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The ball topology and its applications

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Abstract. *We study the coarsest topology on a Banach space X such that every norm-closed ball is closed. We show under certain hypotheses, in particular if X does not contain ℓ_1, that on the closed unit ball of X this topology coincides with a vector topology. This has applications to the study of uniqueness of compact "consistent" topologies on the unit ball and to the uniqueness of preduals. Numerous other applications are given, including an extension of a result of Corson and Lindenstrauss: every weakly compact set can be expressed as an intersection of finite unions of closed balls.*

1. Introduction. This paper is motivated by two general problems. The first is to determine conditions on a Banach space X so that it is isometrically a dual space in a unique way. We say that X has a unique predual (UPD) if there is precisely one closed subspace $E \subset X^*$ so that X can be identified with E^* (or equivalently the closed unit ball X_1 of X is $\sigma(X, E)$–compact). For results on this general problem, see, for example, [4], [5], [6] and [11]. One can rephrase the condition that X has a unique predual; X has (UPD) if there is exactly one compact Hausdorff topology on X_1 induced by a locally convex linear topology on X.

The second problem is to determine conditions on X so that for any Hausdorff linear topology τ on X with X_1 τ–compact we have that $\tau \mid X_1$ agrees with a locally convex

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linear topology on X. This problem was originally studied in [13] and was motivated by the question of the validity of the Krein-Milman Theorem in non-locally convex F-spaces. It was shown by J. W. Roberts [27] that there exists an absolutely convex compact convex set with no extreme points. Thus there exists a Banach space X which is not a dual space but which admits a linear topology τ so that (X_1, τ) is compact. However it is shown in [13] that if X is reflexive or has a separable dual then any linear topology τ so that X_1 is τ−compact agrees with a locally convex topology on X_1. Recently these results have been extended by Sersouri [30].

On comparing the results known for these two problems, one can deduce that for some Banach spaces X (e.g. if X has a separable dual) any two Hausdorff linear topologies which make X_1 compact must agree on X_1.

Let us say that a topology τ on X is *consistent* if the maps

$$g_\lambda(x, y) = \lambda x + (1 - \lambda)y$$

are separately continuous on $X_1 \times X_1$ for $0 \leq \lambda \leq 1$. Motivated by the preceding discussion we seek conditions on X so that X_1 admits exactly one compact Hausdorff consistent topology. In this case we say that X has the *compact uniqueness property (CUP)*. In such spaces the geometry of the unit ball predetermines the compact Hausdorff topology on it. We show in this paper that many of the known conditions for X to have (UPD) in fact force the stronger (CUP); at the same time our methods enable us to produce new conditions implying (UPD) and (CUP).

In order to study this question we introduce the *ball topology* on X denoted by b_X. b_X is the coarsest topology so that every closed ball $B(x, \rho) = \{u : \|u - x\| \leq \rho\}$ is closed in b_X. Thus a point $x_0 \in X$ has a base of neighborhoods of the form

$$V = X \setminus \cup_{i=1}^n B(x_i, \rho_i)$$

where $x_1, \ldots, x_n \in X$, $n \in \mathbf{N}$ and $\|x_0 - x_i\| > \rho_i$. The ball topology seems to have some intrinsic interest and we study it in some detail. Let us note that it was first employed by Corson and Lindenstrauss [2] (cf. Theorem 8.2 below).

Let us make some initial remarks. First we note that b_X is a Hausdorff topology but that it is a T_1-topology. We also note that:

(1) For fixed $y \in X$ the map $x \to x + y$ is b_X-continuous.

(2) For fixed $\lambda > 0$ the map $x \to \lambda x$ is b_X-continuous.

(3) The map $x \to -x$ is b_X-continuous.

It will be convenient to call a topology on a linear space X a *prelinear* topology if it satisfies (1) and (2) above. Thus b_X is a prelinear topology.

We also make some further observations which we use later without comment:

(4) For any $y \in X$ the map $x \to \|x - y\|$ is b_X-lower-semi-continuous (b_X-l.s.c.).

(5) If X is a subspace of E then $b_E \mid X$ is finer than b_X.

(6) If X is separable then b_X satisfies the second axiom of countability ([18] p. 48). In particular every $x \in X$ has a countable base of neighborhoods.

Note that, of course, b_X depends on the norm on X; however, throughout most of the paper, we will suppress mention of the given norm.

The key idea of the paper is to study the restriction of b_X to certain subsets of X, especially the unit ball X_1. In certain circumstances, we are able to show that the restriction of b_X is a "reasonable" topology (e.g. Hausdorff or regular) and from such results we obtain our main theorems. Such an approach was first used in [2].

Let us now summmarize the results of the paper. In Section 2, we discuss some preliminary results on the topology b_X restricted to X_1. We give conditions so that (X_1, b_X) is Hausdorff or compact and discuss the continuity of linear functionals restricted to (X_1, b_X). In Section 3, we prove a technical result characterizing absolutely convex sets C so that (C, b_X) is a regular topological space; we show that this can happen if and only if there is a Hausdorff locally convex linear topology on the span of C whose restriction to C agrees with b_X. We also show that (C, b_X) is regular when C is a Rosenthal set (i.e. contains no basic sequence equivalent to the unit vector basis of ℓ_1).

Our first application is in Section 4. An operator $T : X \to Y$ where Y is an arbitrary topological vector space is called quasi-convex [13] if the continuous affine functionals on $\overline{T(X_1)}$ separate the points of $\overline{T(X_1)}$. We improve considerably on results in [13] by showing that if X does contain ℓ_1 then every operator on X is quasi-convex. Combined with a recent result of Sersouri [30] this characterizes spaces not containing ℓ_1.

In Section 5, we study spaces X for which b_X is *locally linear* i.e. (X_1, b_X) is regular. Any Banach space which does not contain ℓ_1 has this property; more generally if X does not contain ℓ_1 and $X \subset Y \subset X^{**}$ then b_Y is locally linear. An important property of such spaces is that the intersection of all norming subspaces of X^* is itself a norming subspace (the minimal norming subspace). In Section 6, we apply our results to the properties (UPD) and (CUP). This leads to some new results on the property (UPD).

Our methods fail for $C(K)$−spaces; in Section 7, we discuss some partial results on (CUP) for these spaces.

In Sections 8-9 we discuss some other aspects of the ball topology. For example, we extend a result of Corson and Lindenstrauss [2] by showing that a weakly compact set is always "ball-generated" (i.e. can be expressed as an intersection of finite unions of balls). In Section 10 we gather some miscellaneous remarks and open problems.

NOTATION: As above, the closed unit ball of a Banach space X is denoted by X_1. The closed ball center x and radius ρ is denoted by $B(x, \rho)$. All Banach spaces are assumed real. If X and Y are in duality then $\sigma(X, Y)$ denotes the topology on X induced by pointwise convergence on Y. The dual of X is X^*; the weak topology $\sigma(X, X^*)$ is denoted by w and the weak*-topology $\sigma(X^*, X)$ is denoted by w^*. The cardinality of a set A is denoted by $|A|$.

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2. Some general results. Ball continuous linear forms. Our first lemma concerns acceleration of convergence in certain prelinear topologies. It will paly a central role in our study of the ball topology.

LEMMA 2.1. *Let X be a Banach space and let α be any prelinear topology on X. Suppose 0 has a countable base of neighborhoods in the space (X_1, α). Suppose (x_n) is any bounded sequence in X and x is an α−cluster point of (x_n). Then there is a subsequence (u_m) of (x_n) so that if $v_n \in co\{u_i : i \geq n\}$ then we can find a sequence $w_n \in co(\{u_i : i \geq n\} \cup \{x\})$ with $\|w_n - v_n\| \to 0$ and $w_n \to x$ for the topology α.*

PROOF: Let us first note that the continuity properties of addition and scalar multiplication imply that it suffices to prove the lemma when $\|x_n\| \leq 1$ for all n and $x = 0$. Let $(U_i)_{i \geq 1}$ be a base of open α−neighborhoods of 0 in X_1, with $U_1 = X_1$.

Let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and let Λ be the space of all finitely non-zero sequences $\lambda = (\lambda_n)$ in $\mathbf{N}_0$ such that $\sum \lambda_n 2^{-n} \leq 1$. Then set Λ_k to be the subset of $\lambda \in \Lambda$ such that $\lambda_j = 0$ if $j \leq k$ and let Λ_k^n be the set of $\lambda \in \Lambda_k$ such that $\lambda_j = 0$ for $j \geq n+1$.

We shall construct (u_n) by induction so that if

$$w_n = \sum_{k=0}^{\infty} \lambda_k^{(n)} 2^{-k} u_k$$

where $\lambda^{(n)} = (\lambda_k^{(n)}) \in \Lambda_n$ then $w_n \to 0$ (α). This will establish the lemma. Indeed if $v_n \in \text{co}\{u_i : i > n\}$ let us write $v_n = \sum_{i>n} \mu_i u_i$ where $\mu_i \geq 0$, $\sum \mu_i = 1$, and the μ_i are eventually zero. Set $\lambda_i^{(n)} = 0$ for $i \leq n$ and for $i > n$ set $\lambda_i^{(n)} = \sup(k \in \mathbf{N}_0 :\ k.2^{-i} \leq \mu_i)$. Now $\lambda^{(n)} \in \Lambda_n$ and if we define w_n as above then $w_n \in \text{co}(\{u_i : i > n\} \cup \{0\})$ and

$$\|v_n - w_n\| \leq (\sum_{i>n} 2^{-i}) \sup_i \|u_i\|$$

so that the sequence w_n satisfies the conclusions of the theorem.

It suffices to construct (u_n) so that if $1 \leq m \leq n$ and $\lambda \in \Lambda_m^n$ then $\sum \lambda_k 2^{-k} u_k \in U_m$.

Pick $u_1 = x_1$; the conditions are then satisfied since Λ_n^n reduces to the zero sequence for every n and hence for $n = 1$. Next suppose $(u_1, \dots, u_n)$ have been selected to satisfy the inductive hypothesis. If $\lambda \in \Lambda$ then in the topology α,

$$\lim_{p \to \infty} \Big(\sum_{k=1}^{n} \lambda_k 2^{-k} u_k + \lambda_{n+1} 2^{-(n+1)} x_p\Big) = \sum_{k=1}^{n} \lambda_k 2^{-k} u_k .$$

As the set of points $\{\sum_{k=1}^n \lambda_k 2^{-k} u_k : \lambda \in \Lambda\}$ is finite and the range of values of λ_{n+1} is finite we can choose p large enough so that $p > q$ where $u_n = x_q$ and

$$\sum_{k=1}^{n} \lambda_k 2^{-k} u_k + \lambda_{n+1} 2^{-(n+1)} x_p \in U_m$$

whenever $1 \leq m \leq n+1$ and $\lambda \in \Lambda_m^{n+1}$. If we set $u_{n+1} = x_p$ this completes the induction and establishes the lemma.

We shall call a sequence (u_n) which satisfies the conclusion of the Lemma above *convex clustering* at x for the topology α. If we take $\alpha = b_X$ then the conclusion may be strengthened by noting that if $\|w_n - v_n\| \to 0$ and $w_n \to x$ in b_X then $v_n \to x$ in b_X. In fact $v_n \to x$ for b_X is equivalent to the statement that for any $y \in X$ we have $\liminf \|v_n - y\| \geq \|x - y\|$.

