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UNIFORM HOMEOMORPHISMS OF BANACH SPACES AND ASYMPTOTIC STRUCTURE

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ABSTRACT. We give a general result on the behavior of spreading models in Banach spaces which coarse Lipschitz-embed into asymptotically uniformly convex spaces. We use this result to study the uniqueness of the uniform structure in ℓ_p-sums of finite-dimensional spaces for $1 < p < \infty$; in particular we give some new examples of spaces with unique uniform structure.

1. Introduction

It is known that asymptotic smoothness is preserved under uniform homeomorphisms of Banach spaces [11]. In quantitative terms this is measured by the behavior of the convex Szlenk index (Theorem 5.5 of [11]); unfortunately it is not true that one has a precise result on the preservation of the modulus of asymptotic smoothness, even after renorming. Thus, for example, if X and Y are separable uniformly homeomorphic Banach spaces and

$$\overline{\rho}_Y(t) \leq ct^p, \qquad 0 < t < 1,$$

we can only conclude that for any $q < p$ and some equivalent norm on X, one has an estimate

$$\overline{\rho}_X(t) \leq c't^q, \qquad 0 < t < 1.$$

A recent example in [29] shows that we cannot improve this to the case $q = p$. There is a simple application of the ideas of [11] to spreading models in Y. If $(e_n)_{n=1}^{\infty}$ is the basis of a spreading model S of a normalized weakly null sequence in X we have an estimate

$$\|e_1 + \cdots + e_n\|_S \leq C\|e_1 + \cdots + e_n\|_{\ell_{\overline{\rho}_Y}}, \tag{1.1}$$

where the right-hand side represents the norm in the Orlicz sequence space generated by the Orlicz function $\overline{\rho}_Y$. This can be obtained by combining Theorem 4.4 and Theorem 5.5 of [11]. In [30], using simpler arguments, this result is shown to hold more generally (Theorem 6.1) when X coarse Lipschitz-embeds into Y, under the additional hypothesis that Y is reflexive.

Although these results have applications in the nonlinear theory of Banach spaces, it has been a significant drawback that there has been no corresponding result giving a lower bound in terms of asymptotic convexity to the upper bound

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in (1.1). In a recent article [2], results are obtained that suggest one might hope for similar results for asymptotic convexity. Here we justify that hope in Theorem 7.4, where we show that if Y is reflexive and X coarse Lipschitz-embeds in Y, then for some constant $c > 0$ and any spreading model S of a normalized weakly null sequence in X we have an estimate

$$c\|e_1 + \cdots + e_n\|_{\ell_{\overline{\delta}_Y}} \leq \|e_1 + \cdots + e_n\|_S. \tag{1.2}$$

We then use these ideas to study ℓ_p-sums of finite-dimensional spaces. Suppose $1 < p < \infty$. It is a result of Johnson, Lindenstrauss, Preiss and Schechtman [19] that a separable reflexive Banach space X which has two renormings X_1 and X_2 with $\overline{\delta}_{X_1}(t) \sim \overline{\rho}_{X_2}(t) \sim ct^p$ is linearly isomorphic to a subspace of a space $(\sum_{n=1}^\infty E_n)_{\ell_p}$ with each E_n being finite-dimensional. Unfortunately it is shown in [29] that if we take $(G_n)_{n=1}^\infty$ to be a sequence dense in all finite-dimensional normed spaces for Banach-Mazur distance, then $(\sum_{n=1}^\infty G_n)_{\ell_p}$ (see e.g. [22]) is uniformly homeomorphic to $(\sum_{n=1}^\infty G_n)_{T_p}$, where T_p is p-convexified Tsirelson space (see e.g. [8]). This means that being embeddable in an ℓ_p-sum of finite-dimensional spaces is not, in general, invariant under uniform homeomorphisms.

However, under some additional hypotheses, (1.2) and (1.1) can be combined to get such a conclusion. For example it is shown in [11] that if X is uniformly homeomorphic to a subspace (respectively, quotient) of ℓ_p, then X is itself linearly isomorphic to a subspace (respectively, quotient) of ℓ_p when $2 \leq p < \infty$. We show here in Theorem 8.4 that the same conclusion can be obtained when $1 < p < 2$. Let us remark that in [29] we give examples of subspaces X and Y of ℓ_p ($1 < p < \infty$, $p \neq 2$) which are uniformly homeomorphic but not linearly isomorphic.

In [20] it was shown that ℓ_p has unique uniform structure. We extend this result here by showing that $(\sum_{n=1}^\infty \ell_r^n)_{\ell_p}$ has unique uniform structure if $r > \max(p, 2)$ or $1 < r < \min(p, 2)$. A crucial point in these proofs is the role of the uniform approximation property. This mirrors the examples of two uniformly homeomorphic but nonisomorphic subspaces of ℓ_p mentioned above from [29], where one space has the approximation property (but not the uniform approximation property) and the other fails the approximation property.

On the way to obtaining these nonlinear results we require some new results in the linear theory of Banach spaces. If X is a reflexive Banach space, then the condition

$$\|e_1 + \cdots + e_n\|_S \leq Cn^{1/p}$$

for every spreading model of a normalized weakly null sequence is simply the requirement that X has the so-called p-Banach-Saks property. The dual notion that

$$\|e_1 + \cdots + e_n\|_S \geq cn^{1/p}$$

for every spreading model of a normalized weakly null sequence, we call the p-co-Banach-Saks property. If X is a subspace or quotient of L_p when $p > 2$ and has the p-Banach-Saks property, then Johnson [17] showed that X is then also a subspace of a quotient of ℓ_p. If X is a subspace of a quotient of L_p ($p > 2$) and has the p-Banach-Saks property, then Johnson obtained that X is a subspace of a quotient of ℓ_p only under the additional hypothesis that X has the approximation property. We remove this restriction, answering a question of Johnson, and provide dual results for $1 < p < 2$. In fact we give a more general framework for results of this type.

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Foreword: Nigel Kalton, author of this work, suddenly passed away on August 31, 2010. The present article was essentially ready at the time of his death, but some editing work had to be done before it could actually be submitted. Nigel's friends and colleagues are grateful to Gilles Lancien who took care of this editing task with kindness and efficiency.

2. Preliminaries from linear Banach space theory

Our notation for Banach spaces is fairly standard (see e.g. [1, 18, 35]). If X is a Banach space, B_X denotes its closed unit ball and ∂B_X the unit sphere $\{x : \|x\| = 1\}$.

We recall that if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ and X is a Banach space, then the *ultrapower* $X_{\mathcal{U}}$ is defined to be the quotient of $\ell_\infty(X)$ by the subspace of all sequences $(x_n)_{n=1}^\infty$ such that $\lim_{n\in\mathcal{U}} \|x_n\| = 0$. A Banach space is super-reflexive if every ultraproduct is reflexive.

We recall that a separable Banach space X has the *approximation property (AP)* if given any compact subset K of X and $\epsilon > 0$ there is a finite-rank operator $T : X \to X$ with $\|Tx - x\| < \epsilon$ for $x \in K$. X has the *metric approximation property (MAP)* if we can also require $\|T\| \le 1$. Any reflexive Banach space with (AP) has (MAP)(see [35] p. 39). X is said to have the *uniform approximation property* (UAP) if there is a constant K such that for every m there exists n so that if F is a subspace of X of dimension m we can find an operator $T : X \to X$ with rank at most n, $\|T\| \le K$ and $Tx = x$ for $x \in F$. The uniform approximation property was first introduced by Pełczyński and Rosenthal [39]; rather few spaces have this property, but they include the L_p-spaces and reflexive Orlicz spaces [34].

X has a *finite-dimensional decomposition (FDD)* if there is a sequence of finite-rank operators $P_n : X \to X$ such that $P_m P_n = 0$ when $m \ne n$ and $x = \sum_{n=1}^\infty P_n x$ for every $x \in X$. If each P_n has rank one, then X has a *basis.* The (FDD) is called *shrinking* if we also have $x^* = \sum_{n=1}^\infty P_n^* x^*$ for every $x^* \in X^*$. If, in addition, $x = \sum_{n=1}^\infty P_n x$ unconditionally for every $x \in X$, then X has an *unconditional finite-dimensional decomposition (UFDD).* Finally if $\|\sum_{k=1}^n \eta_k P_k\| \le 1$ for every $n \in \mathbb{N}$ and $\eta_k = \pm 1$ for $1 \le k \le n$, then we say that X has a *1-(UFDD).*

We shall say that a Banach space X is *p-uniformly smooth* for $1 < p \le 2$ (or X has a modulus of smoothness of power type p) if for some constant C we have the estimate

$$\frac{1}{2}(\|x_1 + x_2\|^p + \|x_1 - x_2\|^p) \le \|x_1\|^p + C^p \|x_2\|^p, \qquad x_1, x_2 \in X.$$

We say that X is *p-uniformly convex* for $2 \le p < \infty$ (or X has a modulus of convexity of power type p) if for some constant $c > 0$ we have

$$\|x_1\|^p + c^p \|x_2\|^p \le \frac{1}{2}(\|x_1 + x_2\|^p + \|x_1 - x_2\|^p), \qquad x_1, x_2 \in X.$$

We shall frequently deal with ℓ_p-sums of Banach spaces $(X_n)_{n=1}^\infty$. We denote by $(\sum_{n=1}^\infty X_n)_{\ell_p}$ the space of sequences $(x_n)_{n=1}^\infty$ with $x_n \in X_n$ and

$$\|(x_n)_{n=1}^\infty\| = (\sum_{n=1}^\infty \|x_n\|^p)^{1/p} < \infty.$$

If $X_n = X$ is a fixed Banach space, we use the notation $\ell_p(X)$.

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3. Asymptotic moduli in Banach space theory

We now discuss asymptotic uniform smoothness and asymptotic uniform convexity. Let X be a separable Banach space. We define the *modulus of asymptotic uniform smoothness* (due to Milman [36]) $\overline{\rho}(t) = \overline{\rho}_X(t)$ by

$$\overline{\rho}(t) = \sup_{x\in\partial B_X} \inf_E \sup_{y\in\partial B_E} \{\|x+ty\|-1\},$$

where E runs through all closed subspaces of X of finite codimension.

The *modulus of asymptotic uniform convexity* is defined by

$$\overline{\delta}(t) = \inf_{x\in\partial B_X} \sup_E \inf_{y\in\partial B_E} \{\|x+ty\|-1\},$$

where E runs through all closed subspaces of X of finite codimension. Similarly in X^* there is a weak*-modulus of asymptotic uniform convexity defined by

$$\overline{\delta}^*(t) = \inf_{x^*\in\partial B_{X^*}} \sup_E \inf_{y\in\partial B_E} \{\|x^*+ty^*\|-1\},$$

where E runs through all weak*-closed subspaces of X^* of finite codimension.

As shown in [19], if $\overline{\rho}(t) < t$ for some $0 < t \le 1$, then X^* is separable. On the other hand if $\overline{\rho}(t) = 0$ for some $t > 0$, then X is isomorphic to a subspace of c_0 (see [10] and [19]). We say that X is *asymptotically uniformly smooth* if $\lim_{t\to 0}\overline{\rho}(t)/t = 0$. If X is asymptotically uniformly smooth this implies that $\overline{\rho}(t)/t \le Ct^\theta$ for some $0<\theta<1$ (see [32] and [11]). The function $\overline{\rho}$ is clearly convex, while the function $\overline{\delta}$ satisfies the condition that $\overline{\delta}(t)/t$ is increasing so that if we define the convex function

$$\tilde{\delta}(t) = \int_0^t \frac{\overline{\delta}(s)}{s}\,ds,$$

then

$$\overline{\delta}(t/2) \le \tilde{\delta}(t) \le \overline{\delta}(t), \qquad 0<t<\infty$$

so that $\overline{\delta}$ is equivalent to a convex function.

