THE NON-LINEAR GEOMETRY OF BANACH SPACES AFTER NIGEL KALTON

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Dedicated to the memory of Nigel J. Kalton

Abstract. This is a survey of some of the results which were obtained in the last twelve years on the non-linear geometry of Banach spaces. We focus on the contribution of the late Nigel Kalton.

1. Introduction

Four articles among Nigel Kalton's last papers are devoted to the non-linear geometry of Banach spaces ([55], [56], [57], [58]). Needless to say, each of these works is important, for the results and also for the open problems it contains. These articles followed a number of contributions due to Nigel Kalton (sometimes assisted by co-authors) which reshaped the non-linear geometry of Banach spaces during the last decade. Most of these contributions took place after Benyamini-Lindenstrauss' authoritative book [12] was released, and it seems that they are not yet accessible in a unified and organized way. The present survey adresses this need, in order to facilitate the access to Kalton's results (and related ones) and to help trigger further research in this widely open field of research. Nigel Kalton cannot be replaced, neither as a friend nor as the giant of mathematics he was. But his wish certainly was that research should go on, no matter what. This work is a modest attempt to fulfill this wish, and to honor his memory.

Let us outline the contents of this article. Section 2 gathers several tables, whose purpose is to present in a handy way what is known so far about the stability of several isomorphic classes under non-linear isomorphisms or emdeddings. We hope that this section will provide the reader with an easy access to the state of the art. Of course these tables contain a number of question marks, since our present knowledge is far from complete, even for classical Banach spaces. Section 3 displays several results illustrating the non-trivial fact that asymptotic structures are somewhat invariant under non-linear isomorphisms. Section 4 deals with embeddings of special graphs into Banach spaces, and the use of such embeddings for showing the stability of certain properties under non-linear isomorphisms. The non-separable theory is addressed in Section 5. Non-separable spaces behave quite differently from separable ones and this promptly yields to open problems (and even to undecidable ones in ZFC). Section 6 displays the link between coarse embeddings of discrete

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groups (more generally of locally finite metric spaces) into the Hilbert space or super-reflexive spaces, and the classification of manifolds up to homotopy equivalence. This section attempts to provide the reader with some feeling on what the Novikov conjecture is about, and some connections between the non-linear geometry of Banach spaces and the "geometry of groups" in the sense of Gromov. Finally, the last section 7 is devoted to the Lipschitz-free spaces associated with a metric space, their use and their structure (or at least, what is known about it). Sections 2 and 6 contain no proof, but other sections do. These proofs were chosen in order to provide information on the tools we need.

This work is a survey, but it contains some statements (such as Theorem 3.8, Theorem 3.12 or the last Remark in section 5) which were not published before. Each section contains a number of commented open questions. It is interesting to observe that much of these questions are fairly simple to state. Answering them could be less simple. Our survey demonstrates that non-linear geometry of Banach spaces is a meeting point for a variety of techniques, which have to join forces in order to allow progress. It is our hope that the present work will help stimulate such efforts. We should however make it clear that our outline of Nigel Kalton's last papers does not exhaust the content of these articles. We strongly advise the interested reader to consult them for her/his own research.

2. Tables

This section consists of five tables: Table 1 lists a number of classical spaces and check when these Banach spaces are characterized by their metric or their uniform structure. Table 2 displays what is known about Lipschitz embeddings from a classical Banach space into another, and Table 3 does the same for uniform embeddings. Table 4 investigates the stability of certain isomorphism classes (relevant to a classical property) under Lipschitz or uniform homeomorphism. And finally, Table 5 does the same for non-linear embeddability.

References are given within the tables themselves, but in order to improve readability, we almost always used symbols (whose meaning is explained below) rather than using the numbering of the reference list. Questions marks mean of course that to the best of our knowledge, the corresponding question is still open.

Our notation for Banach spaces is standard. All Banach spaces will be real. From the recent textbooks that may be used in the area we mention [4] and [29].

$\clubsuit$= Benyamini-Lindenstrauss book [12].

$\spadesuit$= Kalton recent papers.

$\sharp$ = Mendel-Naor papers [67] and [69].

Δ= Godefroy-Kalton-Lancien papers [36] and [37].

$\Box$= Godefroy-Kalton paper on free spaces [35].

$\heartsuit$= Johnson-Lindenstrauss-Schechtman paper [47].

$\Diamond$= Textbook [29].

$\diamond$= Basic linear theory or topology.

? = Unknown to the authors.

TABLE 1. Spaces determined by weaker structures

Space	Determined by its Lipschitz Structure	Determined by its uniform Structure
ℓ_2	yes	yes ♣
ℓ_p $1 < p < \infty$ $p \neq 2$	yes	yes ♣
ℓ_1	?	?
c_0	yes ♣	?
L_p $1 < p < \infty$ $p \neq 2$	yes ♣	?
L_1	?	?
$C[0,1]$	?	?
$\ell_2(c)$	yes	yes ♣
$c_0(c)$	no ♣	no
ℓ_∞	?	?
$\ell_p \oplus \ell_q$ $1 < p < q < \infty$ $p, q \neq 2$	yes	yes ♣+♠
$\ell_p \oplus \ell_2$ $1 < p < \infty$ $p \neq 2$	yes ♣	?
J James' space	yes ♣+[20]	?

We say that a Banach space X is determined by its Lipschitz (respectively uniform) structure if a Banach space Y is linearly isomorphic to X whenever Y is Lipschitz homeomorphic (respectively uniformly homeomorphic) to X.

Table 2. Lipschitz Embeddings from the 1st column into the 1st row

Space	ℓ_2	ℓ_q $1 < q < \infty$ $q \neq 2$	ℓ_1	c_0	L_q $1 < q < \infty$ $q \neq 2$	L_1	$C[0,1]$	$\ell_2(c)$	$c_0(c)$	ℓ_∞
ℓ_2	yes	no $\clubsuit$	no $\clubsuit$	yes $\clubsuit$	yes $\diamond$	yes $\diamond$	yes $\diamond$	yes $\diamond$	yes $\clubsuit$	yes $\diamond$
ℓ_p $1 < p < \infty$ $p \neq 2$	no $\clubsuit$	yes iff $p = q$ $\clubsuit$	no $\clubsuit$	yes $\clubsuit$	yes iff $q \leq p < 2$ or $p = q$ $\clubsuit$	yes iff $p < 2$ $\clubsuit$	yes $\diamond$	no $\clubsuit$	yes $\clubsuit$	yes $\diamond$
ℓ_1	no $\clubsuit$	no $\clubsuit$	yes	yes $\clubsuit$	no $\clubsuit$	yes $\diamond$	yes $\diamond$	no $\clubsuit$	yes $\clubsuit$	yes $\diamond$
c_0	no $\clubsuit$	no $\clubsuit$	no $\clubsuit$	yes	no $\clubsuit$	no $\clubsuit$	yes $\diamond$	no $\clubsuit$	yes $\diamond$	yes $\diamond$
L_p $1 < p < \infty$ $p \neq 2$	no $\clubsuit$	no $\clubsuit$	no $\clubsuit$	yes $\clubsuit$	yes iff $q \leq p < 2$ or $p = q$ $\clubsuit$	yes iff $p < 2$ $\clubsuit$	yes $\diamond$	no $\clubsuit$	yes $\clubsuit$	yes $\diamond$
L_1	no $\clubsuit$	no $\clubsuit$	no $\clubsuit$	yes $\clubsuit$	no $\clubsuit$	yes	yes $\diamond$	no $\clubsuit$	yes $\clubsuit$	yes $\diamond$
$C[0,1]$	no $\clubsuit$	no $\clubsuit$	no $\clubsuit$	yes $\clubsuit$	no $\clubsuit$	no $\clubsuit$	yes	no $\clubsuit$	yes $\clubsuit$	yes $\diamond$
$\ell_2(c)$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	yes	?	yes $\spadesuit$
$c_0(c)$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\clubsuit$	yes	yes $\spadesuit$
ℓ_∞	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\clubsuit$	?	yes

A Lipschitz embedding of a Banach space X into a Banach space Y is a Lipschitz homeomorphism from X onto a subset (in general non linear) of Y.

TABLE 3. Uniform Embeddings from the 1st column into the 1st row

Space	ℓ_2	ℓ_q $q \in (1,\infty)$ $q \neq 2$	ℓ_1	c_0	L_q $q \in (1,\infty)$ $q \neq 2$	L_1	$C[0,1]$	$\ell_2(c)$	$c_0(c)$	ℓ_∞
ℓ_2	yes	yes $\clubsuit$	yes $\clubsuit$	yes $\clubsuit$	yes $\diamond$	yes $\diamond$	yes $\diamond$	yes $\diamond$	yes $\clubsuit$	yes $\diamond$
ℓ_p $p \in (1,\infty)$ $p \neq 2$	yes iff $p < 2$ $\clubsuit$	yes if $p \leq q$ or $p < 2$ [3]+$\clubsuit$ no if $p > 2$ and $q < p$ $\sharp$	yes iff $p < 2$ $\clubsuit$	yes $\clubsuit$	yes iff $p \leq q$ or $p < 2$ $\sharp$	yes iff $p < 2$ $\clubsuit$	yes $\diamond$	yes iff $p < 2$ $\clubsuit$	yes $\clubsuit$	yes $\diamond$
ℓ_1	yes $\clubsuit$	yes $\clubsuit$	yes	yes $\clubsuit$	yes $\sharp$	yes $\diamond$	yes $\diamond$	yes $\clubsuit$	yes $\clubsuit$	yes $\diamond$
c_0	no $\clubsuit$	no $\spadesuit$ or $\sharp$	no $\clubsuit$	yes	no $\spadesuit$ or $\sharp$	no $\clubsuit$	yes	no $\clubsuit$	yes $\diamond$	yes $\diamond$
L_p $p \in (1,\infty)$ $p \neq 2$	yes iff $p < 2$ $\clubsuit$	no if $p > 2$ and $q < p$ $\sharp$ yes if $p < 2$ $\clubsuit$? if $2 < p \leq q$	yes iff $p < 2$ $\clubsuit$	yes $\clubsuit$	yes iff $p \leq q$ or $p < 2$ $\sharp$	yes iff $p < 2$ $\clubsuit$	yes $\diamond$	yes iff $p < 2$ $\clubsuit$	yes $\clubsuit$	yes $\diamond$
L_1	yes $\clubsuit$	yes $\clubsuit$	yes $\clubsuit$	yes $\clubsuit$	yes $\clubsuit$	yes	yes $\diamond$	yes $\clubsuit$	yes $\clubsuit$	yes $\diamond$
$C[0,1]$	no $\clubsuit$	no $\spadesuit$ or $\sharp$	no $\clubsuit$	yes $\clubsuit$	no $\spadesuit$ or $\sharp$	no $\clubsuit$	yes	no $\clubsuit$	yes $\clubsuit$	yes $\diamond$
$\ell_2(c)$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no	no $\diamond$	yes	?	yes $\spadesuit$
$c_0(c)$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\clubsuit$	yes	yes $\spadesuit$
ℓ_∞	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\diamond$	no $\clubsuit$	?	yes

A uniform embedding of a Banach space X into a Banach space Y is a uniform homeomorphism from X onto a subset of Y. Let us also mention that the same table can be written about coarse embeddings (see the definition in section 3). The set of references to be used is the same except for one paper by Nowak [72], where it is proved that for any $p \in [1,\infty)$, ℓ_2 coarsely embed into ℓ_p.

TABLE 4. Stability under type of homeomorphism

Property	Lipschitz	uniform	
hilbertian	yes	yes ♣	
superreflexivity	yes	yes ♣	
reflexivity	yes ♣	no ♣	
RNP	yes ♣	no ♣	
Asplund	yes ♣	no ♣	
containment of ℓ_1	?	no ♣	
containment of c_0	?	?	
BAP	yes □	no ♡	
Commuting BAP	?	no ♡	
Existence of Schauder basis	?	?	
Existence of M-basis	no ◇	no	
Existence of unconditional basis	no ◇	no	
renorming by Frechet smooth norm	?	no ♣	
renorming by LUR norm	?	?	
renorming by UG norm	no ◇	no	
renorming by WUR norm	?	?	
renorming by AUS norm	yes	yes Δ	

Table 5. Properties shared by embedded spaces

Property	Lipschitz	coarse-Lipschitz	
Hilbertian	yes ♣	yes ♣	
superreflexivity	yes ♣	yes ♣	
reflexivity	yes ♣	no ♣	
RNP	yes ♣	no ♣	
Asplund	no ♣	no ♣	
renorming by Frechet smooth norm	no ♣	no ♣	
renorming by LUR norm	?	?	
renorming by UG norm	no ◇	no	
renorming by WUR norm	?	?	
renorming by AUS norm	no ♣	no	
reflexive+renorming by AUS norm	yes ♣	?	
reflexive+renorming by AUS norm +renorming by AUC norm	yes	yes [8]	

More precisely, the question addressed is the following. If a Banach space X Lipschitz or coarse-Lipschitz (see definition in section 3) embed into a Banach space Y which has one the properties listed in the first column, does X satisfy the same property?

3. Uniform and asymptotic structures of Banach spaces

In this section we will study the stability of the uniform asymptotic smoothness or uniform asymptotic convexity of a Banach space under non linear maps such as uniform homeomorphisms and coarse-Lipschitz embeddings.

