tag{7.2}$$

for a scalar $\operatorname{Res}(T)$.

Proof. The proof is omitted since it is identical to the proof of Theorem 6.32 with the use of Theorem 6.23 replaced exactly by Theorem 5.2(iii). □

To obtain the same corollaries of this trace theorem for closed manifolds as we did for Theorem 6.32 for $\mathbb{R}^d$, we identify the residue locally with the residue on $\mathbb{R}^d$.

7.1. Localised Hodge-Laplacian modulated operators

We emulate the usual treatment of pseudo-differential operators (e.g. [41, Section 4]), in that the symbol of a Hodge-Laplacian modulated operator is defined locally by the restriction to a chart and then patched together using a partition of unity. We extend Theorem 6.23 to a statement involving the cotangent bundle.

Without loss we let X be covered by charts $\{(U_i, h_i)\}_{i=1}^{N}$ such that $h_i(U_i)$ is bounded in $\mathbb{R}^d$. For a chart (U, h) belonging to such an atlas, define $W_h : L_2(h(U)) \to L_2(U)$ by $W_h f = f \circ h$ and $W_h^{-1} : L_2(U) \to L_2(h(U))$ by $W_h^{-1} f = f \circ h^{-1}$. Then W_h, W_h^{-1} are bounded.

Our first step is to confirm that Hodge-Laplacian modulated operators are locally Laplacian modulated operators. If $\phi \in C^\infty(X)$ is a smooth function we denote by M_ϕ the multiplication operator $(M_\phi f)(x) = \phi(x) f(x)$, $f \in L_2(X)$, and note that it is a pseudo-differential operator of order 0.

Proposition 7.7. *Suppose* (U, h) *is a chart of* X *with* $h(U)$ *bounded and* $\phi, \psi \in C^\infty(X)$ *such that* ϕ, ψ *have support in* U.

(i) *If* T *is a Hodge-Laplacian modulated operator then*

$$W_h^{-1} M_\psi T W_h M_{\phi \circ h^{-1}} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$$

is a compactly supported Laplacian modulated operator.

(ii) *If* T' *is a Laplacian modulated operator then*

$$W_h M_{\psi \circ h^{-1}} T' W_h^{-1} M_\phi : L_2(X) \to L_2(X)$$

is a Hodge-Laplacian modulated operator.

Proof. Let $H^s(X)$ denote the Sobolev spaces, $s \in \mathbb{R}$, on X. If V is the pseudo-differential operator with local symbol $(1 + |\xi|^2)^{1/2}$ the Sobolev norms are defined by $\|f\|_s := \|V^s f\|_{L_2}$, [41, Section 7]. We recall that $P : H^s(X) \to H^s(X)$ is continuous, [41, Theorem 7.3], for any zero-order pseudo-differential operator P and $s \in \mathbb{R}$. Since $(1 - \Delta_g)^{-s/2} V^s$ and $V^{-s}(1 - \Delta_g)^{s/2}$, $s \in \mathbb{R}$, are zero-order pseudo-differential operators on X they have bounded extensions.

(i) Let $\tilde{T} = W_h^{-1} M_\psi T W_h M_{\phi\circ h^{-1}}$. Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be such that $\rho(\psi \circ h^{-1}) = \psi \circ h^{-1}$ and $\rho(\phi \circ h^{-1}) = \phi \circ h^{-1}$. Then $M_\rho \tilde{T} M_\rho = \tilde{T}$ and $\tilde{T}$ is compactly supported.

Note that

$$\begin{aligned}\|(1-\Delta_g)^{-d/2} W_h M_{\phi\circ h^{-1}} f\|_{L_2} &= \|(1-\Delta_g)^{-d/2}\phi f \circ h\|_{L_2}\\ &\dot{\leq} \|V^{-d}\phi f \circ h\|_{L_2}\\ &\dot{\leq} \|(1-\Delta)^{-d/2}\phi \circ h^{-1} f\|_{L_2}\\ &\dot{\leq} \|(1-\Delta)^{-d/2} M_{\phi\circ h^{-1}} f\|_{L_2}\\ &\dot{\leq} \|(1-\Delta)^{-d/2} f\|_{L_2}, \quad f \in L_2(\mathbb{R}^d)\end{aligned}$$

since the Sobolev norms on $H^s(U)$ and $H^s(h(U))$ are equivalent norms and $M_{\phi\circ h^{-1}} : H^{-d}(\mathbb{R}^d) \to H^{-d}(\mathbb{R}^d)$ is bounded. By Proposition 5.6, $\tilde{T} = W_h^{-1} M_\psi T (W_h M_{\phi\circ h^{-1}})$ is Laplacian modulated.