LEMMA 2.2. *Let α be a prelinear topology on a Banach space X such that the unit ball X_1 is α–closed. Suppose (u_n) is a sequence which is convex clustering at x for α. Then for every weak*-cluster point x^{**} of (u_n) and every $t \in X$, we have $\|x^{**} - t\| \geq \|x - t\|$.*

PROOF: Clearly every norm-closed ball is also α-closed and the norm is α–lower-semi-continuous. Thus for every $t \in X$,

$$\lim_{n \to \infty} d(t, \mathrm{co}\{u_k : k \geq n\}) \geq \|x - t\|.$$

Indeed, if not, we can find $t \in X$, $\epsilon > 0$ and a sequence $v_k \in \text{co}\{u_i : i \geq k\}$ such that $\|v_k - t\| \leq \|x - t\| - \epsilon$ for every k. Then, if w_k is given as in Lemma 2.1 we have that for large enough k, $\|v_k - w_k\| < \epsilon/2$. Thus $\|w_k - t\| \leq \|x - t\| - \epsilon/2$. This is impossible, since $w_k \to x$ for α and the norm is lower-semi-continuous.

Now let x^{**} be any cluster point of (u_n) in (X^{**}, w^*). For every $n \geq 1$, $\delta > 0$ there exists $y \in \text{co}\{u_k : k \geq n\}$ with $\|y - t\| \leq (1+\delta)\|x^{**} - t\|$. Indeed, if not, by the Hahn-Banach separation theorem, there exists $f \in X^*$ with $\|f\| = 1$ so that for every $k \geq n$ $f(u_k - t) \geq (1+\delta)\|x^{**} - t\|$. But this implies that $f(x^{**} - t) \geq (1+\delta)\|x^{**} - t\|$ which is a contradiction.

We conclude that $\|x^{**} - t\| \geq d(t, \text{co}\{u_k : k \geq n\})$ for all n and the lemma follows.

Note that Lemmas 2.1 and 2.2 apply to $\alpha = b_X$ when X is separable. The following lemma is proved for completeness (see [4]).

LEMMA 2.3. *Let X be a Banach space and suppose $x^{**} \in X^{**}$. The following conditions are equivalent:*

(1) $\|x^{**} - x\| \geq \|x\| \qquad \forall x \in X.$

(2) Ker $x^{**} \cap X_1^*$ is w^*-dense in X_1^*.

PROOF: For any $x \in X$ and any closed subspace $Y \subset X^*$ one has $\|x\,|_Y\,\| = d(x, Y^\perp)$ since Y^* is isometric to $X^{**}/Y^\perp$. If we apply this remark to $Y = \ker x^{**}$ we obtain that (1) is equivalent to $\|x\,|_{\ker(x^{**})}\,\| = \|x\|$ for all $x \in X$. The Hahn-Banach Theorem shows that this is equivalent to (2).

Let us recall that a closed subspace Y of X^* is said to be *norming* if

$$\|x\| = \sup\{y(x) : \; y \in Y, \; \|y\| \leq 1\} \qquad \forall x \in X.$$

Equivalently, Y is norming if $Y \cap X_1^*$ is w^*−dense in X_1^*.

Let us denote by N_X the intersection of all norming subspaces in X^*. N_X is a closed subspace of X^* which can reduce to $\{0\}$ (e.g. if $X = \ell_1$). If N_X is itself norming then X^* has a *minimal norming subspace.*

THEOREM 2.4. *Let X be a Banach space and suppose $f \in X^*$. Then:*

(1) If f is continuous on (X_1, b_X) then $f \in N_X$.

(2) If $f \in N_X$ and X is separable then f is continuous on (X_1, b_X).

PROOF: (1) Suppose $f \in X^*$ is continuous on (X_1, b_X). If Y is a (closed) norming subspace of X^* then the closed balls of X are also $\sigma(X,Y)$−closed and so $\sigma(X,Y)$ is finer than b_X. Therefore f is continuous on $(X_1, \sigma(X,Y))$. Then $\ker(f) \cap X_1$ is $\sigma(X,Y)$−closed; if u is such that $f(u) = 1$ it quickly follows from the Hahn-Banach theorem that for any $\epsilon > 0$ there exists $g \in Y$ with $g(u) = 1$ and $\|g\,|_{\ker(f)}\,\| \leq \epsilon$. Now by the Hahn-Banach theorem there exists $h \in X^*$ with $\|h\| \leq \epsilon$ and $g = h$ on $\ker(f)$. Hence $g - h = \alpha f$ for some α. Further $|\alpha - 1| \leq \epsilon\|u\|$ so that $\|f - g\| \leq \epsilon(\|u\|\|f\| + 1)$. Since Y is closed, $f \in Y$.

(2) If X is separable then (X_1, b_X) satisfies the second axiom of countability. Thus if f is not continuous on (X_1, b_X) there is a sequence (x_n) in X_1 so that $x_n \to 0$ (b_X) and $\lim f(x_n) = \lambda \neq 0$. By Lemma 2.1 we may suppose (x_n) convex clustering at 0 and by Lemma 2.2 we then have, for any w^*-cluster point x^{**}, $\|x^{**} - t\| \geq \|t\|$ for every $t \in X$. By Lemma 2.3, $\ker(x^{**}) = Y$ is a norming subspace of X^*; however, $x^{**}(f) = \lambda$ so that $f \notin N_X$.

In order to classify more precisely those $f \in X^*$ which are b_X−continuous on X_1 we require a rather more technical result.

PROPOSITION 2.5. *Let X be a Banach space and suppose $f \in X^*$. The following conditions are equivalent:*

(1) f is b_X-continuous on X_1.

(2) There is a separable closed subspace E of X so that for every separable closed subspace F of X with $F \supset E$, $f\,|_F$ is b_F−continuous on F_1.

PROOF: (1) $\Rightarrow$ (2) If f is b_X-continuous on X_1, then for every $n \in \mathbf{N}$, there is a finite family of balls $\mathcal{F}_n = \{B_{n,1}, \ldots, B_{n,k(n)}\}$ so that $0 \in V_n = X \setminus \cup_j B_{n,j}$ and $|f(x)| < 1/n$ whenever $x \in V_n \cap X_1$. Let C_n be the set of all centers of the balls of $\mathcal{F}_n$. It is clear that we may take E to be the closed linear span of $\cup C_n$.

(2) $\Rightarrow$ (1). Assume f is not b_X $-$continuous on X_1 and that $E \subset X$ is a separable closed subspace. Fix $\epsilon > 0$ so that $X_1 \cap f^{-1}(-\epsilon, \epsilon) = A$ is not a b_X $-$neighborhood of 0 in X_1.

We inductively define a sequence $x_n \in X_1$. Let $E_0 = E$ and $E_n = \mathrm{span}(E; x_1, \ldots, x_n)$ $(n \geq 1)$; let $E_n^{(0)}$ be a dense countable subset of E_n. For $k \geq 0$ we let $(V_n^k)_{n \geq 1}$ be an ordering of sets of the form $X \setminus \cup_{j=1}^{l} B(u_j, \nu_j)$ where $u_j \in E_k^{(0)}$ $(1 \leq j \leq l)$ and $\nu_j \in \mathbf{Q}$ satisfy $\nu_j < \|u_j\|$.

We now describe the inductive construction. Pick $x_1 \in (V_1^0 \cap X_1) \setminus A$. If $x_1, \ldots, x_{n-1}$ have been determined choose $x_n \in (X_1 \cap V_n^0 \cap V_{n-1}^1 \cap \ldots \cap V_1^{n-1}) \setminus A$.

Let F be the closed linear span of E and $(x_n : n \geq 1)$. The sets $X_1 \cap V_n^0 \cap \ldots \cap V_1^{n-1}$ $(n \geq 1)$ form a base of b_F- neighborhoods of 0. Thus $F_1 \cap f^{-1}(-\epsilon, \epsilon)$ is not a b_F $-$neighborhood of 0 relative to F_1, i.e. f is not b_F-continuous on F_1.

With 2.4 and 2.5 we have now a complete characterization of the elements of X^* which are b_X-continuous on X_1. Let us give some more concrete conditions. We denote by $\mathcal{C}(w^*, w)$ (resp. $\mathcal{C}(w^*, |\cdot|)$) the set of points of continuity of the identity Id:$(X_1^*, w^*) \to (X_1^*, w)$ (resp. Id:$(X_1^*, w^*) \to (X_1^*, \|\cdot\|)$).

THEOREM 2.6. *Let X be a Banach space and suppose $f \in X^*$. If $f \in \mathcal{C}(w^*, w)$ and if f has a countable base of neighborhoods in (X_1^*, w^*), then f is continuous on (X_1, b_X). This is true in particular if:*

(1) $f(x) = \|f\| = \|x\| = 1$ where $x \in X$ is a point of Gateaux-smoothness of the norm of X^{**}.

(2) f belongs to $\mathcal{C}(w^*, |\cdot|)$.

PROOF: Let f be in $\mathcal{C}(w^*, w)$ with a countable basis of w^*-neighborhoods $(V_n)_{n\geq 1}$. We may assume each V_n is defined by a finite number of elements $(x_j^n)_{j\in I_n}$ of X. Let E be the separable closed subspace of X generated by $\{x_j^n : j \in I_n,\ n \geq 1\}$. It is clear that if F is any subspace of X containing E then $f\,|_F$ is still a point of continuity of $\mathrm{Id}:(F_1^*, w^*) \to (F_1^*, w)$. Such a point belongs to all norming subspaces Y of F^* since the weak*-closure of Y_1 is F_1^*; hence, if F is also separable then by Theorem 2.4, $f\,|_F$ is continuous on (F_1, b_F). Proposition 2.5 now shows that f is continuous on (X_1, b_X).

If (1) is satisfied then f is *-exposed in E_1^{***} by x and thus $E_1^{***} \cap (f + E^\perp) = \{f\}$. An easy compactness argument ([4], p.211) shows that this implies $f \in \mathcal{C}(w^*, w)$; moreover, the family $V_n = \{t \in X_1^* : t(x) > 1 - \frac{1}{n}\}$ is a base of neighborhoods of f in (X_1^*, w^*). Finally (2) is clear.

REMARKS 2.7: (1) If X is separable then by 2.6 and the weak*-metrizability of X_1^* every $f \in \mathcal{C}(w^*, w)$ is b_X-continuous on X_1.

(2) If $f(x) = \|f\| = \|x\| = 1$ and x is a point of Frechet-smoothness of the norm of X, then f is b_X-continuous on X_1. This is a special case of 2.6(1) or of 2.6(2); but a direct proof is available. Indeed, for every $y \in X_1$,

$$\|x + \tau y\| - 1 = \tau f(y) + o(|\tau|).$$

Hence, f is a uniform limit on X_1 of a sequence of b_X-l.s.c. functions, namely $g_n(x) = n(\|x + n^{-1}y\| - 1)$. Thus f is l.s.c. on (X_1, b_X). A similar argument shows $(-f)$ is also l.s.c. and so f is actually continuous.

Next we shall investigate the circumstances under which (X_1, b_X) is either Hausdorff or compact. Let us recall that X has the *finite-infinite intersection property* $(IP_{f,\infty})$ (see [21]) if for every collection $(B_\alpha : \alpha \in \mathcal{I})$ of closed balls in X such that $\cap B_\alpha = \emptyset$ there is a finite subset $\mathcal{F} \subset \mathcal{I}$ so that $\cap\{B_\alpha : \alpha \in \mathcal{F}\} = \emptyset$. We also introduce the notation $D(x^{**}) = \{x \in X : \|x - t\| \leq \|x^{**} - t\|\ \forall t \in X\}$ for $x^{**} \in X^{**}$.