3.1. **Notation - Introduction.**

Definition 3.1. Let (M,d) and (N,δ) be two metric spaces and $f : M \to N$ be a mapping.
(a) f is a *Lipschitz isomorphism* (or *Lipschitz homeomorphism*) if f is a bijection and f and f^{-1} are Lipschitz. We denote $M \overset{Lip}{\sim} N$ and we say that M and N are *Lipschitz equivalent*.
(b) f is a *uniform homeomorphism* if f is a bijection and f and f^{-1} are uniformly continuous (we denote $M \overset{UH}{\sim} N$).
(c) If (M,d) is unbounded, we define

$$\forall s>0, \quad Lip_s(f) = \sup\{\frac{\delta((f(x),f(y))}{d(x,y)},\ d(x,y)\geq s\} \ \text{ and } \ Lip_\infty(f) = \inf_{s>0} Lip_s(f).$$

f is said to be *coarse Lipschitz* if $Lip_\infty(f) < \infty$.
(d) f is a *coarse Lipschitz embedding* if there exist $A>0, B>0, \theta\geq 0$ such that

$$\forall x,y\in M \quad d(x,y)\geq\theta \Rightarrow Ad(x,y)\leq \delta(f(x),f(y))\leq Bd(x,y).$$

We denote $M \overset{CL}{\hookrightarrow} N$.

More generally, f is a *coarse embedding* if there exist two real-valued functions ρ_1 and ρ_2 such that $\lim_{t\to+\infty}\rho_1(t)=+\infty$ and

$$\forall x,y\in M \quad \rho_1(d(x,y))\leq \delta(f(x),f(y))\leq \rho_2(d(x,y)).$$

(e) An (a,b)-net in the metric space M is a subset $\mathcal{M}$ of M such that for every $z\neq z'$ in $\mathcal{M}$, $d(z,z')\geq a$ and for every x in M, $d(x,\mathcal{M})<b$.
Then a subset $\mathcal{M}$ of M is a *net* in M if it is an (a,b)-net for some $0<a\leq b$.
(f) Note that two nets in the same infinite dimensional Banach space are always Lipschitz equivalent (see Proposition 10.22 in [12]).
Then two infinite dimensional Banach spaces X and Y are said to be *net equivalent* and we denote $X \overset{N}{\sim} Y$, if there exist a net $\mathcal{M}$ in M and a net $\mathcal{N}$ in N such that $\mathcal{M}$ and $\mathcal{N}$ are Lipschitz equivalent.

Remark. It follows easily from the triangle inequality that a uniformly continuous map defined on a Banach space is coarse Lipschitz and a uniform homeomorphism between Banach spaces is a bi-coarse Lipschitz bijection (see Proposition 1.11 in [12] for details). Therefore if X and Y are uniformly homeomorphic Banach spaces, then they are net equivalent. It has been proved only recently by Kalton in [56] that there exist two net equivalent Banach spaces that are not uniformly homeomorphic. However the finite dimensional structures of Banach spaces are preserved under net equivalence (see Proposition 10.19 in [12], or Theorem 3.3 below).

The main question addressed in this section is the problem of the uniqueness of the uniform (or net) structure of a given Banach space. In other words, whether $X \overset{UH}{\sim} Y$ (or $X \overset{N}{\sim} Y$) implies that X is linearly isomorphic to Y (which we shall denote $X \simeq Y$)? Even in the separable case, the general answer is negative. Indeed Ribe [79] proved the following.

Theorem 3.2. (Ribe 1984) *Let $(p_n)_{n=1}^\infty$ in $(1,+\infty)$ be a strictly decreasing sequence such that $\lim p_n = 1$. Denote $X = (\sum_{n=1}^\infty L_{p_n})_{\ell_2}$. Then $X \overset{UH}{\sim} X \oplus L_1$.*

Therefore reflexivity is not preserved under coarse-Lipschitz embeddings or even uniform homemorphisms. On the other hand, Ribe [80] proved that local properties of Banach spaces are preserved under coarse-Lipschitz embeddings. More precisely.

Theorem 3.3. (Ribe 1978) *Let X and Y be two Banach spaces such that $X \overset{CL}{\hookrightarrow} Y$. Then there exists a constant $K \geq 1$ such that for any finite dimensional subspace E of X there is a finite dimensional subspace F of Y which is K-isomorphic to E.*

Remark. If we combine this result with Kwapien's theorem, we immediately obtain that a Banach space which is net equivalent to ℓ_2 is linearly isomorphic to ℓ_2.

As announced, we will concentrate on some asymptotic properties of Banach spaces. So let us give the relevant definitions.

Definition 3.4. Let $(X, \|\ \|)$ be a Banach space and $t > 0$. We denote by B_X the closed unit ball of X and by S_X its unit sphere. For $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t,x,Y) = \sup_{y \in S_Y} \|x+ty\| - 1 \quad \text{and} \quad \overline{\delta}(t,x,Y) = \inf_{y \in S_Y} \|x+ty\| - 1.$$

Then

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \ \inf_{\dim(X/Y)<\infty} \overline{\rho}(t,x,Y) \quad \text{and} \quad \overline{\delta}_X(t) = \inf_{x \in S_X} \ \sup_{\dim(X/Y)<\infty} \overline{\delta}(t,x,Y).$$

The norm $\|\ \|$ is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0.$$

It is said to be *asymptotically uniformly convex* (in short AUC) if

$$\forall t > 0 \quad \overline{\delta}_X(t) > 0.$$

These moduli have been first introduced by Milman in [70]. We also refer the reader to [48] and [24] for reviews on these.

Examples.

(1) If $X = (\sum_{n=1}^\infty F_n)_{\ell_p}$, $1 \leq p < \infty$ and the F_n's are finite dimensional, then $\overline{\rho}_X(t) = \overline{\delta}_X(t) = (1+t^p)^{1/p} - 1$. Actually, if a separable reflexive Banach space has equivalent norms with moduli of asymptotic convexity and smoothness of power type p, then it is isomorphic to a subspace of an l_p-sum of finite dimensional spaces [48].

(2) For all $t \in (0,1)$, $\overline{\rho}_{c_0}(t) = 0$. And again, if X is separable and $\overline{\rho}_X(t_0) = 0$ for some $t_0 > 0$ then X is isomorphic to a subspace of c_0 [36].

We conclude this introduction by mentioning the open questions that we will comment on in the course of this section.

Problem 1. Let $1 < p < \infty$ and $p \neq 2$. Does $\ell_p \oplus \ell_2$ have a unique uniform or net structure? Does L_p have a unique uniform or net structure?

Problem 2. Assume that Y is a reflexive AUS Banach space and that X is a Banach space which coarse-Lipschitz embeds into Y. Does X admit an equivalent AUS norm?

Problem 3. Assume that Y is an AUC Banach space and that X is a Banach space which coarse-Lipschitz embeds into Y. Does X admit an equivalent AUC norm?

3.2. **The approximate midpoints principle.** Given a metric space X, two points $x, y \in X$, and $\delta > 0$, the approximate metric midpoint set between x and y with error δ is the set:

$$Mid(x, y, \delta) = \left\{ z \in X : \ \max\{d(x,z), d(y,z)\} \leq (1+\delta)\frac{d(x,y)}{2} \right\}.$$

The use of approximate metric midpoints in the study of nonlinear geometry is due to Enflo in an unpublished paper and has since been used extensively, e.g. [15], [38] and [47].

The following version of the approximate midpoint Lemma was formulated in [60] (see also [12] Lemma 10.11).

Proposition 3.5. *Let X be a normed space and suppose M is a metric space. Let $f : X \to M$ be a coarse Lipschitz map. If $Lip_\infty(f) > 0$ then for any $t, \varepsilon > 0$ and any $0 < \delta < 1$ there exist $x, y \in X$ with $\|x - y\| > t$ and*

$$f(\mathrm{Mid}(x, y, \delta)) \subset \mathrm{Mid}(f(x), f(y), (1+\varepsilon)\delta).$$

In view of this Proposition, it is natural to study the approximate metric midpoints in ℓ_p. This is done in the next lemma, which is rather elementary and can be found in [60].

Lemma 3.6. *Let $1 \leq p < \infty$. We denote $(e_i)_{i=1}^\infty$ the canonical basis of ℓ_p and for $N \in \mathbb{N}$, let E_N be the closed linear span of $\{e_i,\ i > N\}$. Let now $x, y \in \ell_p$, $\delta \in (0,1)$, $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$. Then*

(i) There exists $N \in \mathbb{N}$ such that $u + \delta^{1/p}\|v\| B_{E_N} \subset Mid(x, y, \delta)$.

(ii) There is a compact subset K of ℓ_p such that $Mid(x, y, \delta) \subset K + 2\delta^{1/p}\|v\| B_{\ell_p}$.

We can now combine Proposition 3.5 and Lemma 3.6 to obtain

Corollary 3.7. *Let $1 \leq p < q < \infty$.*

Then ℓ_q does not coarse Lipschitz embed into ℓ_p.

Remark. This statement can be found in [60] but was implicit in [47]. It already indicates that, because of the approximate midpoint principle, some uniform asymptotic convexity has to be preserved under coarse Lipschitz embeddings. This idea will be pushed much further in section 3.6.

3.3. **Gorelik principle and applications.** Our goal is now to study the stability of the uniform asymptotic smoothness under non linear maps. The first tool that we shall describe is the Gorelik principle. It was initially devised by Gorelik in [38] to prove that ℓ_p is not uniformly homeomorphic to L_p, for $1 < p < \infty$. Then it was developed by Johnson, Lindenstrauss and Schechtman [47] to prove that for $1 < p < \infty$, ℓ_p has a unique uniform structure. It is important to underline the fact that the Gorelik principle is only valid for certain bijections. In fact, the uniqueness of the uniform structure of ℓ_p can be proved without the Gorelik principle, by using results on the embeddability of special metric graphs as we shall see in section 3.4. Nevertheless, some other results still need the use of the Gorelik principle. This principle is usually stated for homeomorphisms with uniformly continuous inverse (see Theorem 10.12 in [12]). Although it is probably known, we have not found its version for net equivalences. Note that a Gorelik principle is proved in [12] (Proposition 10.20) for net equivalences between a Banach space and ℓ_p. So we will describe here how to obtain a general statement.

Theorem 3.8. *(Gorelik Principle.) Let X and Y be two Banach spaces. Let X_0 be a closed linear subspace of X of finite codimension. If X and Y are net equivalent, then there are continuous maps $U : X \to Y$ and $V : Y \to X$, and constants $K, C, \alpha_0 > 0$ such that:*

$$\forall x \in X \quad \|VUx - x\| \leq C \quad \text{and} \quad \forall y \in Y \quad \|UVy - y\| \leq C \quad \text{and}$$

for all $\alpha > \alpha_0$ there is a compact subset M of Y so that

$$\frac{\alpha}{16K} B_Y \subset M + CB_Y + U(\alpha B_{X_0}).$$

Proof. Suppose that $\mathcal{N}$ is a net of the Banach space X, that $\mathcal{M}$ is a net of the Banach space Y and that $\mathcal{N}$ and $\mathcal{M}$ are Lipschitz equivalent. We will assume as we may that $\mathcal{N}$ and $\mathcal{M}$ are $(1, \lambda)$-nets for some $\lambda > 1$ and that $\mathcal{N} = (x_i)_{i \in I}$, $\mathcal{M} = (y_i)_{i \in I}$ with

$$\forall i, j \in I \quad K^{-1}\|x_i - x_j\| \leq \|y_i - y_j\| \leq K\|x_i - x_j\|,$$

for some $K \geq 1$. Let us denote by $B_E(x, \lambda)$ the open ball of center x and radius λ in the Banach space E. Then we can find a continuous partition of unity $(f_i)_{i \in I}$ subordinate to $(B_X(x_i, \lambda))_{i \in I}$ and a continuous partition of unity $(g_i)_{i \in I}$ subordinate to $(B_Y(y_i, \lambda))_{i \in I}$. Now we set:

$$Ux = \sum_{i \in I} f_i(x) y_i, \ x \in X \quad \text{and} \quad Vy = \sum_{i \in I} g_i(y) x_i, \ y \in Y.$$

The maps U and V are clearly continuous. We shall now state and prove two lemmas about them.

Lemma 3.9. *(i) Let $x \in X$ be such that $\|x - x_i\| \leq r$, then $\|Ux - y_i\| \leq K(\lambda + r)$.*
(ii) Let $y \in Y$ be such that $\|y - y_i\| \leq r$, then $\|Vy - x_i\| \leq K(\lambda + r)$.

Proof. We will only prove (i). If $f_j(x) \neq 0$, then $\|x - x_j\| \leq \lambda$. So $\|x_i - x_j\| \leq \lambda + r$ and $\|y_i - y_j\| \leq K(\lambda + r)$. We finish the proof by writing

$$Ux - y_i = \sum_{j, f_j(x) \neq 0} f_j(x)(y_j - y_i).$$

□

Lemma 3.10. *Let $C = (1 + K + 2K^2)\lambda$. Then*

$$\forall x \in X \quad \|VUx - x\| \le C \quad \text{and} \quad \forall y \in Y \quad \|UVy - y\| \le C.$$

Proof. We only need to prove one inequality. So let $x \in X$ and pick $i \in I$ such that $\|x - x_i\| \le \lambda$. By the previous lemma, we have $\|Ux - y_i\| \le 2K\lambda$ and $\|VUx - x_i\| \le \lambda(K + 2K^2)$. Thus $\|VUx - x\| \le (1 + K + 2K^2)\lambda$. □

We now recall the crucial ingredient in the proof of Gorelik Principle (see step (i) in the proof of Theorem 10.12 in [12]). This statement relies on Brouwer's fixed point theorem and on the existence of Bartle-Graves continuous selectors. We refer the reader to [12] for its proof.

