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# Fully symmetric functionals on a Marcinkiewicz space are Dixmier traces ☆

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**Abstract**

As a consequence of the exposition of Dixmier type traces in the book of A. Connes (1994) [2], we were led to ask how general is this class of functionals within the space of all unitarily invariant functionals on the corresponding Marcinkiewicz ideal $\mathcal{M}_\psi$. In this paper we prove the surprising result that the set of all Dixmier traces on $\mathcal{M}_\psi$ coincides with the set of all fully symmetric functionals on this space.


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## 1. Introduction

For a separable complex Hilbert space $H$, denote by $\mu_n(T)$, $n \in \mathbb{N}$, the singular values of a compact operator $T$ (the singular values are the eigenvalues of the operator $|T| = (T^*T)^{1/2}$ arranged with multiplicity in decreasing order). Let $\Omega$ denote the set of concave functions on $\mathbb{R}_+$

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such that $\lim_{t\to\infty}\psi(t)=\infty$ and $\lim_{t\to+0}\psi(t)=0$. Fixing $\psi\in\Omega$, we define a Banach ideal $(\mathcal{M}_\psi,\|\cdot\|_\psi)$ of compact operators in the algebra $B(H)$ of all bounded linear operators on $H$ by setting

$$\mathcal{M}_\psi := \left\{T\colon\ \|T\|_\psi := \sup_k \frac{1}{\psi(n)}\sum_{k=0}^{n}\mu_k(T)<\infty\right\}.$$

This ideal may be viewed as a noncommutative counterpart of the corresponding sequence space $M_\psi(\mathbb{N})$ of all sequences $x=(x_k)_{k=1}^\infty$ such that

$$\|x\|_{M_\psi}=\sup_n \frac{1}{\psi(n)}\sum_{k=1}^{n}x_k^*<\infty,$$

where $x^*$ denotes the decreasing rearrangement of the sequence $(|x_k|)_{k=1}^n$. Fixing an orthonormal basis $\{\delta_n\}_{n=1}^\infty$ in $H$, we may identify the space $M_\psi(\mathbb{N})$ with the subspace $\mathcal{M}_\psi^d$ of all diagonal matrices $x$ in $\mathcal{M}_\psi$.

Let $\omega$ be an arbitrary dilation-invariant state on $L_\infty(\mathbb{R})$. Consider a subspace in $L_\infty(\mathbb{R})$ generated by all functions taking a constant value $a_n=a_{-n}$ on the interval $[n,n+1)$, $n\in\mathbb{Z}$. This latter subspace is isometrically isomorphic to $\ell_\infty=\ell_\infty(\mathbb{N})$ and we still denote by $\omega$ the restriction of $\omega$ on this subspace. In [3], under the assumption that

$$\lim_{t\to\infty}\frac{\psi(2t)}{\psi(t)}=1,$$

J. Dixmier constructed a non-normal trace (a Dixmier trace) on $\mathcal{M}_\psi$ using the weight

$$Tr_\omega(T):=\omega\left(\left\{\frac{1}{\psi(n)}\sum_{k=0}^{n}\mu_k(T)\right\}_{k=1}^{\infty}\right),\quad 0\leqslant T\in\mathcal{M}_\psi. \tag{1}$$

Motivated by this construction, the concept of fully symmetric functionals was introduced and studied in [4–6]. In the setting of the sequence space $M_\psi(\mathbb{N})$ a positive functional $\varphi\in M_\psi(\mathbb{N})^*$ is called *fully symmetric* if and only if $\varphi(x)\leqslant\varphi(y)$ for any $0\leqslant x,y\in M_\psi(\mathbb{N})$ satisfying the inequality

$$\sum_{k=1}^{n}x_k^*\leqslant\sum_{k=1}^{n}y_k^*,\quad n=1,2,\ldots.$$

It is trivial that every Dixmier trace $Tr_\omega$ restricted to the subspace $\mathcal{M}_\psi^d$ defines a normalized fully symmetric functional on $M_\psi(\mathbb{N})$. It is also not difficult to see (e.g. [4, Theorem 4.5]) that every fully symmetric functional $\varphi\in M_\psi(\mathbb{N})^*$ naturally extends to a unitarily invariant trace on $\mathcal{M}_\psi$.