(ii) We reverse the argument in (i). Note that

$$\begin{aligned}\|(1-\Delta)^{-d/2} W_h^{-1} M_\phi f\|_{L_2} &= \|(1-\Delta)^{-d/2}(\phi f)\circ h^{-1}\|_{L_2}\\ &\dot{\leq} \|V^{-d}\phi f\|_{L_2}\\ &\dot{\leq} \|V^{-d} f\|_{L_2}\\ &\dot{\leq} \|(1-\Delta_g)^{-d/2} f\|_{L_2}, \quad f \in L_2(X)\end{aligned}$$

since the Sobolev norms on $H^s(h(U))$ and $H^s(U)$ are equivalent and $M_\phi : H^s(X) \to H^s(X)$ is bounded. By Proposition 5.6, $W_h M_{\psi\circ h^{-1}} T'(W_h^{-1} M_\phi)$ is Hodge-Laplacian modulated. $\square$

Lemma 7.8. *If T is a Hodge-Laplacian modulated operator and P_1, P_2 are zero-order pseudo-differential operators on X, then $P_1 T P_2$ is a Hodge-Laplacian modulated operator.*

Proof. Since $P_1, P_2 : L_2(X) \to L_2(X)$ are bounded, $P_2 : H^s(X) \to H^s(X)$, $s \in \mathbb{R}$, is bounded, and

$$\|(1-\Delta_g)^{-d/2} P_2 f\|_{L_2} \dot{\leq} \|(1-\Delta_g)^{-d/2} f\|_{L_2}, \quad f \in L_2(X),$$

it follows that $P_1 T P_2$ is Hodge-Laplacian modulated by Proposition 5.6. $\square$

We can now define the coordinate dependent symbol of a Hodge-Laplacian modulated operator.

If (U, h) is a chart of X, let $\sum_{j,k=1}^d g_{jk}(x) dx_j dx_k$ denote the local co-ordinates of the metric for $x \in U$ and $G(x) = [g_{jk}(x)]_{j,k=1}^d$. The $d \times d$ matrix $G(x)$, $x \in U$, is positive and the determinant $|G(x)|$ is a smooth function on X.

Let $\{(U_i, h_i)\}_{i=1}^N$ be an atlas of X where $h_i(U_i) \subset \mathbb{R}^d$. Let $\Psi := \{\psi_j\}_{j=1}^M$ be a smooth partition of unity so that if $K_j := \overline{\mathrm{supp}(\psi_j)}$ then $K_j \cap K_{j'} \neq \emptyset$ implies that there exists an $i \in \{1, \ldots, N\}$ with $K_j \cup K_{j'} \subset U_i$. We will always assume such a partition of unity.

Let T be a Hodge-Laplacian modulated operator. Set

$$T_{jj'} := M_{\psi_j} T M_{\psi_{j'}}.$$

When $K_j \cap K_{j'} \neq \emptyset$,

$$T_{jj'}^h := W_{h_i}^{-1} M_{\psi_j} T W_{h_i} M_{\psi_{j'}\circ h_i^{-1}},$$

where some $i \in \{1, \dots, N\}$ is chosen such that $K_j \cup K_{j'} \subset U_i$, is compactly supported and Laplacian modulated by Proposition 7.7(ii). Let $p^h_{jj'} \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ be the symbol of $T^h_{jj'}$. Define

$$p_{jj'}(x, \xi) := p^h_{jj'}(h_i(x), G^{-1/2}(x)\xi)$$

for each $(x, \xi) \in T^*U_i \cong U_i \times \mathbb{R}^d$. If $K_j \cap K_{j'} = \emptyset$ set $p_{jj'} \equiv 0$.

