LONGHORN NOTES
THE UNIVERSITY OF TEXAS AT AUSTIN
FUNCTIONAL ANALYSIS SEMINAR
1985-1986

Applications of an inequality for H_1-functions

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1. Introduction.

In [5] we showed that if $f \in H_1$ then the limit

$$\sigma = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \log |f(re^{i\theta})| \, d\theta$$

exists and

$$\left|\sigma - f(0)\log|f(0)|\,\right| \le 2(\|f\|_1 - |f(0)|) \; .$$

This inequality was used in the construction of a compact convex subset of a quasi-Banach space which cannot be affinely embedded in L_0. The purpose of this role is to explore some related inequalities and give an application to the study of (UMD)-spaces.

In Section 2 we shall establish the main inequality. We then show the existence of a constant $C = C(p)$ for $0 < p < \infty$ so that if $f \in H_p$ and $f \log|f| \in L_p$ then

$$d(f\log|f|,\ H_p) \le C\,\|f\|_p \; .$$

We then relate our results to recent work of Coifman and Rochberg [4] and Rochberg and Weiss [7].

In Section 3, we apply our results to the so-called (UMD)-Banach spaces introduced and studied by Burkholder ([2], [3]). Our result is that the twisted sum spaces Z_p $(1 < p < \infty)$ introduced in [6] are (UMD)-spaces. The motivation for this result is that the three-space problem for (UMD)-spaces is apparently unresolved; that is, it is unknown whether a Banach space X must be (UMD) if it possesses a closed subspace E so that E and X/E are (UMD)-spaces. The spaces Z_p $(1 < p < \infty)$ are natural candidates for a counterexample since E can be chosen so $Z_p/E \approx E \approx \ell_p$ and Z_p cannot be embedded in any L_r-space ([6]). However, our result shows that these spaces do not provide the expected counterexamples. As remarked in [1], the collection of known (UMD)-spaces is rather small and the addition of a new class is perhaps interesting.

Let us introduce some notation. T will denote the unit circle and Δ the open unit disk in the complex plane. A typical point of T is denoted by $w = e^{i\theta}$. Haar

* Supported by NSF grants DMS-8601401 and DMS-8301099.

measure on $\mathbf{T}$, *i.e.*, $(2\pi)^{-1}\,d\theta$ is also denoted by dm. L_p $(0 < p \le \infty)$ denotes the complex space $L_p(\mathbf{T}, m)$ and $\|\cdot\|_p$ is the norm on L_p. H_p is the closed subspace generated by $\{w^n : n \ge 0\}$. For $1 < p < \infty$ the Riesz projection R is the bounded projection of L_p onto H_p given by

$$Rf = \sum_{n\ge 0} \hat{f}(n) w^n$$

where $(\hat{f}(n))_{n\in\mathbb{Z}}$ are the Fourier coefficients of f, *i.e.*,

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})\,d\theta\ .$$

If X is a complex Banach space then $L_p(X)$ $(1 \le p < \infty)$ is the space $L_p(\mathbf{T}, m, X)$ and the norm on $L_p(X)$ is denoted

$$\|f\|_p = \left(\int \|f\|^p\,dm\right)^{1/p}\ .$$

The vector Fourier coefficients are given by

$$\hat{f}(n) = \int w^{-n} f(w)\,dm(w)\ .$$

$H_p(X)$ is the subspace of $L_p(X)$ so that $\hat{f}(n) = 0$ if $n < 0$. X is a (UMD)-space if the vector Riesz projection $\tilde{R}$ is a bounded projection of $L_p(X)$ onto $H_p(X)$ for some (and hence for all) p with $1 < p < \infty$. Here

$$\tilde{R}f \sim \sum_{n\ge 0} \hat{f}(n) w^n\ .$$

This definition is equivalent to the standard definition by a result of Bourgain [**1**].

The spaces Z_p $(1 < p < \infty)$ were introduced in [**6**]. The (complex) space Z_p consists of all pairs of complex sequences $((u_n),\ (v_n))$ such that

$$\|(u,v)\|_{Z_p} = \left(\sum_{n=1}^{\infty} \left|u_n - v_n \log\frac{\|v\|_p}{|v_n|}\right|^p\right)^{1/p} + \|v\|_p < \infty$$

where

$$\|v\|_p = \left(\sum_{n=1}^{\infty} |v_n|^p\right)^{1/p}\ .$$

Here, as always, $0\log 0 = 0\log\infty = 0$. $\|\cdot\|_{Z_p}$ is a quasi-norm but is equivalent to a norm ([**6**]).