The main result of this paper, Theorem 11, is the very surprising converse statement that the extension of every fully symmetric functional $\varphi\in M_\psi(\mathbb{N})^*$ to $\mathcal{M}_\psi$ coincides with some Dixmier trace $Tr_\omega$. Furthermore, we completely characterize the subclass of all $\psi\in\Omega$ for which the formula (1) yields a Dixmier trace on $\mathcal{M}_\psi$ for any dilation-invariant state $\omega$ (see Theorem 8).

Our main results can be seen as answering a very natural question which arises from the discussion of Dixmier traces in the book of A. Connes [2]. We refer the reader to [2] for an explanation of the remarkable relationship between the Dixmier traces, the Wodzicki residue and geometry, both commutative and noncommutative.

We refer to [4,8,9] for general information concerning the relationship between traces on Banach ideals of compact operators and singular functionals on corresponding symmetric sequence spaces. Our exposition is adjusted to treat general Marcinkiewicz spaces on semifinite von Neumann algebras (the ideal $\mathcal{M}_\psi$ is a special example of a Marcinkiewicz operator space) and thus we consider function (not sequence) spaces $M_\psi$.

## 2. Preliminaries

Let $L_\infty := L_\infty(0,\infty)$ be the space of all bounded Lebesgue measurable functions on $(0,\infty)$.

For a given $\psi \in \Omega$, we define the Marcinkiewicz space $M_\psi$ of real valued measurable functions $x$ on $(0,\infty)$ by setting

$$\|x\|_{M_\psi} := \sup_{t>0} \frac{1}{\psi(t)} \int_0^t x^*(s)\,ds < \infty.$$

Marcinkiewicz spaces are an important example of symmetric (function) spaces and we refer for a detailed exposition of this theory to [10]. Applications of Marcinkiewicz sequence, function and operator spaces in noncommutative geometry are given in [1,2].

**Definition 1.** A positive linear functional $\varphi$ on $M_\psi$ is called fully symmetric if $\varphi(x) \leqslant \varphi(y)$ whenever $0 \leqslant x, y \in M_\psi$ are such that $\int_0^t x^*(s)\,ds \leqslant \int_0^t y^*(s)\,ds$ for all $t > 0$.

Fully symmetric functionals were introduced in [4], where they were termed as “symmetric functionals”.

Let us note that if $\varphi$ is a fully symmetric functional on the sequence space $M_\psi(\mathbb{N})$ it may always be extended to a fully symmetric functional on $M_\psi$ by setting

$$\tilde{\varphi}(x) = \varphi\left(\left(\int_{n-1}^{n} x^*(s)\,ds\right)_{n=1}^{\infty}\right), \quad x \geqslant 0,$$

and extending by linearity (see [4, Theorem 4.5]). The following result is established in [4, Theorem 3.4].

**Theorem 2.** *Let $\psi \in \Omega$. The Marcinkiewicz function space $M_\psi$ admits a non-trivial fully symmetric functional if and only if either*

$$\liminf_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1,$$

*or*

$$\liminf_{t\to 0} \frac{\psi(2t)}{\psi(t)} = 1.$$

For every fully symmetric $\varphi \in M_\psi^*$ the following equality holds (see e.g. [4, proof of Proposition 2.3])

$$\varphi\bigl(s^{-1}\sigma_s x\bigr) = \varphi(x). \tag{2}$$

Here, $\sigma_s$ is the dilation operator on the space of measurable functions on $(0,\infty)$ defined by the formula

$$\sigma_s x(t) := x(t/s), \quad s > 0,\ t > 0.$$

**Definition 3.** A linear functional $\omega \in L_\infty^*$ will be called a dilation-invariant state if

(i) $\omega \geqslant 0$,
(ii) $\omega(1) = 1$,
(iii) $\omega(f) = \omega(\sigma_s f)$ for every $f \in L_\infty$ and $s > 0$.