It is clear that we have that if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, $x \neq 0$ and $(x_n)_{n=1}^\infty$ is a weakly null sequence, then we have

$$\|x\| \lim_{n\in\mathcal{U}} \overline{\delta}(\|x_n\|/\|x\|) \le \lim_{n\in\mathcal{U}} \|x+x_n\| - \|x\| \le \|x\| \lim_{n\in\mathcal{U}} \overline{\rho}(\|x_n\|/\|x\|).$$

This can alternatively be viewed as the statement that

$$\|x\| \lim_{n\to\infty} \overline{\delta}(\|x_n\|/\|x\|) \le \lim_{n\to\infty} \|x+x_n\| - \|x\| \le \|x\| \lim_{n\to\infty} \overline{\rho}(\|x_n\|/\|x\|)$$

whenever all the limits exist. It is clear that if X^* is separable this is an equivalent formulation of the definition.

We remark that it is trivial that if $1<p<\infty$, $\overline{\delta}_{\ell_p}(t) = \overline{\rho}_{\ell_p}(t) = (1+t^p)^{1/p}-1$. We will need the fact that for the corresponding function spaces we have:

Proposition 3.1 ([36]). *Suppose* $1<p<\infty$. *If* $1<p<2$, *then there is a constant* $c=c_p>0$ *such that*

$$\overline{\rho}_{L_p}(t) \le (1+c^pt^p)^{1/p}-1.$$

If $2<p<\infty$, *then there is a constant* $c=c_p>0$ *such that*

$$\overline{\delta}_{L_p}(t) \ge (1+c^pt^p)^{1/p}-1.$$

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Remark. See [36], p. 117. This proposition may be expressed in the following terms. If $1 < p < 2$, then

$$\lim_{n\in\mathcal{U}} \|f + g_n\|_p \le \lim_{n\in\mathcal{U}} (\|f\|^p + c_p^p \|g_n\|^p)^{1/p}$$

whenever $(g_n)_{n=1}^\infty$ is a weakly null sequence in L_p and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Similarly if $2 < p < \infty$,

$$\lim_{n\in\mathcal{U}} \|f + g_n\|_p \ge \lim_{n\in\mathcal{U}} (\|f\|^p + c_p^p \|g_n\|^p)^{1/p}$$

whenever $(g_n)_{n=1}^\infty$ is a weakly null sequence in L_p.

We will also need the following proposition.

Proposition 3.2 ([19])**.** *Let X be a Banach space and suppose $Y = X/E$ is a quotient of X. Then $\overline{\delta}_Y \ge \overline{\delta}_X$ and $\overline{\rho}_Y \le \overline{\rho}_X$.*

There is a natural variant of $\overline{\delta}$ which will be very useful in this paper. We define

$$\hat{\delta}_X(t) = \inf_{x\in\partial B_X} \sup_E \inf_{y\in\partial B_E} \{\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1\},$$

where again E runs through all closed subspaces of X of finite codimension. As with $\overline{\delta}_X$ the function $\hat{\delta}_X(t)/t$ is increasing and so $\hat{\delta}_X$ is equivalent to a convex function. Clearly $\hat{\delta}_X \le \overline{\delta}_X$.

If $(x_n)_{n=1}^\infty$ is a bounded sequence we define $\operatorname{sep}\{x_n\}_{n=1}^\infty = \inf_{m\neq n} \|x_m - x_n\|$.

Proposition 3.3. *Suppose $u, v \in X$ with $\|u - v\| = 1$. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in X and let $t = \operatorname{sep}\{x_n\}_{n=1}^\infty$. Then*

$$\liminf_{n\to\infty} (\|u - x_n\| + \|v - x_n\|) \ge 1 + \hat{\delta}_X(t).$$

Proof. It is clearly enough to show that for all $\nu > 0$ there exists an m with

$$\|u - x_m\| + \|v - x_m\| > 1 + \hat{\delta}_X(t) - \nu.$$

Choose a finite-codimensional subspace E so that if $z \in \partial B_E$, then

$$\frac{1}{2}(\|u - v + tz\| + \|u - v - tz\|) \ge 1 + \hat{\delta}(t) - \frac{1}{2}\nu.$$

Since X/E is finite-dimensional we can find $m \neq n$ so that $d(x_m - x_n, E) < \nu/2$. Hence there exists $z \in \partial B_E$ and $\tau > t$ so that $\|x_m - x_n - \tau z\| < \nu$. Then

$$\frac{1}{2}(\|u - v + (x_m - x_n)\| + \|u - v - (x_m - x_n)\|) \ge 1 + \hat{\delta}(t) - \nu.$$

Now

$$\|u - v + (x_m - x_n)\| \le \|u - x_n\| + \|v - x_m\|$$

and

$$\|u - v - (x_m - x_n)\| \le \|u - x_m\| + \|v - x_n\|$$

so that combining we have either

$$\|u - x_m\| + \|v - x_m\| > 1 + \hat{\delta}_X(t) - \nu$$

or

$$\|u - x_n\| + \|v - x_n\| > 1 + \hat{\delta}_X(t) - \nu.$$

□

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4. The Banach-Saks property and iterated norms

We recall that every bounded sequence $(y_n)_{n=1}^\infty$ in a Banach space has a *spreading subsequence* $(x_n)_{n=1}^\infty$ so that

$$\lim_{(n_1,\ldots,n_m)\to\infty} \|\sum_{j=1}^m a_j x_{n_j}\| = \|\sum_{j=1}^m a_j e_j\|_S$$

exists for all finite scalar sequences $(a_1,\ldots,a_m)$ and defines a seminorm on the space c_{00} of all finitely supported scalar sequences. By this notation we mean that for any $\epsilon > 0$ and $(a_1,\ldots,a_m)$ there exists q so that if $q < n_1 < n_2 < \cdots < n_m$, then

$$\left| \|\sum_{j=1}^m a_j x_{n_j}\| - \|\sum_{j=1}^m a_j e_j\| \right| < \epsilon.$$

As long as $(x_n)_{n=1}^\infty$ is not convergent in norm the seminorm $\|\cdot\|_S$ is a norm. Then $(e_j)_{j=1}^\infty$ is the *spreading model* associated to $(x_n)_{n=1}^\infty$ and is a sequence in the Banach space S obtained by completing c_{00}. If $(x_n)_{n=1}^\infty$ is weakly null we say that $(e_j)_{j=1}^\infty$ is a *weakly null spreading model*; this may not imply that $(e_n)_{n=1}^\infty$ is itself a weakly null sequence in S.

We will be particularly interested in the possible growth rate of $\|\sum_{j=1}^n e_j\|_S$, for a given normalized spreading sequence $(x_n)_{n=1}^\infty$. Note that if $\lim_{n\to\infty}\|\sum_{j=1}^n e_j\| = \infty$, then given any $\nu > 0$ and $k \in \mathbb{N}$, using Ramsey arguments, we can pass to a subsequence and assume that

$$(1-\nu)\|\sum_{j=1}^k e_j\| \le \|\sum_{j=1}^k x_{n_j}\| \le (1+\nu)\|\sum_{j=1}^k e_j\|, \qquad n_1 < n_2 < \cdots < n_k.$$

Lemma 4.1. *Let X be a Banach space and suppose $(e_j)_{j=1}^\infty$ is a spreading model of a normalized sequence $(x_n)_{n=1}^\infty$. Then:*

$$\sup_{\epsilon_j=\pm1} \|\sum_{j=1}^k \epsilon_j e_j\| \le 3\|\sum_{j=1}^k e_j\|$$

and if X is super-reflexive and $(x_n)_{n=1}^\infty$ is weakly null,

$$\|\sum_{j=1}^k e_j\| \le 2\mathbb{E}\|\sum_{j=1}^k \epsilon_j e_j\|,$$

where $(\epsilon_j)_j$ denotes a sequence of independent Rademacher variables.

Proof. Let $\alpha_k = \|\sum_{j=1}^k e_j\|$. Then $\alpha_{k+l} \le \alpha_k + \alpha_l$. Hence $\lim_k \alpha_k/k = \inf_k \alpha_k/k = \theta$ exists. Now for any integer m we have

$$\|\sum_{j=1}^k e_j + \frac{1}{m}\sum_{j=k+1}^{k+ml} e_j\| \le \alpha_{k+l}$$

so that

$$\alpha_k \le \alpha_{k+l} + \frac{1}{m}\alpha_{ml}.$$

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Thus letting $m \to \infty$,

$$\alpha_k \leq \alpha_{k+l} + l\theta \leq \frac{k+2l}{k+l}\alpha_{k+l}.$$

For any k it is clear that

$$\sup_{\epsilon_j=\pm 1} \|\sum_{j=1}^{k} \epsilon_j e_j\| \leq \max_{j\leq k}(\alpha_j + \alpha_{k-j}) \leq 3\alpha_k$$

by the preceding equation.

For the second part we observe that $(e_n)_{n=1}$ is also weakly null and hence 2-unconditional [6]. □

For $\mathbb{M}$ an infinite subset of $\mathbb{N}$, let $\mathcal{G}_k(\mathbb{M})$ denote the space of k-subsets $\{n_1,\dots,n_k\}$ (where $n_1 < n_2 < \cdots < n_k$) of $\mathbb{M}$ regarded as a graph in which $\{m_1, m_2, \dots, m_k\}$ and $\{n_1, n_2, \dots, n_k\}$ are adjacent if they interlace, i.e. $m_1 \leq n_1 \leq m_2 \leq \cdots \leq m_k \leq n_k$ or $n_1 \leq m_1 \leq \cdots \leq n_k \leq m_k$. Let d be the associated least path metric. Let us recall [28] that a Banach space X has property $\mathcal{Q}$ if there is a constant C so that whenever $f : \mathcal{G}_k(\mathbb{N}) \to X$ has Lipschitz constant one, then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that

$$\|f(m_1,\dots,m_k) - f(n_1,\dots,n_k)\| \leq C, \quad (n_1,\dots,n_k),\ (m_1,\dots,m_k) \in \mathcal{G}_k(\mathbb{M}).$$

It is shown in [28] that if either X coarsely embeds in a reflexive space or B_X uniformly embeds in a reflexive space, then X must have property $\mathcal{Q}$.

Proposition 4.2. *Let X be a Banach space with property $\mathcal{Q}$. Then for each spreading model $(e_n)_{n=1}^\infty$ of X there is a constant C so that*

$$\|\sum_{j=1}^{n} e_j\| \leq C\mathbb{E}\|\sum_{j=1}^{n} \epsilon_j e_j\|.$$

Proof. We consider two cases. If $(e_n)_{n=1}^\infty$ is not weakly Cauchy, then $(e_n)_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1 by Rosenthal's theorem [42], and the result is clear. If not, then the sequence $(e_{2j-1} - e_{2j})_{j=1}^\infty$ is 2-unconditional. Hence

$$\|\sum_{j=1}^{k}(e_{2j-1} - e_{2j})\| \leq 2\mathbb{E}\|\sum_{j=1}^{k} \epsilon_j(e_{2j-1} - e_{2j})\| \leq 4\mathbb{E}\|\sum_{j=1}^{k} \epsilon_j e_j\|.$$

Now passing to a suitable subsequence of $(x_n)_{n=1}^\infty$ we can assume that

$$\frac{1}{2}\|\sum_{j=1}^{2k} a_j e_j\| \leq \|\sum_{j=1}^{2k} a_j x_{n_j}\| \leq 2\|\sum_{j=1}^{2k} a_j e_j\|$$

whenever $n_1 < n_2 < \cdots < n_{2k}$ and $|a_j| = 1$.

