LINEARIZATION OF ISOMETRIC EMBEDDINGS BETWEEN BANACH SPACES : AN ELEMENTARY APPROACH

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dedicated to the memory of Nigel J. Kalton (1946-2010)

1. INTRODUCTION

Mazur-Ulam's classical theorem states that any isometric onto map between normed spaces is linear. This result has been generalized by T. Figiel [F] who showed that if Φ is an isometric embedding from a Banach space X to a Banach space Y such that $\Phi(0) = 0$ and $\overline{vect}[\phi(X)] = Y$, there exists a linear quotient map Q such that $\|Q\| = 1$ and $Q \circ \Phi = Id_X$. The third chapter of this short story is [GK] where it is shown that if a quotient map Q from a Banach space Y onto a *separable* Banach space X has a Lipschitz lifting, then it actually has a continuous linear lifting. Combining this statement with Figiel's theorem provides another result from [GK] : if a *separable* Banach space X isometrically embeds into a Banach space Y, there exists an isometric linear embedding from X into Y.

Nigel Kalton had an outstanding ability to set theorems and proofs in their proper frame. His articles are therefore fountains of ideas, irrigating each of the many fields to which he contributed : a non exhaustive survey on these contributions is [G]. The paper [GK] is no exception to this rule, and among other things it prepared the ground for far-reaching extensions, where e.g. the Lipschitz assumption is weakened to the Hölder condition, leading to very different conclusions ([K1], [K2], [K3]). However, some readers could find daunting some arguments from [GK], which would probably turn down undergraduate students.

Since the extension from [GK] of Mazur-Ulam's theorem which is recalled above has a statement which is understandable to any student in mathematics, it seems appropriate to provide an elementary proof. This is the purpose of the present short note, where only basic functional analysis (elementary duality theory) and calculus (culminating at Fubini's theorem for continuous functions on $\mathbf{R}^n$) are used. Moreover this note is fully self-contained : even the generic smoothness of convex fuctions on $\mathbf{R}^n$ is shown, and a detailed proof of Figiel's theorem is provided (following [BL], although Lemma 2 is not stated there). The main result of the note is Theorem 5, whose proof follows the strategy of the proof of ([GK], Corollary 3.2) but in such an elementary way that diagrams, free spaces and infinite-dimensional integration or differentiation arguments are avoided. Thus our approach makes it clear that the core of the proof consists of *finite-dimensional* considerations. We can therefore teach at an undergraduate level Mazur-Ulam's theorem and its natural extensions from [F] and [GK].

2. RESULTS

We begin with a classical application of Baire category theorem.

Proposition 1 :*Let $g : \mathbf{R}^n \to \mathbf{R}$ be a convex function. The function g is continuous on $\mathbf{R}^n$ and it is differentiable at every point of a dense subset of $\mathbf{R}^n$.*

Proof: Pick $x = (x_i)_{1\leqslant i\leqslant n} \in \mathbf{R}^n$ and $\alpha > 0$.
If $\{e_1, e_2, ..., e_n\}$ denotes the canonical basis of $\mathbf{R}^n$, convexity shows that

$$\sup_{\|h\|_1 \leqslant \alpha} g(x+h) - g(x) = \max_{1\leqslant i\leqslant n, |\varepsilon|=1} g(x+\varepsilon\alpha e_i) - g(x).$$

Moreover if C is a non-empty convex subset $\mathbf{R}^n$ such that $(-x) \in C$ for all $x \in C$, and $F : C \to \mathbf{R}$ is a convex function such that $F(0) = 0$ and F is bounded above on C, then

$$\sup_{x\in C} F(x) = \sup_{x\in C} |F(x)|.$$

Thus we have

$$\sup_{\|h\|_1 \leqslant \alpha} |g(x+h)-g(x)| = \max_{1\leqslant i\leqslant n, |\varepsilon|=1} g(x+\varepsilon\alpha e_i) - g(x). \tag{1}$$

It thus follows from the one-dimensional case that g is continuous at every point x of $\mathbf{R}^n$.

For $k \in \mathbf{N}^*$, $1 \leqslant i \leqslant n$ and $t > 0$, let

$$O_{k,i}(t) = \left\{x \in \mathbf{R}^n \; ; \; \frac{g(x+te_i)+g(x-te_i)-2g(x)}{t} < \frac{1}{k}\right\}$$

and

$$V_{k,i} = \bigcup_{t>0} O_{k,i}(t).$$

The sets $V_{k,i}$ are open (as union of open sets).