THEOREM 2.8. *Let X be a Banach space. The following conditions on X are equivalent:*

(1) X_1 is b_X −compact.

(2) X has $IP_{f,\infty}$.

(3) For every $x^{**} \in X^{**}$, $D(x^{**}) \neq \emptyset$.

PROOF: The equivalence of (1) and (2) is simply Alexander's subbase theorem (Kelley [18], p. 139).

(3) $\Rightarrow$ (2). Let $(B_\alpha)_{\alpha \in I}$ be a collection of closed balls so that for every finite $\mathcal{F} \subset I$, $\cap_{\mathcal{F}} B_\alpha \neq \emptyset$. By weak*-compactness there exists x^{**} is the intersection of their weak*-closures. Then if $x \in D(x^{**})$ we have $x \in \cap B_\alpha$.

(2) $\Rightarrow$ (3). If $x^{**} \in X^{**}$ let $\mathcal{B}$ be the collection of balls $B(x, \rho)$ where $\rho > \|x^{**} - x\|$. By the Principle of Local Reflexivity ([22] p.33) $\mathcal{B}$ has the finite-intersection property and so $D(x^{**}) = \cap \mathcal{B} \neq \emptyset$.

THEOREM 2.9. *Let X be a Banach space. Consider the following three statements.*

(1) (X_1, b_X) is Hausdorff.

(2) For every $x \neq y \in X_1$ there is a finite covering of X_1 by closed balls $B_1, \ldots, B_n$ so that no B_i contains both x and y.

(3) $|D(x^{**})| \leq 1$ for every $x^{**} \in X^{**}$.

Then (1) and (2) are equivalent and imply (3). If X is separable then (1), (2) and (3) are equivalent.

PROOF: The equivalence of (1) and (2) is trivial.

(2) $\Rightarrow$ (3). Suppose $\|x^{**}\| \leq 1$ and $x, y \in D(x^{**})$ with $x \neq y$. Then for any covering of X_1 by closed balls $B_1, \ldots, B_n$, x^{**} is in the weak*-closure of some B_i. But then $D(x^{**}) \subset B_i$ and so $\{x, y\} \subset B_i$.

(3) $\Rightarrow$ (1) when X is separable. Since each point has a countable base of neighborhoods, if (X_1, b_X) is not Hausdorff there is a sequence $x_n \in X_1$ with two distinct limits

x, y. Applying Lemma 2.1 twice we may suppose x_n is convex clustering at both x and y. If x^{**} is any weak*-cluster point of the sequence x_n then $\{x,y\} \subset D(x^{**})$ by Lemma 2.2. This produces the desired contradiction and concludes the proof.

We close this section with a simple lemma which will be used frequently.

LEMMA 2.10. *Let X be a Banach space and let C be an absolutely convex subset of X. Then the b_X −closure of C is also absolutely convex.*

PROOF: Assume $x, y \in \overline{C}$ and that $x_\alpha \to x$, $y_\beta \to y$ in b_X where $x_\alpha \in C$, $y_\beta \in C$. Then for fixed β and $0 < \lambda < 1$, $\lambda x_\alpha + (1-\lambda)y_\beta$ converges to $\lambda x + (1-\lambda)y_\beta$ which is therefore in $\overline{C}$. Now letting β vary we similarly obtain that $\lambda x + (1-\lambda)y \in \overline{C}$. As $\overline{C}$ is clearly symmetric this concludes the proof.

3. Localization of the ball topology to an absolutely convex set. In the next theorem we shall suppose that X is a Banach space and that C is a bounded absolutely convex subset of X. The linear span of C is denoted by Y. Y is a normed space if we equip it with the guage functional of C as a norm. With this notation we have the following fundamental result:

THEOREM 3.1. *The following assertions are equivalent:*

(1) There is a Hausdorff locally convex linear topology ν on Y so that $\nu \mid C = b_X \mid C$.

(2) 0 has a base of closed neighborhoods in (C, b_X).

(3) (C, b_X) is regular.

(4) There is a closed subspace M of Y^* which separates Y and so that on C we have $\sigma(Y, M) = b_X$.

(5) If V is neighborhood of 0 in (C, b_X) there is a further neighborhood W of 0 so that $(W + 2W) \cap C \subset V$.

(6) 0 has a base of absolutely convex neighborhoods in (C, b_X).

PROOF: We first establish the equivalence of (2),(5) and (6).

(2) $\Rightarrow$ (6). Let V be a closed neighborhood of 0 in (C, b_X). Then there is a closed neighborhood V_1 of 0 in $(2C, b_X)$ so that $V_1 \cap C \subset V$. Let W be an absolutely convex weak neighborhood of 0 in $2C$ with $W \subset V_1$. By Lemma 2.10, the closure of W in the topology b_X, which we denote W_b, is absolutely convex. Further $W_b \subset V_1$ and thus $W_b \cap C \subset V_1 \cap C \subset V$. It therefore suffices to show that $W_b \cap C$ is a neighborhood of 0 in (C, b_X).

Assume it is not. Then we construct by induction a sequence $(x_n)_{n \geq 1}$ in $C \setminus W_b$ so that $x_i - x_j \notin W_b$ for $i \neq j$. Indeed pick first any $x_1 \in C \setminus W_b$. If $x_1, \dots, x_n$ have been chosen there is a b_X−neighborhood U of 0 so that $(x_k + U) \cap W_b = \emptyset$ for $k = 1, 2, \dots, n$. Assuming W_b is not a neighborhood of 0 in (C, b_X) we have $U \cap C \not\subset W_b$ and we may pick x_{n+1} so that $-x_{n+1} \in (U \cap C) \setminus W_b$. Thus $x_i - x_{n+1} \notin W_b$. This completes the inductive construction of (x_n).

Now C is totally bounded for the weak topology and hence 0 is a closure point of $\{x_i - x_j : i \neq j\}$ for the weak topology. Since W is a weak neighborhood of 0 in $2C$ there exist $i \neq j$ so that $x_i - x_j \in W$. This contradiction completes the proof of this implication.

(6) $\Rightarrow$ (5). Let V be an absolutely convex neighborhood of 0 in (C, b_X). Then there is an absolutely convex neighborhood U of 0 in $(3C, b_X)$ with $U \cap C \subset V$. If $W = \frac{1}{3}U$ then W satisfies (5).

(5) $\Rightarrow$ (2). Let V be a neighborhood of 0 in (C, b_X). Let W be a neighborhood such that $(W + 2W) \cap C \subset V$. Let W_b be the closure of W in b_X. and suppose $x \in W_b \cap C$. Then since $C \subset x - 2C$ we have $x \in W_b \cap (x - 2C)$. However $x - 2W$ is a neighborhood of x in $x - 2C$ and $W \subset C \subset x - 2C$ and so $(x - 2W) \cap W \neq \emptyset$ or $x \in W + 2W$. Thus $W_b \cap C \subset V$ and so 0 has a base of closed neighborhoods in (C, b_X).

(5) + (6) $\Rightarrow$ (1). Let $\mathcal{V}$ be a base of absolutely convex neighborhoods of 0 in (C, b_X). Let $\mathcal{A}$ be the family of sets A of the form

$$A = \bigcup_{N=0}^{\infty} \sum_{n=0}^{N} 2^n V_n$$

where $V_n \in \mathcal{V}$, $n = 0,1,2,\ldots$. It is easily checked that $\mathcal{A}$ is a base of a locally convex topology ν on Y. Note that each V_n is absorbent for Y since the ball topology is coarser than the weak topology of X. We observe that on C, b_X is finer than ν. To complete the proof we show that b_X coincides on C with ν. This will in turn show that ν is Hausdorff since b_X is T_1.

Let $V \in \mathcal{V}$; we will construct $A \in \mathcal{A}$ with $A \cap C \subset V$. Pick first any $W_0 \in \mathcal{V}$ with $(W_0 + 2W_0) \cap C \subset V$. Then inductively pick $(W_n)_{n\geq 1}$ with $W_n \in \mathcal{V}$ and $(W_n + 2W_n) \cap C \subset W_{n-1}$. Let $A = \cup \sum_{n=0}^{N} 2^n W_n$. If $x \in A \cap C$ there exists N so that $x = \sum_{n=0}^{N} 2^n w_n$ with $w_n \in W_n$ for $0 \leq n \leq N$.

For $0 \leq k \leq N$, set

$$v_k = \sum_{n=k}^{N} 2^{n-k} w_n$$

Then for $1 \leq k \leq N-1$ we have $v_k = w_k + 2v_{k+1}$ while $v_N = w_N$. Thus $v_N \in W_N \subset W_{N-1}$ and by an easy induction we obtain $v_k \in W_{k-1}$ for $1 \leq k \leq N$. Thus $v_1 \in W_0$ and $x = w_0 + 2v_1 \in (W_0 + 2W_0) \cap C \subset V$. Hence $A \cap C \subset V$ as required.

(1) $\Rightarrow$ (4). Clearly b_X is coarser than the weak topology on X. Hence if x_n is any sequence in C, 0 is a closure point of the set $\{\frac{1}{2}(x_i - x_j) : i \neq j\}$. This shows that C is precompact for ν and so ν agrees on C with its weak topology and hence with the topology $\sigma(Y, M)$ where M is the collection of all linear functionals on Y whose restrictions to C are b_X-continuous. It is straightforward then to show that M is a closed linear subspace of Y^*. M must also separate the points of Y since ν is Hausdorff.

The remaining implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

DEFINITION 3.2: Let X be a Banach space. An absolutely convex subset C of X is called a *Rosenthal* set if it is bounded and contains no basic sequence equivalent to the unit vector basis of ℓ_1.

Note that by the Rosenthal theorem [29], C is a Rosenthal set if and only if every sequnece in C has a weakly Cauchy subsequence.

THEOREM 3.3. *Let X be a Banach space and let C be an absolutely convex Rosenthal subset of X. Then (C, b_X) is regular and hence satisfies (1)-(6) of Theorem 3.1.*

PROOF: The proof will be carried out in two steps.

STEP 1. *Suppose Y is a separable Banach space not containing ℓ_1, and that X is a separable Banach space. Let $j : Y \to X$ be a bounded linear injection. Then there is a closed separating subspace M of Y^* so that j is a homeomorphism between $(Y_1, \sigma(Y, M))$ and $(j(Y_1), b_X)$.*

PROOF OF STEP 1: We define a subset Z of Y^{**} by

$$Z = \{y^{**} \in Y^{**} : \ker(j^{**}(y^{**})) \cap X_1^* \text{ is } w^* - \text{dense in } X_1^*\}.$$

Since ℓ_1 does not embed into Y, $j^{**}y^{**}$ is of first Baire class on (X_1^*, w^*) and thus $\ker(j^{**}y^{**}) \cap X_1^*$ is a $w^* - G_\delta$ subset of X_1^*. A Baire category argument now shows that Z is a vector subspace of Y^{**} and that Z is weak* sequentially closed (see [4], p.214). Now Y_1^{**} is a pointwise compact set of first Baire class functions on (Y_1^*, w^*) and so by results in [1], $Z \cap Y_1^{**}$ is w^*-closed; thus, by the Banach-Dieudonné theorem, Z is weak*-closed. Let M be the annihilator of Z in Y^*. Note that $M^\perp = Z$ and that M separates Y since $M^\perp \cap Y = Z \cap Y = \{0\}$.