Define $f : \mathcal{G}_k \to X$ by $f(n_1, n_2, \dots, n_k) = x_{n_1} + \cdots + x_{n_k}$. Then f, using the preceding calculation, has Lipschitz constant at most $8\mathbb{E}\|\sum_{j=1}^k \epsilon_j e_j\|$. Hence by property $\mathcal{Q}$ for a suitable constant C independent of k, we can find $n_1 < n_2 < \cdots < n_k < m_1 < \cdots < m_k$ with

$$\|\sum_{j=1}^{k} x_{n_j} - \sum_{j=1}^{k} x_{m_j}\| \leq C\mathbb{E}\|\sum_{j=1}^{k} \epsilon_j e_j\|.$$

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Hence

$$\|\sum_{j=1}^{k} e_j - \sum_{j=k+1}^{2k} e_j\| \le 2C\mathbb{E}\|\sum_{j=1}^{k}\epsilon_j e_j\|.$$

In this case $(e_n)_{n=1}^{\infty}$ has basis constant one and so

$$\|\sum_{j=1}^{k} e_j\| \le 2C\mathbb{E}\|\sum_{j=1}^{k}\epsilon_j e_j\|.$$

□

We say that a norm N on $\mathbb{R}^2$ is absolute if

$$N(a,b) = N(|a|,|b|), \qquad a,b \in \mathbb{R}.$$

For any Lipschitz convex Orlicz function F, the limit $\lim_{t\to\infty} F(t)/t = \theta$ exists and there is a corresponding absolute norm defined by

$$N_F(a,b) = \begin{cases} |a|(1+F(|b|/|a|)), & a \ne 0, \\ \theta|b|, & a = 0. \end{cases}$$

Now suppose N is an absolute norm on $\mathbb{R}^2$ with $N(1,0)=1$. We define the sequence space Λ_N as the completion of c_{00} under the norm defined iteratively by $\|e_1\|_{\Lambda_N} = 1$ and then

$$\|\sum_{j=1}^{n} a_j e_j\|_{\Lambda_N} = N(\|\sum_{j=1}^{n-1} a_j e_j\|_{\Lambda_N}, |a_n|), \qquad n \ge 2.$$

Spaces of this type were first considered in [26]. The space Λ_N coincides with the space h_F, where $F(t) = N(1,t)-1$; here h_F denotes the closure of c_{00} in the Orlicz sequence space ℓ_F. In fact we have

Lemma 4.3. *If $a \in c_{00}$, then*

$$\tfrac{1}{2}\|a\|_{\ell_F} \le \|a\|_{\Lambda_N} \le e\|a\|_{\ell_F}.$$

Proof. Assume $\|a\|_{\ell_F} \le 1$. Then

$$\|e_1 + \sum_{j=1}^{n} a_j e_{j+1}\|_{\Lambda_N} \le \prod_{j=1}^{n}(1+F(|a_j|)) \le e.$$

Conversely if $\|a\|_{\Lambda_N} \le 1$ we have

$$\|e_1 + \sum_{j=1}^{n} a_j e_{j+1}\|_{\Lambda_N} \ge \prod_{j=1}^{n}(1+F(|a_j|/2)) \ge 1 + \sum_{j=1}^{n} F(|a_j|/2)$$

so that $\|a\|_{\ell_F} \le 2$. □

We will need the following proposition:

Proposition 4.4. *Let X be a Banach space with separable dual. Then there exist constants $0 < c < C < \infty$ so that for any spreading model $(e_j)_{j=1}^{\infty}$ of a normalized weakly null sequence we have*

$$c\|\sum_{j=1}^{n} a_j e_j\|_{\ell_{\overline{\delta}}} \le \|\sum_{j=1}^{n} a_j e_j\|_S \le C\|\sum_{j=1}^{n} a_j e_j\|_{\ell_{\overline{\rho}}}. \tag{4.3}$$

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Remark. Of course, the function $\overline{\delta}$ is not necessarily convex but is equivalent to the convex function $\tilde{\delta}$.

Proof. It is easy to check that

$$\|a_1e_1 + \cdots + a_ne_n\|_S \geq \|a_1e_1 + \cdots + a_ne_n\|_{\Lambda_N},$$

where $N(1,t) = 1 + \tilde{\delta}(t)$. Similarly

$$\|a_1e_1 + \cdots + a_ne_n\|_S \leq \|a_1e_1 + \cdots + a_ne_n\|_{\Lambda_{N'}}.$$

where $N'(1,t) = 1 + \overline{\rho}(t)$. Then apply Lemma 4.3. □

The left-hand side of (4.3) can be improved:

Proposition 4.5. *Let X be any Banach space. Then there exists a constant $0 < c < \infty$ so that for any spreading model $(e_j)_{j=1}^{\infty}$ of a normalized sequence we have*

$$c\|\sum_{j=1}^{n} a_je_j\|_{\ell_{\hat{\delta}}} \leq \mathbb{E}\|\sum_{j=1}^{n} \epsilon_ja_je_j\|_S. \tag{4.4}$$

Proof. Let N be the absolute norm such that

$$N(1,t) = 1 + \int_0^t \hat{\delta}(s)\frac{ds}{s}, \quad t \geq 0.$$

Then we prove that

$$\|\sum_{j=1}^{n} a_je_j\|_{\Lambda_N} \leq \mathbb{E}\|\sum_{j=1}^{n} \epsilon_ja_je_j\|_S$$

by induction on n. Assume $n \geq 2$ and the result is known for $n-1$. It is clear that

$$\begin{aligned}
\mathbb{E}\|\sum_{j=1}^{n} \epsilon_ja_je_j\|_S &\geq \mathbb{E}N(\|\sum_{j=1}^{n-1} \epsilon_ja_je_j\|, |a_n|) \\
&\geq N(\mathbb{E}\|\sum_{j=1}^{n-1} \epsilon_ja_j\|, |a_n|) \\
&\geq N(\|\sum_{j=1}^{n-1} a_je_j\|_{\Lambda_N}, |a_n|) \\
&= \|\sum_{j=1}^{n} a_je_j\|_{\Lambda_N}.
\end{aligned}$$

This concludes the proof. □

We say that a Banach space X not containing ℓ_1 has the *p-Banach-Saks property* $(1 < p < \infty)$ if there is a constant C so that for every spreading model $(e_j)_{j=1}^{\infty}$ of a normalized weakly null sequence we have

$$\|\sum_{j=1}^{k} e_j\|_S \leq Ck^{1/p}, \qquad k = 1, 2, \ldots.$$

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This is equivalent to the requirement that there is a constant C' so that every normalized weakly null sequence $(x_n)_n$ has a subsequence $(x_{n_j})_{j=1}^\infty$ such that

$$\|\sum_{j=1}^k x_{n_j}\| \le C'k^{1/p}, \qquad k \in \mathbb{N},\ n_1 < \cdots < n_k.$$

We say that X has the *p-co-Banach-Saks property* $(1 < p < \infty)$ if there is a constant $c > 0$ so that for every spreading model $(e_j)_{j=1}^\infty$ of a normalized weakly null sequence we have

$$\|\sum_{j=1}^k e_j\|_S \ge ck^{1/p}, \qquad k = 1, 2, \ldots.$$

The following proposition follows from Proposition 4.4:

Proposition 4.6. *If $\overline{\rho}(t) \le Ct^p$ for $0 \le t \le 1$, then X has the p-Banach-Saks property. If $\overline{\delta}(t) \ge ct^p$ for $0 \le t \le 1$, then X has the p-co-Banach-Saks property.*

There is a simple duality relationship between these concepts, which we will need:

Proposition 4.7. *Let X be a reflexive space with the p-Banach-Saks property, where $1 < p < \infty$. Then X^* has the q-co-Banach-Saks property, where $q = p/(p-1)$.*

Proof. Let C be the constant of the p-Banach-Saks property for X. Let $(x_n^*)_{n=1}^\infty$ be a normalized weakly null sequence in X^*. We may pick a normalized sequence $(x_n)_{n=1}^\infty$ in X with $x_n^*(x_n) = 1$. Passing to a subsequence we can assume that $\lim_{n\to\infty} x_n = x$ weakly. Then $\|x_n - x\| \le 2$ and so passing to a further subsequence we can assume that for any k,

$$\lim_{(n_1,n_2,\ldots,n_k)\to\infty} \|x_{n_1} + \cdots + x_{n_k} - kx\| \le 2Ck^{1/p}$$

for any k. However

$$\lim_{n_1\to\infty} \ldots \lim_{n_k\to\infty} \langle x_{n_1} + \cdots + x_{n_k} - kx, x_{n_1}^* + \cdots + x_{n_k}^* \rangle = k,$$

which implies that in any spreading model $(e_j)_{j=1}^n$ of $(x_j^*)_{j=1}^\infty$ we must have

$$\|\sum_{j=1}^k e_j\|_S \ge (1/(2C))k^{1/q}.$$

□

Finally let us also introduce a version of the p-co-Banach-Saks property for $p = 1$. We will say that X has the *anti-Banach-Saks property* if there is a constant $c > 0$ so that for every spreading model $(e_j)_{j=1}^\infty$ of a normalized sequence, we have

$$\|\sum_{j=1}^k e_j\|_S \ge ck, \qquad k = 1, 2, \ldots.$$

We make some simple observations about this property.

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Proposition 4.8. *Let X be any Banach space. The following conditions on X are equivalent:*

(i) X has the anti-Banach-Saks property.

(ii) There is a constant c' so that for any spreading model of a normalized sequence $(x_n)_{n=1}^\infty$ we have

$$\mathbb{E}\|\sum_{j=1}^{k}\epsilon_j e_j\|_S \geq c'\theta k,$$

where $\theta = \operatorname{sep}\{x_n\}_{n=1}^\infty$.

Proof. (i) $\implies$ (ii). Let us first note that if X has the anti-Banach-Saks property (with constant c) and $(e_j)_{j=1}^\infty$ is a spreading model of a normalized sequence, then any weak*-cluster point x^{**} of $(e_j)_{j=1}^\infty$ in S^{**} has norm at least c. Indeed, given $\nu > 0$, by Goldstine's theorem we can find $a_1, \dots, a_m \geq 0$ with $\sum_{j=1}^m a_j = 1$ and $\|\sum_{j=1}^m a_j e_j\|_S < \|x^{**}\| + \nu$. Hence for any $k > m$,

$$\|\sum_{i=1}^{k}\sum_{j=1}^{m} a_j e_{i+j}\|_S \leq k(\|x^{**}\| + \nu).$$

Rewriting this we obtain

$$c(k-m) \leq \|\sum_{i=m+1}^{k} e_i\|_S \leq k(\|x^{**}\| + \nu) + 2m.$$

Letting $k \to \infty$ gives $c \leq \|x^{**}\| + \nu$, where $\nu > 0$ is arbitrary.