The function space analogues ZF_p $(1 < p < \infty)$ may be defined analogously. ZF_p is the space of all pairs (f, g) in $L_0(\mathbf{T})$ so that

$$\|(f,g)\|_{ZF_p} = \left(\int \left| f - g\log\frac{\|g\|_p}{|g|}\right|^p dm\right)^{1/p} + \|g\|_p < \infty \ .$$

The spaces Z_p and ZF_p actually arise quite naturally in interpolation theory ([7]).

Finally we note that we use the convention that C is a constant depending on p but not on f, g, h etc., which may vary from line to line.

2. The main inequalities.

For convenience we write $\log_+ x = \max(\log x, 0)$ and $\log_- x = \min(\log x, 0)$ for $x > 0$. If $f \in H_p$ we denoted by f_r $(0 \le r < 1)$ the function

$$f_r(e^{i\theta}) = \tilde{f}(re^{i\theta})$$

where $\tilde{f}$ is the analytic extension of f to Δ.

Our first lemma is a simple technical result which is surely well-known.

Lemma 2.1. *Suppose* $1 \le p < \infty$, *and* $f \in H_p$. *If* $f \log|f| \in L$ *then*

$$\lim_{r\to 1} \|f\log|f| - f_r \log|f_r|\ \|_p = 0 \ .$$

Proof. Since $x \log_- |x|$ is a bounded function, the Bounded Convergence Theorem implies

$$\lim_{r\to 1} \|f\log_-|f| - f_r \log_-|f_r|\ \|_p = 0 \ .$$

Since $f_r \log_+ |f_r| \to f \log_+ |f|$ a.e., it now suffices to find nonnegative functions h_r $(0 \le r < 1)$, $h \in L_p$ so that $\|h_r - h\|_p \to 0$ and $|f_r| \log_+ |f_r| \le h_r$ a.e.

In fact we can set $h = |f| \log_+ |f|$ and define h_r $(0 \le r < 1)$ by

$$h_r(e^{i\theta}) = \int_0^{2\pi} P(r, \theta - \varphi) h(e^{i\varphi}) \frac{d\varphi}{2\pi}$$

where P is the Poisson kernel. As the function $|x| \log_+ |x|$ is convex, Jensen's inequality implies $|f_r| \log_+ |f_r| \le h_r$ and $\|h_r - h\|_p \to 0$ as required.

Our next results restate the inequality proved in [5].

Proposition 2.2. *Suppose* $f \in H_1$ *and* $f \log|f| \in L_1$. *Then*

$$\left|\int f\log|f|\,dm - \tilde{f}(0)\log|\tilde{f}(0)|\right| \le 2\big(\|f\|_1 - |\tilde{f}(0)|\big) \ .$$

Proof. Define $F : \Delta \to H_\infty$ by

$$F(z)(w) = f(zw) \qquad z \in \Delta \quad w \in \mathbf{T} \ .$$

Then F is an analytic function. If we define $\Psi : H_\infty \to \mathbb{R}$ by

$$\Psi(g) = 2\|g\|_1 - \int (Re g) \log |g| \, dm$$

then Ψ is continuous and plurisubharmonic, since the function $2|z| - x \log |z|$ is subharmonic on $\mathbb{C}$. If $|z| = r$,

$$\Psi(F(z)) = 2\|f_r\|_1 - \int (Re f_r) \log |f_r| \, dm$$

and so the expression on the right increases with r. Rearranging we have

$$Re\left(\int f_r \log |f_r| \, dm - \tilde{f}(0) \log |\tilde{f}(0)|\right) \leq 2(\|f_r\|_1 - |\tilde{f}(0)|)$$

and hence, letting $r \to 1$, by Lemma 2.1

$$Re\left(\int f \log |f| \, dm - \tilde{f}(0) \log |\tilde{f}(0)|\right) \leq 2(\|f_r\|_1 - |\tilde{f}(0)|)$$

and this implies the Proposition by considering αf in place of f where $|\alpha| = 1$.