Observe now that if $f : \mathbf{R} \to \mathbf{R}$ is a convex function, and if for a given $x \in \mathbf{R}$ we define $\tau : \mathbf{R}^+ \to \mathbf{R}$ by

$$\tau(t) = [f(x+t)+f(x-t)-2f(x)]/t.$$

then f est differentiable at x if and only if

$$\lim_{t\to 0^+} \tau(t) = 0.$$

It follows that if we let

$$\Delta_i = \bigcap_{k\geqslant 1} V_{k,i}$$

then $\Delta_i = \{x \in \mathbf{R}^n \; ; \; \frac{\partial g}{\partial x_i}(x) \text{ exists}\}$.

Since every convex function of one real variable is differentiable outside a countable set, the sets Δ_i are dense in $\mathbf{R}^n$.

Since every set Δ_i is a countable intersection of open sets, it follows from Baire category theorem that the set

$$\Omega_g = \bigcap_{1\leqslant i\leqslant n} \Delta_i$$

is dense in $\mathbf{R}^n$.

Pick $x \in \Omega_g$. We define a function G by $G(y) = g(y) - g(x) - \sum_{i=1}^{n} (y_i - x_i)\dfrac{\partial g}{\partial x_i}(x)$.

Applying (1) to this function G shows that for any $h \in \mathbf{R}^n$,

$$|G(x+h)| \leqslant \max_{1\leqslant i\leqslant n, |\varepsilon|=1} G(x+\varepsilon\|h\|_1 e_i).$$

It follows that Ω_g is the set of points in $\mathbf{R}^n$ where the function g is differentiable. This concludes the proof of Proposition 1.

Let $\| \,.\, \|$ be a norm on $\mathbf{R}^n$. We denote $\Omega_{\| . \|}$ the set of points where this norm is differentiable. If $x \in \Omega_{\| . \|}$, we denote by $\{\nabla \| \,.\, \|\}(x)$ the differential of the norm at x.

It is easily seen that $\|\{\nabla \| \,.\, \|\}(x)\| = < \{\nabla \| \,.\, \|\}(x), \frac{x}{\|x\|} >= 1$.

Moreover if $z \in \mathbf{R}^n$ and $(x_p)_{p\geqslant 1}$ is a sequence in $\Omega_{\| . \|}$ which converge to z, then

$$\lim_{p\to\infty} < \{\nabla\|\,.\,\|\}(x_p), z >= \|z\|. \tag{2}$$

Lemma 2 :*Let E be a finite-dimensional normed space, with norm $\|\,.\,\|$. Pick $x \in E$ a point of differentiability of the norm $\|\,.\,\|$ with $\|x\| = 1$. Then $\{\nabla\|\,.\,\|\}(x)$ is the only 1-Lipschitz map $\varphi : E \to \mathbf{R}$ such that $\varphi(tx) = t$ for all $t \in \mathbf{R}$.*

Proof: Let $\varphi : E \to \mathbf{R}$ a 1-Lipschitz map such that $\varphi(tx) = t$ for all $t \in \mathbf{R}$. Pick $y \in E$.
For all $t \neq 0$, one has

$$1 = |t\varphi(y) - t\varphi\big((\varphi(y) + 1/t)x\big)| \leqslant \|x - t(y - \varphi(y)x)\|.$$

Therefore the right-hand side function attains its minimum at $t = 0$. Differentiation gives

$$< \{\nabla\|\,.\,\|\}(x), y - \varphi(y)x >= 0$$

and thus $\{\nabla\|\,.\,\|\}(x) = \varphi$.

Lemma 3([F]) : *Let E be a normed space of finite dimension n, let F be a normed space and let $\phi : E \to F$ be an isometry such that $\phi(0) = 0$. We assume that $\overline{vect}[\phi(E)] = F$. Then there exists a unique continuous linear map $T : F \to E$ such that $T \circ \phi = Id_E$, and moreover $\|T\| = 1$.*

Proof: We first consider the one-dimensional case. Let $j : \mathbf{R} \to F$ be an isometry such that $j(0) = 0$. For all $k \in \mathbf{N}$. there exists $x_k^* \in F^*$ with norm 1 such that $< x_k^*, j(k) - j(-k) >= 2k$. It is easily seen that $< x_k^*, j(t) >= t$ for all $t \in [-k, k]$. It follows by weak* compactness that there exists $x^* \in F^*$ with norm 1 such that $< x^*, j(t) >= t$ for all $t \in \mathbf{R}$, and this linear form x^* does the job.