Now suppose $f \in M$. we show that $f \circ j^{-1}$ is continuous on $(j(Y_1), b_X)$. Since X is separable, it is enough to show that if $y_n \in Y_1$, $y \in Y_1$, and $j(y_n) \to j(y)$ for b_X then $f(y_n) \to f(y)$. By Lemma 2.1 we can assume that $(j(y_n))$ is convex clustering at $j(y)$ and by [24] and [29] that y_n converges weak* to y^{**} in Y^{**}. By Lemma 2.2 we have

$$\|j^{**}(y^{**}) - x\| \geq \|j(y) - x\| \qquad \forall x \in X.$$

Now by Lemma 2.3, $\ker(j^{**}(y^{**}-y)) \cap X_1^*$ is weak*-dense in X_1^*. Thus $y^{**}-y \in Z$ and so $f(y^{**}-y)=0$. Thus $f(y_n) \to f(y)$ and $f \circ j^{-1}$ is b_X–continuous on $j(Y_1)$. Thus the map $j^{-1}:(j(Y_1),b_X) \to (Y_1,\sigma(Y,M))$ is continuous.

Conversely suppose y_α is a net in Y_1 converging to some $y \in Y_1$ for $\sigma(Y,M)$. We show that $j(y_\alpha) \to j(y)$ in b_X. It suffices to consider the case when y_α converges to some y^{**} in Y^{**} for the weak* topology. Then $f(y^{**}-y)=0$ for every $f \in M$ and so $y^{**}-y \in M^\perp = Z$. Therefore $\ker(j^{**}(y^{**}-y)) \cap X_1^*$ is w^*-dense in X_1^* and by Lemma 2.3 for every $x \in X$ we have

$$\|j^{**}(y^{**})-x\| \geq \|j(y)-x\|.$$

Thus

$$\|j(y)-x\| \leq \liminf \|j(y_\alpha)-x\|$$

so that $j(y_\alpha)$ converges to $j(y)$ in b_X as required.

STEP 2: COMPLETION OF THE PROOF OF THEOREM 3.3.

By the factorization technique of [3] (cf. [26]) there is a Banach space Y, not containing ℓ_1, and a bounded linear injection $j: Y \to X$ with $j(Y_1) \supset C$.

Let us assume C is not regular in the topology b_X. Then by Theorem 3.1, there is a b_X–neighborhood of 0 relative to C, say V, such that for every b_X–neighborhood W of 0 relative to C we have $(W+2W) \cap C \not\subset V$. We may suppose that

$$V = C \bigcap (X \setminus \bigcup_{i=1}^{n} B(u_i,\nu_i)).$$

Let $E_0 = \operatorname{span}\{u_1,\ldots,u_n\} \subset X$ and let $F_0=\{0\} \subset Y$.

We now construct, by induction two increasing sequences of finite-dimensional subspaces $(E_n)_{n\geq 0}$ in X and $(F_n)_{n \geq 0}$ in Y. Let $E_n^{(0)}$ denote a dense countable subset of E_n (when E_n has been determined). Let $(V_{n,k})_{k\geq 1}$ be an ordering of all sets of the form $X \setminus \cup_{l=1}^m B(x_l,r_l)$ where $x_1,\ldots,x_m \in E_n^{(0)}$, $\|x_l\| > r_l$ and $r_l \in \mathbb{Q}_+$.

Now suppose $E_0, \ldots, E_N$, $F_0, \ldots, F_N$ have been determined. Set

$$W_N = C \cap V_{0,N+1} \cap V_{1,N} \cap \ldots \cap V_{N,1}.$$

There exist $y_{N+1}, z_{N+1} \in Y_1$, so that $j(y_{N+1})$, $j(z_{N+1}) \in W_N$ and $j(y_{N+1} + 2z_{N+1}) \in C \setminus V$. We then set $F_{N+1} = \text{span}\{F_N, y_{N+1}, z_{N+1}\}$ and $E_{N+1} = \text{span}\{E_N, j(y_{N+1}), j(z_{N+1})\}$.

Let E be the closed linear span of $\cup E_n$ and let F be the closed linear span of $\cup F_n$. Note that j maps F into E and both are separable. Now we can apply Step 1. On $j(F) \cap C$ b_E agrees with a linear topology. Now $V \cap j(F)$ is a b_E-neighborhood of 0 in $C \cap j(F)$. Now $(W_N : N \geq 1)$ is a base of b_E-neighborhoods of 0 in C. Hence there exists N so that

$$(W_N + 2W_N) \cap C \cap j(F) \subset V \cap j(F).$$

However $j(y_{N+1})$, $j(z_{N+1}) \in W_N$ and $j(y_{N+1}) + 2j(z_{N+1}) \in C \setminus V$. Clearly $j(y_{N+1}) + 2j(z_{N+1}) \in C \cap j(F)$ which yields a contradiction.

We now state a useful extension lemma.

LEMMA 3.4. *Let C be a bounded absolutely convex subset of a Banach space, and let f be b_X-continuous affine function on C. Then there is a b_X-continuous affine function $\overline{f}$ defined on the b_X-closure $\overline{C}$ of C, so that $\overline{f}(x) = f(x)$ for $x \in V$.*

PROOF: Note that by Lemma 2.10, $\overline{C}$ is absolutely convex.

Now if $x \in \overline{C}$ define $\overline{f}(x) = \liminf_{y \to x} f(y)$. Then $\overline{f}$ is l.s.c on $(\overline{C}, b_X)$ and $\overline{f} \mid C = f$.

We claim that $\overline{f}$ is convex. Suppose first that $x \in C$, $y \in \overline{C}$ and $0 \leq \lambda \leq 1$. Suppose $y_\alpha \in C$ is a net chosen so that $y_\alpha \to y$ (b_X) and $f(y_\alpha) \to \overline{f}(y)$. Then $\lambda x + (1-\lambda)y_\alpha \to \lambda x + (1-\lambda)y$ for b_X and so

$$\begin{aligned} \overline{f}(\lambda x + (1-\lambda)y) &\leq \liminf f(\lambda x + (1-\lambda)y_\alpha) \\ &= \lambda f(x) + (1-\lambda) \liminf f(y_\alpha) \\ &= \lambda \overline{f}(x) + (1-\lambda)\overline{f}(y_\alpha). \end{aligned}$$

Next suppose $x, y \in \overline{C}$. Choose a net $x_\alpha \in C$ so that $x_\alpha \to x$ and $f(x_\alpha) \to \overline{f}(x)$. Repeating the above reasoning we have

$$\begin{aligned}\overline{f}(\lambda x + (1-\lambda)y) &\leq \liminf \overline{f}(\lambda x_\alpha + (1-\lambda)y) \\ &\leq \liminf(\lambda f(x_\alpha) + (1-\lambda)\overline{f}(y)) \\ &= \lambda \overline{f}(x) + (1-\lambda)\overline{f}(y).\end{aligned}$$

Since $x \to -x$ is an affine homeomorphism of $(\overline{C}, b_X)$ the function $\overline{f}(-x)$ is also convex and l.s.c.

Now if $x \in C$, $\overline{f}(x) + \overline{f}(-x) = 2f(0)$. Hence by lower-semi-continuity we have $\overline{f}(x) + \overline{f}(-x) \leq 2f(0)$ for every $x \in \overline{C}$. By convexity we have therefore $\overline{f}(x) + \overline{f}(-x) = 2f(0)$ and thus $\overline{f}(x) = 2f(0) - \overline{f}(-x)$ is both concave and u.s.c. Hence $\overline{f}$ is affine and continuous.

THEOREM 3.5. *Let X be a Banach space and let C be an absolutely convex Rosenthal subset of X. Then the b_X −continuous affine functionals on $\overline{C}$ separate the points of $\overline{C}$. In particular, $\overline{C}$ is Hausdorff for the topology b_X.*

PROOF: Suppose $x, y \in \overline{C}$ with $x \neq y$. Let A be the absolutely convex hull of $C \cup \{x, y\}$. Then A is a Rosenthal set and so by 3.3 and 3.1 there is an affine functional f on A which is continuous for b_X and such that $f(x) \neq f(y)$. f then extends to $\overline{C}$ by 3.4.

4. An application: quasi-convex operators. Let X be a Banach space and suppose E is a topological vector space (not assumed to be locally convex). In [13] an operator $T : X \to E$ is called *quasi-convex* if the set $C = \overline{T(X_1)}$ has the property that the continuous affine functions on C separate the points of C.

It is shown in [13] that if X is reflexive then every operator is quasi-convex, while if X^* is separable every compact operator is quasi-convex. An example of Roberts [27] shows that there exists a compact absolutely convex subset of L_p, $(p < 1)$ with no extreme points and hence there is a Banach space B and a compact operator $T : B \to L_p$ which is not quasi-convex.

Recently A. Sersouri [30] has improved the results of [13] by showing that if X is separable and does not contain ℓ_1 then every compact operator on X is quasi-convex. He also shows that if X is a Banach space containing ℓ_1 then there is a non-quasi-convex operator on X. However from [17] it can be shown that every compact operator on a $C(K)$−space is quasi-convex. We remark that there is a compact semi-embedding of ℓ_1 into L_p $(p < 1)$ which is not quasi-convex [15].

Our main result combined with the work of Sersouri shows that a Banach space X has the property that every operator on X is quasi-convex if and only if X contains no copy of ℓ_1.

THEOREM 4.1. *Let X be a Banach space not containing ℓ_1 and let E be a Hausdorff topological vector space. Then any continuous operator $T : X \to E$ is quasi-convex.*

PROOF: We may assume that E is complete. Let C be the closure of $T(X_1)$ and let Z be the linear span of C equipped with the guage functional of C. Then Z is a Banach space and the ball topology b_Z is coarser than the topology induced on Z by the topology of E.

Since X does not contain ℓ_1, $T(X_1)$ is a Rosenthal set in Z. The b_Z −closure of $T(X_1)$ includes the E−closure, i.e. C. By Theorem 3.5, the b_Z −continuous affine functionals on C separate the points of C. The result now follows.

5. Banach spaces for which the ball topology is locally linear. We shall say that the ball topology b_X is locally linear if the unit ball X_1 is regular for the b_X-topology and hence satisfies the equivalent conditions of Theorem 3.1.

PROPOSITION 5.1. *The ball topology is locally linear if and only if the set of $f \in X^*$ which are b_X −continuous on X_1 is a norming subspace of X^*.*

PROOF: Let $M \subset X^*$ be the set of linear functionals which are b_X −continuous on X_1.

Then M is a closed linear subspace of X^*. If b_X is locally linear then $\sigma(X,M)$ agrees with b_X on X_1. If $\lambda \geq 1$, $\lambda^{-1}X_1$ is $\sigma(X,M)$-closed in X_1 and hence X_1 is $\sigma(X,M)$−closed in λX_1; thus X_1 is $\sigma(X,M)$−closed and so M is norming. Conversely if M is norming then $\sigma(X,M)$ is finer than b_X so that $\sigma(X,M) = b_X$ on X_1.

THEOREM 5.2. *If (X, b_X) is locally linear then X^* contains a minimal norming subspace.*

If X is separable then (X, b_X) is locally linear if and only if X^* contains a minimal norming subspace.

PROOF: Let N_X denote the intersection of all norming subspaces in X^* and let M denote the subspace of all linear functionals which are b_X −continuous on X_1. If b_X is locally linear then $N_X \subset M$ by Proposition 5.1, but by Theorem 2.4 we have $N_X \supset M$. Hence $N_X = M$ is norming.