Hence under the conditions of (ii), applying the above reasoning to $(e_{2j-1} - e_{2j})$, we find a weak*-cluster point z^{**} of this sequence with $\|z^{**}\| \geq c\theta$. It follows that there exists $\varphi \in S^*$ with $\|\varphi\| = 1$ and $\lim_{j\to\infty}\varphi(e_{2j}) = \alpha$ and $\lim_{j\to\infty}\varphi(e_{2j-1}) = \beta$, where $\beta - \alpha \geq c\theta$. By considering translates we deduce the existence of $\psi \in S^*$ with $\|\psi\| \leq 1$ and $\psi(e_j) = \frac{1}{2}(\beta - \alpha)(-1)^j$. From this it is clear using the properties of the spreading model that for any choice of sign ϵ_j we have:

$$\|\sum_{j=1}^{k}\epsilon_j e_j\|_S \geq \frac{1}{2}c\theta k.$$

(ii) $\implies$ (i). Let $(e_j)_{j=1}^\infty$ be a spreading model of an arbitrary normalized sequence. If $\|e_1 - e_2\|_S \leq 1/2$, then

$$\|e_1 + \cdots + e_k\|_S \geq k/2, \qquad k = 1, 2, \dots.$$

Otherwise

$$\mathbb{E}\|\sum_{j=1}^{k}\epsilon_j e_j\|_S \geq \tfrac{1}{2}c' k$$

and so, using Lemma 4.1,

$$\|\sum_{j=1}^{k} e_j\|_S \geq \tfrac{1}{6}c' k.$$

$\square$

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5. ℓ_p-SUMS OF FINITE-DIMENSIONAL SPACES

The special properties of ℓ_p-sums of finite-dimensional spaces have been studied in detail by many authors. Many of the ideas in this section originated in the early work of Johnson and Zippin on the spaces C_p ([16], [22] and [23]). See also [35].

For $1 \le p < \infty$, we shall say that a separable Banach space X has *property* $(\tilde{m}_p)$ if it is isomorphic to a closed subspace of a space $(\sum_{n=1}^\infty E_n)_{\ell_p}$, where $(E_n)_{n=1}^\infty$ is a sequence of finite-dimensional spaces. This terminology is motivated by the definition of property (m_p) for $1 < p < \infty$. We recall that a Banach space X has property (m_p) [31] if for every $x \in X$ and every weakly null sequence $(x_n)_{n=1}^\infty$ such that the limits exist we have

$$\lim_{n\to\infty} \|x + x_n\|^p = \|x\|^p + \lim_{n\to\infty} \|x_n\|^p.$$

Condition (m_p) is exactly equivalent to the conditions $\overline{\rho}_X(t) = \overline{\delta}_X(t) = (1+t^p)^{1/p} - 1$. It is clear that if X has $(\tilde{m}_p)$ for $1 < p < \infty$ it has an equivalent norm with property (m_p).

There are several characterizations of property $(\tilde{m}_p)$ for $1 < p < \infty$. The following result is due to Johnson, Lindenstrauss, Preiss and Schechtman [19], Proposition 2.11 (see also [37] for another isomorphic version).

Theorem 5.1. *Suppose $1 < p < \infty$ and let X be a separable reflexive Banach space. In order that X has $(\tilde{m}_p)$ it is necessary and sufficient that it is isomorphic to a space Y with $\overline{\rho}_Y(t) \le Ct^p$ for $0 \le t \le 1$ and to a space Z with $\overline{\delta}_Z(t) \ge ct^p$ for $0 \le t \le 1$, where $0 < c, C < \infty$.*

On the other hand we have the following theorem. Part (i) is proved in [31], Theorem 3.2 and its proof; part (ii) follows from (i) by duality.

Theorem 5.2. *Suppose $1 < p < \infty$. If X is a separable Banach space with property (m_p), then:*

(i) X is linearly isomorphic to a quotient of a space $(\sum_{n=1}^\infty E_n)_{\ell_p}$, where $(E_n)_{n=1}^\infty$ is a sequence of finite-dimensional subspaces of X.

(ii) X is linearly isomorphic to a subspace of a space $(\sum_{n=1}^\infty E_n)_{\ell_p}$, where $(E_n)_{n=1}^\infty$ is a sequence of finite-dimensional quotients of X.

Note that it is actually shown in [31], Theorem 3.2 that if X has property (m_p), then for every $\epsilon > 0$, X is $(1+\epsilon)-$isomorphic to a subspace of a space $(\sum_{n=1}^\infty E_n)_{\ell_p}$ with each $\dim E_n < \infty$.

The fact that in Theorem 5.2 (ii) one requires that the $(E_n)_{n=1}^\infty$ be quotients rather than subspaces is an inconvenience which can be rectified if X has the approximation property. Results of this nature go back to the early work of Johnson and Zippin [23], who proved such a result for the special case of $C_p = (\sum_{n=1}^\infty G_n)_{\ell_p}$, where $(G_n)_{n=1}^\infty$ is a sequence dense in all finite-dimensional spaces in the sense of Banach-Mazur distance.

Proposition 5.3. *Suppose $1 < p < \infty$ and X is a separable Banach space with property $(\tilde{m}_p)$ and the approximation property. Then there is a sequence of finite-rank operators $A_n : X \to X$ such that $A_jA_k = 0$ for $|k - j| > 1$,*

$$x = \sum_{n=1}^\infty A_n x, \qquad x \in X$$

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and for some constant C we have

$$C^{-1}\|x\| \leq (\sum_{k=1}^{\infty}\|A_kx\|^p)^{1/p} \leq C\|x\|, \qquad x \in X.$$

Proof. We may assume that X is a subspace of a space $Z = (\sum_{n=1}^{\infty} G_n)_{\ell_p}$. Let $S_n : Z \to Z$ be the partial sum operators associated to the canonical (FDD) of Z. Let $S_0 = 0$. It follows from [16] that X has the commuting metric approximation property and so (see [7]), we may find a sequence of finite-rank operators $R_n : X \to X$ that are finite-rank operators such that $x = \lim_{n\to\infty} R_nx$ for $x \in X$ and $x^* = \lim_{n=1}^{\infty} R_n^*x^*$ for $x^* \in X^*$ and $R_mR_n = R_nR_m = R_n$ if $m > n$. Consider the operators S_n, R_n in $\mathcal{K}(X, Z)$. Then we have $\lim_{n\to\infty} x^{**}(S_n^* - R_n^*)z^* = 0$ for $x^{**} \in X$ and $z^* \in Z^*$. This implies by Corollary 3 of [25] that $(S_n - R_n)$ converges weakly to 0.

Now fix $(\epsilon_k)_{k=1}^{\infty}$ with $\epsilon_k > 0$ and such that $\sum_{k=1}^{\infty}\epsilon_k < 1/8$. It follows from Mazur's Theorem that we can find an increasing sequence of integers $(m_k)_{k=0}^{\infty}$ with $m_0 = 0$ and nonnegative $(a_j)_{j=1}^{\infty}$ with

$$\sum_{j=m_{k-1}+1}^{m_k} a_j = 1, \qquad k = 1, 2, \ldots$$

and

$$\|\sum_{j=m_{k-1}+1}^{m_k} a_j(S_j - R_j)\|_{X\to Z} < \epsilon_k, \qquad k = 1, 2, \ldots.$$

We define $V_k = \sum_{j=m_{k-1}+1}^{m_k} a_jR_j$ and $T_k = \sum_{j=m_{k-1}+1}^{m_k} a_jS_j$ with $V_0 = T_0 = 0$. Let $A_k = V_k - V_{k-1}$ and $B_k = T_k - T_{k-1}$. Then $A_jA_k = 0$ if $|j - k| \geq 1$,

$$\|A_k - B_k\| \leq \epsilon_k + \epsilon_{k-1}.$$

Hence for $x \in X$,

$$\left|(\sum_{k=1}^{\infty}\|A_kx\|^p)^{1/p} - (\sum_{k=1}^{\infty}\|B_kx\|^p)^{1/p}\right| \leq \frac{1}{4}\|x\|.$$

On the other hand,

$$\frac{1}{2}\|x\| \leq (\sum_{k=1}^{\infty}\|B_kx\|^p) \leq 2\|x\|, \qquad x \in X.$$

□

Theorem 5.4. *Suppose $1 < p < \infty$. Suppose X is a separable Banach space with property $(\tilde{m}_p)$ and the approximation property. If X is a complemented subspace in a Banach space Y and $(E_i)_{i\in I}$ is a directed family of finite-dimensional subspaces of Y with $\bigcup_{i\in I} E_i$ dense in Y, then X is isomorphic to a complemented subspace of a space $(\sum E_{i_n})_{\ell_p}$ for some sequence $(i_n)_{n=1}^{\infty}$ in I.*

In particular there is a sequence of finite-dimensional subspaces, $(F_n)_{n=1}^{\infty}$ of X such that X is linearly isomorphic to a complemented subspace of $(\sum_{n=1}^{\infty} F_n)_{\ell_p}$.

Proof. Let (A_n) be the finite rank operators given by the previous proposition. We may embed $A_n(X)$ in a finite-dimensional subspace of Y, H_n, say, such that

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$d(H_n, E_{i_n}) \leq 2$ for a suitable choice of i_n. Let $P : Y \to X$ be a bounded projection and define $Q : (\sum_{n=1}^{\infty} H_n)_{\ell_p} \to X$ by

$$Q(h_k)_{k=1}^{\infty} = \sum_{j=1}^{\infty} \sum_{|j-k|\leq 1} A_j P h_k.$$

Notice that if $(x_k)_k$ is a finitely nonzero sequence with $x_k \in A_k(X)$ we have an estimate for $j = 0, 1, 2$:

$$\begin{aligned}\|\sum_{k=1}^{\infty} x_{3k-j}\| &= \|\sum_{k=1}^{\infty}(A_{3k-j+1} + A_{3k-j} + A_{3k-j-1})x_{3k-j}\| \\ &\leq C(\sum_{k=1}^{\infty} \|A_{3k-j+1}x_{3k-j}\|^p + \|A_{3k-j}x_{3k-j}\|^p + \|A_{3k+j+1}x_{3k-j}\|^p)^{1/p} \\ &\leq 3C^2(\sum_{k=1}^{\infty} \|x_{3k-j}\|^p)^{1/p}.\end{aligned}$$

Hence if $h = (h_k)_{k=1}^{\infty}$ is finitely nonzero,

$$\|Qh\| \leq 9C(\sum_{j=1}^{\infty} \|A_j(\sum_{|k-j|\leq 1} Ph_k)\|^p)^{1/p} \leq 3^3 C^2 \|P\|\|h\|$$

so that Q extends to a bounded operator.

Define $J : X \to (\sum_{n=1}^{\infty} H_n)_{\ell_p}$ by $Jx = (A_n x)_{n=1}^{\infty}$. Then J is bounded and $QJ = Id_X$. □

Our final result will be useful when studying uniform homeomorphisms.

Theorem 5.5. *Suppose $1 < p < \infty$ and that X is a separable Banach space with property $(\tilde{m}_p)$. Let $(E_n)_{n=1}^{\infty}$ be a sequence of finite-dimensional subspaces of X such that for some constant $\lambda \geq 1$ and every m, n there is a subspace $F_{m,n}$ of X such that $F_{m,n}$ is $\lambda-$complemented in X and $d(F_{m,n}, \ell_p^m(E_n)) \leq \lambda$. Then $(\sum_{n=1}^{\infty} E_n)_{\ell_p}$ is isomorphic to a complemented subspace of X.*

Proof. We can assume X has (m_p) (and so X^* has (m_q)). We first show that given any finite-dimensional subspaces $G \subset X$, $H \subset X^*$ and $n \in \mathbb{N}$ there exist operators $A : E_n \to X$ and $B : X \to E_n$ with $BA = I_{E_n}$, $\|A\|, \|B\| \leq 2\lambda$, and $A(E_n) \subset H^{\perp}$, $B^*(E_n^*) \subset G^{\perp}$.

Let $d_e = \dim E_n$, $d_g = \dim G$ and $d_h = \dim H$. Fix $m > 2^8\lambda^4 d_h(d_g + d_h)d_e$. By hypothesis there exist operators $S : \ell_p^m(E_n) \to X$ and $T : X \to \ell_p^m(E_n)$ with $TS = I_{\ell_p^m(E_n)}$ and $\|S\|, \|T\| \leq \lambda$. If we write $S(u_j)_{j=1}^m = \sum_{j=1}^m S_j u_j$ and $Tx = (T_j x)_{j=1}^m$, then $T_j S_j = I_{E_n}$.