The next lemma is a standard calculation by power series which we omit.

Lemma 2.3. *Suppose $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 > 0$. Then*

(1)
$$\int_0^{2\pi} \log |a + be^{i\varphi}| \frac{d\varphi}{2\pi} = \log\left(\max(|a|,\ |b|)\right)$$

(2)
$$\int_0^{2\pi} e^{i\varphi} \log |a + be^{i\varphi}| \frac{d\varphi}{2\pi} = \begin{cases} \frac{1}{2} \frac{\bar{b}}{\bar{a}} & |b| \leq |a| \\ \frac{1}{2} \frac{a}{b} & |a| \leq |b| \end{cases}$$

Theorem 2.4. *Suppose $1 \leq p < \infty$. Then there is a constant $C = C_p$ so that if $f \in H_p$ with $f \log |f| \in L_p$ and if $g \in H_{q,0}$, where q is the conjugate index for p, then*

$$\left|\int fg \log |f| \, dm\right| \leq C_p \|f\|_p \|g\|_q .$$

Proof. Since the integral is homogeneous in both f and g it will suffice to suppose $\|f\|_p = \|g\|_q = 1$ in each case.

Case 1: $p = 1$

By Proposition 2.2,

$$\left|\int fg \log |fg| \, dm\right| \leq 2$$

but $\|g \log |g|\|_\infty \le e^{-1}$ so that

$$\left| \int fg \log |f| \, dm \right| \le C_1 = 2 + e^{-1} \ .$$

Case 2: $p = 2$

Lemma 2.1 allows us to specialize to the case when $f, g \in H_\infty$. For $0 \le \varphi \le 2\pi$ set

$$h_\varphi = e^{-i\varphi} f + e^{i\varphi} g \ .$$

Observe that

$$\int_0^{2\pi} \|h_\varphi\|_2^2 \, \frac{d\varphi}{2\pi} = \|f\|_2^2 + \|g\|_2^2 = 2 \ .$$

By 2.2, applied to h_φ^2, since $\tilde{h}_\varphi(0) = e^{-i\varphi} \tilde{f}(0)$,

$$\left| \int h_\varphi^2 \log |h_\varphi| \, dm - \tfrac{1}{2} e^{-2i\varphi} \tilde{f}(0) \log |\tilde{f}(0)| \right| \le \|h_\varphi\|^2 - |\tilde{f}(0)| \ .$$

Integrating over φ we obtain

$$\left| \int_0^{2\pi} \int h_\varphi^2 \log |h_\varphi| \, dm \, \frac{d\varphi}{2\pi} \right| \le 2 \ .$$

Define $G \in L_\infty$ pointwise by

$$G = \frac{1}{2\pi} \int h_\varphi^2 \log |h_\varphi| \, d\varphi \ .$$

Then by Fubini's theorem

$$\left| \int G \, dm \right| \le 2 \ .$$

We now estimate G pointwise, by expanding h_φ^2. First we have

$$\left| \int_0^{2\pi} e^{-2i\varphi} f^2 \log |h_\varphi| \, \frac{d\varphi}{2\pi} \right| \le \tfrac{1}{2} |f|^2$$

by Lemma 2.3. Similarly

$$\left| \int_0^{2\pi} e^{2i\varphi} g^2 \log |h_\varphi| \, \frac{d\varphi}{2\pi} \right| \le \tfrac{1}{2} |g|^2 \ .$$

For the remaining term

$$\int_0^{2\pi} fg \log |h_\varphi| \, \frac{d\varphi}{2\pi} = fg \log H$$

where $H = \max(|f|, |g|)$. Thus

$$|G - 2fg \log H| \leq \tfrac{1}{2}(|f|^2 + |g|^2)$$

and so

$$\left| \int fg \log H \, dm \right| \leq \tfrac{3}{2} .$$

However $0 \leq |f| \log \frac{H}{|f|} \leq e^{-1} H$ so that

$$\left| \int fg \log \frac{H}{|f|} \, dm \right| \leq e^{-1} \left| \int H|g| \, dm \right| .$$