We thus define a chart dependent function on the cotangent bundle $T^*(X, g)$, which we call the *coordinate dependent symbol* of T, by

$$p_T^{(\Psi,g)} := \sum_{j,j'=1}^{M} p_{jj'}.$$

Definition 7.9. If $T_{jj'} \in \operatorname{Com}\mathcal{L}_{1,\infty}(L_2(X))$ when $K_j \cap K_{j'} = \emptyset$, then we say T is *Ψ-localised.*

A codisc bundle $D^*(r)(X, g)$ of radius $r > 0$ is the subbundle of the cotangent bundle $T^*(X, g)$ with fibre over $x \in X$ given by

$$D^*_x(r)(X, g) \cong \{\xi \in \mathbb{R}^d \,|\, |G^{-1/2}(x)\xi| \le r\}.$$

Let dv be the density on $T^*(X, g)$ which corresponds locally to $dG^{-1/2}(x)\xi dx$.

Theorem 7.10. *If (X, g) is a closed d-dimensional Riemannian manifold, T is Hodge-Laplacian modulated, and T is Ψ-localised with respect to an atlas and partition of unity Ψ as above, then*

$$\sum_{k=1}^{n} \lambda_k(T) - \frac{1}{(2\pi)^d} \int_{D^*(n^{1/d})(X,g)} p_T^{(\Psi,g)}(v)dv = O(1). \tag{7.3}$$

Proof. Let $\Psi = \{\psi_j\}_{j=1}^M$ and set $T_{jj'} := M_{\psi_j} T M_{\psi_{j'}}$. By Lemma 7.8 $T_{jj'}$ is Hodge-Laplacian modulated such that $T = \sum_{j,j'=1}^M T_{jj'}$. We recall from Corollary 5.12 that

$$\sum_{k=1}^{n} \lambda_k(T) - \sum_{j,j'=1}^{M} \sum_{k=1}^{n} \lambda_k(T_{jj'}) = O(1). \tag{7.4}$$

By the assumption that T is Ψ-localised then $T_{jj'} \in \operatorname{Com}\mathcal{L}_{1,\infty}$ when $K_j \cap K_{j'} = \emptyset$. Hence

$$\sum_{k=1}^{n} \lambda_k(T_{jj'}) = O(1)$$

by Theorem 3.3 when $K_j \cap K_{j'} = \emptyset$. Thus (7.4) is valid when removing those $T_{jj'}$ such that $K_j \cap K_{j'} = \emptyset$.

When $K_j \cap K_{j'} \neq \emptyset$,

$$T_{jj'} : L_2(U_i) \to L_2(U_i)$$

and so

$$T^h_{jj'} : L^2(h_i(U_i)) \to L^2(h_i(U_i)),$$

for the chosen $i \in \{1 \ldots, N\}$ such that $K_j \cap K_{j'} \subset U_i$, is compactly supported and Laplacian modulated by Proposition 7.7(i). Since the Hilbert spaces $L_2(U_i)$ and $L_2(h_i(U_i))$ are equivalent it is an easy result that

$$\lambda_k(T_{jj'}) = \lambda_k(T^h_{jj'}) \tag{7.5}$$

with the same multiplicity and the same ordering.

Note that,

$$\begin{aligned}
&\int_{D^*(n^{1/d})(X,g)} (\chi_{K_j} p_{jj'} \chi_{K_{j'}})(v)dv \\
&\quad = \int_{h_i(U_i)} \int_{|G^{-1/2}(x)\xi| \le n^{1/d}} p^h_{jj'}(x, G^{-1/2}(x)\xi) dG^{-1/2}(x)\xi dx \\
&\quad = \int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p^h_{jj'}(x, \xi) d\xi dx.
\end{aligned}$$

Therefore

$$\int_{D^*(n^{1/d})(X,g)} p_T^{(\Psi,g)}(v)dv = \sum_{j,j} \int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p^h_{jj'}(x, \xi) d\xi dx \tag{7.6}$$

where the sum is over those j, j' with $K_j \cap K_{j'} \neq \emptyset$.