We clearly have $\|\sum_{j=1}^m \theta_j T_j\|_{X\to E_n} \leq \lambda$ for all $\theta_j = \pm 1$. Since $\mathcal{L}(G, E_n)$ is $\sqrt{d_g d_e}$-isomorphic to a Hilbert space we have

$$\sum_{j=1}^m \|T_j\|_{G\to E_n}^2 \leq d_g d_e \lambda^2.$$

Similarly

$$\sum_{j=1}^m \|S_j^*\|_{H\to E_n^*}^2 \leq d_h d_e \lambda^2.$$

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Thus there exists j so that

$$\|T_j\|^2_{G\to E_n} + \|S_j\|^2_{H\to E_n^*} \le m^{-1}(d_g+d_h)d_e\lambda^2.$$

Now we can find two projections, P and Q on X with $\|P\| \le \sqrt{d}_g$ and $\|Q\| \le 2\sqrt{d}_h$ so that $P(X) = G$ and $Q^*(X^*) = H$. Now consider the operator $T_j(I-P)(I-Q)S_j$. We have

$$\|QS_j\| = \|S_j^*Q^*\| \le 2\sqrt{d_h}\|S_j^*\|_{H\to E_n^*} \le 2\lambda d_e^{1/2}d_h^{1/2}m^{-1/2}(d_g+d_h)^{1/2} \le 1/(8\lambda)$$

and

$$\|T_jP\| \le \lambda d_g^{1/2}d_e^{1/2}m^{-1/2}(d_g+d_h) \le 1/(8\lambda).$$

Hence

$$\|I_{E_n} - T_j(I-P)(I-Q)S_j)\| \le 1/4 + 1/(64\lambda^2) < 1/3.$$

Hence there is an operator $D : E_n \to E_n$ with $\|D\| \le 3/2$ so that

$$T_j(I-P)(I-Q)S_jD = I_{E_n}.$$

Let $B = T_j(I-P)$ and $A = (I-Q)S_jD$; then $\|A\|, \|B\| \le 2\lambda$. This completes the proof of our claim.

Since X has (m_p) and X^* has (m_q), it now follows that we can use an inductive construction to find two sequences of operators $A_n : E_n \to X$ and $B_n : X \to E_n$ so that

$$\|\sum_{n=1}^{\infty} A_nu_n\| \le 4\lambda(\sum_{n=1}^{\infty}\|u_n\|^p)^{1/p}, \qquad u_n \in E_n,\ n = 1,2,\dots$$

and

$$\|\sum_{n=1}^{\infty} B_n^*u_n^*\| \le 4\lambda(\sum_{n=1}^{\infty}\|u_n^*\|^q)^{1/q}, \qquad u_n^* \in E_n^*,\ n = 1,2,\dots$$

and $B_nA_n = I_{E_n}$.

Hence we may define $A : (\sum_{n=1}^\infty E_n)_{\ell_p} \to X$ and $B : X \to (\sum_{n=1}^\infty E_n)_{\ell_p}$ by $A((u_n)_{n=1}^\infty) = \sum_{n=1}^\infty A_nu_n$ and $Bx = \sum_{n=1}^\infty B_nx$ and we have $BA = I_{(\sum_{n=1}^\infty E_n)_{\ell_p}}$ and $\|A\|, \|B\| \le 4\lambda$. ∎

6. Subspaces and quotients of L_p

We now introduce a definition which will be useful to us later. This idea was first used in the work of Haydon, Raynaud and Levy on ultraproducts ([33] and [13]).

Let us say that a Banach space Y has a *random L_p-norm* if there is a (nonlinear) map $V : Y \to Z$, where Z is an abstract L_p-space such that:

$$Vy \ge 0, \qquad y \in Y,$$
$$V(\alpha y) = |\alpha|Vy, \qquad y \in Y,\ \alpha \in \mathbb{R},$$
$$V(y_1+y_2) \le Vy_1 + Vy_2, \qquad y \in Y,$$

and

$$\|Vy\|_p = \|y\|, \qquad y \in Y.$$

V is then called the random L_p−norm on Y. V is easily verified to be continuous and hence if Y is separable we can replace Z by $L_p[0,1]$. If $r > p$ we say that V is of type r if there is a constant C such that for any $y_1, y_2 \in Y$ we have

$$\frac{1}{2}(V(y_1+y_2) + V(y_1-y_2)) \le ((Vy_1)^r + C^r(Vy_2)^r)^{1/r}.$$

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Theorem 6.1. *Suppose $1 < p < r \le 2$. Let Y be a separable Banach space with a random L_p-norm of type r, and let X be any quotient of Y. Then if X has the p-co-Banach-Saks property, then X has property $(\tilde{m}_p)$.*

Proof. Let $V : Y \to L_p[0,1]$ be the random L_p-norm. Let us use $|E|$ to denote the Lebesgue measure of a measurable set E. First we define for any $0 < \theta < 1$,

$$\|f\|_{p,\theta} = \sup_{|E|\le\theta} \|\chi_E f\|_p, \qquad f \in L_p.$$

We also let $Q : Y \to X$ be the quotient map.

Next observe that Y (and hence X) is super-reflexive. Indeed if C is the constant in the definition of a random L_p-norm of type r,

$$\frac{1}{2}(\|y_1+y_2\|^p+\|y_1-y_2\|^p) \le \int_0^1 ((Vy_1(s))^r+C^r(Vy_2(s))^r)^{p/r}\,ds \le \|y_1\|^p+C^p\|y_2\|^p.$$

This implies that Y is p-uniformly smooth. Further if $\|y\| = 1$ and $\|z\| = t$ is such that $y^*(z) = 0$, where $\|y^*\| = y^*(y) = 1$, then

$$\|y + tz\| - 1 \le \|y + tz\| + \|y - tz\| - 2 \le \|y + tz\|^p + \|y - tz\|^p - 2 \le 2C^pt^p.$$

Hence $\overline{\rho}_Y(t) \le 2C^pt^p$ for $0 \le t \le 1$. This implies that also by Proposition 3.2, $\overline{\rho}_X(t) \le 4C^pt^p$. To prove the theorem it therefore suffices by Theorem 5.1 to show that $\overline{\delta}_X(t) \ge at^p$ for $0 \le t \le 1$ for some $a > 0$.

Let us suppose that X has the p-co-Banach-Saks property with constant $c > 0$. Suppose $\|x\| = 1$ and $(x_n)_{n=1}^\infty$ is a weakly null sequence with $\|x_n\| = t \le 1$. We will show that

$$\liminf_{n\to\infty} \|x + x_n\|^p \ge 1 + 2^{-p}C^{-p}c^pt^p. \tag{6.5}$$

If this false we can pass to a subsequence and suppose that

$$\lim_{n\to\infty} \|x + x_n\|^p = 1 + b^p,$$

where $b < \frac{c}{2C}t$. If $0 < \lambda < 1$ is chosen so that $b < \frac{\lambda c}{2C}$, we can pass to a further subsequence and suppose that

$$\|x_{n_1} + \cdots + x_{n_k}\| \ge \lambda c k^{1/p} t, \qquad n_1 < n_2 < \cdots < n_k$$

and that (using Lemma 4.1)

$$\mathbb{E}\|\sum_{j=1}^k \epsilon_j x_{n_j}\| \ge \frac{\lambda c}{2} k^{1/p} t, \qquad n_1 < n_2 < \cdots < n_k.$$

Pick $z_n \in Y$ so that $Qz_n = x + x_n$ and $\|z_n\| = \|x + x_n\|$. Passing to a yet further subsequence we can suppose that $(z_n)_{n=1}^\infty$ converges weakly to some $y \in Y$; then let $y_n = z_n - y$. Thus $Qy = x$ and $Qy_n = x_n$. In particular

$$(\mathbb{E}\|\sum_{j=1}^k \epsilon_j y_{n_j}\|^p)^{1/p} \ge \frac{\lambda c}{2} k^{1/p} t, \qquad n_1 < n_2 < \cdots < n_k.$$

Now suppose $0 < \theta < 1$. Note that we have a crude estimate $\|y\|, \|z_n\| \le 2$ and hence $\|y_n\| \le 4$. For each n, let E_n be a measurable subset of $[0,1]$ with measure $|E_n| = \theta$ and $\|Vy_n\|_{p,\theta} = \|\chi_{E_n} Vy_n\|_p$. Let us denote $\tilde{E}_n = [0,1] \setminus E_n$. Then $\|Vy_n\chi_{\tilde{E}_n}\|_\infty \le 4\theta^{-1/p}$ and so $\|Vy_n\chi_{\tilde{E}_n}\|_r \le 4\theta^{-1/p}(1-\theta)^{1/r}$.

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Now

$$(\mathbb{E}\|\sum_{j=1}^k \epsilon_j y_{n_j}\|^p)^{1/p} = (\mathbb{E}\|V(\sum_{j=1}^k \epsilon_j y_{n_j}\|_p^p)^{1/p}$$
$$\leq C\|(\sum_{j=1}^k (Vy_{n_j})^r)^{1/r}\|_p.$$

We first estimate:

$$\|(\sum_{j=1}^k (Vy_{n_j})^r \chi_{\tilde{E}_{n_j}})^{1/r}\|_p \leq \|(\sum_{j=1}^k (Vy_{n_j})^r \chi_{\tilde{E}_{n_j}})^{1/r}\|_r$$
$$\leq 4\theta^{-1/p}(1-\theta)^{1/r} k^{1/r}.$$

On the other hand

$$\|(\sum_{j=1}^k (Vy_{n_j})^r \chi_{E_{n_j}})^{1/r}\|_p \leq \|(\sum_{j=1}^k (Vy_{n_j})^p \chi_{E_{n_j}})^{1/p}\|_p$$
$$\leq (\sum_{j=1}^k \|Vy_{n,j}\|_{p,\theta}^p)^{1/p}.$$

Combining we have that

$$\frac{1}{2}\lambda c k^{1/p} t \leq C(\sum_{j=1}^k \|Vy_{n_j}\|_{p,\theta}^p)^{1/p} + 4C\theta^{-1/p}(1-\theta)^{1/r}k^{1/r}.$$

Since this holds for any $n_1 < \cdots < n_k$ we conclude that (for any $0 < \theta < 1$),

$$\liminf_{n\to\infty} \|Vy_n\|_{p,\theta} \geq \frac{\lambda ct}{2C}. \tag{6.6}$$

Choose b_1 so that $b < b_1 < \frac{\lambda ct}{2C}$. Then we pick for each n a set G_n of minimal measure so that $\|\chi_{G_n} Vy_n\|_p = b_1$. Then from (6.6) we have $\lim_{n\to\infty} |G_n| = 0$.