Proposition 3.11. *Let X_0 be a finite-codimensional subspace of X. Then, for any $\alpha > 0$ there is a compact subset A of $\frac{\alpha}{2}B_X$ such that for every continuous map $\phi : A \to X$ satisfying $\|\phi(a) - a\| \le \frac{\alpha}{4}$ for all $a \in A$, we have that $\phi(A) \cap X_0 \neq \emptyset$.*

We are now ready to finish the proof of Theorem 3.8. Fix $\alpha > 0$ such that $\alpha > \max\{8C, 96K\lambda\}$ and $y \in \frac{\alpha}{16K}B_Y$ and define $\phi : A \to X$ by $\phi(a) = V(y + Ua)$. The map ϕ is clearly continuous and we have that for all $a \in A$:

$$\|\phi(a) - a\| \le \|V(y + Ua) - VUa\| + \|VUa - a\| \le \frac{\alpha}{8} + \|V(y + Ua) - VUa\|.$$

Now, pick i so that $\|Ua - y_i\| \le \lambda$ and j so that $\|y + Ua - y_j\| \le \lambda$. Then $\|VUa - x_i\| \le 2K\lambda$ and $\|V(y + ua) - x_j\| \le 2K\lambda$. But

$$\|x_i - x_j\| \le K\|y_i - y_j\| \le K\|y_i - Ua\| + K\|Ua + y - y_j\| + K\|y\| \le (2\lambda + \|y\|)K.$$

So

$$\|V(y + Ua) - VUa\| \le 6K\lambda + K\|y\| \le 6K\lambda + \frac{\alpha}{16} \le \frac{\alpha}{8}.$$

Thus $\|\phi(a) - a\| \le \frac{\alpha}{4}$.

So it follows from Proposition 3.11 that there exists $a \in A$ such that $\phi(a) \in X_0$. Besides, $\|a\| \le \frac{\alpha}{2}$ and $\|\phi(a) - a\| \le \frac{\alpha}{4}$, so $\phi(a) = V(y + Ua) \in \alpha B_{X_0}$. But we have that $\|UV(y+Ua) - (y+Ua)\| \le C$. So if we consider the compact set $M = -U(A)$, we have that $y \in M + CB_Y + U(\alpha B_{X_0})$. This finishes the proof of Theorem 3.8.

□

We can now apply the above Gorelik principle to obtain the net version of a result appeared in [37]. This result is new.

Theorem 3.12. *Let X and Y be Banach spaces. Assume that X is net equivalent to Y and that X is AUS. Then Y admits an equivalent AUS norm. More precisely, if $\overline{\rho}_X(t) \le Ct^p$ for $C > 0$ and $p \in (1, \infty)$, then, for any $\varepsilon > 0$, Y admits an equivalent norm $\| \ \|_\varepsilon$ so that $\overline{\rho}_{\| \ \|_\varepsilon}(t) \le C_\varepsilon t^{p-\varepsilon}$ for some $C_\varepsilon > 0$.*

The proof is actually done by constructing a sequence of dual norms as follows:

$$\forall y^* \in Y^* \quad |y^*|_k = \sup\Big\{\frac{\langle y^*, Ux - Ux'\rangle}{\|x - x'\|},\ \|x - x'\| \ge 4^k\Big\}.$$

For k large enough they are all equivalent. Then for N large enough the predual norm of the norm defined by

$$\forall y^* \in Y^* \quad \|y^*\|_N = \frac{1}{N}\sum_{k=k_0+1}^{k=k_0+N} |y^*|_k,$$

is the dual of an equivalent AUS norm on Y with the desired modulus of asymptotic smoothness. The proof follows the lines of the argument given in [37] but uses the above version of Gorelik principle.

Remarks.

(1) It must be pointed out that the quantitative estimate in the above result is optimal as it follows from a remarkable example obtained by Kalton in [57].

(2) If the Banach spaces X and Y are Lipschitz equivalent and X is AUS then the norm defined on Y^* by:

$$|y^*| = \sup\Big\{\frac{\langle y^*, Ux - Ux'\rangle}{\|x - x'\|},\ x \neq x'\Big\}$$

is a dual norm of a norm $|\ |$ on Y such that $\overline{\rho}_{|\ |}(t) \leq c\overline{\rho}_X(ct)$ for some $c > 0$. When X is a subspace of c_0, this implies that Y is isomorphic to a subspace of c_0. Finally, when $X = c_0$, one gets that X is isomorphic to c_0 (see [36]).

(3) Other results were originally derived from the Gorelik principle. We have chosen to present them in the next subsections as consequences of more recent and possibly more intuitive graph techniques introduced by Kalton and Randrianarivony in [60] and later developed by Kalton in [58].

3.4. **Uniform asymptotic smoothness and Kalton-Randrianarivony's graphs.** The fundamental result of this section is about the minimal distortion of some special metric graphs into a reflexive and asymptotically uniformly smooth Banach space. These graphs have been introduced by Kalton and Randrianarivony in [60] and are defined as follows. Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$ and fix $a = (a_1, .., a_k)$ a sequence of non zero real numbers. We denote

$$G_k(\mathbb{M}) = \{\overline{n} = (n_1, .., n_k),\ n_i \in \mathbb{M}\ \ n_1 < .. < n_k\}.$$

Then we equip $G_k(\mathbb{M})$ with the distance

$$\forall \overline{n}, \overline{m} \in G_k(\mathbb{M}), \quad d_a(\overline{n}, \overline{m}) = \sum_{j,\ n_j \neq m_j} |a_j|.$$

Note also that it is easily checked that $\overline{\rho}_Y$ is an Orlicz function. Then, we define the Orlicz sequence space:

$$\ell_{\overline{\rho}_Y} = \{a \in \mathbb{R}^{\mathbb{N}},\ \exists r > 0\ \ \sum_{n=1}^{\infty} \overline{\rho}_Y\Big(\frac{|a_n|}{r}\Big) < \infty\},$$

equipped with the Luxemburg norm

$$\|a\|_{\overline{\rho}_Y} = \inf\{r > 0,\ \sum_{n=1}^{\infty} \overline{\rho}_Y\Big(\frac{|a_n|}{r}\Big) \leq 1\}.$$

Theorem 3.13. (Kalton-Randrianarivony 2008) *Let Y be a reflexive Banach space, $\mathbb{M}$ an infinite subset of $\mathbb{N}$ and $f : (G_k(\mathbb{M}), d_a) \to Y$ a Lipschitz map. Then for any $\varepsilon > 0$, there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ such that:*

$$diam\ f(G_k(\mathbb{M}')) \leq 2eLip(f)\|a\|_{\overline{\rho}_Y} + \varepsilon.$$

The proof is done by induction on k and uses iterated weak limits of subsequences and a Ramsey argument. Such techniques will be displayed in the next two sections.

Remark. The reflexivity assumption is important. Indeed, by Aharoni's Theorem the spaces $(G_k(\mathbb{N}), d_a)$ Lipschitz embed into c_0 with a distortion controlled by a uniform constant. But $\|a\|_{\overline{\rho}_{c_0}} = \|a\|_\infty$, while diam $G_k(\mathbb{M}') = \|a\|_1$.

As it is described in [58] one can deduce the following.

Corollary 3.14. *Let X be a Banach space and Y be a reflexive Banach space. Assume that X coarse Lipschitz embeds into Y. Then there exists $C > 0$ such that for any normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X and any sequence $a = (a_1, .., a_k)$ of non zero real numbers, there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that:*

$$\|\sum_{i=1}^k a_i x_{n_i}\| \leq C\|a\|_{\overline{\rho}_Y}, \quad \textit{for every } \overline{n} \in G_k(\mathbb{M}).$$

Proof. The result is obtained by applying Theorem 3.13 to $f = g \circ h$, where g is a coarse-Lipschitz embedding from X into Y and $h : (G_k(\mathbb{N}), d_a) \to X$ is defined by $h(\overline{n}) = \lambda \sum_{i=1}^k a_i x_{n_i}$ for some large enough $\lambda > 0$. □

In fact, this is stated in [58] in the following more abstract way.

Corollary 3.15. *Let X be a Banach space and Y be a reflexive Banach space. Assume that X coarse Lipschitz embeds into Y. Then there exists $C > 0$ such that for any spreading model $(e_i)_i$ of a normalized weakly null sequence in X (whose norm is denoted $\|\ \|_S$) and any finitely supported sequence $a = (a_i)$ in $\mathbb{R}$:*

$$\|\sum a_i e_i\|_S \leq C\|a\|_{\overline{\rho}_Y}.$$

3.5. **Applications.** The first consequence is the following.

Corollary 3.16. *Let $1 \leq q \neq p < \infty$.*

Then ℓ_q does not coarse Lipschitz embed into ℓ_p.

Proof. If $q < p$, this follows immediately from the previous results.
If $q > p$, this is Corollary 3.7. □

Then we can deduce the following result, proven in [47] under the assumption of uniform equivalence.

Theorem 3.17. (Johnson, Lindenstrauss and Schechtman 1996)
Let $1 < p < \infty$ and X a Banach space such that $X \overset{N}{\sim} \ell_p$. Then $X \simeq \ell_p$.

Proof. Suppose that $X \overset{N}{\sim} \ell_p$, with $1 < p < \infty$. We may assume that $p \neq 2$. Then the ultra-products $X_{\mathcal{U}}$ and $(\ell_p)_{\mathcal{U}}$ are Lipschitz isomorphic and it follows from the classical Lipschitz theory that X is isomorphic to a complemented subspace of $L_p = L_p([0,1])$. Now, it follows from Corollary 3.16 that X does not contain any isomorphic copy of ℓ_2. Then we can conclude with a classical result of Johnson and Odell [49] which asserts that any infinite dimensional complemented subspace of L_p that does not contain any isomorphic copy of ℓ_2 is isomorphic to ℓ_p. $\square$

Remark. These linear arguments are taken from [47]. Note that the key step was to show that that X does not contain any isomorphic copy of ℓ_2. In the original paper [47] this relied on the Gorelik principle. We have chosen to present here a proof using this graph argument. In fact, more can be deduced from this technique.

Corollary 3.18. *Let $1 \leq p < q < \infty$. and $r \geq 1$ such that $r \notin \{p,q\}$.*
Then ℓ_r does not coarse Lipschitz embed into $\ell_p \oplus \ell_q$.

Proof. When $r > q$, the argument is based on a midpoint technique. If $r < p$, it follows immediately from Corollary 3.14. So we assume now that $1 \leq p < r < q < \infty$ and $f = (g,h) : \ell_r \to \ell_p \oplus_\infty \ell_q$ is a coarse-Lipschitz embedding. Applying the midpoint technique to the coarse Lipschitz map g and then Theorem 3.13 to the map $h \circ \varphi$ with φ of the form $\varphi(\overline{n}) = u + \tau k^{-1/r}(e_{n_1} + .. + e_{n_k})$, where (e_n) is the canonical basis of ℓ_r and $\tau > 0$ is large enough, leads to a contradiction. $\square$

We can now state and prove the main result of [60].

Theorem 3.19. *If $1 < p_1 < .. < p_n < \infty$ are all different from 2, then $\ell_{p_1} \oplus ... \oplus \ell_{p_n}$ has a unique net structure.*

Proof. We will only sketch the proof for $\ell_p \oplus \ell_q$, with $1 < p < q < \infty$ such that $2 \notin \{p,q\}$. Assume that X is Banach space such that $X \overset{N}{\sim} \ell_p \oplus \ell_q$. The key point is again to show that X does not contain any isomorphic copy of ℓ_2. This follows clearly from the above corollary. To conclude the proof, we need to use a few deep linear results. The cases $1 < p < q < 2$ and $2 < p < q$, were actually settled in [47] for uniform homeomorphisms. So let us only explain the case $1 < p < 2 < q$. As in the proof of Theorem 3.17, we obtain that $X \subseteq_c L_p \oplus L_q$. Since $\ell_2 \subsetneq X$ and $q > 2$, a theorem of Johnson [45] insures that any bounded operator from X into L_q factors through ℓ_q and therefore that $X \subseteq_c L_p \oplus \ell_q$. Then we notice that L_p and ℓ_q are totally incomparable, which means that they have no isomorphic infinite dimensional subspaces. We can now use a theorem of Èdelšteĭn and Wojtaszczyk [26] to obtain that $X \simeq F \oplus G$, with $F \subseteq_c L_p$ and $G \subseteq_c \ell_q$. First it follows from [75] that G is isomorphic to ℓ_q or is finite dimensional. On the other hand, we know that $\ell_2 \subsetneq F$, and by the Johnson-Odell theorem [49] F is isomorphic to ℓ_p or finite dimensional. Summarizing, we have that X is isomorphic to ℓ_p, ℓ_q or $\ell_p \oplus \ell_q$. But we already know that ℓ_p and ℓ_q have unique net structure. Therefore X is isomorphic to $\ell_p \oplus \ell_q$. $\square$

Remark. Let $1 < p < \infty$ and $p \neq 2$. It is clear that the above proof cannot work for $\ell_p \oplus \ell_2$. As we already mentioned, it is unknown whether $\ell_p \oplus \ell_2$ has a unique uniform structure. The same question is open for L_p, $1 < p < \infty$ (see Problem 1).

However, let us indicate a few other results from [60] that can be derived from Theorem 3.13. The following theorem is related to a recent result of [41] stating that if $2 < p < \infty$, a subspace X of L_p which is not isomorphic to a subspace of $\ell_p \oplus \ell_2$ contains an isomorphic copy of $\ell_p(\ell_2)$.

