

THE ORLICZ-PETTIS THEOREM

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1. Introduction

It is my aim in this talk to give a summary of certain aspects of the development of the Orlicz-Pettis theorem since its origins fifty years ago. During the course of its evolution this theorem has evolved almost beyond recognition, and the techniques developed for its study have themselves helped to illuminate a number of ideas in functional analysis.

This will be only a very partial survey; in such a short time it would be impossible to cover adequately all the directions taken by recent research. It will reflect also my personal interests in the area. The Orlicz-Pettis theorem has been an important catalyst in the development of the theory of (non-locally convex) F-spaces.

2. The basic Orlicz-Pettis theorem

Let us start by giving two equivalent formulations of the classical Orlicz-Pettis theorem.

Theorem 1 (a) Let X be a Banach space and let $\sum x_n$ be series in X with the property that for every subseries $\sum x_{n_k}$ there exists $x \in X$, with

$$x^*(x) = \sum_{k=1}^{\infty} x^*(x_{n_k}) \quad x^* \in X^* .$$

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Then $\sum x_n$ converges in X [Briefly: a weakly subseries convergent series is convergent].

(b) Let X be a Banach space, and let A be a σ -algebra of sets. Suppose $\mu: A \rightarrow X$ is set function such that $x^* \circ \mu$ is a countably additive measure for all $x^* \in X^*$. Then μ is a countably additive vector measure. [Briefly: a weak countably additive measure is countably additive for the norm].

With the restriction that X is weakly sequentially complete, Theorem 1 (a) was proved by Orlicz in 1929 [22]. However 1 (a) was known to Orlicz without this restriction (see [2]). The first accessible proof was given by Pettis in 1938 ([24]), and it was Pettis who pointed out the applications of the result to vector measures, in particular 1 (b).

The proof of the theorem by both Orlicz and Pettis depends on a special property of the Banach space ℓ_1 , now known as the Schur property. Schur had proved a result for summability methods which translates to the statement that if $\{x_n\}$ is a weakly convergent sequence in ℓ_1 then $\{x_n\}$ converges in norm. If we identify $\ell_1^* = \ell_\infty$, then in fact it is necessary only to check that $\lim_{n \rightarrow \infty} x^*(x_n)$ exists for all $x^* \in m_0$ where $m_0 \subset \ell_\infty$ is the subspace of finitely-valued sequences.

Let us now sketch a proof of Theorem 1 (a). First observe that we can assume X separable, and next can reduce the problem to showing the impossibility of

$$\|x_n\| \geq 1 \quad n \in \mathbb{N}.$$

Pick $x_n^* \in X^*$ with $\|x_n^*\| \leq 1$ and

$$x_n^*(x_n) = 1 \quad n \in \mathbb{N}.$$

Since X is separable, we can pass to a subsequence and suppose for some

$$x_0^* \in X^*$$

$$\lim_{n \rightarrow \infty} x_n^*(x) = x_0^*(x) \quad x \in X.$$

Suppose $t = \{t_n\}_{n=1}^{\infty} \in m_0$; then for some $y_t \in X$

$$x^*(y_t) = \sum_{n=1}^{\infty} t_n x_n^*(x_n) \quad x^* \in X^*$$

and so

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} t_n (x_m^*(x_n) - x_0^*(x_n)) = 0.$$

It is easy to check that for each m

$$\sum_{n=1}^{\infty} |x_m^*(x_n) - x_0^*(x_n)| < \infty$$

and so by the Schur property of ℓ_1

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} |x_m^*(x_n) - x_0^*(x_n)| = 0$$

In particular

$$\lim_{n \rightarrow \infty} |x_n^*(x_n) - x_0^*(x_n)| = 0$$

and $x_0^*(x_n) \rightarrow 1$ contradicting the convergence of $\sum x_0^*(x_n)$.

A later rather more brutal Banach space style proof was given by Bessaga and Pelczynski in 1958 ([4]), exploiting the properties of basic sequences. Still more recently Uhl [29] has given another proof based on the Pettis-measurability theorem, showing a rather curious tie-up between these two at first sight unrelated results.

3. Further developments in the context of Banach spaces

As far as Banach spaces are concerned the main thrust of research into improving the Orlicz-Pettis theorem has been to attempt to replace the assumption in 1 (b) that $x^* \circ \mu$ is countably additive for all $x^* \in X^*$ by some smaller collection of linear functionals. The definitive result here is due to Diestel and Faires (1974, [7]) and it perhaps suggests that the role of the weak topology in the Orlicz-Pettis theorem is exaggerated.