The following result is very similar to [5, Lemma 3.4] and is given here only for convenience of the reader.

**Proposition 4.** *Let $\omega$ satisfy* (i) *and* (ii) *of Definition* 3 *and let $f \in L_\infty$ be such that $\omega(f) = a$ and such that either $f - a \leqslant 0$ or else $f - a \geqslant 0$. Then the following hold:*

(i) $\omega(fg) = \omega(f)\omega(g)$, $\forall g \in L_\infty$;
(ii) $\omega(1/f) = 1/\omega(f)$, *if, in addition*, $1/f \in L_\infty$.

**Proof.** It is sufficient to prove only (i) since (ii) is a special case of (i). Due to assumptions we have

$$\bigl|\omega\bigl((f-a)g\bigr)\bigr| \leqslant \omega\bigl(|f-a|\|g\|_\infty\bigr) = \pm\|g\|_\infty \omega(f-a) = 0. \qquad \square$$

## 3. Main results

Let $D_\psi$ be the linear span of all positive non-increasing functions from $M_\psi$. Observe that $x \in D_\psi$ if and only if $x = y^* - z^*$, for $y, z \in M_\psi$.

**Lemma 5.** *Let $\varphi$ be a fully symmetric functional on $M_\psi$. If $x, y \in D_\psi$ are such that $\int_0^t x(s)\,ds \leqslant \int_0^t y(s)\,ds$, for every $t > 0$ then $\varphi(x) \leqslant \varphi(y)$.*

**Proof.** Let $x = u_1 - u_2$ and $y = v_1 - v_2$ with $u_1, u_2, v_1, v_2 \in M_\psi$ being positive non-increasing functions. We have

$$\int_0^t u_1(s) - u_2(s)\,ds \leqslant \int_0^t v_1(s) - v_2(s)\,ds, \quad \forall t > 0$$

and so

$$\int_0^t u_1(s) + v_2(s)\,ds \leqslant \int_0^t v_1(s) + u_2(s)\,ds, \quad \forall t > 0.$$

Since $u_1 + v_2$ and $v_1 + u_2$ are positive and non-increasing functions and $\varphi$ is fully symmetric, we have $\varphi(u_1 + v_2) \leqslant \varphi(v_1 + u_2)$ and $\varphi(x) \leqslant \varphi(y)$. $\square$

**Theorem 6.** *Let $\omega$ be a dilation-invariant state and suppose $\psi \in \Omega$. Then the following conditions on $\omega$ are equivalent:*

(i) $\omega(\frac{\psi(at)}{\psi(t)}) = 1$ *for some $a > 0$ and $a \neq 1$.*
(ii) $\omega(\frac{\psi(at)}{\psi(t)}) = 1$ *for every $a > 0$.*
(iii) $\omega(\frac{t\psi'(t)}{\psi(t)}) = 0$.

**Proof.** We start with the observation that if (i) holds for some $a > 0$, then it also holds for $1/a$. Indeed, since $f := \psi(at)/\psi(t)$, $t > 0$ satisfies the assumption of Proposition 4 we have

$$\omega\left(\frac{\psi(t)}{\psi(at)}\right) = 1$$

and by the dilation invariance of $\omega$ we obtain (i) for $a$ replaced with $1/a$.