Theorem 3.20. *Let $1 < p < \infty$ and $p \neq 2$.*
Then $\ell_p(\ell_2)$ and therefore L_p do not coarse Lipschitz embed into $\ell_p \oplus \ell_2$.

It follows from Ribe's counterexample that reflexivity is not preserved under uniform homeomorphisms. However, the following is proved in [8].

Theorem 3.21. *Let X be a Banach space and Y be a reflexive Banach space with an equivalent AUS norm. Assume that X coarse Lipschitz embeds into Y. Then X is reflexive.*

The proof has three ingredients: a result of Odell and Schlumprecht [73] asserting that Y can be renormed in such a way that $\overline{\rho}_Y \leq \overline{\rho}_{\ell_p}$, James' characterization of reflexivity and Theorem 3.13.

Remark. Note that Theorems 3.21 and 3.13 seem to take us very close to the solution of Problem 2. See also Corollary 4.6 below.

3.6. **Uniform asymptotic convexity.** Until very recently, there has been no corresponding result about the stability of convexity. The only thing that could be mentioned was the elementary use of the approximate midpoint principle that we already described. In a recent article [58], Nigel Kalton made a real breakthrough in this direction. Let us first state his general result.

Theorem 3.22. *Suppose X and Y are Banach spaces and that there is a coarse Lipschitz embedding of X into Y. Then there is a constant $C > 0$ such that for any spreading model $(e_k)_{k=1}^\infty$ of a weakly null normalized sequence in X (whose norm will be denoted $\| \ \|_S$), we have:*

$$\|e_1 + .. + e_k\|_{\overline{\delta}_Y} \leq C\|e_1 + .. + e_k\|_S.$$

We will not prove this in detail. We have chosen instead to show an intermediate result whose proof contains one of the key ingredients. Let us first describe the main idea. We wish to use the approximate midpoints principle. But, unless the space is very simple or concrete (like ℓ_p spaces), the approximate midpoint set is difficult to describe. Kalton's strategy in [58] was, in order to prove the desired inequality, to define an adapted norm on an Orlicz space associated with any given weakly null sequence in X. Then, by composition, he was able to reduce the question to the study of a coarse-Lipschitz map from that space to the Orlicz space associated with the modulus of asymptotic convexity of Y. Finally, as in ℓ_p, the approximate midpoints are not so difficult to study in an Orlicz space. Before stating the result, we need some preliminary notation.

Let ϕ be an Orlicz Lipschitz function. Then, $N_\phi(1,t) = 1 + \phi(|t|)$ (for $t \in \mathbb{R}$) extends to a norm N_ϕ^2 defined on $\mathbb{R}^2$. We now define inductively a norm on $\mathbb{R}^j$ by $N_\phi^j(x_1,..,x_j) = N_\phi^2(N_\phi^{j-1}(x_1,..,x_{j-1}),x_j)$ for $j \geq 2$. These norms are compatible and define a norm Λ_ϕ on c_{00}. One can check that $\frac{1}{2}\| \ \|_\phi \leq \| \ \|_{\Lambda_\phi} \leq \| \ \|_\phi$.

We also need to introduce the following quantities:

$$\hat{\delta}_X(t) = \inf_{x\in\partial B_X} \sup_E \inf_{y\in\partial B_E} \{\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1\}$$

where again E runs through all closed subspaces of X of finite codimension. The function $\hat{\delta}_X(t)/t$ is increasing and so $\hat{\delta}_X$ is equivalent to the convex function

$$\tilde{\delta}_X(t) = \int_0^t \frac{\hat{\delta}_X(s)}{s}\, ds.$$

Theorem 3.23. *Let $(\varepsilon_i)_{i=1}^\infty$ be a sequence of independent Rademacher variables on a probability space Ω. Assume that X and Y are Banach spaces and that there is a coarse Lipschitz embedding of X into Y. Then, there is a constant $c > 0$ such that given any (x_n) weakly null normalized θ-separated sequence in X and any integer k, there exist $n_1 < .. < n_k$ so that:*

$$c\theta\|e_1 + \cdots + e_k\|_{\tilde{\delta}_Y} \leq \|\varepsilon_1 x_{n_1} + \cdots + \varepsilon_k x_{n_k}\|_{L^1(\Omega,X)},$$

where $(e_k)_{k=1}^\infty$ is the canonical basis of c_{00}.

Proof. For $k \in \mathbb{N}$, let $\sigma_k = \sup\Big\{\|\sum_{j=1}^k \varepsilon_j x_{n_j}\|_{L^1(\Omega,X)},\ \ n_1 < n_2 < \cdots < n_k\Big\}$. For each k, define the Orlicz function F_k by

$$F_k(t) = \begin{cases} \sigma_k t/k, & 0 \leq t \leq 1/\sigma_k \\ t + 1/k - 1/\sigma_k, & 1/\sigma_k \leq t < \infty. \end{cases} \tag{3.1}$$

We introduce an operator $T : c_{00} \to L_1(\Omega;X)$ defined by $T(\xi) = \sum_{j=1}^\infty \xi_j\varepsilon_j \otimes x_j$. We omit the proof of the fact that for all $\xi \in c_{00}$: $\|T\xi\| \leq 2\|\xi\|_{F_k} \leq 4\|\xi\|_{\Lambda_{F_k}}$.
Assume now that $f : X \to Y$ is a map such that $f(0) = 0$ and

$$\|x-z\| - 1 \leq \|f(x) - f(z)\| \leq K\|x-z\| + 1, \qquad x, z \in X.$$

We then define $g : (c_{00}, \Lambda_{F_k}) \to L_1(\Omega;Y)$ by $g(\xi) = f \circ T\xi$. It can be easily checked that g is coarse-Lipschitz and that $Lip_\infty(g) > 0$. So we can apply the approximate midpoint principle to the map g and obtain that for τ as large as we wish, there exist $\eta,\ \zeta \in c_{00}$ with $\|\eta - \zeta\|_{\Lambda_{N_k}} = 2\tau$ such that

$$g(Mid(\eta,\zeta,1/k)) \subset Mid(g(\eta), g(\zeta), 2/k).$$

Let $m \in \mathbb{N}$ so that $\eta,\zeta \in span\,\{e_1,\ldots,e_{m-1}\}$. It follows from the definition of Λ_{N_k} that for $j \geq m$: $\xi + \tau\sigma_k^{-1}e_j \in \mathrm{Mid}(\eta,\zeta,1/k)$, where $\xi = \frac{1}{2}(\eta+\zeta)$.
Thus the functions

$$h_j = f \circ (\sum_{i=1}^{m-1} \xi_i\varepsilon_i \otimes x_i + \tau\sigma_k^{-1}\varepsilon_j \otimes x_j), \quad j \geq m$$

all belong to $Mid(g(\eta), g(\zeta), 2/k)$. Since the ε_i's are independent so do the functions

$$h'_j = f(\sum_{i=1}^{m-1} \xi_i \varepsilon_i \otimes x_i + \tau \sigma_k^{-1} \varepsilon_m \otimes x_j), \quad j \geq m.$$

Therefore, for all $j \geq m$:

$$\|g(\eta) - h'_j\| + \|g(\zeta) - h'_j\| - \|g(\eta) - g(\zeta)\| \leq 2k^{-1}\|g(\eta) - g(\zeta)\|.$$

We shall now use without proof the following simple property of $N = N^2_{\tilde{\delta}_Y}$: for any bounded t-separated sequence in Y and any $z \in Y$,

$$\liminf_{n\to\infty}(\|y - y_n\| + \|z - y_n\|) \geq N(\|y - z\|, t). \tag{3.2}$$

Note that for any $\omega \in \Omega$ we have

$$\|h'_i(\omega) - h'_j(\omega)\| \geq \theta\tau\sigma_k^{-1} - 1, \qquad i > j \geq m.$$

Hence, using (3.2), integrating and using Jensen's inequality we get

$$\liminf_{j\to\infty}(\|g(\eta) - h'_j\| + \|g(\zeta) - h'_j\|) \geq N(\|g(\eta) - g(\zeta)\|, \theta\tau\sigma_k^{-1} - 1).$$

Now $\|g(\eta) - g(\zeta)\| \leq 8K\tau + 1$ and $N(t, 1) - t$ is a decreasing function so

$$N_Y(8K\tau + 1, \theta\tau\sigma_k^{-1} - 1) - (8K\tau + 1) \leq \frac{2}{k}\|g(\eta) - g(\zeta)\| \leq 2(8K\tau + 1)k^{-1}.$$

Multiplying by $(8K\tau + 1)^{-1}$ and letting τ tend to $+\infty$ we obtain that

$$\tilde{\delta}(\frac{\theta}{16K\sigma_k}) \leq \frac{2}{k} \quad \text{and therefore} \quad \tilde{\delta}(\frac{\theta}{32K\sigma_k}) \leq \frac{1}{k}$$

or

$$\|e_1 + \cdots + e_k\|_{\tilde{\delta}_Y} \leq 32K\theta^{-1}\sigma_k.$$

□

We will end this section by stating two theorems proved by Kalton in [58]. Their proofs use, among many other ideas, the results we just explained on the stability of asymptotic uniform convexity under coarse-Lipschitz embeddings.

Theorem 3.24. (Kalton 2010) *Suppose $1 < p < \infty$. Then*
(i) If X is a Banach space which can be coarse Lipschitz embedded in ℓ_p, then X is linearly isomorphic to a closed subspace of ℓ_p.
(ii) If X is a Banach space which is net equivalent to a quotient of ℓ_p then X is linearly isomorphic to a quotient of ℓ_p.
(iii) If X can be coarse Lipschitz embedded into a quotient of ℓ_p then X is linearly isomorphic to a subspace of a quotient of ℓ_p.

Remark. On the other hand, Kalton constructed in [57] two subspaces (respectively quotients) of ℓ_p $(1 < p \neq 2 < \infty)$ which are uniformly homeomorphic but not linearly isomorphic.

Problem 4. It is not known whether a Banach space which is net equivalent to a subspace (respectively a quotient) of c_0 is linearly isomorphic to a subspace (respectively a quotient) of c_0.

As we have already seen, a Banach space Lipschitz-isomorphic to a subspace of c_0 is linearly isomorphic to a subspace of c_0 [36]. It is not known if the class of Banach spaces linearly isomorphic to a quotient of c_0 is stable under Lipschitz-isomorphisms. However, this question was almost solved by Dutrieux who proved in [23] that if a Banach space is Lipschitz-isomorphic to a quotient of c_0 and has a dual with the approximation property, then it is linearly isomorphic to a quotient of c_0.

Finally, let us point one last striking consequence of more general results from [58].

Theorem 3.25. (Kalton 2010) *Suppose* $1 < p, r < \infty$ *are such that* $p < \min(r, 2)$ *or* $p > \max(r, 2)$*, then the space* $(\sum_{n=1}^{\infty} \ell_r^n)_{\ell_p}$ *has a unique net structure.*

4. Embeddings of special graphs into Banach spaces

In this section we will study special metric graphs or trees that are of particular importance for the subject. More precisely we will study the Banach spaces in which they embed. This will allow us to characterize some linear classes of Banach spaces by a purely metric condition of the following type: given a metric space M (generally a graph), what are the Banach spaces X so that $M \overset{Lip}{\hookrightarrow} X$. Or, given a family $\mathcal{M}$ of metric spaces, what are the Banach spaces X for which there is a constant $C \geq 1$ so that for all M in $\mathcal{M}$, $M \overset{C}{\hookrightarrow} X$ (i.e. M Lipschitz embeds into X with distorsion at most C).

Most of the time these linear classes were already known to be stable under Lipschitz or coarse-Lipschitz embeddings, when such characterizations were proved. However, we will show one situation (see Corollary 4.6) where this process yields new results about such stabilities.

The section will be organized by the nature of the linear properties that can be characterized by such embedding conditions.

4.1. **Embeddings of special metrics spaces and local properties of Banach spaces.** We already know (see Theorem 3.3) that local properties of Banach spaces are preserved under coarse Lipschitz embeddings. This theorem gave birth to the "Ribe program" which aims at looking for metric invariants that characterize local properties of Banach spaces. The first occurence of the "Ribe program" is Bourgain's metric characterization of superreflexivity given in [15]. The metric invariant discovered by Bourgain is the collection of the hyperbolic dyadic trees of arbitrarily large height N. We denote $\Delta_0 = \{\emptyset\}$, the root of the tree. Let $\Omega_i = \{-1, 1\}^i$, $\Delta_N = \bigcup_{i=0}^{N} \Omega_i$ and $\Delta_\infty = \bigcup_{i=0}^{\infty} \Omega_i$. Then we equip Δ_∞, and by restriction every Δ_N, with the hyperbolic distance ρ, which is defined as follows. Let s and s' be two elements of Δ_∞ and let $u \in \Delta_\infty$ be their greatest common ancestor. We set

$$\rho(s, s') = |s| + |s'| - 2|u| = \rho(s, u) + \rho(s', u).$$

Bourgain's characterization is the following:

Theorem 4.1. (Bourgain 1986) *Let X be a Banach space. Then X is not superreflexive if and only if there exists a constant $C \geq 1$ such that for all $N \in \mathbb{N}$, $(\Delta_N, \rho) \overset{C}{\hookrightarrow} X$.*

Remark. It has been proven by Baudier in [7] that this is also equivalent to the metric embeddability of the infinite hyperbolic dyadic tree (Δ_∞, ρ). It should also be noted that in [15] and [7], the embedding constants are bounded above by a universal constant.