Finally, using (7.5) and (7.6), and where the sums are over those j, j' with $K_j \cap K_{j'} \neq \emptyset$,

$$\begin{aligned}
&\sum_{k=1}^n \lambda_k(T) - \frac{1}{(2\pi)^d} \int_{D^*(n^{1/d})(X,g)} p_T^{(\Psi,g)}(v)dv \\
&\quad = \left(\sum_{k=1}^n \lambda_k(T) - \sum_{j,j} \sum_{k=1}^n \lambda_k(T_{jj'}) \right) \\
&\qquad + \sum_{j,j} \left(\sum_{k=1}^n \lambda_k(T^h_{jj'}) - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p^h_{jj'}(x, \xi) dx d\xi \right) \\
&\quad = O(1) + O(1)
\end{aligned}$$

by (7.4) and by Theorem 6.23. $\square$

Suppose T and U are Hodge-Laplacian modulated operators. If $p_T^{(\Psi,g_1)}$ represents a coordinate dependent symbol of T with respect to a metric g_1 and some atlas and partition of unity Ψ as above, and $p_U^{(\Omega,g_2)}$ a coordinate dependent symbol of U with respect to a metric g_2 and some atlas and partition of unity Ω as above, then we write $p_T^{(\Psi,g_1)} \sim p_U^{(\Omega,g_2)}$ if

$$\int_{D^*(n^{1/d})(X,g_1)} p_T^{(\Psi,g_1)}(v)dv - \int_{D^*(n^{1/d})(X,g_2)} p_U^{(\Omega,g_2)}(v)dv = O(1).$$

The relation $\sim$ is easily checked to be an equivalence relation on coordinate dependent symbols.

Definition 7.11. A Hodge-Laplacian modulated operator T is called *localised* if $\phi T \psi \in \operatorname{Com} \mathcal{L}_{1,\infty}(L_2(X))$ for every pair of functions $\phi, \psi \in C^\infty(X)$ with $\phi\psi = 0$.

The next result says that every localised Hodge-Laplacian modulated operator can be assigned a coordinate and metric independent "principal" symbol.

Corollary 7.12. *Let T be a localised Hodge-Laplacian modulated operator. Then $p_T^{(\Psi,g_1)} \sim p_T^{(\Omega,g_2)}$ for every coordinate dependent symbol of T.*

Proof. If T is localised then T is Ψ-localised and Ω-localised. The eigenvalues of T are coordinate and metric independent, therefore

$$\begin{aligned}
&\frac{1}{(2\pi)^d}\left|\int_{D^*(n^{1/d})(X,g_1)} p_T^{(\Psi,g_1)}(v)dv - \int_{D^*(n^{1/d})(X,g_2)} p_T^{(\Omega,g_2)}(v)dv\right| \\
&\quad \le \left|\sum_{k=1}^n \lambda_k(T) - \frac{1}{(2\pi)^d}\int_{D^*(n^{1/d})(X,g_1)} p_T^{(\Psi,g_1)}(v)dv\right| \\
&\qquad + \left|\sum_{k=1}^n \lambda_k(T) - \frac{1}{(2\pi)^d}\int_{D^*(n^{1/d})(X,g_2)} p_T^{(\Omega,g_2)}(v)dv\right|.
\end{aligned}$$

The result follows by Theorem 7.10. □

Due to the result of Theorem 7.10 we will only be concerned with coordinate dependent symbols up to the equivalence $\sim$. As a result of Lemma 7.2 and Corollary 7.12 we may fix a metric g and we may take any coordinate dependent symbol to act as a representative for the symbol when discussing localised Hodge-Laplacian modulated operators. To this end we let p_T denote a coordinate dependent symbol $p_T^{(\Psi,g)}$, dropping explicit reference to the coordinates and the metric.

We can now prove the desired result that links the residue for Hodge-Laplacian operators to a formula involving the "principal" symbol.