On Y consider the seminorm $z \to \int (Vy(s))^{p-1} Vz(s)\, ds$. Then by the Hahn-Banach theorem there is a linear functional $y^* \in Y^*$ with $y^*(y) = \|Vy\|^p = \|y\|^p$ and

$$y^*(z) \leq \int (Vy(s))^{p-1} Vz(s)\, ds, \qquad z \in Y.$$

In particular

$$\|y\|^p = \lim_{n\to\infty} y^*(y+y_n) \leq \liminf_{n\to\infty} \int (Vy(s))^{p-1} V(y+y_n)(s)\, ds.$$

Now

$$\int_{G_n} (Vy(s))^{p-1} V(y+y_n)(s)\, ds \leq \left(\int_{G_n} (Vy(s))^p ds\right)^{1-1/p} \|y+y_n\|$$

so that

$$\lim_{n\to\infty} \int_{G_n} (Vy(s))^{p-1} V(y+y_n)(s)\, ds = 0.$$

Hence

$$\liminf_{n\to\infty} \int_{\tilde{G}_n} (Vy(s))^{p-1} V(y+y_n)(s)\, ds \geq \|y\|^p.$$

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This implies by Hölder's inequality that

$$\liminf_{n\to\infty} \|\chi_{\tilde{G}_n} V(y+y_n)\|_p \geq \|y\| \geq 1.$$

On the other hand

$$\|\chi_{G_n} Vy\|_p + \|\chi_{G_n} V(y+y_n)\|_p \geq \|\chi_{G_n} Vy_n\|_p = b_1$$

so that

$$\liminf_{n\to\infty} \|\chi_{G_n} V(y+y_n)\|_p \geq b_1.$$

Hence

$$\liminf_{n\to\infty} \|V(y+y_n)\|_p^p \geq 1 + b_1^p.$$

However

$$\lim_{n\to\infty} \|V(y+y_n)\|_p^p = \lim_{n\to\infty} \|x+x_n\|^p = 1 + b^p.$$

This contradiction shows that $\overline{\delta}_X(t) \geq (1 + 2^{-p}C^{-p}c^p t^p) - 1$ for $0 \leq t \leq 1$ and concludes the proof. □

Corollary 6.2. *(i) Suppose $1 < p < 2$ and that X is a subspace (respectively quotient space) of L_p with the p-co-Banach-Saks property. Then X is a subspace (respectively quotient space) of ℓ_p.*

(ii) Suppose $2 < p < \infty$ and that X is a subspace (respectively quotient space) of L_p with the p-Banach-Saks property. Then X is a subspace (respectively quotient space) of ℓ_p.

Proof. (ii) is due to Johnson [17]. For (i) we observe that X has $(\tilde{m}_p)$ by Theorem 6.1 and so X^* has $(\tilde{m}_q)$, where $1/p + 1/q = 1$. Hence we can apply (ii) to deduce that X^* is a quotient (respectively a subspace) of ℓ_q and then use duality. □

The second part of the next theorem was proved by Johnson [17] with an additional hypothesis that X is the quotient of a subspace of L_p with the approximation property. The theorem answers a question raised by Johnson (Problem IV.2) in that paper.

Theorem 6.3. *Suppose $1 < p < \infty$ and that X is a subspace of a quotient of L_p.*

(i) If $1 < p < 2$ and X has the p-co-Banach-Saks property, then X is isomorphic to a subspace of a quotient of ℓ_p.

(ii) If $2 < p < \infty$ and X has the p-Banach-Saks property, then X is isomorphic to a subspace of a quotient of ℓ_p.

Proof. (i) By Theorem 6.1, X has property $(\tilde{m}_p)$ and hence by Theorem 5.2, X embeds into $(\sum_{n=1}^\infty E_n)_{\ell_p}$, where the E_n's are finite-dimensional subspaces of quotients of L_p and hence also of ℓ_p. Thus X is a subspace of a quotient of ℓ_p.

(ii) By Theorem 4.7, X^* has the q-co-Banach-Saks property, where $1/p + 1/q = 1$; hence by (i), X^* is a subspace of a quotient of ℓ_q and the result follows by duality. □

Let us now consider the analogue of these results when $p = 1$. Let us recall that a Banach space X has the strong Schur property if there is a constant $c > 0$ so that if $(x_n)_{n=1}^\infty$ is a sequence in X with $\operatorname{sep}\{x_n\}_{n=1}^\infty = \delta > 0$, then there is a subsequence with

$$\|\sum_{j=1}^k a_j x_{n_j}\| \geq c \sum_{j=1}^k |\alpha_j|.$$

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This concept was considered first (implicitly) by Johnson and Odell [21] and then by Bourgain and Rosenthal [5], who gave examples of subspaces of L_1 with the strong Schur property but failing to have the Radon-Nikodým Property.

An alternative formulation of the strong Schur property is given in [27]. X has the strong Schur property if there is a constant c so that for every bounded sequence $(x_n)_{n=1}^{\infty}$ there exists $x^* \in X^*$ with $\|x^*\| = 1$ and

$$\limsup_{n\to\infty} x^*(x_n) \geq c \limsup_{n\to\infty} \|x_n\|.$$

Theorem 6.4. *Let X be a closed subspace of L_1. The following conditions on X are equivalent:*

(i) X has the anti-Banach-Saks property.

(ii) X has the strong Schur property.

(iii) For some $c > 0$ we have $\hat{\delta}_X(t) \geq ct$.

Proof. That (iii) implies (ii) follows from Proposition 4.5. It is clear that (ii) implies (i). It remains to show that (i) implies (iii). The argument is a variation on Theorem 6.1. By Proposition 4.8 there is a constant $c > 0$ so that for every normalized sequence $(g_n)_{n=1}^{\infty}$ with $\operatorname{sep}\{g_n\}_{n=1}^{\infty} = \alpha$ in X we can pass to a subsequence $(f_n)_{n=1}^{\infty}$ with

$$\mathbb{E}\|\sum_{j=1}^{k} \epsilon_j f_{n_j}\| \geq c\alpha k.$$

Let us fix such a sequence $(f_n)_{n=1}^{\infty}$. Suppose $0 < \theta < 1$ and pick $E_n \subset [0,1]$ so that $|E_n| = \theta$ and $\|\chi_{E_n} f_n\|_1 = \|f_n\|_{1,\theta}$. Then for any $n_1 < n_2 < \cdots < n_k$,

$$\|(\sum_{j=1}^{k} \chi_{\tilde{E}_{n_j}} |f_{n_j}|^2)^{1/2}\|_1 \leq k^{1/2}\theta^{-1}$$

so that

$$\mathbb{E}\|\sum_{j=1}^{k} \epsilon_j \chi_{\tilde{E}_{n_j}} f_{n_j}\|_1 \leq k^{1/2}\theta^{-1}.$$

Hence

$$\mathbb{E}\|\sum_{j=1}^{k} \epsilon_j \chi_{E_{n_j}} f_{n_j}\|_1 \geq c\alpha k - \theta^{-1}k^{1/2}$$

so that

$$\sum_{j=1}^{k} \|f_{n_j}\|_{1,\theta} \geq c\alpha k - \theta^{-1}k^{1/2}.$$

In particular $\liminf_{n\to\infty} \|f_n\|_{1,\theta} \geq c\alpha$.

Now if $f \in L_1$ with $\|f\|_1 = 1$ and $t > 0$, we have

$$\begin{aligned}\frac{1}{2}(\|f + tf_n\|_1 + \|f - tf_n\|_1) &\geq t\int_{E_n} |f_n(s)|\,ds + \int_{\tilde{E}_n} |f(s)|\,ds \\ &= 1 + t\|f_n\|_{1,\theta} - \int_{E_n} |f(s)|\,ds.\end{aligned}$$

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Hence

$$\liminf_{n\to\infty} \frac{1}{2}(\|f+tf_n\|_1 + \|f-tf_n\|_1) \geq 1 + c\alpha t - \sup_{|E|\leq\theta} \int_E |f|\,ds.$$

As $\theta > 0$ is arbitrary we have $\hat{\delta}_X(t) \geq ct$. □

7. Mappings on Orlicz spaces and applications

We refer to [3] for background on nonlinear theory. However, we need to recall some definitions and notation. Let (M,d) and (N,δ) be two unbounded metric spaces. We define for $f : M \to N$:

$$\forall t > 0 \quad \omega_f(t) = \sup\{\delta(f(x), f(y)),\ x, y \in M,\ d(x,y) \leq t\}.$$

We say that f is *uniformly continuous* if $\lim_{t\to 0} \omega_f(t) = 0$. The map f is said to be *coarsely continuous* if $\omega_f(t) < \infty$ for some $t > 0$.

Let us now introduce

$$Lip_s(f) = \sup_{t\geq s} \frac{\omega_f(t)}{t}, \quad \text{for } s > 0$$

and

$$L(f) = \sup_{s>0} Lip_s(f), \qquad Lip_\infty(f) = \inf_{s>0} Lip_s(f).$$

A map is Lipschitz if and only if $L(f) < \infty$. We will say that it is *coarse Lipschitz* if $Lip_\infty(f) < \infty$. Clearly, a coarse Lipschitz map is coarsely continuous. If f is bijective, we will say that f is a *uniform homeomorphism* (respectively, *coarse homeomorphism, Lipschitz homeomorphism, coarse Lipschitz homeomorphism*) if f and f^{-1} are uniformly continuous (respectively, coarsely continuous, Lipschitz, coarse Lipschitz). Finally we say that f is a *coarse Lipschitz embedding* if it is a coarse Lipschitz homeomorphism from M onto $f(M)$.

It is well known that if X and Y are Banach spaces, then for any map $f : X \to Y$, ω_f is a subadditive function. It follows that any uniform homeomorphism $f : X \to Y$ is a coarse Lipschitz homeomorphism.

Given a metric space X, two points $x, y \in X$, and $\nu > 0$, the approximate metric midpoint set between x and y with error ν is the set:

$$\text{Mid}(x,y,\nu) = \left\{ z \in X :\ \max\{d(x,z), d(y,z)\} \leq (1+\nu)\frac{d(x,y)}{2} \right\}.$$

The use of metric midpoints in the study of nonlinear geometry is due to Enflo in an unpublished paper and has since been used elsewhere, e.g. [4], [12] and [20].

The following version of the Midpoint Lemma was formulated in [30] (see also [3], Lemma 10.11). Note that completeness of X is not needed.

Lemma 7.1. *Let X be a normed space and suppose M is a metric space. Let $f : X \to M$ be a coarse Lipschitz map. If $Lip_\infty(f) > 0$, then for any $t, \epsilon > 0$ and any $0 < \nu < 1$, there exist $x, y \in X$ with $\|x - y\| > t$ and*

$$f(\text{Mid}(x,y,\nu)) \subset \text{Mid}(f(x), f(y), (1+\epsilon)\nu).$$

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Lemma 7.2. *Let X be a normed space and suppose $(x_n)_{n=1}^\infty$ is a normalized sequence in X. Define $T : c_{00} \to X$ by $T\xi = \sum_{k=1}^\infty \xi_k x_k$. Let*

$$\sigma_k = \sup\left\{ \|\sum_{j=1}^k \theta_j x_{n_j}\|, \qquad n_1 < n_2 < \cdots < n_k,\ \theta_j = \pm 1 \right\}.$$

For each $k \in \mathbb{N}$ define

$$F_k(t) = \begin{cases} \sigma_k t/k, & 0 \le t \le 1/\sigma_k, \\ t + 1/k - 1/\sigma_k, & 1/\sigma_k \le t < \infty. \end{cases} \tag{7.7}$$

Then for any $\xi \in c_{00}$ we have

$$\|T\xi\| \le 2\|\xi\|_{\ell_{F_k}}.$$

Proof. Note first that for any set $A \subset \mathbb{N}$ with $|A| = m$ we have

$$\|\sum_{j\in A} \xi_j x_j\| \le \max_{j\in A} |\xi_j|\sigma_m.$$

Let $a \in \ell_1$ with $\sum_{j=1}^\infty F_k(|\xi_j|) \le 1$ and let $(\xi_j^*)_{j=1}^\infty$ be the decreasing rearrangement of $(|\xi_j|)_{j=1}^\infty$. Now $F_k(\xi_k^*) \le 1/k$ and hence $\xi_k^* \le 1/\sigma_k$. Then

$$\begin{aligned}
\|T\xi\| &\le \sum_{j=1}^\infty (\xi_j^* - \xi_{j+1}^*)\sigma_j \\
&\le \sum_{j=1}^k (\xi_j^* - \xi_{j+1}^*)\sigma_j + \frac{\sigma_k}{k} \sum_{j=k+1}^\infty j(\xi_j^* - \xi_{j+1}^*) \\
&= \sum_{j=1}^k \xi_j^*(\sigma_j - \sigma_{j-1}) + \frac{\sigma_k}{k} \sum_{j=k+1}^\infty \xi_j^* \\
&\le \sum_{j=1}^k (F_k(\xi_j^*) + \sigma_k^{-1})(\sigma_j - \sigma_{j-1}) + \sum_{j=k+1}^\infty F_k(\xi_j^*) \\
&\le 1 + \sum_{j=1}^\infty F_k(\xi_j^*) \\
&\le 2.
\end{aligned}$$