We also wish to mention that Johnson and Schechtman [51] recently characterized the super-reflexivity through the non embeddability of other graphs such as the "diamond graphs" or the Laakso graphs. We will only give an intuitive description of the diamond graphs. D_0 is made of two connected vertices (therefore at distance 1), that we shall call T (top) and B (bottom). D_1 is a diamond, therefore made of four vertices T, B, L (left) and R (right) and four edges : $[B, L]$, $[L, T]$, $[T, R]$ and $[R, B]$. Assume D_N is constructed, then D_{N+1} is obtained by replacing each edge of D_N by a diamond D_1. The distance on D_{N+1} is the path metric of this new discrete graph. The graph distance on a diamond D_N will be denoted by d. The result is the following.

Theorem 4.2. (Johnson, Schechtman 2009) *Let X be a Banach space. Then X is not super-reflexive if and only if there is a constant $C \geq 1$ such that for all $N \in \mathbb{N}$, $(D_N, d) \overset{C}{\hookrightarrow} X$.*

The metric characterization of the linear type of a Banach space has been initiated by Enflo in [28] and continued by Bourgain, Milman and Wolfson in [16]. Let us first describe a concrete result from [16]. For $1 \leq p \leq 2$ and $n \in \mathbb{N}$, H_p^n denotes the set $\{0,1\}^n$ equipped with the metric induced by the ℓ_p norm. The metric space H_1^n is called the Hamming cube. One of their results is the following.

Theorem 4.3. (Bourgain, Milman, Wolfson 1986) *Let X be a Banach space and $1 \leq p \leq 2$. Define p_X to be the supremum of all r's such that X is of linear type r. Then, the following assertions are equivalent.*

(i) $p_X \leq p$.

(ii) There is a constant $C \geq 1$ such that for all $n \in \mathbb{N}$, $H_p^n \overset{C}{\hookrightarrow} X$.

In particular, X is of trivial type if and only if $H_1^n \overset{C}{\hookrightarrow} X$, for all $n \in \mathbb{N}$ and for some universal constant $C \geq 1$.

The fundamental problem of defining a notion of type for metric spaces is behind this result. Of course we expect such a notion to coincide with the linear type for Banach spaces and to be stable under reasonable non linear embedddings. This program was achieved with the successive definitions of the Enflo type [28], the Bourgain-Milman-Wolfson type [16] and finally the scaled Enflo type introduced by Mendel and Naor in [68]. An even more difficult task was to define the right notion of metric cotype. This was achieved by Mendel and Naor in [69]. We will not address this subject in this survey, but we strongly advise the interested reader to study these fundamental papers.

Let us also describe a simpler metric characterization of the Banach spaces without (linear) cotype. First let us recall that a metric space (M,d) is called *locally finite* if all its balls of finite radius are finite. It is of *bounded geometry* if for any $r>0$ there exists $C(r)\in\mathbb{N}$ such that the cardinal of any ball of radius r is less than $C(r)$. We will now construct a particular metric space with bounded geometry. For $k,n\in\mathbb{N}$, denote

$$M_{n,k}=knB_{\ell_\infty^n}\cap n\mathbb{Z}^n.$$

Let us enumerate the $M_{n,k}$'s: $M_1,...,M_i,..$ with $M_i=M_{n_i,k_i}$ in such a way that $\mathrm{diam}(M_i)$ is non decreasing. Note that $\lim_i diam(M_i)=+\infty$. Then let M_0 be the disjoint union of the M_i's ($i\geq 1$) and define on M_0 the following distance:
If $x,y\in M_i$, $d(x,y)=\|x-y\|_\infty$, where $\|\ \|_\infty$ is the $\ell_\infty^{n_i}$ norm.
If $x\in M_i$ et $y\in M_j$, with $i<j$, set $d(x,y)=F(j)$, where F is built so that F is increasing and

$$\forall j\ \ \forall i\leq j\ \ F(j)\geq\frac{1}{2}\,diam(M_i).$$

Note that $\lim_i F(i)=+\infty$. We leave it to the reader to check that (M_0,d) is a metric space with bounded geometry. We can now state the following.

Theorem 4.4. *Let X be a Banach space. The following assertions are equivalent.*

(i) X has a trivial cotype.

(ii) X contains uniformly and linearly the ℓ_∞^n's.

(iii) There exists $C\geq 1$ such that for every locally finite metric space M, $M\overset{C}{\hookrightarrow}X$.

(iv) There exists $C\geq 1$ such that for every metric space with bounded geometry M, $M\overset{C}{\hookrightarrow}X$.

(v) There exists $C\geq 1$ such that $M_0\overset{C}{\hookrightarrow}X$.

(vi) There exists $C\geq 1$ such that for every finite metric space M, $M\overset{C}{\hookrightarrow}X$.

Proof. The equivalence between (i) and (ii) is part of classical results by Maurey and Pisier [64].

(ii)⇒ (iii) is due to Baudier and the second author [9].

(iii)⇒ (iv) and (iv)⇒ (v) are trivial.

For any $k,n\in\mathbb{N}$, the space M_0 contains the space $M_{n,k}=knB_{\ell_\infty^n}\cap n\mathbb{Z}^n$ which is isometric to the $\frac{1}{k}$-net of $B_{\ell_\infty^n}$: $B_{\ell_\infty^n}\cap\frac{1}{k}\mathbb{Z}^n$. But, after rescaling, any finite metric space is isometric to a subset of $B_{\ell_\infty^n}$, for some $n\in\mathbb{N}$. Thus, for any finite metric space M and any $\varepsilon>0$, there exist $k,n\in\mathbb{N}$ so that M is $(1+\varepsilon)$-equivalent to a subset of $M_{k,n}$. The implication (v)⇒ (vi) is now clear.

The proof of (vi)⇒ (ii) relies on an argument due to Schechtman [82]. So assume that (vi) is satisfied and let us fix $n\in\mathbb{N}$. Then for any $k\in\mathbb{N}$, there exists a map $f_k:(\frac{1}{k}\mathbb{Z}^n\cap B_{\ell_\infty^n},\|\ \|_\infty)\to X$ such that $f_k(0)=0$ and

$$\forall x,y\in\frac{1}{k}\mathbb{Z}^n\cap B_{\ell_\infty^n}\quad \|x-y\|_\infty\leq\|f_k(x)-f_k(y)\|\leq K\|x-y\|_\infty.$$

Then we can define a map $\lambda_k : B_{\ell_\infty^n} \to \frac{1}{k}\mathbb{Z}^n \cap B_{\ell_\infty^n}$ such that for all $x \in B_{\ell_\infty^n}$: $\|\lambda_k(x) - x\|_\infty = d(x, \frac{1}{k}\mathbb{Z}^n \cap B_{\ell_\infty^n})$. We now set $\varphi_k = f_k \circ \lambda_k$.
Let $\mathcal{U}$ be a non trivial ultrafilter. We define $\varphi : B_{\ell_\infty^n} \to X_{\mathcal{U}} \subseteq X_{\mathcal{U}}^{**}$ by $\varphi(x) = (\varphi_k(x))_{\mathcal{U}}$. It is easy to check that φ is a Lipschitz embedding. Then it follows from results by Heinrich and Mankiewicz on weak*-Gâteaux differentiabilty of Lipschitz maps [42] that ℓ_∞^n is K-isomorphic to a linear subspace of $X_{\mathcal{U}}^{**}$. Finally, using the local reflexivity principle and properties of the ultra-product, we get that ℓ_∞^n is $(K+1)$-isomorphic to a linear subspace of X.

□

4.2. **Embeddings of special graphs and asymptotic structure of Banach spaces.** We will start this section by considering the countably branching hyperbolic trees. For a positive integer N, $T_N = \bigcup_{i=0}^N \mathbb{N}^i$, where $\mathbb{N}^0 := \{\emptyset\}$. Then $T_\infty = \bigcup_{N=1}^\infty T_N$ is the set of all finite sequences of positive integers. The hyperbolic distance ρ is defined on T_∞ as follows. Let s and s' be two elements of T_∞ and let $u \in T_\infty$ be their greatest common ancestor. We set

$$\rho(s,s') = |s| + |s'| - 2|u| = \rho(s,u) + \rho(s',u).$$

The following result, that appeared in [8], is an asymptotic analogue of Bourgain's characterization of super-reflexivity given in Theorem 4.1 above.

Theorem 4.5. *Let X be a reflexive Banach space. The following assertions are equivalent.*

(i) There exists $C \geq 1$ such that $T_\infty \overset{C}{\hookrightarrow} X$.

(ii) There exists $C \geq 1$ such that for any N in $\mathbb{N}$, $T_N \overset{C}{\hookrightarrow} X$.

(iii) X does not admit any equivalent asymptotically uniformly smooth norm <u>or</u> X does not admit any equivalent asymptotically uniformly convex norm.

We will only mention one application of this result.

Corollary 4.6. *The class of all reflexive Banach spaces that admit both an equivalent AUS norm and an equivalent AUC norm is stable under coarse Lipschitz embeddings.*

Proof. Assume that X coarse Lipschitz embeds in a space Y which is reflexive, AUS renormable and AUC renormable. First, it follows from Theorem 3.21 that X is reflexive. Now the conclusion is easily derived from Theorem 4.5. □

Note that this class coincide with the class of reflexive spaces X such that the Szlenk indices of X and X^* are both equal to the first infinite ordinal ω (see [37]).

Problem 5. We do not know if the class of all Banach spaces that are both AUS renormable and AUC renormable is stable under coarse Lipschitz embeddings, net equivalences or uniform homeomorphisms.

Problem 6. We now present a variant of Problem 2. As we already indicated, we do not know if the class of reflexive and AUS renormable Banach spaces is stable under coarse Lipschitz embeddings. The important results by Kalton and Randrianarivony on the stability of the asymptotic uniform smoothness are based on the use of particular metric graphs, namely the graphs $G_k(\mathbb{N})$ equipped with the distance:

$$\forall \overline{n},\overline{m} \in G_k(\mathbb{N}),\quad d(\overline{n},\overline{m}) = |\{j,\ n_j \neq m_j\}|.$$

It seems interesting to try to characterize the Banach spaces X such that there exists a constant $C \geq 1$ for which $G_k(\mathbb{N}) \overset{C}{\hookrightarrow} X$, for all $k \in \mathbb{N}$. In particular, one may ask whether a reflexive Banach space which is not AUS renormable always contains the $G_k(\mathbb{N})$'s with uniform distortion (the converse implication is a consequence of Kalton and Randrianarivony's work). A positive answer would solve Problem 2.

4.3. **Interlaced Kalton's graphs.** Very little is known about the coarse embeddings of metric spaces into Banach spaces and about coarse embeddings between Banach spaces (see Definition 3.1. for coarse embeddings). For quite some time it was not even known if a reflexive Banach space could be universal for separable metric spaces and coarse embeddings. This was solved negatively by Kalton in [53] who showed the following.

Theorem 4.7. (Kalton 2007) *Let X be a separable Banach space. Assume that c_0 coarsely embeds into X. Then one of the iterated duals of X has to be non separable. In particular, X cannot be reflexive.*

The idea of the proof is to consider a new graph metric δ on $G_k(\mathbb{M})$, for $\mathbb{M}$ infinite subset of $\mathbb{N}$. We will say that $\overline{n} \neq \overline{m} \in G_k(\mathbb{M})$ are adjacent (or $\delta(\overline{n},\overline{m}) = 1$) if they interlace or more precisely if

$$m_1 \leq n_1 \leq .. \leq m_k \leq n_k \ \text{ or } \ n_1 \leq m_1 \leq .. \leq n_k \leq m_k.$$

For simplicity we will only show that X cannot be reflexive. So let us assume that X is a reflexive Banach space and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = w - \lim_{n_k \in \mathcal{U}} f(n_1,..,n_{k-1},n_k).$$

Note that for $1 \leq i \leq k$, $\partial^i f$ is a bounded map from $G_{k-i}(\mathbb{N})$ into X and that $\partial^k f$ is an element of X. Let us first state without proof a series of basic lemmas about the operation ∂.

Lemma 4.8. *Let $h : G_k(\mathbb{N}) \to \mathbb{R}$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that*

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad |h(\overline{n}) - \partial^k h| < \varepsilon.$$

Lemma 4.9. *Let $f : G_k(\mathbb{N}) \to X$ and $g : G_k(\mathbb{M}) \to X^*$ be two bounded maps. Define $f \otimes g : G_{2k}(\mathbb{N}) \to \mathbb{R}$ by*

$$(f \otimes g)(n_1,..,n_{2k}) = \langle f(n_2,n_4,..,n_{2k}), g(n_1,..,n_{2k-1})\rangle.$$

Then $\partial^2(f \otimes g) = \partial f \otimes \partial g$.

Lemma 4.10. *Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that*

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f *is the modulus of continuity of* f*.*

Lemma 4.11. *Let* $\varepsilon > 0$*,* X *be a separable reflexive Banach space and* I *be an uncountable set. Assume that for each* $i \in I$*,* $f_i : G_k(\mathbb{N}) \to X$ *is a bounded map. Then there exist* $i \neq j \in I$ *and an infinite subset* $\mathbb{M}$ *of* $\mathbb{N}$ *such that*

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|f_i(\overline{n}) - f_j(\overline{n})\| \leq \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon.$$

We are now ready for the proof of the theorem. As we will see, the proof relies on the fact that c_0 contains uncountably many isometric copies of the $G_k(\mathbb{N})$'s with too many points far away from each other (which will be in contradiction with Lemma 4.11).