Theorem 7.13. *If (X, g) is a closed d-dimensional Riemannian manifold and T is localised and Hodge-Laplacian modulated with symbol p_T, then*

$$\mathrm{Res}(T) = \left[\frac{d}{\log(1+n)}\int_{D^*(n^{1/d})(X,g)} p_T(v)dv\right]$$

where $[\cdot]$ denotes an equivalence class in ℓ_∞/c_0, or, more specifically,

$$\sum_{j=1}^n (Te_j, e_j) = \frac{1}{(2\pi)^d}\int_{D^*(n^{1/d})(X,g)} p_T(v)dv + O(1). \tag{7.7}$$

Proof. The first display clearly follows from the second display. By Theorem 7.10

$$\sum_{k=1}^n \lambda_k(T) - \frac{1}{(2\pi)^d}\int_{D^*(n^{1/d})(X,g)} p_T(v)dv = O(1)$$

and by Theorem 5.2(iii)

$$\sum_{k=1}^n \lambda_k(T) - \sum_{j=1}^n (Te_j, e_j) = O(1).$$

The result is shown. □

7.2. Residues of Hodge-Laplacian modulated operators

We give examples of Hodge-Laplacian modulated operators and compute their residue using Theorem 7.13.

Example 7.14 (*Pseudo-Differential Operators*)**.** To show a pseudo-differential operator $P : C^\infty(X) \to C^\infty(X)$ of order $-d$ is Hodge-Laplacian modulated we use the following lemma.

Lemma 7.15. *The operator* $(1 - \Delta_g)^{-d/2}$ *is Hodge-Laplacian modulated.*

Proof. For brevity, set $V := (1 - \Delta_g)^{-d/2}$. Let $\{s_n\}_{n=1}^\infty$ be the singular values of V, where $s_n \le Cn^{-1}$ by Weyl's formula for a constant $C > 0$, as explained. Then

$$\|V(1+tV)^{-1}\|_{\mathcal{L}_2}^2 = \sum_{n=1}^\infty s_n^2(1+ts_n)^{-2} \le \sum_{n=1}^\infty (C^{-1}n+t)^{-2} \dot{\le} t^{-1}, \quad t \ge 1.$$

Hence V is V-modulated. □

Proposition 7.16. *Let* $P : C^\infty(X) \to C^\infty(X)$ *be a pseudo-differential operator of order* $-d$*. Then the extension* $P : L_2(X) \to L_2(X)$ *is localised and Hodge-Laplacian modulated.*

Proof. For brevity set $V := (1 - \Delta_g)^{-d/2}$. Then V is Hodge-Laplacian modulated. Thus $P_0 V$ is Hodge-Laplacian modulated for every zero-order pseudo-differential operator P_0 by Lemma 7.8. By the pseudo-differential calculus PV^{-1} is zero order, [41, Sections 3 and 6], hence $P = (PV^{-1})V$ is Hodge-Laplacian modulated.

The operator P is localised since, $M_\psi P M_\phi \in \mathcal{L}_1 \subset \mathrm{Com}\,\mathcal{L}_{1,\infty}$ when $\psi, \phi \in C^\infty(X)$ and $\psi\phi = 0$ by the definition of pseudo-differential operators on a manifold, [41, Sections 4 and 27]. □

It follows from Theorem 7.13 that the residue of a pseudo-differential operator of order $-d$ can be calculated from its principal symbol.

Example 7.17 (*Noncommutative Residue*)**.** The cosphere bundle S^*X of (X, g) is the subbundle of $T^*(X, g)$ with fibre over $x \in X$ given by

$$S_x^*X \cong \{\xi \in \mathbb{R}^d \,|\, |G^{-1/2}(x)\xi| = 1\}.$$

The cosphere bundle has a density ds equating locally to $dx ds_x$ where ds_x is the volume element of the fibre.