A simple example shows that we cannot simply demand that $x^* \circ \mu$ is countably additive for a total subspace of X^* . Let $P N$ be the power set of N and let $\mu: P N \rightarrow \ell_\infty$ be defined by

$$\mu(A) = \chi_A$$

(the characteristic function of A). Then $x^* \circ \mu$ is countably additive if $x^* \in \ell_1 \subset (\ell_\infty)^*$, but μ is not countably additive. The Diestel-Faires theorem shows that this is the "only" such example.

Theorem 2 Let X be a Banach space containing no copy of ℓ_∞ , and let A be a σ -algebra of sets. Let $\mu: A \rightarrow X$ have the properties that $x^* \circ \mu$ is countably additive for x^* in some total subspace H of X^* . Then μ is countably additive.

Theorem 2 depends on two deep results.

Theorem 3 (Grothendieck [14]) m_0 is a barrelled subspace of ℓ_∞ ;
equivalently if $(\phi_i: i \in I)$ is any collection of finitely additive bounded set functions on $P N$ with

$$\sup_{i \in I} |\phi_i(A)| < \infty \quad A \in P N$$

then

$$\sup_{A \in P N} \sup_{i \in I} |\phi_i(A)| < \infty$$

Theorem 4 (Rosenthal [26]) If X is a Banach space not containing ℓ_∞ and $T: \ell_\infty \rightarrow X$ is a bounded linear operator, then T is weakly compact.

Now a proof of the Diestel-Faires theorem can be built as follows. We can suppose $A = \mathcal{P} N$; μ induces a linear map $T_0: m_0 \rightarrow X$. The hypotheses make T_0 a linear map with closed graph; but now Theorem 3 allows us to use the Closed Graph Theorem, since m_0 is a barrelled normed space. Hence T_0 is bounded and induces a bounded linear extension $T: \ell_\infty \rightarrow X$. Then by Theorem 4, T is weakly compact. From here it is an easy step to show that $x^* \circ \mu$ is countably additive for every $x^* \in X^*$ and then apply the original Orlicz-Pettis theorem.

The appearance of the Closed Graph Theorem is significant here, as we shall see later when we leave the secure surrounds of Banach spaces.

4. Locally convex spaces

The extension of the Orlicz-Pettis theorem to generally locally convex spaces was achieved by McArthur in 1967 [21], after earlier results by Grothendieck [15]; another proof was given by Robertson [25].

Theorem 5 Let X be a locally convex topological vector space and let A be a σ -algebra of sets. Suppose $\mu: A \rightarrow X$ is a weakly countably additive measure. Then μ is countably additive.

Subsequent work in this area has largely been directed to pushing countable additivity further than the original topology on X ; see for example [3]. For some recent developments see Dierolf [5] and Graves [12].

5. Abelian groups

We have already seen an interaction between the Closed Graph theorem and the Orlicz-Pettis theorem. In fact, Theorem 5 can be understood as a closed graph-type result: the identity map from X with the weak topology into X with original topology has closed graph and Theorem 5 answers its "continuity" at least on subseries convergent series.

In 1970, I became interested in this link after establishing a mild

variation of Ptak's Closed Graph Theorem. In [16], I showed that if E is a Mackey space whose dual E^* is weak* sequentially complete and F is a separable Frechet space then every linear map $T: E \rightarrow F$ with closed graph is continuous. The hypotheses on E are satisfied if we take $E = m_0$ with the Mackey topology induced by ℓ_1 . Then the theorem can be used to obtain a weaker form of the Diestel-Faires theorem (Theorem 2), valid for separable Banach spaces.

At the same time, Stiles [27] established the Orlicz-Pettis theorem (Theorem 1) for F -spaces (complete metric linear spaces) with a Schauder basis. Of course, the basis assures the non-triviality of the dual space, but there is no longer any close relationship between the weak topology and the metric topology on the space. In this sense, Stiles's result was a very important and significant departure from earlier results. On seeing Stiles's paper, I was prompted to examine whether some form of Orlicz-Pettis theorem could be proved in F -spaces using the Closed Graph Theorem.

In fact, Polish abelian groups proved to be the appropriate setting. The main step was to establish the suitability as a domain space in the Closed Graph Theorem for groups of the integer-valued analogue of m_0 with its Mackey topology induced by ℓ_1 . The main result was ([17]).

Theorem 6 Let G be an abelian Polish group and let α be any weaker Hausdorff group topology on G . Suppose $\mu: A \rightarrow G$ is α -countably additive. Then μ is countably additive.