Let $A = \{w : |f(w)| > |g(w)|\}$ and $B = \mathbf{T} \setminus A$. Then

$$\begin{aligned} \int H|g| \, dm &\leq \left(\int_A |f|^2 \, dm \right)^{1/2} \left(\int_A |g|^2 \, dm \right)^{1/2} + \int_B |g|^2 \, dm \\ &\leq \sigma + 1 - \sigma^2 \end{aligned}$$

where $\sigma = (\int_A |g|^2 \, dm)^{1/2}$. Thus since $0 \leq \sigma \leq 1$

$$\left| \int fg \log \frac{H}{|f|} \, dm \right| \leq \tfrac{5}{4} e^{-1}$$

and

$$\left| \int fg \log |f| \, dm \right| \leq C_2$$

where $C_2 = \frac{3}{2} + \frac{5}{4} e^{-1}$.

<u>Case 3: $1 < p < 2$.</u>

Again we suppose $f, g \in H_\infty$. By using inner-outer factorization we can write $f = IF$ where I is inner and F is outer. Now let

$$\begin{aligned} f_0 &= F^{\frac{p}{2}-1} f \\ g_0 &= F^{1-\frac{p}{2}} g \end{aligned}$$

Then $\|f_0\|_2 = 1$ and

$$\begin{aligned} \|g_0\|_2^2 &= \int |F|^{2-p} |g|^2 \, dm \\ &\leq \left(\int |F|^p \, dm \right)^{\frac{2}{p}-1} \left(\int |g|^q \, dm \right)^{2/q} = 1 . \end{aligned}$$

Thus by Case 2

$$\left| \int f_0 g_0 \log|f_0|\, dm \right| \le C_2$$

and so

$$\left| \int fg \log|f|\, dm \right| \le C_p = \frac{2}{p} C_2$$

Case 4: $2 < p < \infty$.

Again we assume f, g bounded. Then

$$\left| \int fg \log|g|\, dm \right| \le C_q$$

(by Case 3). However

$$\left| \int fg \log|fg|\, dm \right| \le 2$$

by 2.2. Thus

$$\left| \int fg \log|f|\, dm \right| \le 2 + C_q = C_p\ .$$

Corollary 2.5. *Suppose* $1 \le p < \infty$. *Then if* $f \in H_p$ *and* $f \log|f| \in L_p$

$$d(f\log|f|\ ,\ H_p) \le C_p \|f\|_p\ .$$

Proof. This is simply the Hahn-Banach Theorem.

The analogue of Corollary 2.5 holds also for $0 < p < 1$.

Theorem 2.6. *Suppose* $0 < p < 1$. *Then there is a constant* C_p *so that if* $f \in H_p$ *is bounded then*

$$d(f\log|f|,\ H_p) \le C_p \|f\|_p\ .$$

Proof. Let $f = IF$ where I is inner and F is outer. Then $IF^p \in H_1$ and there exists by 2.5, $g \in H_1$ with

$$\begin{aligned} \|pIF^p \log|f| - g\|_1 &\le C_1 \|IF^p\|_1 \\ &= C_1 \|f\|_p^p\ . \end{aligned}$$

Then $p^{-1}F^{1-p}g \in H_p$ and

$$f\log|f| - p^{-1}F^{1-p}g = p^{-1}F^{1-p}(pIF^p \log|f| - g)\ .$$

By Hölder's inequality,

$$\begin{aligned} \int \left|f\log|f| - p^{-1}F^{1-p}g\right|^p dm &\le (p^{-1}C_1\|f\|_p^p)^p \left(\int |F|^p\, dm\right)^{1-p} \\ &= (p^{-1}C_1\|f\|_p)^p \end{aligned}$$

so that the theorem is proved with $C_p = p^{-1}C_1$.

We conclude this section by relating our ideas to some known results in harmonic analysis. In [7] (in particular Proposition 3.35) Rochberg and Weiss use interpolation theory to show that operators such as the Riesz projection or the Hilbert transform satisfy certain nonlinear commutation relations with the function $x \log|x|$. We derive from Corollary 2.5 a direct proof for the Riesz projection (and hence for the Hilbert transform). Note also that Theorem 2.7 implies Corollary 2.5, but without the specific constants.