Take now $\phi : E \to F$ as above. Pick any $x \in E$ where the norm $\|\,.\,\|$ is differentiable. By the one-dimensional case, there exists $f_x^* \in F^*$ with norm 1 such that $< f_x^*, \phi(tx) >= t$ for all $t \in \mathbf{R}$. Lemma 2 shows that $f_x^* \circ \phi = \{\nabla\|\,.\,\|\}(x)$.

It follows from Proposition 1 and (2) that for any $z \in E\backslash\{0\}$, there is $x' \in \Omega_{\|\,.\,\|}$ such that $\{\nabla\|\,.\,\|\}(x')(z) \neq 0$.
It follows that we can find points $x_1, x_2, \ldots, x_n$ in $\Omega_{\|\,.\,\|}$ such that the set of linear forms $\big(\{\nabla\|\,.\,\|\}(x_i)\big)_{1\leqslant i\leqslant n}$ is a basis of E^*.
We denote by $(z_j)_{1\leqslant j\leqslant n}$ the dual basis in E, such that

$$\{\nabla\|\,.\,\|\}(x_i)(z_j) = \delta_{i,j}.$$

For all $1 \leqslant i \leqslant n$, there exists $f_{x_i}^* \in F^*$ such that

$$\{\nabla\|\,.\,\|\}(x_i) = f_{x_i}^* \circ \phi.$$

We define $T : F \to E$ by

$$T(y) = \sum_{i=1}^{n} f_{x_i}^*(y) z_i.$$

The map T is linear and continuous, and $T \circ \phi = Id_E$.
Uniqueness of such a map T follows immediately from $\overline{vect}[\phi(E)] = F$.
Moreover, for all $x' \in \Omega_{\|\,.\,\|}$, one has

$$f_{x'}^{*} = \{\nabla \| \,.\, \|\}(x') \circ T \tag{3}$$

since these continuous linear forms coincide on the dense set $vect[\phi(E)]$. If we pick any $y \in F$ and we apply (2) to $z = T(y)$, it follows from (3) that $\|y\| \leqslant \|z\|$ and thus $\|T\| = 1$.

Theorem 4 ([F]) : *Let X be a separable infinite-dimensional Banach space. Let F be a normed space and let $\phi : X \to F$ be an isometry such that $\phi(0) = 0$. We assume that $\overline{vect}[\phi(X)] = F$. Then there exists a unique continuous linear map $T : F \to X$ such that $T \circ \phi = Id_X$, and moreover $\|T\| = 1$.*

Proof: We write

$$X = \overline{\bigcup_{k \geqslant 1} E_k}$$

where $(E_k)_{k \geqslant 1}$ is an increasing sequence of finite-dimensional subspaces. We let $F_k = vect[\Phi(E_k)]$. By Lemma 3, there exists a unique continuous linear map $T_k : F_k \to E_k$ such that $T_k(\Phi(x)) = x$ for all $x \in E_k$, and moreover $\|T_k\| = 1$.

Uniqueness implies that we can consistently define $T : \bigcup_{k \geqslant 1} F_k \to X$ by $T(y) = T_k(y)$ if $y \in F_k$, and $\|T\| = 1$ since $\|T_k\| = 1$ for all k. Finally our assumption implies that $F = \overline{\bigcup_{k \geqslant 1} F_k}$ and T can be extended to F since it takes values in the complete space X.

Remarks : 1) Assuming X separable in Theorem 4 is a matter of convenience. The same argument works for any Banach space X, written as the union of the directed set of its finite-dimensional subspaces.

2) Theorem 4 immediately implies Mazur-Ulam's theorem : every onto isometry $\Phi : X \to Y$ between Banach spaces such that $\Phi(0) = 0$ is linear.

Theorem 5 ([GK]) : *Let X be a separable Banach space. Let Y be a Banach space, and let $Q : Y \to X$ a continuous linear map, such that there exists an M-Lipschitz map $\mathscr{L} : X \to Y$ such that $Q \circ \mathscr{L} = Id_X$. Then there exists a continuous linear map $S : X \to Y$ such that $Q \circ S = Id_X$ and $\|S\| \leqslant M$.*

Proof: Let $Lip_0(X)$ be the space of Lipschitz functions f from X to $\mathbf{R}$ such that $f(0) = 0$, equipped with its natural norm

$$\|f\|_L = sup\Big\{\frac{|f(x) - f(y)|}{\|x - y\|} \ ;\ (x, y) \in X^2,\ x \neq y\Big\}.$$

The dual space X^* is a subspace of $Lip_0(X)$ and $\|x^*\| = \|x^*\|_L$ pour tout $x^* \in X^*$.