If X is separable we have $N_X = M$ in general by Theorem 2.4 and so the converse also follows.

Our next theorem lists some easy examples of Banach spaces for which b_X is a locally linear topology.

THEOREM 5.3. *If X is a Banach space satisfying any one of the following conditions then b_X is locally linear and hence X^* has a minimal norming subspace.*

(1) The norm on X is Frechet-smooth on a dense set.

(2) $X = \mathcal{L}(E)$ where E is a reflexive Banach space.

(3) X is the dual of a Banach space with RNP.

(4) X contains no subspace isomorphic to ℓ_1.

PROOF: (1) This follows from Theorem 2.6 (and Remark 2.7) combined with Proposition 5.1.

(2) This follows from 2.6 and [10], combined with Proposition 5.1.

(3) Let $X = E^*$ where E has RNP. If $f \in E \subset X^*$ is a strongly exposed point of E_1 then f is continuous on (X_1, b_X) by Theorem 2.6. It follows that every $f \in E$ is continuous on (X_1, b_X) [25] and so b_X is locally linear by Proposition 5.1.

(4) Theorem 3.3.

We remark that it was shown in [4] that if X is a separable Banach space not containing ℓ_1 then X^* has a minimal norming subspace. Thus (4) of the above theorem removes the separability assumption. We shall, however, extend (4) to prove a much more general result.

THEOREM 5.4. *Let E be a Banach space such that E^* contains a norming subspace Y which contains no subspace isomorphic to ℓ_1. Let X be any closed subspace of E^* with $X \supset Y$. Then b_X is locally linear, and X^* has a minimal norming subspace.*

PROOF: Let $M \subset X^*$ be the subspace of functionals continuous on (X_1, b_X). We show that M is norming. Indeed suppose $x \in X, \|x\| = 1$ and $\epsilon > 0$. Let $Z = \text{span}\{Y, x\}$. Then b_Z is locally linear by Theorem 5.3(4). Hence there exists $\phi \in Z_1^*$ with $\phi(x) > 1 - \epsilon$ and so that ϕ is b_Z−continuous on Z_1; in particular ϕ is continuous for the restriction of b_X to Z_1.

Now b_X is coarser than the topology $\sigma(E^*, E)$ on X and Y_1 is $\sigma(E^*, E)$−dense in E_1^*. Hence the b_X−closure of Z_1 is X_1. By Lemma 3.4, ϕ extends to a b_X−continuous affine function on X_1; since $\phi(0) = 0$ this implies that there is a linear functional f on X with $f = \phi$ on Z and $f \mid X_1$ is b_X−continuous. Clearly if $u \in X_1$ then $|f(u)| \leq \sup_{z \in Z_1} |f(z)| \leq 1$ so that $f \in X_1^*$. Hence $f \in M$, $\|f\| \leq 1$ and $f(x) \geq 1 - \epsilon$. Thus M is norming and the proof is complete.

We emphasize a special case (see Theorem 9.3 later):

COROLLARY 5.5. *Let Y be a Banach space not containing ℓ_1 and let X be a subspace of Y^{**} with $X \supset Y$. Then b_X is locally linear and X^* contains a minimal norming subspace.*

PROOF: Take $E = Y^*$.

Let us recall that a Banach space X has a *unique predual* (X has (UPD)) if there is exactly one norming subspace $M \subset X^*$ so that X_1 is $\sigma(X,M)-$compact. If X^* has a minimal norming subspace and X is a dual space then it is easy to see that X has a unique predual. Thus the results of this section can be employed to show that a wide range of spaces have (UPD). However we shall examine the more general question of the existence of compact topologies on X_1 consistent with the convex structure in Section 6.

Let us conclude with an application to the set $C_X = \{x^{**} \in X^{**} : \|x^{**} - x\| \geq \|x\| \ \forall x \in X\}$. By Lemma 2.3, C_X is the set of $x^{**} \in X^{**}$ so that $\ker x^{**}$ is norming. It is shown in [4] that C_X is a weak*-closed subspace of X^{**} whenever X is separable and does not contain ℓ_1.

THEOREM 5.6. *Let X be a Banach space. Consider the following statements:*

(1) b_X is locally linear.

(2) C_X is a weak*-closed linear subspace of X^{**}.

(3) C_X is a linear subspace of X^{**}.

In general (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If X is separable then (1), (2) and (3) are equivalent.*

PROOF: As usual let N_X be the intersection of all norming subspaces of X.

(1) $\Rightarrow$ (2). Clearly $C_X \subset N_X^\perp$. Suppose $x^{**} \in N_X^\perp$ and $x \in X$. Then there is a net $x_\alpha \in X$ so that $\|x_\alpha - x\| \leq \|x^{**} - x\|$ and $x_\alpha \to x^{**}$ (w^*). Clearly, $x_\alpha \to 0$ for the topology $\sigma(X, N_X)$ and hence in b_X. Thus $\|x\| \leq \liminf \|x_\alpha - x\| \leq \|x^{**} - x\|$ and so $x^{**} \in C_X$. Hence $C_X = N_X^\perp$ is a weak*-closed linear subspace.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1) for X separable. If b_X is not locally linear then by Theorem 3.1 there exist sequences $(x_n), (y_n)$ so that $\|x_n\| \leq 1$, $\|y_n\| \leq 1$, $\|x_n + 2y_n\| \leq 1$ and $x_n \to 0$ (b_X), $y_n \to 0$ (b_X) but $x_n + 2y_n \notin V$ for some fixed b_X- neighborhood, V, of 0.

By passing to subsequences we may suppose that both (x_n) and (y_n) are convex clustering in the ball topology at 0 (Lemma 2.1). It then follows that for any weak*-cluster points x^{**}, y^{**} of $(x_n), (y_n)$, respectively we have $x^{**}, y^{**} \in C_X$ by Lemma 2.2. Hence $x^{**} + 2y^{**} \in C_X$. Now fix $x \in X$. Then

$$\|x\| \leq \|x^{**} + 2y^{**} - x\| \leq \liminf \|x_n + 2y_n - x\|$$

so that $(x_n + 2y_n)$ converges to 0 in the ball topology contrary to assumption.

REMARKS 5.7: In particular is a weak*-closed linear subspace of X^{**} if X contains no copy of ℓ_1, removing the separability assumption in [4].

6. Compact topologies on the unit ball of X. Let C be a convex subset of a linear space E. For $0 \leq \lambda \leq 1$ define the map $g_\lambda : C \times C \to C$ by $g_\lambda(x,y) = \lambda x + (1-\lambda)y$. Let τ be a topology on C. Then we shall say that τ is *consistent* if the map g_λ is separately continuous for $0 \leq \lambda \leq 1$ and *strongly consistent* if g_λ is jointly continous for $0 \leq \lambda \leq 1$.

We observe that compact Hausdorff convex sets with a strongly consistent topology have been studied in the literature. Let us mention the theorem of Lawson [20] and Roberts [28] that a compact Hausdorff convex set with a strongly consistent topology such that each point has a base of convex neighborhoods is affinely homeomorphic to a compact convex subset of a locally convex linear topological space.

As we have noted, a Banach space X has a unique predual (UPD) if there is exactly one compact Hausdorff topology on X_1 induced by a locally convex linear topology on X. In view of the Lawson-Roberts theorem one can rephrase this: X has (UPD) if X_1 admits exactly one compact Hausdorff strongly consistent topology so that each point has a base of convex neighborhoods.

We extend these ideas naturally by defining X to have the *compact uniqueness property (CUP)* if there is exactly one compact Hausdorff consistent topology on X_1. If X is a dual space then (CUP) implies (UPD), but in general there is no obvious relationship between these properties.

LEMMA 6.1. *Let X be a Banach space and suppose τ is a compact Hausdorff consistent topology on X_1. Define a topology τ^* on X by $U \in \tau^*$ if and only if $\lambda U \cap X_1 \in \tau$ for every $\lambda > 0$. Then τ^* is a prelinear topology on X which is finer than b_X and $\tau^* \mid X_1 = \tau$. (We do not assert that τ^* is Hausdorff.)*

PROOF: First note that if $0 < \lambda < 1$ then $h_\lambda(x) = \lambda x$ is a homeomorphism of X_1 onto λX_1 in τ. Thus the sets λX_1 are $\tau-$closed for $0 < \lambda < 1$.

Now we show that $\tau^* \mid X_1 = \tau$. Clearly τ^* is coarser than τ on X_1. Conversely suppose $V = V_0 \subset X_1$ is $\tau-$open. For any $\tau-$open set W the set $\frac{1}{2}W$ is $\tau-$open in $\frac{1}{2}X_1$ and so there is a $\tau-$open set W' with $W' \cap \frac{1}{2}X_1 = \frac{1}{2}W$ or $2W' \cap X_1 = W$. Hence, by induction we can find a sequence $(V_n)_{n\geq 0}$ of $\tau-$open sets in X_1 so that $V_n = 2V_{n+1} \cap X_1$ for $n \geq 0$. By induction we have $2^n V_n \cap 2^k X_1 = 2^k V_k$ for $n \geq k$. Thus the sequence $2^n V_n$ is increasing. Let $U = \cup 2^n V_n$. Then $U \cap 2^k X_1 = 2^k V_k$ for $k \geq 0$, and, in particular $U \cap X_1 = V$.

Now suppose $\lambda > 0$. Pick k so that $\lambda > 2^{-k}$ and write $\mu = \lambda^{-1}2^k$. Then $\lambda U \cap X_1 = \mu^{-1}(2^{-k}U \cap \mu X_1)$. Now $2^{-k}U \cap X_1 = V_k$ so that $\lambda U \cap X_1 = \mu^{-1}(V_k \cap \mu X_1) = h_\mu^{-1}(V_k \cap \mu X_1)$ is $\tau-$open. Thus U is τ^*-open. We thus have proved that τ and τ^* agree on X_1.

To show τ^* is prelinear note that the map $x \to \lambda x$ is obviously τ^*-continuous for $\lambda > 0$. Also τ^* is the finest topology agreeing with itself on the sets mX_1 for $m \in \mathbf{N}$. If $x \in X_1$ the map $y \to \frac{1}{2}(x + y)$ is continuous on each mX_1 and hence is τ^*-continuous. It follows easily that τ^* is prelinear.

Finally since λX_1 is τ-closed for $0 < \lambda \leq 1$, the set X_1 is τ^*-closed and hence every ball is τ^*-closed so that τ^* is finer than the ball topology b_X.

The following lemma is essentially implied by the work of Waelbroeck [34] (cf. also Turpin [33]). However, as our formulation is rather different, we include a proof for completeness.