Proposition 7.18. *Let* $P : C^\infty(X) \to C^\infty(X)$ *be a classical pseudo-differential of order* $-d$ *and with principal symbol* p_{-d}*. Then* P *is localised and Hodge-Laplacian modulated and the residue of* P *is the scalar value*

$$\mathrm{Res}(P) = \mathrm{Res}_W(P) := \int_{S^*X} p_{-d}(s)ds$$

where Res_W *denotes the noncommutative residue.*

Proof. By Proposition 7.16 P is localised and Hodge-Laplacian modulated. Both Res and Res_W evidently depend only on the principal symbol and hence we can work locally. The result then follows immediately from Proposition 6.16. Note also that this implies, from the proof of Theorem 7.13 and Proposition 6.16, that

$$\sum_{j=1}^{n}(Pe_j, e_j) = \frac{1}{d(2\pi)^d}\int_{S^*X} p_{-d}(s)ds \log n + O(1). \quad \square$$

An immediate corollary (the proof is omitted) is the following spectral formulation of the noncommutative residue of classical pseudo-differential operators. The first equality was observed by Fack, [17, p. 359], and proven in [2, Corollary 2.14].

Corollary 7.19 (*Spectral Formula for the Noncommutative Residue*). *Let $P : C^\infty(X) \to C^\infty(X)$ be a classical pseudo-differential operator of order $-d$, $\{\lambda_n(P)\}_{n=1}^\infty$ denote the non-zero eigenvalues of P ordered so that $|\lambda_n(P)|$ is decreasing, and $(e_n)_{n=1}^\infty$ an orthonormal basis of eigenvectors of the Hodge-Laplacian, $-\Delta_g e_n = s_n e_n$, $n \in \mathbb{N}$, ordered such that the eigenvalues $s_1 \le s_n \le \cdots$ are increasing. Then, if Res_W is the noncommutative residue,*

$$d^{-1}(2\pi)^{-d}\mathrm{Res}_W(P) = \lim_{n\to\infty}\frac{1}{\log(1+n)}\sum_{j=1}^{n}\lambda_j(P) = \lim_{n\to\infty}\frac{1}{\log(1+n)}\sum_{j=1}^{n}(Pe_j, e_j).$$

Example 7.20 (*Integration of Square Integrable Functions*). If $f \in L_2(X)$ let $M_f : L_\infty(X) \to L_2(X)$ be defined by $(M_f h)(x) = f(x)h(x)$, $h \in L_\infty(X)$.

Proposition 7.21. *If $f \in L_2(X)$ then there is a localised Hodge-Laplacian modulated operator T_f such that $M_f(1-\Delta_g)^{-d/2} - T_f \in \mathcal{L}_1$ and*

$$\mathrm{Res}(T_f) = \mathrm{Vol}\,\mathbb{S}^{d-1}\int_{X,g} f(x)\,dx.$$

Proof. Let $\{\psi_j\}_{j=1}^M$ be a partition of unity as in the previous section. For brevity, let $V := (1-\Delta_g)^{-d/2}$. Set $V_{jj'} := M_{\psi_j} V M_{\psi_{j'}}$ and $V_{jj'}^h = W_{h_i}^{-1} M_{\psi_j} V W_{h_i} M_{\psi_{j'}\circ h^{-1}}$. Set $T_{jj'} := M_f V_{jj'}$ and $T_{jj'}^h = M_{f\circ h^{-1}} V_{jj'}^h$.

If $K_j \cap K_{j'} \neq \emptyset$ we can find a chart (U_i, h_i) so that $K_j \cup K_{j'} \subset U_i$. Note that $V_{jj'}^h$ is a pseudo-differential operator of order $-d$ on $\mathbb{R}^d$ compactly supported in $h_i(U_i)$. Then

$$T_{jj'}^h = M_{f\circ h^{-1}}(1-\Delta)^{-d/2} P_0$$

where $P_0 = (1-\Delta)^{d/2} V_{jj'}^h$ is zero-order. By Proposition 6.11 $M_{f\circ h^{-1}}(1-\Delta)^{-d/2}$ is Laplacian modulated and by Remark 6.8 $M_{f\circ h^{-1}}(1-\Delta)^{-d/2}P_0$ is Laplacian modulated. Hence $T_{jj'}^h$ is Laplacian modulated and compactly supported in $h(U_i)$. It follows from Proposition 7.7(ii) that $T_{jj'}$ is Hodge-Laplacian modulated.