Proof of Theorem 4.7. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_{k=1}^{\infty}$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \quad s_A(n) = \sum_{k \leq n,\ k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \quad f_A(\overline{n}) = \sum_{i=1}^{k} s_A(n_i).$$

Then the $h \circ f_A$'s form an uncountable family of bounded maps from $G_k(\mathbb{N})$ to X. It therefore follows from Lemma 4.11 that there are two distinct infinite subsets A and B of $\mathbb{N}$ and another infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that:

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|h \circ f_A(\overline{n}) - h \circ f_B(\overline{n})\| \leq \omega_{h \circ f_A}(1) + \omega_{h \circ f_B}(1) + 1 \leq 2\omega_h(1) + 1.$$

But, since $A \neq B$, there is $\overline{n} \in G_k(\mathbb{M})$ with $\|f_A(\overline{n}) - f_B(\overline{n})\| = k$. By taking arbitrarily large values of k we deduce that h cannot be a coarse embedding. $\square$

Remarks.

(1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

(2) It is now easy to adapt this proof in order to obtain the stronger result stated in Theorem 4.7. Indeed, one just has to change the definition of the operator ∂ as follows. If $f : G_k(\mathbb{N}) \to X$ is bounded, define $\partial f : G_{k-1}(\mathbb{N}) \to X^{**}$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = w^* - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

We leave it to the reader to rewrite the argument.

(3) On the other hand, Kalton proved in [52] that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that X does not contain any subspace linearly isomorphic to c_0.

(4) We will see in the next section that Kalton recently used a similar operation ∂ and the same graph distance on $G_k(\omega_1)$, where ω_1 is the first uncountable ordinal (see [55]) in order to study uniform embeddings into ℓ_∞.

Problem 7. In view of this result, the metric graphs $(G_k(\mathbb{N}), \delta)$ are clearly of special importance. It is a natural question to characterize the Banach spaces containing the spaces $(G_k(\mathbb{N}), \delta)$ with uniformly bounded distortion.

In [53] Kalton pushed the idea behind the proof of Theorem 4.7 much further and introduced the following abstract notions in order to study the coarse or uniform embeddings into reflexive Banach spaces.

Let (M, d) be a metric space, $\varepsilon > 0$ and $\eta \geq 0$. We say that M has *property* $\mathcal{Q}(\varepsilon, \eta)$ if for every $k \in \mathbb{N}$ and every map $f : (G_k(\mathbb{N}), \delta) \to (M, d)$ with $\omega_f(1) \leq \eta$ there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that:

$$d(f(\sigma), f(\tau)) \leq \varepsilon \quad \sigma < \tau, \quad \sigma, \tau \in G_k(\mathbb{M}).$$

Then $\Delta_M(\varepsilon)$ is the supremum of all $\eta \geq 0$ so that M has *property* $\mathcal{Q}(\varepsilon, \eta)$ and Kalton proves the following general statement.

Theorem 4.12. *Let M be a metric space and X be a reflexive Banach space.*
(i) If M embeds uniformly into X, then $\Delta_M(\varepsilon) > 0$, for all $\varepsilon > 0$.
(ii) If M embeds coarsely into X, then $\lim_{\varepsilon \to +\infty} \Delta_M(\varepsilon) = +\infty$.

Let us now turn to the case when our metric space is a Banach space that we shall denote E. Then it easy to see that the function Δ_E is linear. We denote $\mathcal{Q}_E$ the constant such that for all $\varepsilon > 0$, $\Delta_E(\varepsilon) = \mathcal{Q}_E \varepsilon$. Finally, we say that E has the *$\mathcal{Q}$-property* if $\mathcal{Q}_E > 0$. It follows from Lemma 4.10 that a reflexive Banach space has the $\mathcal{Q}$-property. Thus we have:

Corollary 4.13. *If a Banach space E fails the $\mathcal{Q}$-property, then E does not coarsely embed into a reflexive Banach space and B_E does not uniformly embed into a reflexive Banach space.*

The fact that c_0 fails the $\mathcal{Q}$-property follows from Theorem 4.7 but it is actually an ingredient of its proof. Then Kalton continues his study of the links between reflexivity and the $\mathcal{Q}$-property. Let us mention without proof a few of the many interesting results obtained in [53].

(1) A non reflexive Banach space with the alternating Banach-Saks property (in particular with a non trivial type) fails the $\mathcal{Q}$-property.

(2) The James space J and its dual fail the $\mathcal{Q}$-property.

(3) However, there exists a quasi-reflexive but non reflexive Banach space with the $\mathcal{Q}$-property.

Problem 8. Is there a converse to Corollary 4.13? More precisely: if E is a separable Banach space with the $\mathcal{Q}$-property, does B_E uniformly embed into a reflexive Banach space or does E coarsely embed into a reflexive Banach space? The answer is unknown for the space constructed in the above statement (3).

5. Nonseparable spaces

We collect here a few recent results obtained by Nigel Kalton on nonseparable Banach spaces together with some related open problems. All the results presented in this section are taken from Kalton's paper [55]. They mainly concern embeddings of nonseparable Banach spaces into ℓ_∞. We start with a positive result.

Theorem 5.1. *If X has an unconditional basis and is of density character at most c (the cardinal of the continuum), then it is Lipschitz embeddable into ℓ_∞.*

Sketch of the main ideas in the proof. Assume that the basis is $1-$ unconditional and that it is indexed by the set $\mathbb{R}$ of real numbers. Denote by $(e_t^*)_{t\in\mathbb{R}}$ the biorthogonal functionals of the basis. If $x \in X$, we write $x(t) = e_t^*(x)$. Suppose that $a, b, c \in \mathbb{Q}^n$. We write typically, $a = (a_1, a_2, \ldots, a_n)$ and denote by $-a = (-a_1, -a_2, \ldots, -a_n)$. Define then a subset $U(a, b, c) \subset \mathbb{R}^n$ by $(t_1, t_2, \ldots, t_n) \in U(a, b, c)$ if $b_j < t_j < c_j$ for $j = 1, 2, \ldots, n$, $t_1 < t_2 < \ldots, t_n$ and

$$\|\sum_{j=1}^n a_j e_{t_j}^*\|_{X^*} \le 1.$$

For $t \in \mathbb{R}$ write $t_+ = \max(t, 0)$ and define $f_{(a,b,c)} : X \to \mathbb{R}$ by $f_{(a,b,c)}$ is identically 0 if $U(a, b, c)$ is empty and otherwise

$$f_{(a,b,c)}(x) = \sup\Big\{\sum_{j=1}^n (a_j x(t_j))_+,\ (t_1, t_2, \ldots, t_n) \in U(a, b, c)\Big\}.$$

Finally define the map

$$F(x) = \big(f_{(a,b,c)}(x)\big)_{(a,b,c)\in\bigcup_n Q^n}$$

It can then be shown that F is a Lipschitz embedding of X into ℓ_∞. $\square$

Problem 9. Let X be reflexive of density $\le c$. Is X Lipschitz embeddable in ℓ_∞?

We now proceed with other spaces of density $\le c$. Let I be a set of cardinality c. It is easy to show, using almost disjoint families, that the space $c_0(I)$ is isometric to a subspace of ℓ_∞/c_0, and it follows that there is no linear continuous injective map from ℓ_∞/c_0 into ℓ_∞. But by Theorem 5.1 above, $c_0(I)$ Lipschitz embeds into ℓ_∞. This was shown much earlier [2] using the space JL_∞, and it can also be seen by applying Theorem VI. 8. 9 in [22] to any separable compact space K with weight c and with some finite derivative empty. Hence the linear argument does not extend to the non-linear case. However, Kalton showed:

Theorem 5.2. *$C[0, \omega_1]$ or ℓ_∞/c_0 cannot be uniformly embedded into ℓ_∞.*

Before discussing the main ideas in the proof of this result, we need some preparation. For $n \ge 0$, let $\Omega_n = \Omega_1^{[n]}$ be the collection of all $n-$subsets of $\Omega_1 = [1, \omega_1)$. For $n = 0$, $\Omega_0 = \{\emptyset\}$. We write a typical element of Ω_n in the form $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. If $n \ge 1$, and $A \subset \Omega_n$, we define $\partial A \subset \omega_1^{[n-1]}$ by $\{\alpha_1, \ldots, \alpha_{n-1}\} \in \partial A$ if and only if $\{\beta : \{\alpha_1, \ldots, \alpha_{n-1}, \beta\} \in A\}$ is uncountable. If $n = 1$, this means that $\emptyset \in \partial A$ if and only if A is uncountable.

We will say that $A \subset \Omega_n$ is *large* if $\emptyset \in \partial^n A$. Otherwise A is *small*. We will say that $A \subset \Omega_n$ is *very large* if its complement is small. Then one can show the following Ramsey type result.

Lemma 5.3. *If A is a very large subset in Ω_n, then there is an uncountable set $\Theta \subset \Omega_1$ such that $\Theta^{[n]} \subset A$.*

We will now make Ω_n into a graph by declaring $\alpha \neq \beta$ to be adjacent if they interlace, namely if

$$\alpha_1 \leq \beta_1 \leq \cdots \leq \alpha_n \leq \beta_n \quad \text{or} \quad \beta_1 \leq \alpha_1 \leq \cdots \leq \beta_n \leq \alpha_n,$$

and we define d to be the least path metric on Ω_n, which then becomes a metric space.

We write $\alpha < \beta$ if $\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots \beta_n$. If $\alpha < \beta$, then $d(\alpha, \beta) = n$ so Ω_n has diameter n.

The next lemma relies on basic properties of the ordered set Ω_1.

Lemma 5.4. *(i) If A and B are large sets in Ω_n, then there exist $\alpha \in A$ and $\beta \in B$ so that α and β interlace.*

(ii) If f is a Lipschitz map from Ω_n into $\mathbb{R}$, with Lipschitz constant L, then there is $\xi \in \mathbb{R}$ so that $\{\alpha,\ |f(\alpha) - \xi| > L/2\}$ is small.

It yields:

Proposition 5.5. *If f is a Lipschitz map from $(\Omega_n, d) \to \ell_\infty$ with Lipschitz constant L, then there is $\xi \in \ell_\infty$ and an uncountable subset Θ of Ω_1 so that*

$$\|f(\alpha) - \xi\| \leq L/2, \qquad \alpha \in \Theta^{[n]}.$$

Sketch of the proof of Theorem 5.2. For the case of $C[0, \omega_1]$, let $(x_\mu)_{\mu \leq \omega_1}$ be defined by $x_\mu = \chi_{[0,\mu]}$. Assume that $X = C[0, \omega_1]$ uniformly homeomorphically embeds into ℓ_∞ and let $f : B_X \to \ell_\infty$ be a uniformly homeomorphic embedding.
Under the notation above, for each n, consider the map $f_n : \Omega_n \to \ell_\infty$ given by

$$f_n(\alpha) = f(\frac{1}{n}\sum_{j=1}^{n} x_{\alpha_j})$$

If α and β interlace, then by a telescopic argument

$$\|\frac{1}{n}\sum_{j=1}^{n}(x_{\beta_j} - x_{\alpha_j})\| \leq \frac{2}{n}$$

and from the definition of the distance in Ω_n we get that f_n has Lipschitz constant $\psi_f(\frac{2}{n})$, where ψ_f is the modulus of uniform continuity of f.

By Proposition 5.5, we may pick an uncountable subset Θ_n of Ω_1 so that

$$\|f_n(\alpha) - f_n(\beta)\| \leq \psi_f(\frac{2}{n}), \qquad \alpha, \beta \in \Theta_n.$$

Hence

$$\|\frac{2}{n}\sum_{j=1}^{n} x_{\beta_j} - \frac{2}{n}\sum_{j=1}^{n} x_{\alpha_j}\| \leq \psi_g(\psi_f(\frac{2}{n})), \qquad \alpha, \beta \in \Theta_n$$

Pick now $\alpha_1 < \alpha_2 < \cdots < \alpha_n \in \Theta_n$. If $\nu > \mu > \alpha_n$, we can find in $\beta_1, ..\beta_n$ in Θ_n so that $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \nu$. Then

$$\|x_\nu - x_\mu\| \leq \|\frac{1}{n}\sum_{j=1}^{n} x_{\beta_j} - \frac{1}{n}\sum_{j=1}^{n} x_{\alpha_j}\| \leq \psi_g(\psi_f(\frac{2}{n}))$$

Thus

$$\theta(\mu) := \sup_{\sigma > \mu} \|x_\sigma - x_\mu\| \leq \psi_g(\psi_f(\frac{2}{n})), \qquad \mu > \alpha_n.$$

Applying this for every n, since $\lim_{n\to\infty} \psi_g(\psi_f(\frac{2}{n})) = 0$, we get $\theta(\mu) = 0$ eventually, which is not true.

For ℓ_∞/c_0, Theorem 5.2 follows from the case of $C[0,\omega_1]$ and from the result that $C[0,\omega_1]$ is linearly isometric to a subspace of ℓ_∞/c_0 [74].

□

Note that Theorem 5.2 implies that there is no quasi-additive Lipschitz projection from ℓ_∞ onto c_0 (see [12]). As another application of Theorem 5.2, Kalton obtains the following fundamental example.

Theorem 5.6. *There is a (nonseparable) Banach space Z that is not a uniform retract of its second dual.*

Before starting on discussing the ideas in the proof, let us include the following useful lemma of independent interest.