Hence $\|T\|_{\ell_{F_k} \to X} \le 2$. □

Theorem 7.3. *Let X and Y be two Banach spaces such that X coarse Lipschitz-embeds into Y. Then there is a constant $c > 0$ so that given any normalized sequence $(x_n)_{n=1}^\infty$ with $\operatorname{sep}\{x_n\}_{n=1}^\infty = \theta > 0$ in X and any integer k there exist $n_1 < \cdots < n_k$ so that*

$$c\theta\|e_1 + \cdots + e_k\|_{\ell_{\hat{\delta}_Y}} \le \mathbb{E}\|\epsilon_1 x_{n_1} + \cdots + \epsilon_k x_{n_k}\|.$$

Proof. We may assume that for some constant K we have a map $f : X \to Y$ such that $f(0) = 0$ and

$$\|x - z\| - 1 \le \|f(x) - f(z)\| \le K\|x - z\| + 1, \qquad x, z \in X.$$

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Let

$$\sigma_k = \sup\left\{ \mathbb{E}\|\sum_{j=1}^{k} \epsilon_j x_{n_j}\|, \qquad n_1 < n_2 < \cdots < n_k \right\}.$$

Let $\sigma_0 = 0$. Then $(\sigma_k)_{k=0}^{\infty}$ is a monotone increasing sequence.

For each k, define the Orlicz function F_k by (7.7). We let N_k be the absolute norm on $\mathbb{R}^2$ such that

$$N_k(1,t) = 1 + F_k(t), \quad t \geq 0.$$

We also define an absolute norm on $\mathbb{R}^2$ by

$$N_Y(1,t) = 1 + \int_0^t \hat{\delta}_Y(s)\frac{ds}{s}, \quad t \geq 0.$$

Let us note, for future use, the following property of N_Y. If $y, z \in Y$ and $(y_n)_{n=1}^{\infty}$ is any bounded sequence in Y, then

$$\liminf_{n\to\infty}(\|y - y_n\| + \|z - y_n\|) \geq N_Y(\|y - z\|, \operatorname{sep}\{y_n\}_{n=1}^{\infty}). \tag{7.8}$$

This is an immediate consequence of Proposition 3.3.

We define an operator $T : c_{00} \to L_1(\Delta; X)$ by

$$T(\xi) = \sum_{j=1}^{\infty} \xi_j \epsilon_j \otimes x_j.$$

Combining Lemmas 4.3 and 7.2 we have

$$\|T\xi\| \leq 4\|\xi\|_{\Lambda_{N_k}}.$$

We then consider the map $g : c_{00} \to L_1(\Delta; Y)$ defined by $\xi \to f \circ T\xi$. This is well-defined because $f(0) = 0$ and $T\xi$ is a simple function so that there are no measurability problems. We have an estimate

$$\|g(\xi) - g(\eta)\| \leq 4K\|\xi - \eta\|_{\Lambda_{N_k}} + 1, \qquad \xi, \eta \in c_{00}.$$

We also have $\|g(te_1)\| = \frac{1}{2}(\|f(tx_1)\| + \|f(-tx_1)\|) \geq t - 1$ so that $\operatorname{Lip}_{\infty}(g) \geq 1$.

We apply the Midpoint Lemma (Lemma 7.1) to $g : (c_{00}, \|\cdot\|_{\Lambda_{N_k}}) \to L_1(\Delta, Y)$ with $\nu = 1/k$. For any $\tau_0 > 0$ we can find $\tau > \tau_0$ and points $\eta,\ \zeta \in c_{00}$ with $\|\eta - \zeta\|_{\Lambda_{N_k}} = 2\tau$ such that

$$g(\operatorname{Mid}(\eta, \zeta, 1/k)) \subset \operatorname{Mid}(g(\eta), g(\zeta), 2/k).$$

Let $\xi = \frac{1}{2}(\eta + \zeta)$.

There exists $m \in \mathbb{N}$ so that $\eta, \zeta \in [e_1, \ldots, e_{m-1}]$. Thus if $j \geq m$ we have, from the iterative nature of the norm on Λ_{N_k}, $\xi + \tau\sigma_k^{-1} e_j \in \operatorname{Mid}(\eta, \zeta, 1/k)$.

Thus the functions

$$h_j = f(\sum_{i=1}^{m-1} \xi_i \epsilon_i \otimes x_i + \tau\sigma_k^{-1} \epsilon_j \otimes x_j)$$

all belong to $\operatorname{Mid}(g(\eta), g(\zeta), 2/k)$ for $j \geq m$. Since both $g(\eta)$ and $g(\zeta)$ depend only on the first $m - 1$ coordinates of Δ, this implies that the same is true for the functions

$$h'_j = f(\sum_{i=1}^{m-1} \xi_i \epsilon_i \otimes x_i + \tau\sigma_k^{-1} \epsilon_m \otimes x_j).$$

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The functions h_j' now depend on the first m coordinates of Δ. In particular

$$\|g(\eta) - h_j'\| + \|g(\zeta) - h_j'\| - \|g(\eta) - g(\zeta)\| \le 2k^{-1}\|g(\eta) - g(\zeta)\|. \tag{7.9}$$

Note that for any $s \in \Delta$ we have

$$\|h_i'(s) - h_j'(s)\| \ge \theta\tau\sigma_k^{-1} - 1, \qquad i > j \ge m.$$

Hence, using (7.8), we have

$$\liminf_{j\to\infty} \|g(\eta)(s) - h_j'(s)\| + \|g(\zeta)(s)) - h_j'(s)\| \ge N_Y(\|g(\eta)(s) - g(\zeta)(s))\|, \theta\tau\sigma_k^{-1} - 1)$$

as long as $\tau > \sigma_k/\theta$.

Integrating (note the integral is simply a finite sum in this case),

$$\begin{aligned}\liminf_{j\to\infty}(\|g(\eta) - h_j'\| + \|g(\zeta) - h_j'\|) &\ge \int_\Delta N_Y(\|g(\eta)(s)) - g(\zeta)(s))\|, \theta\tau\sigma_k^{-1} - 1)ds \\ &\ge N_Y(\|g(\eta) - g(\zeta)\|, \theta\tau\sigma_k^{-1} - 1).\end{aligned}$$

Now $\|g(\eta) - g(\zeta)\| \le 8K\tau + 1$ and since $N_Y(t,1) - t$ is a decreasing function we conclude that

$$\liminf_{j\to\infty}(\|g(\eta)-h_j'\|+\|g(\zeta)-h_j'\|-\|g(\eta)-g(\zeta)\|) \ge N_Y(8K\tau+1, \theta\tau\sigma_k^{-1}-1)-(8K\tau+1).$$

Hence, by (7.9),

$$N_Y(8K\tau + 1, \theta\tau\sigma_k^{-1} - 1) - (8K\tau + 1) \le \frac{2}{k}\|g(\eta) - g(\zeta)\| \le 2(8K\tau + 1)k^{-1}.$$

We simplify this as

$$N_Y\left(1, \frac{\theta - \sigma_k\tau^{-1}}{\sigma_k(8K + \tau^{-1})}\right) \le 1 + \frac{2}{k}.$$

Now we can let $\tau \to \infty$ and deduce that

$$N_Y(1, \frac{\theta}{8K\sigma_k}) \le 1 + \frac{2}{k}.$$

This implies that

$$\hat\delta(\frac{\theta}{16K\sigma_k}) \le \frac{2}{k}$$

and hence

$$\hat\delta(\frac{\theta}{32K\sigma_k}) \le \frac{1}{k}$$

or

$$\|e_1 + \cdots + e_k\|_{\ell_{\hat\delta_Y}} \le 32K\theta^{-1}\sigma_k.$$

□

Our next theorem combines Theorem 7.3 with Theorem 6.1 from [30]. Note of course that reflexivity of Y is not used for the left-hand inequality, and the right-hand inequality could be improved to

$$\|a_1e_1 + \cdots + a_ke_k\|_S \le C\|a_1e_1 + \cdots + a_ke_k\|_{\ell_{\overline{\delta}_Y}}$$

for any $a_1, \dots, a_k$. However the theorem as stated shows that we have both an upper and lower estimate for the behavior of weakly null spreading sequences in X.

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Theorem 7.4. *Suppose X and Y are Banach spaces. Suppose there is a coarse Lipschitz embedding of X into Y and Y is reflexive. Then, there is a constant C so that for any spreading model of a weakly null sequence in X we have:*

$$\frac{1}{C}\|e_1+\cdots+e_k\|_{\ell_{\overline{\delta}_Y}} \le \|e_1+\cdots+e_k\|_S \le C\|e_1+\cdots+e_k\|_{\ell_{\overline{\rho}_Y}}, \qquad k\in\mathbb{N}. \tag{7.10}$$

In particular if $\overline{\delta}_Y(t)>0$ for any $t>0$, then there is a constant C so that every normalized weakly null sequence $(x_n)_{n=1}^\infty$ has a subsequence $(x_n)_{n\in\mathbb{M}}$ so that

$$\frac{1}{C}\|e_1+\cdots+e_k\|_{\ell_{\overline{\delta}_Y}} \le \|x_{n_1}+\cdots+x_{n_k}\| \le C\|e_1+\cdots+e_k\|_{\ell_{\overline{\rho}_Y}}, \qquad k\in\mathbb{N}. \tag{7.11}$$

Proof. The left-hand side follows from Theorem 7.3 and Proposition 4.2. For the right-hand side, suppose $f : X\to Y$ is a coarse Lipschitz embedding. We may assume that

$$\|x-y\|-1\le\|f(x)-f(y)\|\le K\|x-y\|+1.$$

Consider the space $\mathcal{P}_k$ of k-subsets of $\mathbb{N}$ with the metric

$$d(\{m_1,\ldots,m_k\},\{n_1,\ldots,n_k\}) = |\{k:\ m_k\neq n_k\}|.$$

Let $(x_n)_{=1}^\infty$ be a normalized spreading sequence generating the spreading model $\{e_n\}_{n=1}^\infty$. Then, for any $\lambda>0$, the map

$$F_\lambda(\{n_1,\ldots,n_k\}) = f(\lambda(x_{n_1}+\cdots+x_{n_k}))$$

is Lipschitz with constant at most $2(\lambda+1)$. Hence, if $\nu>0$ we can find an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that if $\{m_1,\ldots,m_k,n_1,\ldots,n_k\}\subset\mathbb{M}$ we have

$$\|F_\lambda(m_1,\ldots,m_k)-F_\lambda(n_1,\ldots,n_k)\|\le 2e(\lambda+1)\|e_1+\cdots+e_k\|_{\ell_{\overline{\rho}_Y}}+\nu.$$

Hence

$$\|x_{m_1}+\cdots+x_{m_k}-x_{n_1}-\cdots-x_{n_k}\|\le 2e(1+1/\lambda)\|e_1+\cdots+e_k\|_{\ell_{\overline{\rho}_Y}}+(2+\nu)/\lambda$$

and thus, letting $n_1,\ldots,n_k\to\infty$, $\lambda\to\infty$ and $\nu\to 0$, we have

$$\|x_{m_1}+\cdots+x_{m_k}\|\le 2e\|e_1+\cdots+e_k\|_{\ell_{\overline{\rho}_Y}}$$

so that the right-hand side follows.