LEMMA 6.2. *τ^* is a linear topology if and only if τ is strongly consistent.*

PROOF: One direction is clear. For the other, let us suppose that τ is strongly consistent. We will show that the map $(\lambda, x) \to \lambda x$ is jointly continuous on $[-1,1] \times (X_1, \tau)$. For, if not, there is a net $x_\alpha \in X_1$ and $\lambda_\alpha \in [-1,1]$ such that $x_\alpha \to x$ and $\lambda_\alpha \to \lambda$ but $\lambda_\alpha x_\alpha \to y$ in τ where $y \neq \lambda x$. But then $\frac{1}{2}(\lambda_\alpha x_\alpha) \to \frac{1}{2}y$. Hence $\frac{1}{2}((\lambda_\alpha - \lambda)x_\alpha + \lambda x_\alpha) \to \frac{1}{2}y$. Now $\frac{1}{2}(\lambda x_\alpha + (-\lambda)x_\alpha) \to 0$ and $|\lambda| x_\alpha \to |\lambda| x$ by the consistency of τ; hence, the strong consistency of τ implies that the only possible limit point of λx_α is λx and hence by compactness $\lambda x_\alpha to \lambda x$. For some β we have $|\lambda_\alpha - \lambda| \leq 1$ for $\alpha \geq \beta$. Let u be any cluster point of the net $\{(\lambda_\alpha - \lambda)x_\alpha : \alpha \geq \beta\}$. Then by the strong consistency of τ we have $\frac{1}{2}(u + \lambda x) = \frac{1}{2}y$. However for any given $\epsilon > 0$ we have $(\lambda_\alpha - \lambda)x_\alpha \in \epsilon X_1$ eventually. Hence $u \in \epsilon X_1$ for every $\epsilon > 0$ and hence $u = 0$ leading to a contradiction and proving joint continuity.

From this follows that 0 has a base of balanced absorbent sets in τ; further, by joint continuity of the map $(x, y) \to \frac{1}{2}(x + y)$ given any $\tau-$neighborhood V we can a further $\tau-$neighborhood W with $W + W \subset 2V$.

Consider sets of the form $A = \cup_{n=0}^{\infty} \sum_{k=0}^{n} 4^k V_k$, where each V_k is a $\tau-$ neighborhood of 0 in X_1. Such sets A form a base at 0 for a linear topology ν on X.

Suppose each V_k is $\tau-$open and let $A_n = \sum_{k=0}^{n} 4^k V_k$. then

$$A_n \cap 2^{2n-1} X_1 = \bigcup_{x \in A_{n-1}} (x + 4^n V_n) \cap 2^{2n-1} X_1$$

for $n \geq 1$.

If $V_n = W_n \cap X_1$ where W_n is τ^*-open then

$$(x + 4^n W_n) \cap 2^{2n-1} X_1 = (x + (4^n W_n \cap 4^n X_1)) \cap 2^{2n-1} X_1$$

for every $x \in A_{n-1}$ since then $\|x\| < \frac{1}{3} 4^n$. Hence $A_n \cap 2^{2n-1} X_1$ is relatively τ^*-open in $2^{2n-1} X_1$. Thus $A_n \cap \lambda X_1$ is relatively τ^*-open in λX_1 for $0 \leq \lambda \leq 2^{2n-1}$. Hence $A \cap \lambda X_1$

is relatively τ^*−open in λX_1 for every $\lambda \geq 0$. Thus A is τ^*−open and this implies that ν is coarser than τ^*.

Conversely suppose U is a τ^*−open neighborhood of 0. We construct a sequence V_k, $k \geq 0$ of τ−open neighborhoods of 0 by induction so that $\overline{A_n} \subset U$ where $A_n = \sum_{k=0}^n 4^k V_k$ and the closure in taken in τ^*.

For $n = 0$ this is possible since a compact Hausdorff space is regular. For $n > 0$ we utilize the remark that addition is jointly continuous for τ^* on norm bounded sets and so if $C_1, \ldots C_m$ are norm bounded and τ^*-compact then $C_1 + \cdots + C_m$ is τ^*-compact. Suppose then (V_k) have been constructed for $k \leq n$ with $\overline{A_n} \subset U$. If the next inductive step cannot be carried out then for every neighborhood W of 0 in (X_1, τ) we would have $\overline{A_n} + 4^{n+1}\overline{W} \not\subset U$. Then by the compactness of $4^{n+2}X_1$ we would have the existence of a point $x \in \overline{A_n} + 4^{n+1}\overline{W}$ for all such W. A simple compactness argument (since addition is jointly continuous for τ^* on norm-bounded sets) now shows that $x \in \overline{A_n}$, a contradiction. This show the induction can be carried out and leads to the conclusion that U is a ν−neighborhood of 0. Since τ^* is prelinear this rapidly shows that $\tau^* = \nu$.

REMARK: If 0 has a base of convex neighborhoods in τ then τ^* is locally convex; this proves a special case (for absolutely convex sets) of the Lawson-Roberts theorem quoted above. In general (see Proposition 7.16 of [16]) 0 has a base of p−convex neighborhoods for any $p < 1$ and so τ^* is locally p−convex for every $p < 1$.

THEOREM 6.3. *Let X be a Banach space so that (X_1, b_X) is Hausdorff. Then the following conditions on X are equivalent:*

(1) X is a dual space.

(2) X has (UPD).

(3) X has (CUP).

(4) X has $(IP_{f,\infty})$.

(5) There is a norm-one projection of X^{**} onto X.

PROOF: The implications (2) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (3). By (4) X_1 is b_X −compact (Theorem 2.8). Now if τ is any consistent compact Hausdorff topology on X_1 then $\tau^* \geq b_X$ and hence on X_1 $\tau = \tau^* = b_X$.

(3) $\Rightarrow$ (4). If τ is any consistent topology for which X_1 is compact then in τ^* every ball is compact and the finite-intersection property implies $(IP_{f,\infty})$.

(4) $\Rightarrow$ (2). Again by Theorem 2.8, X_1 is compact for the ball topology and therefore regular. Thus b_X is locally linear and Theorem 3.1 implies the existence of a locally convex vector topology ρ on X so that $\rho \mid X_1 = b_X$. Thus X is a dual space and since (3) holds we have (2).

Let us consider some special cases of Theorem 6.3.

COROLLARY 6.4. *Let X be a separable Banach space. If $D(x^{**}$ has exactly one member for every $x^{**} \in X^{**}$ then X is a dual space and has both (CUP) and (UPD).*

PROOF: By Theorems 2.8 and 2.9.

We also note that if b_X is a locally linear topology then (X_1, b_X) is Hausdorff and so the results of Section 4 apply.

COROLLARY 6.5. *A dual space has (CUP) if either (a) the norm on X is Frechet-smooth on a dense set, (b) $X = \mathcal{L}(E)$ where E is reflexive or (c) X is the dual of a space with (RNP).*

COROLLARY 6.6. *Let E be a Banach space not containing ℓ_1 and let X be a closed subspace of E^{**} containg E. Then the following are equivalent:*

(1) X is a dual space.

(2) X has (CUP).

(3) X has $(IP_{f,\infty})$.

(4) X has (UPD).

(5) There is a norm-one projection of X^{**} onto X.

In particular these equivalences apply to the case $X = E$.

REMARKS: (1) Let X be a separable Banach space not containing ℓ_1. In [5] it is shown that X is a dual space if and only if X has $(IP_{f,\infty})$; in [6] it is shown that X^{**} has (UPD). A. Sersouri [30] has shown that there is at most one strongly consistent compact topology on X_1. The above Corollary improves all these results; note that no assumptions of separability are now required.

(2) It is not known if $(IP_{f,\infty})$ already implies the existence of a norm-one projection of X^{**} onto X in general.

7. C(K)-spaces. If K is compact Hausdorff space, the ball topology on $C(K)$ is Hausdorff if and only if K has a dense set of isolated points (e.g. $K = \beta\mathbf{N}$, $C(K) = \ell_\infty$). However a classical result of Grothendieck [11] asserts that $C(K)$ has (UPD) if and only if $C(K)$ is a dual space (if and only if K is hyperstonean). In this section we give some weaker analogues of the results of Section 6 for $C(K)$-spaces. The first result is a direct consequence of results in [17].

THEOREM 7.1. *Let K be comapct Hausdorff. Then any strongly consistent compact Hausdorff topology on the closed unit ball B of $C(K)$ is locally convex. Hence if B admits a strongly consistent compact Hausdorff topology, it is unique.*

PROOF: By Lemma 6.2, we may suppose that $C(K)$ admits a Hausdorff vector topology τ^* such that B is τ^*-compact. It suffices to show that the τ^*-continuous affine functionals on B separate the points of B. Suppose $f_1, f_2 \in B$ with $f_1 \neq f_2$. Since τ^* is locally p-convex for any $p < 1$ (Lemma 6.2 and the following remarks) there is a quasi-Banach space X and a linear operator $S : C(K) \to X$ so that S is τ^*-continuous and $Sf_1 \neq Sf_2$. The proof is completed by showing that the operator S is quasi-convex in the sense of Section 4. For then there exists a continuous affine functional h on $S(B)$ with $h(Sf_1) \neq h(Sf_2)$ and $h \circ S$ is τ^*-continuous on B.

First we note that S is compact on $C(K)$ and $S(B)$ is closed. By a representation theorem due to Thomas [32] (cf. also [14]) there is a regular X-valued vector measure μ defined on the Borel sets $\mathcal{B}$ of K so that $S\phi = \int \phi \, d\mu$ for all $\phi \in C(K)$. In particular the set $\overline{\text{co}}\mu(\mathcal{B})$ coincides with the closed set $S\{\phi : 0 \leq \phi \leq 1\}$ and is compact. Now by [17] Theorem 5.1 μ has a control measure and by [13] Theorem 4.1 the set $\overline{\text{co}}\mu(\mathcal{B})$ is quasi-convex so that the operator S is also quasi-convex.

The last part follows from the cited result of Grothendieck.

In view of this result it is appropriate to ask if dual $C(K)$-spaces have (CUP). However the argument given above does not seem to extend to topologies which are merely consistent. We can however prove a partial result, which depends on the following proposition. Note that the ball topology is not Hausdorff on the ball of $C(K)$ in general.

PROPOSITION 7.2. *Let K be a compact Hausdorff space and let Ω be a comeager subset of K. Let $(f_n)_{n\geq 1}$ be a bounded sequence in $C(K)$ which converges pointwise on Ω to some $f \in C(K)$. Then f is the unique limit of (f_n) for the ball topology.*

PROOF: It is easy to see that $f_n \to f$ in the ball topology. For uniqueness, first define, for any $k, N \geq 1$

$$\Omega_{N,k} = \{x \in K : \ |f_m(x) - f_n(x)| \leq \frac{1}{k} \ \ \forall m,n \geq N\}.$$

For any k, $\Omega \subset \cup_{N\geq 1}\Omega_{N,k}$ and each $\Omega_{N,k}$ is closed. Thus by Baire's theorem the set $\Omega \cap (\cup_{N\geq 1}\text{int}\Omega_{N,k})$ is dense in Ω. Again, by Baire's theorem, the set

$$\Omega' = \bigcap_{k\geq 1}\left(\bigcup_{N\geq 1}\text{int}\Omega_{N,k}\right)\bigcap \Omega$$

is dense in Ω.

Suppose $x \in \Omega'$ and $\epsilon > 0$ are given. Then there is an open set V containing x, and $N \geq 1$ such that:

$$|f_n(x') - f_m(x')| < \frac{\epsilon}{3} \quad \forall x' \in V, \ \forall m,n \geq N.$$

We can further find an open neighborhood V' of x contained in V so that

$$|f_N(x') - f_N(x)| < \frac{\epsilon}{3} \qquad \forall x' \in V'.$$

Then we have

$$|f_m(x') - f(x)| < \epsilon \qquad \forall x' \in V',\ \forall m \geq N.$$

Now ϕ be any limit of the sequence (f_n). For any $x \in \Omega'$ and $\epsilon > 0$ we construct V' as above and then there is a function $h \in C(K)$ with supp $h \subset V'$ and $0 \leq h(x') \leq h(x) = 1$ fro $x' \in V'$. For any $k \geq 0$,

$$\|\phi + kh\| \leq \liminf \|f_n + kh\|.$$

If $x' \in V'$ then for $n \geq N$,

$$\begin{aligned} f_n(x') + kh(x') &\leq f(x) + \epsilon + kh(x) \\ &\leq f(x) + \epsilon + k. \end{aligned}$$

Choose any $k_0 > 2\sup \|f_n\|$. For every n,

$$\sup_{V'} |f_n + k_0 h| > \sup \|f_n\| \geq \sup_{k \setminus V'} |f_n + k_0 h|.$$

Thus

$$\|\phi + k_0 h\| \leq \liminf \sup_{V'} |f_n + k_0 h| \leq f(x) + k_0 + \epsilon$$

and so

$$\phi(x) \leq f(x) + \epsilon.$$

As $\epsilon > 0$ is arbitrary and Ω' is dense in K we conclude that $\phi \leq f$; by a similar argument we have $\phi \geq f$.