Lemma 5.7. *Let X be a Banach space and let $Q : Y \to X$ be a quotient mapping. In order that there is a uniformly continuous selection $f : B_X \to Y$ of the quotient mapping Q, it is sufficient that for some $0 < \lambda < 1$ there is a uniformly continuous map $\phi : S_X \to Y$ with $\|Q(\phi(x)) - x\| \leq \lambda$ for $x \in S_X$.*

Proof. We extend ϕ to B_X to be positively homogeneous and ϕ remains uniformly continuous. Define $g(x) = x - Q(\phi(x))$, so that g is also positively homogeneous. Then $\|g(x)\| \leq \lambda\|x\|$ and so $\|g^n(x)\| \leq \lambda^n\|x\|^n$ for $x \in B_X$. Let $g^0(x) = x$. Let

$$f(x) = \sum_{n=0}^{\infty} \phi(g^n(x)).$$

The series converges uniformly in $x \in B_X$ and so f is uniformly continuous. Furthermore,

$$Qf(x) = \sum_{n=0}^{\infty}(g^n(x) - g^{n+1}(x)) = x.$$

□

Sketch of the proof of Theorem 5.6. The space Z that we shall consider was constructed by Benyamini in [11]. Consider the quotient map $Q : \ell_\infty \to \ell_\infty/c_0$. For each n, pick a maximal set D_n in the interior of B_{ℓ_∞/c_0} so that $\|x - x'\| \geq \frac{1}{n}$

for $x, x' \in D_n$ and $x \neq x'$. Then for each n define a map $h_n : D_n \to B_{\ell_\infty}$ with $Qh_n(x) = x$ for $x \in D_n$ and denote by Y_n the space ℓ_∞ with the equivalent norm

$$\|y\|_{Y_n} = \max\big\{\frac{1}{n}\|y\|_{\ell_\infty}, \|Qy\|_{\ell_\infty/c_0}\big\}.$$

Note that Q remains a quotient map for the usual norm on ℓ_∞/c_0.

Let $Z = (\sum Y_n)_{c_0}$ and assume that there is a uniformly continuous retraction of $B_{Z^{**}}$ onto B_Z. Then it follows that there is a sequence of retractions $g_n : Y^{**} \to Y_n$ which is equi–uniformly continuous, i.e their moduli of uniform continuity satisfy

$$\psi_{g_n}(t) \leq \psi(t) \quad 0 < t \leq 2, \quad \text{with} \quad \lim_{t\to 0} \psi(t) = 0.$$

Consider the map $h_n : D_n \to Y_n$. If $x \neq x' \in D_n$, then

$$\|h_n(x) - h_n(x')\| \leq \max\big\{\frac{2}{n}, \|x - x'\|\big\} \leq 2\|x - x'\|.$$

Since $B_{Y_n^{**}}$ is a 1-absolute Lipschitz retract, there is an extension $f_n : B_{\ell_\infty/c_0} \to B_{Y_n^{**}}$ of h_n with $\mathrm{Lip}(f_n) \leq 2$. Now, if $x \in B_{\ell_\infty/c_0}$, there is $x' \in D_n$ with $\|x - x'\| < \frac{2}{n}$. Thus

$$\|g_n(f_n(x)) - g_n(f_n(x'))\| \leq \psi(\frac{4}{n})$$

and hence

$$\|Q(g_n(f_n(x)) - x\| \leq \psi(\frac{4}{n}) + \frac{2}{n}.$$

Then for n large enough, we have

$$\psi(\frac{4}{n}) + \frac{2}{n} < 1.$$

By Lemma 5.7, this means that there is a uniformly continuous selection of the quotient map $Q : B_{\ell_\infty/c_0} \to Y_n$. Thus B_{ℓ_∞/c_0} uniformly embeds into Y_n which is isomorphic to ℓ_∞ and this is a contradiction with Theorem 5.2.

□

Problem 10. Does there exist a separable, or at least a weakly compactly generated (WCG) Banach space X which is not a Lipschitz retract of its bidual?

We note that if X is Lipschitz embedded in ℓ_∞, then X admits a countable separating family of Lipschitz real valued functions on X. Let us also recall that Bourgain proved in [14] that ℓ_∞/c_0 has no equivalent strictly convex norm. This somehow suggests the following problem.

Problem 11. If X is Lipschitz embeddable into ℓ_∞, does X have an equivalent strictly convex norm?

We will finish this section by discussing a few more examples of nonisomorphic nonseparable spaces that have unique Lipschitz structure. We will discuss one way of getting many examples by using the so called pull-back construction in the theory of exact sequences (see [19]). First, we need some preparation.

A diagram $0 \to Y \to X \to Z \to 0$ of Banach spaces and operators is said to be an exact sequence if the kernel of each arrow coincides with the image of the preceding one. Hence, in the diagram above, the second arrow denotes an injection and the third one is a quotient map. This means, by the open mapping theorem, that Y is isomorphic to a closed subspace of X and that the corresponding quotient is isomorphic to Z. We say that X is a twisted sum of Y and Z or an extension of Y by Z.

We say that the exact sequence *splits* if the second arrow i admits a linear retraction (i.e. an arrow r from X into Y so that $ri = Id_Y$) or equivalently if the third arrow q admits a linear section, or selection, i.e. if there is an arrow s from Z into X such that $qs = Id_Z$. This implies that then X is isomorphic to the direct sum $Y \oplus Z$.

Let $A : U \to Z$ and $B : V \to Z$ be two operators. The *pull-back* of $\{A, B\}$ is the space $PB = \{(u, v) : Au = Bv\} \subset U \times V$ considered with the canonical projections of $U \times V$ onto U and V respectively.

If $0 \to Y \to X \to Z \to 0$ is an exact sequence with quotient map q and $T : V \to Z$ is an operator and PB denotes the pull-back of of the couple $\{q, T\}$, then the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & X \arrow[r] & Z \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r] & PB \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, "T"] & 0 \end{tikzcd}$$

is commutative with exact rows. It follows that the pull-back sequence splits if and only if the operator T can be lifted to X, i.e. there exists an operator $\tau : V \to X$ such that $q\tau = T$.

Kalton's Lemma [55] says that if the quotient map in the first row admits a Lipschitz section then so does the quotient map in the second row.

We now consider the following pull-back diagram.

$$\begin{tikzcd} 0 \arrow[r] & c_0 \arrow[r] & JL_\infty \arrow[r] & c_0(I) \arrow[r] & 0 \\ 0 \arrow[r] & c_0 \arrow[r] \arrow[u, "Ic_0"] & JL_2 \arrow[r] \arrow[u, "\hat{T}"] & \ell_2(I) \arrow[r] \arrow[u, "T"] & 0 \\ 0 \arrow[r] & c_0 \arrow[r] \arrow[u, "Ic_0"] & CC \arrow[r] \arrow[u, "\hat{S}"] & \ell_\infty \arrow[r] \arrow[u, "S"] & 0 \end{tikzcd}$$

In this diagram, the operator T is the inclusion map and the operator S is the Rosenthal quotient map [81]. Therefore we get that the space JL_2 of Johnson and Lindenstrauss is an example of a non WCG space that it is Lipschitz isomorphic to $\ell_2(I) \oplus c_0$ and Lipschitz embeds into ℓ_∞. However, it is known that it does not linearly isomorphically embeds into ℓ_∞ (see [46]).

The space CC in the third row is then an example of a space that is Lipschitz homeomorphic to $\ell_\infty \oplus c_0$, but is not linearly isomorphic to it. Note also that CC is linearly isomorphic to a subspace of ℓ_∞ because JL_∞ linearly embeds into ℓ_∞ and CC is a subspace of $JL_\infty \oplus \ell_\infty$ (see [18] for details).

Similarly, Kalton obtained for instance that any nonseparable WCG space that contains an isomorphic copy of c_0 fails to have unique Lipschitz structure.

Problem 12. Does every reflexive (superreflexive) space have unique Lipschitz structure?

Problem 13. Does ℓ_∞ have unique Lipschitz structure?

Remark: It turns out that the undecidability of the continuum hypothesis (and of related statements) casts a shadow on the Lipschitz classification of non-separable spaces. Let I be a set of cardinality c. We assume that the space $c_0(I)$ is Lipschitz-isomorphic to a WCG Banach space X. Does it follow that X is linearly isomorphic to $c_0(I)$? It follows from [36] that the answer is positive if $c < \aleph_{\omega_0}$, where $\aleph_{\omega_0}$ denotes the ω_0-th cardinal. On the other hand, it follows from [10] and [65] that the answer is negative if $c > \aleph_{\omega_0}$ (see also [6] for a related result on the failure of a non-separable Sobczyk theorem at the density $\aleph_{\omega_0}$). Since (ZFC) does not decide if c is strictly less or strictly more than $\aleph_{\omega_0}$, the above question is undecidable in (ZFC).

6. Coarse embeddings into Banach spaces and geometric group theory

Two topological spaces M and N are homotopically equivalent if there exist continuous maps $f : M \to N$ and $g : N \to M$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identity map on the space on which they operate. For instance, any topological *vector* space is homotopically equivalent to a point (in other words, is contractible).

We consider the case where M and N are real compact manifolds. In order to understand their geometry, it is of course important to find quantities which are invariant under homotopy equivalence. A basic example is the signature of the manifold, which can be defined as follows when the dimension $n = 4k$ is divisible by 4. If d_j denotes the exterior derivative acting on the differential forms of degree j, then $F_k = Ker(d_{2k})$ is the vector space of closed forms of degree $2k$ and $E_k = Im(d_{2k-1})$ is its subspace of exact forms of degree $2k$. If ω_1 and ω_2 are two forms in F_k and if we define Q by

$$Q(\omega_1, \omega_2) = \int_M \omega_1 \wedge \omega_2$$

then Q is a bilinear symmetric form, and an easy computation shows that the value of Q depends only upon the classes of ω_1 and ω_2 in the finite-dimensional quotient space $H_k(M) = F_k/E_k$. Therefore Q defines a quadratic form on $H_k(M)$, and its signature is called the signature of the manifold M. This signature is invariant under homotopy equivalence. Actually, a theorem of Novikov asserts (for simply connected manifolds) that the signature is the only homotopy invariant which can be computed in terms of quantities called Pontryagin polynomials.

We now recall the basics of geometric group theory. Let G be a finitely generated group, and S be a finite set generating G. We can equip G with the word distance

d_S associated to S, as follows: if $\|g\|_S$ is the minimal length of a word written with elements of S and S^{-1} representing g, then we define the left-invariant distance d_S on G by $d_S(g_1, g_2) = \|g_1^{-1}g_2\|_S$. If for instance G is the group $\mathbb{Z}^n$ and S is its generating set consisting of the unit vector basis, then d_S coincide with the distance induced by the ℓ_1 norm on $\mathbb{R}^n$. It is easily checked that if S and S' are two finite generating sets, then the identity map is a coarse Lipschitz isomorphism between the metric spaces (G, d_S) and $(G, d_{S'})$. A property of finitely generated groups is said to be *geometric* if it depends only upon the space (G, d_S) up to coarse Lipschitz equivalence, Many natural properties of groups (such as amenability, hyperbolicity, or being virtually nilpotent, i.e. containing a nilpotent subgroup of finite index) turn out to be geometric.

We denote again by M a real compact manifold. Novikov's conjecture asserts that certain "higher signatures" are homotopy invariants. We would have to introduce several highly non-trivial concepts before providing a precise statement of this conjecture and this is not the purpose of this survey. However it is easy to describe the link between Novikov's conjecture and geometric group theory. Indeed, let $\pi_1(M)$ be the first homotopy group of the manifold M. Mikhail Gromov conjectured a link between the geometry of the finitely generated group $\pi_1(M)$ and the Novikov conjecture, and this conjecture was confirmed by Yu: Novikov's conjecture (and even the stronger coarse Baum-Connes conjecture) holds true if the group $\pi_1(M)$ equipped with the word distance coarsely embeds into the Hilbert space [83]. This important result was later generalized by Kasparov and Yu [61] who showed that the Hilbert space can be replaced in this statement by any super-reflexive space. This is indeed a generalization since the spaces ℓ_p with $p > 2$ do not coarsely embed into the Hilbert space [50]. Note that conversely, it is an open question to know if the separable Hilbert space coarsely embeds into every infinite-dimensional Banach space, in other words if coarse embedding into ℓ_2 is the strongest possible property of that kind. A word of warning is needed here: what we call "coarse embedding" is often called (after Gromov) "uniform embedding" in the context of differential geometry. In this survey however, the word uniform bears another meaning.

Let us recall that a metric space E is called locally finite if every ball of E is finite, and it has *bounded geometry* if for any $r > 0$, the cardinality of subsets of diameter less than r is uniformly bounded. The left invariance of the distance d_S shows that any finitely generated group has bounded geometry. Therefore the question occurs to decide which finitely generated group, and more generally which space with bounded geometry coarsely embeds into a super-reflexive space. For instance, could it be that every space with bounded geometry coarsely embeds into a super-reflexive space?

This question is now negatively answered. A first example of a locally finite space which does not coarsely embed into the Hilbert space is obtained in [21], using in particular a construction of Enflo [27]. Then Gromov shows [39] through a random approach the existence of finitely generated groups G such that the metric space (G, d_S) coarsely contains a sequence of expanders E_i - that is, a sequence of graphs such that the first positive eigenvalue of the Laplacian is uniformly bounded below - such that the girth of E_i, namely the length of the shortest closed curve, increases to infinity. As shown in [63], embeddings of expanders into the Hilbert space have maximal distortion. It follows that G cannot be coarsely embedded into a super-reflexive space, and since such a group can be realized as an homotopy group (see [40]) it cuts short hopes to prove the full Novikov conjecture through the coarse embedding approach (see also [43]).