The second part (7.11) is an equivalent statement. □

Remark. If X and Y are uniformly homeomorphic one can relax the assumption that Y is reflexive. This follows from results in [11]. If we assume $\lim_{t\to 0}\overline{\rho}_Y(t)/t=0$, then the Szlenk Index of Y is ω_0 and hence by Theorem 5.5 so is the Szlenk index of X; furthermore the convex Szlenk indices of these spaces are equivalent and the argument is similar to that of Theorem 5.8 of [11], which treats the special case $\overline{\rho}_Y(t)\le ct^p$.

8. Applications to uniform and coarse homeomorphisms

The first proposition is well known and goes back to work of Ribe [40] and [41] (who considered only the uniform case).

Proposition 8.1. *Let X and Y be separable Banach spaces and suppose there is a coarse Lipschitz embedding of X into Y. Then X is finitely representable in Y and hence isomorphic to a subspace of any ultraproduct $Y_{\mathcal{U}}$.*

Proof. There is a Lipschitz embedding of X into $Y_{\mathcal{U}}$ and hence a linear embedding into $Y_{\mathcal{U}}^{**}$ ([3], p.176). □

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In order to apply our results we need to use the ultraproduct technique which goes back to the classic paper of Heinrich and Mankiewicz [15]. The next result summarizes these ideas.

Theorem 8.2. *Let X and Y be separable Banach spaces which are coarsely (or uniformly) homeomorphic. Assume Y is super-reflexive. Then given any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we can find separable closed subspaces X_1 of $X_{\mathcal{U}}$ and Y_1 of $Y_{\mathcal{U}}$, such that:*

(i) $X \subset X_1,\ Y \subset Y_1$.

(ii) X_1 is complemented in $X_{\mathcal{U}}$ and Y_1 is complemented in $Y_{\mathcal{U}}$.

(iii) X_1 and Y_1 are linearly isomorphic.

Proof. The argument is standard. Since Y is super-reflexive, so is X ([40], [3]). First one notes that $X_{\mathcal{U}}$ and $Y_{\mathcal{U}}$ are Lipschitz isomorphic (and both are reflexive). Then it is possible (using the separable complementation property) to find separable 1-complemented subspaces, $X \subset X_1 \subset X_{\mathcal{U}}$ and $Y \subset Y_0 \subset Y_{\mathcal{U}}$ so that X_1 and Y_0 are Lipschitz isomorphic. But this implies that X_1 is isomorphic to a complemented subspace Y_1 of Y_0 (see [15], [3]). □

Theorem 8.3. *Suppose $1 < p < \infty$ and that $X = (\sum_{n=1}^{\infty} E_n)_{\ell_p}$, where $(E_n)_{n=1}^{\infty}$ is a sequence of finite-dimensional spaces. Suppose either that:*

(i) $1 < p < r \le 2$ and the spaces $(E_n)_{n=1}^{\infty}$ are uniformly r-uniformly smooth, or

(ii) $2 \le r < p < \infty$ and the spaces $(E_n)_{n=1}^{\infty}$ are uniformly r-uniformly convex.

Suppose Y coarse Lipschitz-embeds into a quotient of X. Then Y has property $\tilde{m}_p$.

Proof. In either case X (and hence Y) is super-reflexive.

(i) We start with the observation that if $V : X \to \ell_p$ is defined by $V((x_n)_{n=1}^{\infty}) = (\|x_n\|)_{n=1}^{\infty}$, then V is a random L_p-norm. By our assumptions this is a random L_p-norm of type r. It follows that we can induce a random L_p-norm of type r, $\tilde{V} : X_{\mathcal{U}} \to (\ell_p)_{\mathcal{U}}$. Now Y embeds into a quotient of $X_{\mathcal{U}}$, and hence is a quotient of a subspace of $X_{\mathcal{U}}$. However, by Theorem 7.4, Y has the p-co-Banach-Saks property. By Theorem 6.1 this implies that Y has property $\tilde{m}_p$.

(ii) In this case we argue similarly that $(X^*)_{\mathcal{U}} = (X_{\mathcal{U}})^*$ has a random L_q-norm of type s where $1/p+1/q = 1/r+1/s = 1$. In this case Y^* is a quotient of (a separable subspace) of $(X^*)_{\mathcal{U}}$. We again use Theorem 7.4 to deduce that Y has the p-Banach-Saks property. By Proposition 4.7 this means that Y^* has the q-co-Banach-Saks property. By Theorem 6.1, Y^* has property $\tilde{m}_q$, and so Y has property $(\tilde{m}_p)$. □

Theorem 8.4. *Suppose $1 < p < \infty$. Then*

(i) If X is a Banach space which can be coarse Lipschitz-embedded in ℓ_p, then X is linearly isomorphic to a closed subspace of ℓ_p.

(ii) If X is a Banach space which is coarsely homeomorphic to a quotient of ℓ_p, then X is linearly isomorphic to a quotient of ℓ_p.

(iii) If X can be coarse Lipschitz-embedded into a quotient of ℓ_p, then X is linearly isomorphic to a subspace of a quotient of ℓ_p.

Proof. Suppose first that X can be coarse Lipschitz-embedded into a quotient of ℓ_p. Then it is a special case of Theorem 8.3 that X has property $(\tilde{m}_p)$.

(i) In this case for $2 \le p < \infty$ the result is proved in [11]. If $1 < p < 2$, then X^* is isomorphic to a quotient of L_q where $1/p + 1/q = 1$ and has property $(\tilde{m}_q)$;

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in particular it has the q-Banach-Saks property and by [17], X^* is isomorphic to a quotient of ℓ_q; i.e. X is isomorphic to a subspace of ℓ_p.

(ii) Again this is proved for $2 \le p < \infty$ in [11]. If $1 < p < 2$, then X^* is isomorphic to a subspace of L_q which has property $(\tilde{m}_q)$ and hence contains no subspace isomorphic to ℓ_2. By the classical result of Kadets and Pełczyńksi [24] this implies that X^* is isomorphic to a subspace of ℓ_q. Hence X is isomorphic to a quotient of ℓ_p.

(iii) In this case, X is isomorphic to a subspace of a quotient of L_p. Since X has property (m_p) we can use Theorem 5.2 to embed X in an ℓ_p-sum, $(\sum_{n=1}^\infty E_n)_{\ell_p}$, where the spaces E_n are finite-dimensional and uniformly quotients of X and hence into a subspace of a quotient of ℓ_p. Thus X is isomorphic to a subspace of a quotient of ℓ_p. □

Remark. Of course if X is uniformly homeomorphic to ℓ_p, then X is linearly isomorphic to ℓ_p [20]. In [29] we show that for every $1 < p < \infty$ there are two uniformly homeomorphic subspaces (respectively, quotients) of ℓ_p which are not isomorphic. We do not know if Theorem 8.4 holds for subspaces or quotients of c_0. In the Lipschitz category there are corresponding results proved in [10] and [9] (except note in [9] for the case of quotients one needs an extra hypothesis that X^* has the approximation property).

Theorem 8.5. *(i) Suppose $1 < p < r \le 2$ and that Z is an r-uniformly smooth Banach space with the (UAP). Suppose $(E_n)_{n=1}^\infty$ is an increasing sequence of uniformly complemented finite-dimensional subspaces of Z. Then $X = (\sum_{n=1}^\infty E_n)_{\ell_p}$ has unique coarse (or uniform) structure.*

(ii) Suppose $2 \le r < p < \infty$ and that Z is an r-uniformly convex Banach space with the (UAP). Suppose $(E_n)_{n=1}^\infty$ is an increasing sequence of uniformly complemented finite-dimensional subspaces of Z. Then $X = (\sum_{n=1}^\infty E_n)_{\ell_p}$ has unique coarse (or uniform) structure.

Proof. Let us start by observing that, in both cases (i) and (ii), X is linearly isomorphic to $\ell_p(X)$. Indeed if $(n_k)_{k=1}^\infty$ is any sequence of natural numbers such that $\{n_k = j\}$ is infinite for each j and $n_k \le k$, then $(\sum_{k=1}^\infty E_{n_k})_{\ell_p}$ is complemented in X; hence $\ell_p(X)$ is isomorphic to a complemented subspace of X. Hence for some Banach space W, we have $X \approx \ell_p(X) \oplus W \approx \ell_p(X) \oplus \ell_p(X) \oplus W \approx \ell_p(X)$. Next we observe that X is isomorphic to a complemented subspace of $\ell_p(Z)$ and so has the (UAP) by Theorem 9.4 of [14].

Now suppose Y is coarsely homeomorphic to X. Since X is super-reflexive we can apply Theorem 8.2 to deduce that Y is super-reflexive and has the approximation property. By Theorem 8.3, Y has property $(\tilde{m}_p)$. We can therefore apply Theorem 5.4. It follows that Y is isomorphic to a complemented subspace of a space $(\sum_{n=1}^\infty F_n)_{\ell_p}$, where each F_n can be assumed to be of the form $(\sum_{j=1}^k E_j)_{\ell_p}$ for some k. This implies that Y is isomorphic to a complemented subspace of X.

To complete the proof we use Theorem 5.5. Since X is isomorphic to a complemented subspace of an ultraproduct $Y_{\mathcal{U}}$ of Y it follows that there is a constant λ so that for each m, n the finite-dimensional subspace $\ell_p^m(E_n)$ is λ-isomorphic to a λ-complemented subspace of Y. Hence $X = (\sum_{n=1}^\infty E_n)_{\ell_p}$ is isomorphic to a complemented subspace of Y. Now by the standard Pełczyński decomposition trick, this means (since $X \approx \ell_p(X)$) that X is isomorphic to Y. □

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The following corollary extends the result of Johnson, Lindenstrauss and Schechtman [20] of the uniqueness of the uniform structure of ℓ_p for $1 < p < \infty$.

Corollary 8.6. *Suppose* $1 < p, r < \infty$. *The spaces* $(\sum_{n=1}^{\infty} \ell_r^n)_{\ell_p}$ *have unique uniform structure if either* $1 < p < \min(r, 2)$ *or* $p > \max(r, 2)$.

Note that for the case $r = 2$, Corollary 8.6 reduces to the result of Johnson, Lindenstrauss and Schechtman [20] since $(\sum_{n=1}^{\infty} \ell_2^n)_{\ell_p} \approx \ell_p$ (see [38]). As pointed out in the Introduction for every $1 < p < \infty$ we can find two nonisomorphic subspaces (respectively, quotients) of ℓ_p which are uniformly homeomorphic (see [29]).

Theorem 8.7. *Let* X *be a subspace of* L_1 *with the strong Schur property. Suppose* Y *coarse Lipschitz-embeds into* X*; then* Y *also has the strong Schur property.*

Proof. By Theorem 7.3 and Proposition 4.8 it is clear that Y has the anti-Banach-Saks property. We also have that Y Lipschitz-embeds into an ultraproduct of X and hence into L_1. Thus Y linearly embeds into L_1^{**} and hence into L_1. Finally we apply Theorem 6.4. □

If X is uniformly homeomorphic to a subspace of ℓ_1, then X is linearly isomorphic to a subspace of L_1; the above theorem implies that X has the strong Schur property, but we do not know if X linearly embeds into ℓ_1. If X is Lipschitz isomorphic to a subspace of ℓ_1, then one can deduce that X linearly embeds into ℓ_1 by exploiting the Radon-Nikodým property and differentiability arguments (see [3]).

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