The shortest and neatest proof of Theorem 6 is due to Drewnowski ([8], see also [10]). His approach was to establish first

Theorem 7 Let (G, β) be a separable abelian topological group and let α be a weaker Hausdorff group topology on G . Suppose β has a base of α -closed neighborhoods of 0 . Then any α -countably additive measure is β -countably additive.

The proof of Theorem 7 is a simple argument based on the Baire Category Theorem. Now Theorem 6 is proved by taking γ to be the maximal group topology on G weaker than the original metric topology for which μ is

countably additive. Define γ^* to be that topology whose base at 0 consists of all γ -closed metric neighborhoods of 0. Use Theorem 7 to deduce $\gamma = \gamma^*$ and the basic Closed Graph Theorem for groups to deduce that γ is the metric topology (cf. [18] p. 213). In fact a version of Theorem 6 may be proved for non-abelian groups ([8]).

In 1973 Anderssen and Christensen [1] extended Theorem 6 to groups with an analytic topology (i.e. abelian topological groups which are the continuous images of Polish spaces). They showed it is necessary only to have the identity from (G, α) into G a Borel map, and this is automatic if G has an analytic topology. Continuing this line, recently Labuda [20], Pachl [23] and Graves [13] have established:

Theorem 8 (Graves-Labuda-Pachl) Let G be a complete abelian topological group and let α be a Hausdorff group topology on G weaker than the original so that the identity $i: (G, \alpha) \rightarrow G$ is universally measurable. Then any α -countably additive G -valued measure is countably additive.

It is worth briefly sketching the ideas of the proof as given by Graves [13]. It can be shown to be sufficient to prove the result when i is universally Lusin-measurable, and for a measure μ on $P N$. If we topologize $P N$ as $\{0,1\}^\omega$ it is compact metric, and considered as the Cantor group it has a Haar measure λ say. Now $\mu: P N \rightarrow G$ is universally Lusin measurable. It will suffice (since G is complete) to show that $\mu\{n\} = e_n \rightarrow 0$; and we may again reduce to the case where $e_n \notin V$ for some V of 0, and prove the result by contradiction.

Now choose $M \subset P N$ to be compact such that $\lambda(M) > \frac{1}{2}$ and $\mu|_M$ is continuous. For each n , let $M_n^- = \{A \in M, n \notin A\}$ and $M_n^+ = \{A: n \in A, A \cup \{n\} \in M\}$. Then

$$\lambda(M_n^-) + \lambda(M_n^+) = \lambda(M)$$

$$> \frac{1}{2}$$

$$= \lambda\{A: n \notin A\}$$

Hence there exists $A_n \in M_n^- \cup M_n^+$. In particular $A_n \in M$ and so by passing to a subsequence we may suppose $A_n \rightarrow A_\infty \in M$ and $A_n \cup \{n\} \rightarrow A_\infty \in M$. Thus

$$e_n = \mu\{n\} = \mu(A_n \cup \{n\}) - \mu(A_n)$$

$$\rightarrow 0.$$

This contradiction is sufficient to prove the result.

It seems to me that Theorem 8 is probably the final word on this line of development.

6. F-spaces

Interesting problems arise in the attempt to extend the Diestel-Faires Theorem (Theorem 2) to general non-locally convex F-spaces. One might hope that if X is an F-space containing no copy of ℓ_∞ and $\mu: A \rightarrow X$ is a countably additive measure for some weaker Hausdorff vector topology then μ is countably additive. Recently this reasonable hope was exploded by Turpin [28] who showed it to be false when X is a certain Orlicz sequence space $\ell_\phi(I)$ over an uncountable index set I (it is of course true if I is countable by Theorem 6).

If we analyze the breakdown of Theorem 2, we find that an analogue of Theorem 4 is true in this setting; an important extension of Rosenthal's result was obtained by Drewnowski ([9] and [11]) and this would suffice for the argument. In place of Theorem 3 however one would require m_0 ultrabarrelled and this is false ([6]). Thus we find that Theorem 2 fails in this setting because it is not possible to show that μ is bounded and then use operators on ℓ_∞ . This is exactly what happens in Turpin's example.

Turpin does raise the question of what happens in the case when X is locally bounded. This again reduces to a question concerning the space m_0 : is it "p-barrelled" for $0 < p < 1$.

In the context of groups, Labuda [19] has proved substitutes for the Diestel-Faires theorem involving the idea of copies of rings of sets.

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