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DECOMPOSITIONS OF BANACH LATTICES INTO DIRECT SUMS

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ABSTRACT. We consider the problem of decomposing a Banach lattice Z as a direct sum $Z = X \oplus Y$ where X and Y are complemented subspaces satisfying a condition of incomparability (e.g. every operator from Y to X is strictly singular). We treat both the atomic and nonatomic cases. In particular we answer a question of Wojtaszczyk by showing that $L_1 \oplus L_2$ has unique structure as a nonatomic Banach lattice.

One of the most important problems in the theory of Banach lattices, which is still open, is whether any complemented subspace of a Banach lattice must be linearly isomorphic to a Banach lattice. The main difficulty seems to lie in the fact that most of the criteria for a Banach space to be isomorphic to a lattice do not really distinguish between lattices and their complemented subspaces.

We do not actually treat this question in the present paper but rather consider the situation $Z = X \oplus Y$, where Z is a Banach lattice and X and Y two complemented subspaces which are assumed to satisfy different conditions that make them "distinct" in some or another sense. This line of research was initiated by P. Wojtaszczyk [**28**] (and also by I. S. Edelstein and P. Wojtaszczyk [**3**]) who proved that if Z has a normalized unconditional basis $\{z_n\}_{n=1}^\infty$ (i.e. it is a separable atomic lattice) so that every linear operator from Y into X is compact then $\{z_n\}_{n=1}^\infty$ splits into two disjoint parts which are respectively equivalent to bases of X and Y. In particular, both X and Y have unconditional bases. The proof of this result is based on a fundamental theorem from [**28** and **3**], which is mentioned below as Theorem A. We give here a different proof which does not make use of Theorem A but instead is based on a simple "change of signs" result from [**2**], which is described below as Theorem B. We also consider the case when the compactness assumption above is replaced by the total incomparability of X and Y for which we prove a similar result provided X and Y have unconditional bases. Unfortunately, the most interesting case when every operator from Y into X is assumed to be strictly singular (which was raised as an open problem in [**28**]) remains unsolved. We conclude the section devoted to the atomic case with a simple theorem on block bases of a space with unconditional basis $\{z_n\}_{n=1}^\infty$ whose span is complemented. Such a block basis splits into two disjoint parts, the first equivalent to a subsequence of $\{z_n\}_{n=1}^\infty$

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and the second equivalent to a sequence in the complement of the span of the block basis.

In the continuous case we show that a nonatomic Banach lattice Z which has some nontrivial cotype cannot be split into totally incomparable infinite dimensional subspaces; thus if $Z \approx X \oplus Y$ with X, Y totally incomparable then either $\dim X < \infty$ or $\dim Y < \infty$. The same result, under the assumption that every operator $T: X \to Y$ is strictly singular, is false as the example $L_1 \oplus L_2$ shows. However, in this case we give some partial results which suggest that the general result may be true if Z has some nontrivial type.

We conclude by studying the structure of Banach lattices of the form $L_1 \oplus X$, where X is reflexive. We show that if Z is a Banach lattice isomorphic to $L_1 \oplus Y$, where Y has some nontrivial type then Z has a band decomposition $Z \approx Z_0 \oplus Z_0^{\perp}$, where Z_0 is an AL-space and $Z_0^{\perp}$ is isomorphic to Y. As a consequence the Banach space $L_1 \oplus L_2$ has a unique structure as a nonatomic Banach lattice; this answers a question raised by P. Wojtaszczyk [**28**].

0. Preliminaries. In order to make the paper as self-contained as possible, we quote in this section some results that are used very often. We begin with a result which is crucial all throughout the article; instead of presenting the two available versions separately we incorporate them as one theorem.

THEOREM A [**3**, **28**]. *Let X and Y be two Banach spaces and suppose that they are either totally incomparable (i.e. no infinite dimensional subspace of X is isomorphic to a subspace of Y) or that every bounded linear operator T from Y into X is strictly singular (i.e. there exists no infinite dimensional subspace of Y so that T restricted to Y is an isomorphism). Let V be a complemented subspace of $X \oplus Y$. Then there exists an automorphism ψ of $X \oplus Y$ such that $\psi(V) = X_1 \oplus Y_1$, where $X_1 = \psi(V) \cap X$ and $Y_1 = \psi(V) \cap Y$ are complemented subspaces of X, respectively Y.*

The first section of the paper is devoted to spaces with an unconditional basis. The standard material on such spaces can be found e.g. in [**15**]. We quote here a decomposition theorem which is used several times.