THEOREM 7.3. *Let μ be a probability measure and let τ be a consistent compact metric topology on the ball of $L_\infty(\mu)$. Then τ coincides with the weak*-topology $\sigma(L_\infty, L_1)$ (and, of course, L_1 is separable).*

PROOF: Let K be the spectrum of $L_\infty(\mu)$. By Lemma 6.1 there is a prelinear topology τ^* on $C(K) = L_\infty(\mu)$ which induces τ on the closed unit ball B, and τ^* is finer than the ball topology.

We prove first that the L_1−norm topology on the ball B is finer than τ. Indeed suppose $f_n \in B$ and $\|f_n - f\|_1 \to 0$. It suffices to show that for some subsequence g_n we have $g_n \to f$ fro τ. In fact we may pick g_n so that $g_n \to f$, μ−a.e. or equivalently on a comeager subset of K. Thus, by Proposition 7.2, f is the only limit point of any subsequence of g_n for the ball topology and thus $g_n \to f$ for the topology τ.

To complete the proof we suppose $f_n \to f$ (τ) and for some $\sigma(L_\infty, L_1)$-neighborhood V of f we have $f_n \notin V$ for all n. We may suppose that (f_n) is τ^*−convex clustering at f by Lemma 2.1. Since B is weakly compact in L_1 we may further suppose that f_n converges to some g for $\sigma(L_1, L_\infty)$ by Eberlein's theorem. Now there is a sequence $g_n \in \text{co}\{f_i : i \geq n\}$ so that $\|g_n - g\|_1 \to 0$. Hence there is a sequence $h_n \in \text{co}(\{f_i : i \geq n\} \cup \{f\})$ with $\|g_n - h_n\|_\infty \to 0$ and $h_n \to f$ in τ. However $\|h_n - g\|_1 \to 0$ and so we must have $f = g$. Now clearly $f_n \to g$ for the weak*-topology which contradicts the fact that $f_n \notin V$. This shows that the identity map from (B, τ) to $(B, \sigma(L_\infty, L_1))$ is continuous and completes the proof.

8. Ball-generated subsets of a Banach space. Let X be a Banach space and let A be a subset of X. We shall say that A is *ball-generated* if there is a family of sets $\{F_i : i \in I\}$ so that each F_i is a finite union of closed balls and $A = \cap F_i$.

Alternatively, A is ball-generated if and only if A is closed for the ball topology. However the definition above seems more suggestive. Our first result extends a result of

Corson and Lindenstrauss [2] who showed that a weakly compact subset of a separable reflexive space is ball-generated. Their proof is essentially to show that the unit ball is Hausdorff for the ball topology in such a space.

THEOREM 8.1. *Let X be a Banach space. Then any weakly compact subset of X is ball-generated.*

PROOF: Let W be weakly compact and let A be its closed absolutely convex hull, which is also weakly compact. Let C be the closure of A in the ball topology. By Theorem 3.5, the ball topology is Hausdorff on C. Since b_X is coarser than the weak topology, we conclude that W is compact for the ball topology and hence closed relative to C. Thus W is ball-generated.

THEOREM 8.2. *Let W be a subset of a Banach space which is ball-generated for every equivalent norm on X. Then W is weakly compact.*

PROOF: Suppose W is not weakly compact; since it is clearly bounded and norm-closed there exists a net w_α in W with a weak* limit $x^{**} \in X^{**} \setminus X$. Suppose $x_0 \in X \setminus W$. Let $M \subset X^*$ be the subspace of all x^* such that $x^{**}(x^*) = x^*(x_0)$. We define a new norm on X by

$$|||x||| = \sup\{|x^*(x)| : \ \|x^*\| \leq 1, \ x^* \in M\}.$$

Clearly $|||x||| \leq \|x\|$. Conversely if $x \in X$ there exists by the Hahn-Banach theorem $\phi \in X^{**}$ with $\|\phi\| = |||x|||$ and $\phi \mid M = x \mid M$. Thus $\phi = x + \lambda(x^{**} - x_0)$ for some real λ. Hence $d(x, E) \leq |||x|||$ where E is the span of $(x^{**} - x_0)$. Thus $|||.|||$ is an equivalent norm on X.

Now if $x^* \in M$ then $\lim_\alpha x^*(w_\alpha) = x^{**}(x^*) = x^*(x_0)$. Hence if $x \in X$, $|||x - x_0||| \leq \liminf |||x - w_\alpha|||$. Thus x_0 is a closure point of W in the ball topology induced by the new norm and W is not closed for this topology.

We may also ask when every weakly closed and bounded set is ball-generated.

THEOREM 8.3. *Let X be a Banach space. Consider the following statements.*

(1) Every norm-closed bounded convex set is ball-generated.

(2) Every weakly closed bounded set is ball-generated.

(3) X^* has no proper norming subspace.

Then (1) and (2) are equivalent and imply (3). If X is separable (1), (2) and (3) are equivalent.

PROOF: Clearly (2) implies (1). If (1) holds and $f \in X^*$ then for every closed interval $I \subset \mathbf{R}$, $f^{-1}(I) \cap X_1$ is b_X-closed and so f is continuous on (X_1, b_X). Thus by Theorem 2.4, $N_X = X^*$, i.e. X^* has no proper norming subspace and so (1) implies (3). Further b_X agrees with the weak topology on X_1 whence (1) also implies (2). If X is separable and (3) holds then Theorem 2.4 implies that each $f \in X^*$ is b_X-continuous on X_1, and so b_x agrees with the weak topology on X_1 whence (3) implies (2).

REMARKS 8.4: (1) Separability is probably not a necessary assumption for the full equivalence in Theorem 8.3. Any Banach space with the property that $\mathcal{C}(w^*, |\cdot|)$ separates X^{**} has the property that every weakly closed bounded set is ball-generated. This condition holds in most of the interesting examples.

(2) It is shown in [10] that for any Banach space X the condition that X^* contains no proper norming subspace is equivalent to

$$\bigcap_{x \in X} B(x, \|x^{**} - x\|) = \{x^{**}\}$$

for every $x^{**} \in X^{**}$. This property allows one to show, for example that if X^* has no proper norming subspace and X has (MAP) then X^* has (MAP) [10].

(3) If the weak topology agrees with b_X on X_1, then by the w^*-density of X_1 in X_1^{**} and by Lemma 3.4, $b_{X^{**}}$ agrees with the weak*-toplogy on X_1^{**}. In particular $b_{X^{**}}$ is

locally linear and the results of Section 6 apply to X^{**} (in particular X^{**} has (UPD) and (CUP)).

(4) Theorem 8.3 applies to spaces with the Mazur intersection property, for example to spaces with a Frechet-smooth norm (for recent results on the Mazur property see [35]). It applies also to Hahn-Banach smooth separable spaces, e.g. separable spaces which are M-ideals in their biduals, since $\mathcal{C}(w^*, w)$ is the unit sphere of the dual space (by [4]). Finally if X and Y are reflexive the space $\mathcal{K}(X, Y)$ of compact operators from X to Y equipped with operator norm satisfies 8.3 (e.g. by [10]).

We now characterize reflexive spaces.

THEOREM 8.5. *Let X be a Banach space. Then X is reflexive if and only if (1) every closed bounded convex set is ball-generated and (2) X has $(IP_{f,\infty})$.*

PROOF: If X is reflexive then (1) follows from Theorem 8.2 and (2) from weak compactness. Conversely if (1) holds then by the argument in 8.3, b_X agrees with the weak topology on X_1, while Theorem 2.8 implies that X_1 is b_X –compact.

Note that conditions (1) and (2) hold for any equivalent norm if they hold for just one norm. Also if X is non-reflexive and 1-complemented in X^{**} then 8.5 implies that X contains a closed convex set which is not ball-generated. This result (an improvement of [4] and [31]) means that these Banach spaces are somehow "rough" (see Section 10).

9. Characterizations of separable Banach spaces not containing ℓ_1. We already know (Theorem 3.5) that if X is a Banach space not containing ℓ_1 then the topology b_X is locally linear. We will show that, conversely, if X is separable and contains ℓ_1, there exists a norm on X such that X_1 is "very far" from being Hausdorff and *a fortiori* very far from being locally linear.

LEMMA 9.1. *Let X be a separable Banach space. The following are equivalent:*

(1) For every non-empty b_X-open subsets O_1 and O_2 of X_1, $O_1 \cap O_2 \neq \emptyset$.

(2) If $\{B_1, \ldots, B_n\}$ is any finite covering of X_1 with closed balls there exists j so that $X_1 \subset B_j$.

(3) For every finite-dimensional subspace F of X and every $\epsilon > 0$ there exists x with $\|x\| = 1$ such that $\|t + x\| \geq (1 - \epsilon)(\|t\| + \|x\|)$ for every $t \in F$.

(4) There exists $x^{**} \in X^{**} \setminus \{0\}$ such that $\|x^{**} + x\| = \|x^{**}\| + \|x\|$ for every $x \in X$.

PROOF: (1) $\Rightarrow$ (4). Let $(O_n)_{n\geq 1}$ be a base of the ball topology on X_1. By (1), we have $O_1 \cap O_2 \cap \ldots \cap O_n \neq \emptyset$ for every n. Hence we can pick $x_n \in O_1 \cap O_2 \cap \ldots \cap O_n$ for every n. It is clear that any $x \in X_1$ is a b_X −cluster point of any subsequence of (x_n). Thus by Lemma 2.1 and a diagonal argument, there is a sequence (t_n) in X_1 which is b_X −convex clustering to every $x \in X_1$. Let x^{**} be a w^*-cluster point of (t_n) in X_1^{**}. By Lemma 2.2 we have

$$\|t - x^{**}\| \geq \|t - x\| \qquad \forall t \in X \quad \forall x \in X_1.$$

In particular, if $x = -t/\|t\|$, we have

$$\|t - x^{**}\| \geq \|t\| + 1.$$

Since $\|x^{**}\| \leq 1$, the result is proved.

(4) $\Rightarrow$ (3). This is clear by the Principle of Local Reflexivity.

(3) $\Rightarrow$ (2). Let $\{B(x_j, \rho_j)\}_{1\leq j\leq n}$ be a covering of X_1 by closed balls. We must show that for some j, we have $\rho_j \geq \|x_j\| + 1$. Suppose the contrary; by (3) we can find x with $\|x\| = 1$ so that $\|x - x_j\| > \rho_j$ for every j. Then x belongs to no $B(x_j, \rho_j)$ which is a contradiction.