If $K_j \cap K_{j'} = \emptyset$ the operator $M_{\overline{\psi_{j'}}} V M_{\overline{\psi_j}} : L_1(X) \to C^\infty(X) \subset L_2(X)$ has a smooth kernel and is hence nuclear. Since $M_f : L_2 \to L_1$ is bounded we obtain $M_{\overline{\psi_{j'}}} V M_{\overline{\psi_j}} M_f \in \mathcal{L}_1$. By taking the adjoint $T_{jj'} \in \mathcal{L}_1$.

Let $T_f = T := \sum_{jj'} T_{jj'}$ where $K_j \cap K_{j'} \neq \emptyset$. Then T is Hodge-Laplacian modulated and obviously localised. Set $S := \sum_{jj'} T_{jj'}$ where $K_j \cap K_{j'} = \emptyset$. Then $S \in \mathcal{L}_1$. We have that $M_f(1-\Delta_g)^{-d/2} = T + S$ so the first statement is shown.

Since T is localised we need only work locally to determine the residue. If $K_j \cap K_{j'} \neq \emptyset$ so that $K_j \cup K_{j'} \subset U_i$ we examine the compactly supported Laplacian modulated operator

$T^h_{jj'} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$. The symbol $p^h_{jj'}$ of $T^h_{jj'}$ is given by $p^h_{jj'}(x,\xi) = f \circ h^{-1}(x) q_{jj'}(x,\xi)$ where $q_{jj'}(x,\xi)$ is the symbol of the pseudo-differential operator $M_{\psi_j} V M_{\psi_{j'}}$ in local co-ordinates. We recall that

$$q_{jj'}(x,\xi) - (\psi_j\psi_{j'})(h^{-1}(x))|G^{1/2}(x)\xi|^{-d} \in S^{-d-1}_{\text{base}}$$

by Lemma 6.18. Since $f \in L_2(X) \subset L_1(X)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} \left| p^h_{jj'}(x,\xi) - (f\psi_j\psi_{j'})(h^{-1}(x))|G^{1/2}(x)\xi|^{-d}\right| dx\,d\xi \\
&\quad \dot{\le} \int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} |f(x)|\langle\xi\rangle^{-d-1} dx\,d\xi \;\dot{\le}\; \|f\|_{L_1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} p^h_{jj'}(x,\xi)dx\,d\xi \\
&\quad = \int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} (f\psi_j\psi')(h^{-1}(x))|G(x)^{-1/2}\xi|^{-d} d\xi dx + O(1) \\
&\quad = \int_{\mathbb{R}^d}\int_{|G(x)^{1/2}\xi|\le n^{1/d}} (f\psi_j\psi')(h^{-1}(x))|G(x)|^{1/2}|\xi|^{-d} dx d\xi + O(1) \\
&\quad = \int_{\mathbb{R}^d}\int_{|\xi|\le n^{1/d}} (f\psi_j\psi')(h^{-1}(x))|G(x)|^{1/2}|\xi|^{-d} dx d\xi + O(1) \\
&\quad = \frac{\operatorname{Vol}\mathbb{S}^{d-1}}{d}\int_{X,g} f(x)\psi_j(x)\psi_{j'}(x)dx \log n + O(1)
\end{aligned}$$

where, in the second last equality, we used (6.13). Hence

$$\begin{aligned}
\operatorname{Res}(T) &= \sum_{j,j'} \operatorname{Res}(T_{jj'}) \\
&= \operatorname{Vol}\mathbb{S}^{d-1}\int_{X,g} f(x)\left(\sum_{j,j}\psi_j(x)\psi_{j'}(x)\right)dx = \operatorname{Vol}\mathbb{S}^{d-1}\int_{X,g} f(x)dx. \quad \square
\end{aligned}$$

7.3. Traces of localised Hodge-Laplacian modulated operators

In this section we obtain Connes' trace theorem and other results for closed Riemannian manifolds as corollaries of Theorem 7.6.