(i) $\Longrightarrow$ (iii). We observe that for a suitable constant $C = C(a)$ we have $\psi(t) \leqslant C\psi(at)$ for all $t$. Then by concavity of $\psi$

$$\frac{at\psi'(at)}{\psi(at)} \leqslant C\frac{at\psi'(at)}{\psi(t)} \leqslant \frac{Ca}{a-1}\frac{\psi(at) - \psi(t)}{\psi(t)}.$$

Since $\omega$ is a dilation-invariant state, we deduce that

$$\omega\left(\frac{t\psi'(t)}{\psi(t)}\right) \leqslant \frac{Ca}{a-1}\omega\left(\frac{\psi(at) - \psi(t)}{\psi(t)}\right) = 0.$$

(iii) $\Longrightarrow$ (ii). Suppose $a > 1$. Then

$$\frac{\psi(at)}{\psi(t)} \leqslant 1 + \frac{(a-1)t\psi'(t)}{\psi(t)}$$

and so

$$\omega\left(\frac{\psi(at)}{\psi(t)}\right) \leqslant 1.$$

Since $\psi$ is increasing this gives

$$\omega\left(\frac{\psi(at)}{\psi(t)}\right) = 1.$$

Now if $a < 1$ we obtain the same result by the remarks at the beginning of the proof.

(ii) $\Longrightarrow$ (i) is trivial. $\square$

**Definition 7.** A dilation-invariant state $\omega \in L_\infty^*$ is said to be a $\psi$-compatible state if and only if the three equivalent conditions of Theorem 6 hold, i.e. if

$$\omega\left(\frac{\psi(at)}{\psi(t)}\right) = 1, \quad \text{for all } a > 0. \tag{3}$$

For many $\psi$ (e.g. if $\lim_{t\to 0}\psi(2t)/\psi(t) = \lim_{t\to\infty}\psi(2t)/\psi(t) = 1$) there is the case that every dilation-invariant state is $\psi$-compatible. We next give a precise condition for this to happen.

**Theorem 8.** *In order that every dilation-invariant state is $\psi$-compatible it is necessary and sufficient that for every $\epsilon > 0$ there exists a constant $C = C(\epsilon)$ so that*

$$\psi(st) \leqslant C s^\epsilon \psi(t), \quad t > 0,\ s > 1. \tag{4}$$

**Proof.** Assume (4) holds. Then for fixed $\epsilon > 0$ and any $n$ we have $\psi(2^n t) \leqslant C 2^{n\epsilon}\psi(t)$. Hence for any dilation-invariant state $\omega$,

$$\omega\left(\log\frac{\psi(2^n t)}{\psi(t)}\right) \leqslant \log C + \epsilon n \log 2.$$

However

$$\log\frac{\psi(2^n t)}{\psi(t)} = \sum_{k=1}^n \log\frac{\psi(2^k t)}{\psi(2^{k-1}t)}$$

and so by dilation-invariance we have

$$n\omega\left(\log\frac{\psi(2t)}{\psi(t)}\right) \leqslant \log C + \epsilon n$$

so that, letting $n \to \infty$ and then $\epsilon \to 0$,

$$\omega\left(\log\frac{\psi(2t)}{\psi(t)}\right) = 0.$$

Next note that

$$1 - \frac{\psi(t)}{\psi(2t)} \leqslant \log\frac{\psi(2t)}{\psi(t)}$$

and so

$$\omega\left(\frac{\psi(t/2)}{\psi(t)}\right) = \omega\left(\frac{\psi(t)}{\psi(2t)}\right) = 1.$$

Thus $\omega$ is $\psi$-compatible.