On the other hand, the problem remains open to decide which metric spaces coarsely embed into Banach spaces of given regularity. Any metric space with bounded geometry coarsely embeds into a reflexive space [17]. This result is widely generalized in [53] where it is shown that every stable metric space (where "stable" means that the order of limits can be permuted in $\lim_k \lim_n d(x_k, y_n)$ each time all limits exist) can be coarsely embedded into a reflexive space. This is indeed extending [17] since every metric space whose balls are compact, and thus every locally finite metric space, is stable. Moreover, it follows from Theorem 4.4 that any locally finite metric space Lipschitz embeds into the following very simple reflexive space: $(\sum_{n=1}^{\infty} \ell_\infty^n)_{\ell_2}$ (note that this space is both AUC and AUS). On the other hand, Theorem 4.7 states that c_0 does not coarsely embed into a reflexive space, nor into a stable metric space (by the above, or directly by [78]). Note that the important stable Banach space L^1 coarsely embeds into the Hilbert space [50],[77].

As seen before, coarse embeddings of special graphs bear important consequences on the non-linear geometry of Banach spaces. Sequences of expanders shed light on the geometry of groups and its applications to homotopy invariants. It is plausible that such expanders could provide several interesting examples in geometry of Banach spaces.

For the record, we recall the

Problem 14. Does ℓ_2 coarsely embed into every infinite dimensional Banach space?

7. Lipschitz-free spaces and their applications

Let M be a pointed metric space, that is, a metric space equipped with a distinguished point denoted 0. The space $\mathrm{Lip}_0(M)$ is the space of real-valued Lipschitz functions on M which vanish at 0. Let $\mathcal{F}(M)$ be the natural predual of $\mathrm{Lip}_0(M)$, whose w^*-topology coincide on the unit ball of $\mathrm{Lip}_0(M)$ with the pointwise convergence on M. The Dirac map $\delta : M \to \mathcal{F}(M)$ defined by $\langle g, \delta(x)\rangle = g(x)$ is an isometric embedding from M to a subset of $\mathcal{F}(M)$ which generates a dense linear subspace. This predual $\mathcal{F}(M)$ is called in [35] the *Lipschitz-free space over M*. When M is separable, $\mathcal{F}(M)$ is separable as well since $\delta(M)$ spans a dense subspace. Although Lipschitz-free spaces over separable metric spaces constitute a class of separable Banach spaces which are easy to define, the structure of these spaces is very poorly understood to this day. Improving our understanding of this class is a fascinating research program. Note that if we identify (through the Dirac map) a metric space M with a subset of $\mathcal{F}(M)$, any Lipschitz map from M to a metric space N extends to a continuous linear map from $\mathcal{F}(M)$ to $\mathcal{F}(N)$. So Lipschitz maps become linear, but of course the complexity is shifted from the map to the free space: this may explain why the structure of Lipschitz-free spaces is not easy to analyze. A first example is provided by the real line, whose free space is isometric to L_1. Actually, metric spaces M whose free space is isometric to a subspace of L_1 are characterized in [31] as subsets of metric trees equipped with the least path metric. On the other hand, the free space of the plane $\mathbb{R}^2$ does not embed isomorphically into L^1 [71].

Banach spaces X are in particular pointed metric spaces (pick the origin as distinguished point) and we can apply the previous construction. Note that the isometric embedding $\delta : X \to \mathcal{F}(X)$ is of course non linear since there exist Lipschitz functions on X which are not affine.

This Dirac map has a linear left inverse $\beta : \mathcal{F}(X) \to X$ which is the quotient map such that $x^*(\beta(\mu)) = \langle x^*, \mu \rangle$ for all $x^* \in X^*$; in other words, β is the extension to $\mathcal{F}(X)$ of the barycenter map. This setting provides canonical examples of Lipschitz-isomorphic spaces. Indeed, if we let $Z_X = Ker(\beta)$, it follows easily from $\beta\delta = Id_X$ that $Z_X \oplus X = \mathcal{G}(X)$ is Lipschitz-isomorphic to $\mathcal{F}(X)$.

Following [35], let us say that a Banach space X has the *lifting property* if there is a continuous linear map $R : X \to \mathcal{F}(X)$ such that $\beta R = Id_X$, or equivalently, if for Y and Z Banach spaces and $S : Z \to Y$ and $T : X \to Y$ continuous linear maps, the existence of a Lipschitz map $\mathcal{L}$ such that $T = S\mathcal{L}$ implies the existence of a continuous linear operator L such that $T = SL$. A diagram-chasing argument shows that $\mathcal{G}(X)$ is linearly isomorphic to $\mathcal{F}(X)$ if and only if X has the lifting property [35]. It turns out that all non-separable reflexive spaces, and also the spaces $\ell_\infty(\mathbb{N})$ and $c_0(\Gamma)$ when Γ is uncountable, fail the lifting property and this provides canonical examples of pairs of Lipschitz but not linearly isomorphic spaces.

On the other hand, the following result is proved in [35]:

Theorem 7.1. *: Every separable Banach space X has the lifting property.*

Proof. We will actually give two proofs. First, one can pick a Gaussian measure γ whose support is dense in X and use the result that $(\delta * \gamma)$ is Gâteaux-differentiable. Then in the above notation $R = (\delta * \gamma)'(0)$ satisfies $Id_X = \beta R$.

The second proof is essentially self-contained. It consists into replacing the Gaussian measure by a cube measure, and this will be useful later. It underlines the simple fact that being separable is equivalent to being "compact-generated". Again, we use differentiation, but only in the directions which are normal to the faces of the cube.

Let $(x_i)_{i \geq 1}$ be a linearly independent sequence of vectors in X such that

$$\overline{span}\,[(x_i)_{i \geq 1}] = X$$

and $\|x_i\| = 2^{-i}$ for all i. Let $H = [0,1]^{\mathbb{N}}$ be the Hilbert cube and $H_n = [0,1]^{\mathbb{N}_n}$ be the copy of the Hilbert cube where the factor of rank n is omitted; that is, $\mathbb{N}_n = \mathbb{N} \backslash \{n\}$. We denote by λ (resp. λ_n) the natural probability measure on H (resp. H_n) obtained by taking the product of the Lebesgue measure on each factor.

We denote $E = span\,[(x_i)_{i \geq 1}]$ and $R : E \to \mathcal{F}(X)$ the unique linear map which satisfies for all $n \geq 1$ et all $f \in Lip_0(X)$

$$R(x_n)(f) = \int_{H_n} \big[f(x_n + \sum_{j=1, j \neq n}^{\infty} t_j x_j) - f(\sum_{j=1, j \neq n}^{\infty} t_j x_j)\big]\, d\lambda_n(t)$$

Pick $f \in \mathrm{Lip}_0(X)$. If the function f is Gâteaux-differentiable, Fubini's theorem shows that for all $x \in E$

$$R(x)(f) = \int_H < \{\nabla f\}(\sum_{j=1}^{\infty} t_j x_j), x > d\lambda(t)$$

Thus $|R(x)(f)| \leq \|x\| \, \|f\|_L$ in this case. But since X is separable, any $f \in \mathrm{Lip}_0(X)$ is a uniform limit of a sequence (f_j) of Gâteaux-differentiable functions such that $\|f_j\|_L \leq \|f\|_L$. It follows that

$$\|R\| \leq 1.$$

We may now extend R to a linear map $\overline{R} : X \to \mathcal{F}(X)$ such that $\|\overline{R}\| = 1$ and it is clear that $\overline{R}(x)(x^*) = x^*(x)$ for all $x \in X$ and all $x^* \in X^*$. Hence $\beta\overline{R} = Id_X$.

□

The above proof follows [35]. We refer to [32] for an elementary approach along the same lines, which uses only finite-dimensional arguments and is accessible at the undergraduate level.

The lifting property for separable spaces forbids the existence of a *separable* Banach space X such that $\mathcal{F}(X)$ and $\mathcal{G}(X)$ are not linearly isomorphic, but on the other hand it shows that if there exists an isometric embedding from a separable Banach space X into a Banach space Y, then Y contains a linear subspace which is isometric to X. Indeed a theorem due to Figiel [30] states that if $J : X \to Y$ is an isometric embedding such that $J(0) = 0$ and $\overline{span}[J(X)] = Y$ then there is a linear quotient map with $\|Q\| = 1$ and $QJ = Id_X$, and then the lifting property provides a linear contractive map R such that $QR = Id_X$, and this map R is a linear isometric embedding. We note that $P = RQ$ is a contractive projection from Y onto $R(X)$. This remark is developed further in [33] where it is shown that the existence of a non-linear isometric embedding from X into Y is a very restrictive condition on the couple (X, Y).

Nigel Kalton constructed the proper frame for showing the gap which separates Hölder maps from Lipschitz ones [52]. If $(X, \| \ \|)$ is a Banach space and $\omega : [0, +\infty) \to [0, +\infty)$ is a subadditive function such that $\lim_{t\to 0} \omega(t) = \omega(0) = 0$ and $\omega(t) = t$ if $t \geq 1$, then the space $\mathrm{Lip}_\omega(X)$ of $(\omega \circ d)$-Lipschitz functions on X which vanish at 0 has a natural predual denoted $\mathcal{F}_\omega(X)$, and the barycentric map $\beta_\omega : \mathcal{F}_\omega(X) \to X$ (whose adjoint is the canonical embedding from X^* to $\mathcal{F}_\omega(X)$) is still a linear quotient map such that $\beta_\omega \delta = Id_X$. However, the Dirac map $\delta : X \to \mathcal{F}_\omega(X)$ is now uniformly continuous with modulus ω - e.g. α- Hölder when $\omega(t) = \max(t^\alpha, t)$ with $0 < \alpha < 1$. Uniformly continuous functions fail the differentiability properties that Lipschitz functions enjoy, and thus one can expect that this part of the theory is more "distant" from the linear theory than the Lipschitz one. It is indeed so, and [52, Theorem 4.6], reads as follows.

Theorem 7.2. *If ω satisfies $\lim_{t\to 0} \frac{\omega(t)}{t} = \infty$, then $\mathcal{F}_\omega(X)$ is a Schur space - that is, weakly convergent sequences in $\mathcal{F}_\omega(X)$ are norm convergent.*

It follows from Theorem 7.2 that the uniform analogue of the lifting property fails unless X has the (quite restrictive) Schur property. Moreover, $\mathcal{F}_\omega(X)$ is (3ω)-uniformly homeomorphic to $[X \oplus Ker(\beta_\omega)]$ and as soon as $\lim_{t\to 0} \frac{\omega(t)}{t} = 0$ and X fails the Schur property we obtain canonical pairs of uniformly (even Hölder) homeomorphic separable Banach spaces which are not linearly isomorphic. We refer to [79, 47] for other examples of such pairs.

Along with Hölder maps between Banach spaces, one may as well consider Lipschitz maps between quasi-Banach spaces, and this is done in [5] where similar methods provide examples of separable quasi-Banach spaces which are Lipschitz but not linearly isomorphic.

We now observe that the proof (with cube measures) of Theorem 7.1 provides the existence of compact metric spaces whose free space fails the approximation property (in short, A.P.). This has been observed in [34].

Theorem 7.3. *There exists a compact metric space K whose free space $\mathcal{F}(K)$ fails the approximation property.*

Proof. We use the notation of the proof of Theorem 7.1. Let C be the closed convex hull of the sequence $(x_i)_{i\geq 1}$, and let $K = 2C$. It is easily seen that the map R takes its values in the closed subspace $\mathcal{F}(K)$ of $\mathcal{F}(X)$, and so does $\overline{R}$. It follows that X is isometric to a 1-complemented subspace of $\mathcal{F}(K)$, through the projection $\overline{R}Q$. If this construction is applied to a Banach space X which fails A.P. , then $\mathcal{F}(K)$ fails A. P. as well since A. P. is carried to complemented subspaces.

□

Problem 15. Let X be a separable Banach space, and Y a Banach space which is Lipschitz-isomorphic to X. Does it follow that Y is linearly isomorphic to X?

This question amounts to know if every separable Banach space is determined by its metric structure. It is open for instance if $X = \ell_1$ or if $X = C(K)$ with K a countable compact metric space, unless $C(K)$ is isomorphic to c_0. Note that by the above the answer to this question is negative if we drop the separability assumption, or if we replace Lipschitz by Hölder, or if we replace Banach by quasi-Banach.

Problem 16. Is the Lipschitz-free space $\mathcal{F}(\ell_1)$ over ℓ_1 complemented in its bidual?

A motivation for this question is that if $\mathcal{F}(\ell_1)$ is complemented in its bidual, it follows that every space X which is Lipschitz-isomorphic to ℓ_1 is complemented in its bidual, and then [12, Corollary 7.7] shows that X is linearly isomorphic to ℓ_1.

Problem 17. Theorem 7.3 leads to the question of knowing for which compact spaces K the space $\mathcal{F}(K)$ has A. P. or its metric version M. A. P. So far, very little is known on this topic, which is related with the existence of linear extension operators for Lipschitz functions (see [13], [34]).

Problem 18. Let M be an arbitrary uniformly discrete metric space, that is, there exists $\theta > 0$ such that $d(x, y) \geq \theta$ for all $x \neq y$ in M. Does $\mathcal{F}(M)$ have the B. A. P? Note that A. P. holds by ([52, Proposition 4.4]). A positive answer to this question would imply that every separable Banach space X is approximable, that is, the identity Id_X is the pointwise limit of an equi-uniformly continuous sequence of maps with relatively compact range. By [56, Theorem 4.6], it is indeed so for X and X^* when X^* is separable, and in particular every separable reflexive space is approximable. On the other hand, a negative answer to this question would provide an equivalent norm on ℓ_1 failing M. A. P. and this would solve a famous problem in approximation theory, by providing the first example of a dual space - namely, ℓ_∞ equipped with the corresponding dual norm - with A. P. (and even B. A. P.) but failing M. A. P.

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