(2) $\Rightarrow$ (1). Let $O_1 = X_1 \setminus (B_1 \cup B_2 \cup \ldots \cup B_k)$ and $O_2 = X_1 \setminus (B_{k+1} \cup B_{k+2} \cup \ldots \cup B_n)$, where $B_1, \ldots, B_n$ are closed balls. If $O_1 \cap O_2 = \emptyset$ then $\{B_1, \ldots, B_n\}$ is a covering of X_1 and (2) implies that either $O_1 = \emptyset$ or $O_2 = \emptyset$. Since sets of this form are a basis for the topology on X_1, the result is proved.

It is now an immediate consequence of [9], Théorème III.1 and [8], Theorem 2.4, that we have

THEOREM 9.2. *Let X be a Banach space. Then X contains ℓ_1 if and only if there exists an equivalent norm on X which satisfies the equivalent conditions of 9.1.*

The above result concerns one given equivalent norm on X. There is no hope of having such a result for all equivalent norms on X. Indeed it is easily seen (with 2.4 and [22],p. 12) that on any separable Banach space X, there exists an equivalent norm such that b_X is locally linear. However, one has:

THEOREM 9.3. *Let X be a separable Banach space. The following are equivalent:*

(1) X does not contain ℓ_1.

(2) For every closed subspace Y of X^{**} with $X \subset Y \subset X^{**}$, Y_1 is b_Y −Hausdorff.

(3) For every closed subspace Y of X^{**} with $X \subset Y \subset X^{**}$, b_Y is locally linear.

PROOF: (1) ⇒ (3) follows from Corollary 5.5.

(3) ⇒ (2) is obvious.

(2) ⇒ (1). If X is separable and contains ℓ_1 then by Maurey's theorem [23], there exists $x^{**} \in X^{**} \setminus \{0\}$ such that

$$\|x^{**} + x\| = \|x^{**} - x\| \quad \forall x \in X.$$

Let $Y = \text{span}(X \cup \{x^{**}\})$. We claim that Y_1 is not b_Y −Hausdorff. Indeed let $S : Y \to Y$ be the linear isometry such that $Sx = x$ for $x \in X$ and $S(x^{**}) = -x^{**}$. Then S is b_Y −continous since it is an isometric isomorphism. Let (x_α) be a net in X_1 which converges w^* and hence also in b_Y to x^{**}. Then Sx_α converges to $-x^{**}$ in b_Y and thus b_Y is not Hausdorff.

REMARKS 9.4:

(1) Let us mention as after 8.5, that the conditions (2) and (3) of 9.3 are satisfied for one norm if and only if they are satisfied for every equivalent norm.

(2) The above proof of (2) $\Rightarrow$ (1) together with Theorem 5.4 provides an alternative proof of one of the directions in Maurey's theorem [23] (actually, the easier one): if there exists $x^{**} \in X^{**} \setminus \{0\}$ such that $\|x^{**} - x\| = \|x^{**} + x\|$ for every $x \in X$, then X contains ℓ_1.

(3) By [3] there exists a lattice locally uniformly convex norm on c_0. Let X be the dual space of c_0 with this norm. Then b_X coincides with the w^*−topology. However X is a weakly sequentially complete lattice and so there is an isometric symmetry S on X^{**} such that $\ker(S - I) = X$ and $\ker(S + I) = c_0^{\perp}$. This shows that the only continuous linear forms on $(X_1^{**}, b_{X^{**}})$ are elements of c_0. Hence we have in particular an example of an absolutely convex set $C = (X_1, b_{X^{**}})$ so that the $b_{X^{**}}$− continuous linear forms on C separate C but not its closure X_1^{**}. This should be compared with Theorem 3.5.

10. Miscellaneous remarks. (1) We first observe that the ball topology may be used to introduce a comparison between norms on a given Banach space. If X is a Banach space and ν_1 and ν_2 are two equivalent norms on X then we say that ν_1 is *smoother* than ν_2 if the unit ball $B(\nu_2) = \{x : \nu_2(x) \leq 1\}$ is ball generated in (X, ν_1). equivalently ν_1 is smoother than ν_2 if and only if the associated ball topology b_{ν_1} is finer than b_{ν_2}.

If X contains no subspace isomorphic to ℓ_1 then ν_1 is smoother than ν_2 if and only if $N(\nu_1) \supset N(\nu_2)$ where $N(\nu)$ is the intersection of all norming subspaces of (X^*, ν^*). For separable spaces we can compare norms via the subsets $C(\nu)$ where

$$C(\nu) = \{x^{**} \in X^{**} : \nu^{**}(x^{**} - x) \geq \nu(x) \ \forall x \in X\}.$$

THEOREM 10.1. *If X is separable and ν_1 is smoother than ν_2 if and only if $C(\nu_1) \subset C(\nu_2)$.*

PROOF: Suppose ν_1 is smoother than ν_2 and that $x^{**} \in C(\nu_1)$. For $x \in X$ select a net $x_\alpha \in X$ so that $x_\alpha \to x^{**}$ (w^*) and $\nu_2(x_\alpha - x) \leq \nu_2^{**}(x^{**} - x)$ for every α. Then for $y \in X$,

$$\liminf \nu_1(x_\alpha - y) \geq \nu_1^{**}(x^{**} - y) \geq \nu_1(y)$$

so that $x_\alpha \to 0$ in b_{ν_1}. Thus $x_\alpha \to 0$ in b_{ν_2} and so

$$\nu_2^{**}(x^{**} - x) \geq \liminf \nu_2(x_\alpha - x) \geq \nu_2(x)$$

so that $x^{**} \in C(\nu_2)$.

Conversely, suppose $C(\nu_1) \subset C(\nu_2)$. It suffices to show that b_{ν_1} is finer than b_{ν_2} on bounded sets. By Lemmas 2.1 and 2.2 it suffices to show that if (x_n) is a bounded sequence which is b_{ν_1}–convex clustering at x then $x_n \to x$ in b_{ν_2}. If x^{**} is any weak* cluster point of x_n we have $x^{**} - x \in C(\nu_1)$ by Lemma 2.1. Hence $x^{**} - x \in C(\nu_2)$. Now if $y \in X$

$$\liminf \nu_2(x_n - y) \geq \nu_2^{**}(x^{**} - y) \geq \nu_2(x - y)$$

so that $x_n \to x$ in b_{ν_2}.

Let us give an application to convolutions of norms. If ν_1 and ν_2 are two equivalent norms on X we define for $1 \leq p < \infty$

$$\nu_1 *_p \nu_2(x) = \sup\{|x^*(x)| : \ \nu_1^*(x^*)^p + \nu_2(x^*)^p \leq 1\}.$$

PROPOSITION 10.2. *If X is separable or does not contain ℓ_1, then for $1 \leq p < \infty$, $\nu_1 *_p \nu_2$ is smoother than both ν_1 and ν_2.*

PROOF: If N is a norming subspace of $(X^*, (\nu_1 *_p \nu_2)^*)$ i.e. of $(X^*, ((\nu_1^*)^p + (\nu_2^*)^p)^{1/p})$ then it is easy to verify that N is norming in both (X^*, ν_1^*) and (X^*, ν_2^*). In fact if $x^* \in X^*$ there is a net $x_\alpha^* \in N$ with

$$\nu_1^*(x_\alpha^*)^p + \nu_2^*(x_\alpha^*)^p \leq \nu_1^*(x^*)^p + \nu_2^*(x^*)^p$$

and $x_\alpha^* \to x^*$ (w^*). By passing to a subnet we may suppose that $\lim_\alpha \nu_j^*(x_\alpha^*)$ exists for $j = 1, 2$. Clearly we must have $\nu_j(x^*) = \lim_\alpha \nu_j^*(x_\alpha^*)$ for $j = 1, 2$ and hence N is norming in both spaces.

Now if X contains no copy of ℓ_1 the conclusion is immediate. If X is separable and $x^{**} \in C(\nu_1 *_p \nu_2)$ then ker x^{**} is norming in $(X^*, (\nu_1 *_p \nu_2)^*)$ and so $x^* \in C(\nu_j)$ for $j = 1, 2$ so that the result follows by the preceding theorem.

COROLLARY 10.3. *If X is separable or does not contain ℓ_1, and $\nu_1 * \nu_2$ is a dual norm then both ν_1 and ν_2 are dual norms with the same predual.*

If X does not contain ℓ_1 then it may be shown that ν is a dual norm if and only if ν is minimal for the ordering induced by smoothness, i.e. if ν_1 is less smooth than ν then ν and ν_1 induce the same ball topologies. In general this is false; if we equip ℓ_1 with the dual norm of an l.u.c norm on c_0 then this norm is strictly smoother than the usual norm on ℓ_1.

(2) Next we consider the question of when the $b_{X^{**}}$-topology restricts to the b_X-topology on X. This happens in a variety of situations, in particular when the b_X-topology agrees with the weak topology on X_1 (see Theorem 8.3). It may also be shown to be true for separable stable Banach spaces (see [19]). In fact $b_X = b_{X^{**}}|X$ if X is separable and satisfies a weak form of stability, i.e. for every pair (x_n), (y_n) of bounded sequences in X there exist $u_n \in \text{co}\{x_k : k \geq n\}$, $v_n \in \text{co}\{y_k : k \geq n\}$ and

$$(*) \qquad \lim_{n\to\infty} \lim_{k\to\infty} \|u_n + v_k\| = \lim_{k\to\infty} \lim_{n\to\infty} \|u_n + v_k\|.$$

However (*) can be shown to imply that X is weakly sequentially complete (by the same argument as [12]). Thus the example of c_0 shows that (*) does not characterize the property $b_X = b_{X^{**}}|X$, We do not know if any weakly sequentially complete Banach space has an equivalent norm satisfying (*).

(3) An *admissible* subset A of a Banach space X is one such that every x^{**} in the the weak*-closure of $A \subset X^{**}$ satisfies $|D(x^{**}| \leq 1$ ([7]). If X is separable then Theorem 2.9 can be adapted to show that b_X is Hausdorff on A. If τ is a prelinear topology on X so that X_1 is closed and A is compact then $\tau = b_X$ on A (see Section 6).

(4) There is an analogue of the characterization of weakly compact sets (8.2) for Rosenthal sets. If a set A is b_X –Hausdorff for every equivalent norm then A is a Rosenthal set (see [8], [9]).

(5) We conclude this work with some questions.

QUESTION A: Let C be an absolutely convex bounded subset of a Banach space X. If C is b_X –Hausdorff, is C b_X-regular?

A positive answer to Question A would imply, for separable X, a positive answer to the next question.

QUESTION B: Let X be a Banach space such that the intersection N_X of all norming subspaces of X^* separates X. Is N_X a norming subspace of X^*?

QUESTION C: Let C be a b_X –regular subset of X and let $\overline{C}$ be its b_X –closure. Is $\overline{C}$ b_X –regular? In particular what happens if C is a Rosenthal set? (See 3.5 and 9.4(3).)

QUESTION D: Let X be a Banach space with (UPD). Does X have (CUP) in general? What happens if X is a W*-algebra?

QUESTION E: Let X be an Asplund space. Does there exist an equivalent norm on X so that X^* has no proper norming subspace? Conversely if X^* has no proper norming subspace is X an Asplund space? For the separable case these questions have trivial positive answers.

QUESTION F: Suppose $(X^{**}, b_{X^{**}})$ is locally linear; does it follow that X cannot contain ℓ_1? Note that $(X^{**}, b_{X^{**}})$ is locally linear if and only if it is Hausdorff. (Compare 9.3.)

QUESTION G: Let $X = L_1[0,1]$. Is the usual norm minimal in the ordering introduced in this section?

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