Conversely assume every $\omega$ is $\psi$-compatible. If (4) fails for some $\epsilon > 0$ there are sequences $(t_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ with $0 < t_n < \infty,\ 1 < a_n < \infty$ so that

$$\psi(a_n t_n) \geqslant n a_n^{\epsilon} \psi(t_n).$$

Note that we must have $a_n \to \infty$. Now define $\omega \in L_\infty^*$ by

$$\omega(f) = \lim_{\mathcal{U}} \frac{1}{\log a_n} \int_{t_n}^{a_n t_n} f(s) \frac{ds}{s},$$

where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$. It is straightforward that $\omega$ is a dilation-invariant state and

$$\omega\left(\frac{t\psi'(t)}{\psi(t)}\right) \geqslant \epsilon.$$

This is a contradiction. $\square$

Suppose that the conditions of Definition 7 hold. By Theorem 2, the space $M_\psi$ admits not-trivial singular fully symmetric functionals. Indeed, in this case

$$\inf_{t>0} \frac{\psi(2t)}{\psi(t)} \leqslant \omega\left(\frac{\psi(2t)}{\psi(t)}\right) = 1$$

and since $\frac{\psi(2t)}{\psi(t)} \geqslant 1$, we have either

$$\liminf_{t\to 0} \frac{\psi(2t)}{\psi(t)} = 1, \quad \text{or} \quad \liminf_{t\to \infty} \frac{\psi(2t)}{\psi(t)} = 1.$$

On the other hand, the condition on $\psi$ above implies (through Theorems 2 and 11) that the set of $\psi$-compatible dilation-invariant state is non-empty.

**Definition 9.** Let $\omega$ be a dilation-invariant state. The functional given by

$$\tau_\omega(x) = \omega\left(\frac{1}{\psi(t)} \int_0^t x^*(s)\, ds\right), \quad 0 < x \in M_\psi$$

is called a Dixmier functional (trace) if $\tau_\omega(x + y) = \tau_\omega(x) + \tau_\omega(y)$, $0 \leqslant x, y \in M_\psi$.

It is easy to check, that if $\tau_\omega$ is linear, then it is fully symmetric and $\omega$ is $\psi$-compatible, by (2). The following proposition proves the converse (linearity of $\tau_\omega$).

**Proposition 10.** *Let $\omega \in L_\infty^*$ be a $\psi$-compatible dilation-invariant state. The functional $\tau_\omega$ extends to a fully symmetric normalized linear functional on $M_\psi$.*

**Proof.** Since $\omega$ is $\psi$-compatible we have

$$\omega\left(\left(\frac{\psi(2t)}{\psi(t)}-1\right)\right)=0$$

and hence for any $0\leqslant x\in M_\psi$ we have by Proposition 4

$$\omega\left(\left(\frac{\psi(2t)}{\psi(t)}-1\right)\frac{1}{\psi(2t)}\int_0^{2t}x^*(s)\,ds\right)=0,$$

or, equivalently

$$\omega\left(\frac{1}{\psi(2t)}\int_0^{2t}x^*(s)\,ds\right)=\omega\left(\frac{1}{\psi(t)}\int_0^{2t}x^*(s)\,ds\right).$$

Let $0\leqslant x, y\in M_\psi$. We have

$$\int_0^{t/2}x^*(s)+y^*(s)\,ds\leqslant\int_0^{t}(x+y)^*(s)\,ds\leqslant\int_0^{t}x^*(s)+y^*(s)\,ds$$

for every $t>0$ (see e.g. [10, Chapter 2]). Thus,

$$\tau_\omega(x+y)\leqslant\tau_\omega\big(x^*+y^*\big)=\tau_\omega(x)+\tau_\omega(y).$$

On the other hand by the remarks above and since $\omega$ is dilation invariant, we have

$$\begin{aligned}\tau_\omega(x)+\tau_\omega(y)&=\tau_\omega\big(x^*+y^*\big)\\&\leqslant\omega\left(\frac{1}{\psi(t)}\int_0^{2t}(x+y)^*(s)\,ds\right)\\&=\omega\left(\frac{1}{\psi(2t)}\int_0^{2t}(x+y)^*(s)\,ds\right)\\&=\omega\left(\frac{1}{\psi(t)}\int_0^{t}(x+y)^*(s)\,ds\right).\end{aligned}$$

Combining gives the fact that $\tau_\omega(x+y)=\tau_\omega(x)+\tau_\omega(y)$ and so $\tau_\omega$ extends to a linear functional on $M_\psi$. $\square$

The following theorem is our main result. Recall, that if $\varphi$ is positive, then $\|\varphi\|=1$ if and only if $\varphi(\psi')=1$.