Let $(x_i)_{i \geqslant 1}$ be a linearly independent sequence of vectors in X such that $\overline{vect}[(x_i)_{i \geqslant 1}] = X$ and $\|x_i\| = 2^{-i}$ for all i.

We let $E_k = vect[\{x_i \ ;\ 1 \leqslant i \leqslant k\}]$.

We denote $R_k : E_k \to Lip_0(X)^*$ the unique linear map which satisfies for all $1 \leqslant n \leqslant k$ et all $f \in Lip_0(X)$

$$R_k(x_n)(f) = \int_{[0,1]^{k-1}} \big[f(x_n + \sum_{j=1, j \neq n}^{k} t_j x_j) - f(\sum_{j=1, j \neq n}^{k} t_j x_j)\big] dt_1 dt_2 ... dt_{n-1} dt_{n+1} ... dt_k. \tag{4}$$

Pick $f \in Lip_0(X)$. Let f_k be the restriction of f to E_k. If the function f_k est continuously differentiable, Fubini's theorem shows that for all $x \in E_k$

$$R_k(x)(f) = \int_{[0,1]^k} < \{\nabla f_k\}(\sum_{j=1}^{k} t_j x_j), x > dt_1 dt_2 ... dt_k. \tag{5}$$

Thus $|R_k(x)(f)| \leqslant \|x\|.\|f\|_L$ if f_k is continuously differentiable on E_k. But classically, convolutions with a sequence of smooth functions on E_k shows that any $f \in Lip_0(X)$ is a uniform limit of a sequence f_j of functions whose restrictions to E_k are continuously differentiable, and such that $\|f_j\|_L \leqslant \|f\|_L$. Hence (5) shows that

$$\|R_k\| \leqslant 1$$

.

Observe now that (4) shows that if $1 \leqslant n \leqslant k$, then

$$\|R_{k+1}(x_n) - R_k(x_n)\| \leqslant 2\|x_{k+1}\|.$$

It follows that for all $x \in E_k$, the sequence $(R_l(x))_{l\geqslant k}$ converges in the Banach space $Lip_0(X)^*$.

We let $C = \bigcup_{k\geqslant 1} E_k$ and we define for all $x \in C$

$$R(x) = \lim_{l\to\infty} R_l(x).$$

Clearly R is a linear map from C to $Lip_0(X)^*$ such that $\|R\| = 1$. Moreover $R_k(x_n)(x^*) = x^*(x_n)$ for all $x^* \in X^*$, and thus $R(x)(x^*) = x^*(x)$ for all $x \in C$ and all $x^* \in X^*$.

Since C is dense in X, it follows that there exists a linear map $\overline{R} : X \to Lip_0(X)^*$ such that $\|\overline{R}\| = 1$ and $\overline{R}(x)(x^*) = x^*(x)$ for all $x \in X$ and all $x^* \in X^*$.

In the notation of the theorem, we may and do assume that $\mathscr{L}(0) = 0$. We now define a linear map $S : X \to Y^{**}$ by the equation

$$< S(x), y^* >=< \overline{R}(x), y^* \circ \mathscr{L} > \tag{6}$$

Since weak* convergence in B_{Y^*} implies uniform convergence on compact sets, (6) shows that $S(x)$ is weak* continuous on B_{Y^*} and thus S takes its values into Y. Moreover, for any $x^* \in X^*$,

$$< x^*, QS(x) >=< S(x), Q^*(x^*) >=< \overline{R}(x), x^* \circ Q \circ \mathscr{L} >=< \overline{R}(x), x^* >=< x^*, x >$$

and thus $Q \circ S = Id_X$. Finally, $\|\overline{R}\| = 1$ shows that $\|S\| \leqslant M$.

Theorem 6 ([GK]) : *Let X be a separable Banach space. If there exists an isometry Φ from X into a Banach space Y, then Y contains a closed linear subspace which is linearly isometric to X.*

Proof: We may and do assume that $\Phi(0) = 0$ and that $\overline{vect}[\Phi(X)] = Y$. By Lemma 3 and Theorem 4, there is a quotient map $Q : Y \to X$ of norm 1 such that $Q\circ\Phi = Id_X$. We can therefore apply Theorem 5 with $\mathscr{L} = \phi$, and this shows the existence of $S : X \to Y$ with $\|S\| = 1$ and $Q \circ S = Id_X$. It is now clear that S is a linear isometry from X into Y.

Remark : In sharp contrast with what happens in Theorem 4, the separability of X is crucially important in Theorems 5 and 6, which both fail for instance when X is a non-separable Hilbert space ([GK], Theorem 4.3).

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