Corollary 7.22 (*Connes' Trace Theorem*)**.** *Let (X,g) be a closed d-dimensional Riemannian manifold. Suppose $P : C^\infty(X) \to C^\infty(X)$ is a classical pseudo-differential operator of order $-d$ with noncommutative residue* $\operatorname{Res}_W(P)$. *Then (the extension) $P \in \mathcal{L}_{1,\infty}(L_2(X))$ and*

$$\tau(P) = \frac{1}{d(2\pi)^d}\operatorname{Res}_W(P),$$

for every trace τ on $\mathcal{L}_{1,\infty}$ such that $\tau(\operatorname{diag}\{n^{-1}\}_{n=1}^\infty) = 1$.

Proof. That P is localised and Hodge-Laplacian, and satisfies (7.2) for the scalar $\operatorname{Res}_W(P)$, is given by Proposition 7.18. The results follow from Theorem 7.6. $\square$

As before, the qualifier classical cannot be omitted from Connes' trace theorem.

Corollary 7.23 (*Non-Measurable Pseudo-Differential Operators*)**.** *Let (X, g) be a closed d-dimensional Riemannian manifold. There exists a pseudo-differential operator $Q' : C^\infty(X) \to C^\infty(X)$ of order $-d$ such that the value $\mathrm{Tr}_\omega(Q')$ depends on the dilation invariant state ω.*

Proof. Let Q' be such that Q' vanishes outside a chart (U, h), and in local coordinates Q' is the operator Q (suitably scaled) of Corollary 6.34. Then the value $\mathrm{Tr}_\omega(Q')$ depends on the state ω. □

The final result is a stronger variant of one of our results in [33]. In the cited paper we showed the following result for Dixmier traces associated to zeta function residues. The proof employing the methods of this paper is completely different.

Corollary 7.24 (*See [33], Theorem 2.5*)**.** *Let (X, g) be a closed d-dimensional Riemannian manifold. Let $f \in L_2(X)$ and $M_f : L_\infty(X) \to L_2(X)$ be defined by $(M_f h)(x) = f(x)h(x)$, $h \in L_\infty(X)$. Then $M_f(1 - \Delta_g)^{-d/2} \in \mathcal{L}_{1,\infty}(L_2(X))$ and*

$$\tau(M_f(1 - \Delta_g)^{-d/2}) = \frac{\mathrm{Vol}\,\mathbb{S}^{d-1}}{d(2\pi)^d} \int_{X,g} f(x)dx,$$

for any trace τ on $\mathcal{L}_{1,\infty}$ such that $\tau(\mathrm{diag}\{n^{-1}\}_{n=1}^\infty) = 1$.

Proof. Proposition 7.21 provides the result that $M_f(1-\Delta_g)^{-d/2} = T_f + S$ where $T_f \in \mathcal{L}_{1,\infty}$ (by Theorem 7.6 since T_f is Hodge-Laplacian modulated) and $S \in \mathcal{L}_1$. Hence $M_f(1 - \Delta_g)^{-d/2} \in \mathcal{L}_{1,\infty}$. Also

$$\tau(M_f(1 - \Delta_g)^{-d/2}) = \tau(T_f)$$

for every trace τ on $\mathcal{L}_{1,\infty}$. Note that, for the operator T_f, Eq. (7.2) is satisfied for the scalar $\mathrm{Vol}\,\mathbb{S}^{d-1} \int_{X,g} f(x)dx$ by the proof of Proposition 7.21. By Theorem 7.6

$$\tau(T_f) = \frac{\mathrm{Vol}\,\mathbb{S}^{d-1}}{d(2\pi)^d} \int_{X,g} f(x)dx$$

for every τ. □

Remark 7.25. Our final remark is that the residue of Hodge-Laplacian modulated operators, Definition 7.4, is an extensive generalisation of Wodzicki's noncommutative residue. Definition 7.1 is a global definition requiring no reference to local behaviour. Therefore "non-local" Hodge-Laplacian modulated operators can exist and they admit a residue and, in theory, calculable trace. Whether there are any interesting possibilities behind this observation we do not know yet.

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