**Theorem 11.** *If $\varphi$ is a normalized fully symmetric functional on $M_\psi$ then $\varphi$ is a Dixmier functional.*

**Proof.** Define a map $T : D_\psi \to L_\infty$ by the formula

$$(Tx)(t) = \frac{1}{\psi(t)} \int_0^t x(s)\,ds.$$

Since $T$ is injection, one can define a linear functional $\omega_0$ on $R := T(D_\psi)$ by the formula $\omega_0(Tx) = \varphi(x)$.

Note that

$$\int_0^t x(s)\,ds \leqslant \|x\|_{M_\psi} \int_0^t \psi'(s)\,ds, \quad x \in D_\psi.$$

Hence, by Lemma 5,

$$\varphi(x) \leqslant \|x\|_{M_\psi}, \quad x \in D_\psi$$

or

$$\omega_0(f) \leqslant \|f\|_{L_\infty}, \quad f \in R.$$

This means that $\|\omega_0\|_{L_\infty^*} \leqslant 1$. Let $\omega_1 \in L_\infty^*$ be an extension of $\omega_0$ with $\|\omega_1\|_{L_\infty^*} = 1$. Since $\omega_0(1) = 1$, we have $\omega_1 \geqslant 0$.

Let $\tilde{R}$ be the smallest subspace of $L_\infty$ containing $R$ which is $\sigma_a$-invariant for every $a > 0$. We will show that $\omega_1$ is $\sigma_a$-invariant on $\tilde{R}$. For this it suffices to show that if $f \in R$ and $a > 0$ we have $\omega_1(\sigma_a f) = \omega_1(f)$. To this end, we need only to show that $\omega_1(\sigma_a Tx) = \omega_1(Tx)$ for every $x = x^* \in M_\psi$. Then

$$\begin{aligned}\bigl|\sigma_a Tx - T\bigl(a^{-1}\sigma_a x\bigr)\bigr| &= \left|\frac{\psi(t)}{\psi(t/a)} - 1\right| \frac{1}{\psi(t)} \int_0^t x(s/a)\,ds/a \\ &\leqslant \left|\frac{\psi(t)}{\psi(t/a)} - 1\right| \|\sigma_a\|/a\|x\|_{M_\psi}.\end{aligned}$$

Let us mention, that for $x = a\psi'(at)$, due to (2), we have $\omega_0(\frac{\psi(at)}{\psi(t)}) = \varphi(a\psi'(at)) = 1$. So, $\omega_1$ is $\psi$-compatible by Theorem 6. Since, $|\frac{\psi(t)}{\psi(t/a)} - 1| = \operatorname{sign}(a-1)(\frac{\psi(t)}{\psi(t/a)} - 1)$ we have $\omega_1(|\frac{\psi(t)}{\psi(t/a)} - 1|) = 0$. Hence

$$\omega_1\bigl(\sigma_a Tx - T\bigl(a^{-1}\sigma_a x\bigr)\bigr) = 0$$

and, consequently, $\omega_1(\sigma_a Tx) = \varphi(x) = \omega_1(Tx)$, $x = x^* \in M_\psi$.

It follows by the invariant form of the Hahn–Banach Theorem [7, p. 157] that there is a dilation-invariant extension $\omega$ of $\omega_1$ from $\tilde{R}$ to $L_\infty$ such that $\|\omega\|_{L_\infty^*} \leqslant 1$. Since $\omega_0(1) = 1$ it follows that $\omega(1) = 1$ and hence $\omega$ is a dilation-invariant state. Thus $\tau_\omega$ is a Dixmier functional and clearly $\varphi = \tau_\omega$. $\square$

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