Theorem 2.7. *For $1 < p < \infty$ there is a constant $C = C(p)$ so that if $f \in L_r$ where $r > p$ then*

$$\|R(f\log|f|) - (Rf)\log|Rf|\,\|_p \le C\,\|f\|_p\ .$$

Proof. In this argument C will denote a constant depending only on p but which may vary from line to line. Let $u = Rf$, $v = f - Rf$. Then

$$\big|u\log|u| + v\log|v| - f\log|f|\big| \le \frac{2}{e}(|u| + |v|)$$

so that

$$\begin{aligned}\|u\log|u| + v\log|v| - f\log|f|\|_p &\le \frac{4}{e}\|R\|\ \|f\|_p\\ &= C\,\|f\|_p\ .\end{aligned}$$

By 2.5 there exists $g \in H_p$ with

$$\|u\log|u| - g\|_p \le C\,\|u\|_p \le C\,\|f\|_p$$

and h with $\bar{h} \in H_{p,0}$ and

$$\|v\log|v| - h\|_p \le C\,\|f\|_p\ .$$

Thus

$$\|R(f\log|f|) - g\|_p \le C\,\|f\|_p$$

and hence

$$\|R(f\log|f|) - u\log|u|\,\|_p \le C\,\|f\|_p$$

as required.

Finally we note a real-variables version of 2.2. Let ReH_1 be the space of real functions $f \in L_1$ so that there is a unique $g \in L_1$ such that $f + ig \in H_{1,0}$. Set

$$\|f\|_{ReH_1} = \|f + ig\|_1\ .$$

Proposition 2.8. *If $f \in ReH_1$ and $f \in L_p$ for some $p > 1$ then*

$$\left|\int f\log|f|\,dm\right| \le 3\|f\|_{ReH_1}$$

Proof. In fact

$$\left|\int (f+ig)\log(f^2+g^2)^{1/2}\,dm\right| \le 2\|f\|_{ReH_1}$$

and hence

$$\left|\int f\log(f^2+g^2)^{1/2}\,dm\right| \le 2\|f\|_{ReH_1}\,.$$

Now

$$|f|\log\frac{(f^2+g^2)^{1/2}}{|f|} \le (f^2+g^2)^{1/2}$$

and the Proposition follows.

Let f^* be the radial maximal function defined by

$$f^*(e^{i\theta}) = \sup_{r<1}\left|\int_0^{2\pi} P(r,\theta-\varphi)f(e^{i\varphi})\frac{d\varphi}{2\pi}\right|.$$

Then $\|f\|_{ReH_1}$ is equivalent to $\|f^*\|_1$ and

$$\left|f\log|f| - f\log f^*\right| \le \frac{1}{e}f^*$$

so that we easily deduce

Proposition 2.9. *There is a constant C so that if $f\in ReH_1\cap L_p$ for some $p>1$*

$$\left|\int f\log f^*\,dm\right| \le C\,\|f\|_{ReH_1}\,.$$

A very similar result is proved by Coifman and Rochberg [4]. They show

$$\left|\int f\log(|f|^*)\,dm\right| \le C\,\|f\|_{ReH_1}\,.$$

The relationship between this and Proposition 2.9 is not clear at the present.

3. Applications to (UMD)-spaces.

Let us suppose X is a Banach space with a closed subspace E so that E and X/E are (UMD)-spaces. It is unknown whether, in general, X is a UMD-space. Our next proposition gives a simple criterion.

Proposition 3.1. *Suppose $1<p<\infty$ is fixed. Let X be a Banach space with a closed subspace E so that E and X/E are* UMD*-spaces. Then X is a* (UMD)*-space if and only if the natural quotient map $\tilde{q}: L_p(X)\to L_p(X/E)$ maps $H_p(X)$ onto $H_p(X/E)$.*

Proof. Suppose X is (UMD). If $f\in H_p(X/E)$ there exists $g\in L_p(X)$ with $\tilde{q}g=f$. If the vector Riesz projection $\tilde{R}$ is bounded then $\tilde{q}\tilde{R}g=f$.

Conversely let $f \in L_p(X)$, and assume $\tilde{q}$ maps $H_p(X)$ onto $H_p(X/E)$. Then $\tilde{q}f \in L_p(X/E)$ and so $\tilde{q}f = g_1 + g_2$ where $g_1 \in H_p(X/E)$ and $g_2(w^{-1}) \in H_{p,0}(X/E)$. By assumption we can find $h_1, h_2 \in L_p(X)$ so that $h_1 \in H_p(X)$, $h_2(w^{-1}) \in H_{p,0}(X)$ and $\tilde{q}h_j = g_j \quad (j = 1, 2)$. Now $f - h_1 - h_2 \in L_p(E)$ and so $f - h_1 - h_2 = h_3 + h_4$ where $h_3 \in H_p(E)$ and $h_4(w^{-1}) \in H_{p,0}(E)$. Writing $f = (h_1 + h_3) + (h_2 + h_4)$ we are done.

Our next Proposition is the vector version of Corollary 2.5.

Proposition 3.2. *If $1 \le p < \infty$ there is a constant $C = C(p)$ so that if X is a reflexive Banach space and $f \in H_p(X)$ satisfies $f \log \|f\| \in L_p(X)$ then*

$$d(f \log \|f\| \ , \ H_p(X)) \le C \, \|f\|_p \ .$$

Proof. Since X is reflexive, $L_p(X)^* = L_q(X^*)$. Thus there exists $g \in H_p(X)^\perp = H_{q,0}(X^*)$ so that $\|g\|_q = 1$ and

$$d(f \log \|f\| \ , \ H_p(X)) = \int \langle f, g \rangle \log \|f\| \, dm \ .$$

Let $h \in H_p$ be an outer function with $|h| = \|f\|$ a.e. Then $\langle h^{-1} f, g \rangle \in H_{q,0}$ and

$$\int |\langle h^{-1} f, g \rangle|^q \, dm \le \int \|g\|^q \, dm = 1 \ .$$

Thus

$$\left| \int \langle h^{-1} f, g \rangle h \log |h| \, dm \right| \le C \, \|h\|_p = C \, \|f\|_p$$

by Theorem 2.4, and the proof is complete.

Theorem 3.3. *For $1 < p < \infty$, Z_p is a (UMD)-space.*

Proof. We show that the natural quotient map $\tilde{q} : L_p(Z_p) \to L_p(\ell_p)$ maps $H_p(Z_p)$ onto $H_p(\ell_p)$. To do this it suffices to show that there is a constant C so that if $f \in H_p(\ell_p)$ is bounded and vanishes off a finite number of co-ordinates in ℓ_p then there exists $g \in H_p(Z_p)$ with $\tilde{q}g = f$ and $\|g\|_p \le C \, \|f\|_p$.

Let

$$f(w) = \big(f_k(w)\big)_{k=1}^\infty$$

where $f_k \in H_\infty$ and $f_k = 0$ for $k \ge N$.

By 3.2 there exists $g \in H_p(\ell_p)$, so that

$$\big\| f \log \|f\| - g \big\|_p \le C \, \|f\|_p \ .$$

Clearly we may suppose

$$g(w) = \big(g_k(w)\big)_{k=1}^\infty$$

where $g_k = 0$ for $k \geq N$.

By 2.5 select $h_k \in H_p$ with $h_k = 0$ for $k \geq N$ and

$$\left\| f_k \log |f_k| - h_k \right\|_p \leq C \left\| f_k \right\|_p .$$

Thus

$$\int \left| f_k \log |f_k| - h_k \right|^p dm \leq C^p \int |f_k|^p \, dm$$

and by summing

$$\int \sum_{k=1}^{N} \left| f_k \log |f_k| - h_k \right|^p dm \leq C^p \left\| f \right\|_p^p .$$

Now define $G \in H_p(Z_p)$ by

$$G(w) = \left(g_k(w) - h_k(w) \, , \; f_k(w) \right)_{k=1}^{\infty}$$

$$\|G\|^p \leq C \left(\sum_{k=1}^{N} \left| g_k - h_k - f_k \log \frac{\|f\|}{|f_k|} \right|^p + |f_k|^p \right)$$

$$\leq C \left(\sum_{k=1}^{N} \left| g_k - f_k \log \|f\| \right|^p + \sum_{k=1}^{N} \left| f_k \log |f_k| - h_k \right|^p + \sum_{k=1}^{N} |f_k|^p \right)$$

Thus

$$\|G\|_p \leq C \, \|f\|_p$$

and clearly $\tilde{q} G = f$.

Remark. Similar arguments show that ZF_p is a (UMD)-space for $1 < p < \infty$.

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