THEOREM B [**2**]. *Let X and Y be two Banach spaces and Q a bounded linear projection from $X \oplus Y$ onto a subspace V with a finite or infinite K-unconditional basis $\{v_n\}_{n\in\tau}$ so that*

$$Q(u) = \sum_{n\in\tau} v_n^*(u) v_n,$$

where $\{v_n^*\}_{n\in\tau}$ are functionals biorthogonal to $\{v_n\}_{n\in\tau}$. Let P_X and P_Y denote the corresponding projections from $X \oplus Y$ onto X, respectively Y. Fix $0 < \alpha < 1$ and split τ into two disjoint subsets τ_X and τ_Y so that $v_n^*(QP_X v_n) \geq \alpha$, for $n \in \tau_X$ and $v_n^*(QP_Y v_n) \geq 1 - \alpha$, for $n \in \tau_Y$.

Then there exists a constant $M = M(\|Q\|, K, \alpha)$ so that, for any choice of $1 \leq p \leq \infty$, $\{v_n\}_{n\in\tau_X}$ and $\{v_n\}_{n\in\tau_Y}$ are respectively M-equivalent to $\{r_n(t) P_X v_n\}_{n\in\tau_X}$ and $\{r_n(t) P_Y v_n\}_{n\in\tau_Y}$, considered as elements of $L_p(X)$ and $L_p(Y)$. Moreover, $[r_n(t) P_X v_n]_{n\in\tau_X}$ and $[r_n(t) P_Y v_n]_{n\in\tau_Y}$ are M-complemented in $L_p(X)$, respectively $L_p(Y)$.

Here, as usual, $\{r_n(t)\}_{n=1}^{\infty}$ denotes the sequence of the Rademacher functions and $L_p(X)$ stands for $L_p([0,1],X)$.

The spaces with an unconditional basis can be viewed as atomic lattices. In the second and third sections, we study nonatomic and also general lattices. The standard material on this topic is available, for instance, in [**16** and **26**]. Of particular interest are the order continuous Banach lattices, i.e. those in which any decreasing net $\{z_\gamma\}$ whose g.l.b. is 0 satisfies $\lim_\gamma z_\gamma = 0$. Order continuous Banach lattices with a weak unit admit a very useful representation theorem (cf. e.g. [**16**, 1.b.14]).

THEOREM C. *Let Z be an order continuous Banach lattice with a weak unit. Then there exist a probability space (Ω,Σ,μ), an (not necessarily closed) ideal $\tilde{Z}$ of $L_1(\Omega,\Sigma,\mu)$ and a lattice norm $\|\cdot\|_{\tilde{Z}}$ on $\tilde{Z}$ such that*

(i) *Z is order isometric to $\tilde{Z}$.*

(ii) *$\tilde{Z}$ is dense in $L_1(\Omega,\Sigma,\mu)$ and $L_\infty(\Omega,\Sigma,\mu)$ is dense in $\tilde{Z}$.*

(iii) $\|f\|_1 \le \|f\|_{\tilde{Z}} \le 2\|f\|_\infty$; $f \in L_\infty(\Omega,\Sigma,\mu)$.

(iv) *The dual of the above order isometry maps Z^* onto the Banach lattice $\tilde{Z}^*$ of all g for which*

$$\|g\|_{\tilde{Z}^*} = \sup\left\{\int_\Omega fg\,d\mu; \|f\|_{\tilde{Z}} \le 1\right\} < \infty$$

and $g(f) = \int_\Omega fg\,d\mu$.

A lattice of functions as above will be called a Banach function space.

We point out that a Banach lattice which contains no subspace isomorphic to c_0 is order continuous. In fact, such a lattice has the stronger property that any increasing net $\{z_\gamma\}_{\gamma\in\Gamma}$ with $\sup_{\gamma\in\Gamma}\|z_\gamma\| < \infty$ converges to its l.u.b.

As we have already mentioned, the third section studies embedding of L_1-spaces into Banach lattices. One of the most important tools used in this section is a representation theorem for operators from $L_1(0,1)$ to a Banach lattice containing no copy of c_0, which is presented and used extensively in [**10** and **11**]. Rather than discussing these results in detail, we collect here only those facts that are needed in the sequel. We also quote a theorem concerning the so-called sign-preserving operators.

An operator T from $L_1(0,1)$ into a Banach space Z is called sign-preserving provided there exist a $\delta > 0$ and a subset A of $[0,1]$ of positive measure so that $\|T(\psi)\| \ge \delta$, whenever $\psi \in L_1(0,1)$ has mean zero and $|\psi| = \chi_A$ (χ_A denotes the characteristic function of A).

THEOREM D. *Let Z be a Banach lattice containing no isomorphic copy of c_0.*

(i) [**11**] *If T is an isomorphism from $L_1(0,1)$ into Z then there exists also an order isomorphism S from $L_1(0,1)$ into Z.*

(ii) [**11, 25**] *An operator S from $L_1(0,1)$ into a $L_1(\Omega,\Sigma,\mu)$-space is a lattice homomorphism if and only if there exist measurable functions $0 \le a\colon\Omega \to \mathbf{R}$ and $\sigma\colon\Omega\to[0,1]$ so that*

$$S(\psi)(\omega) = a(\omega)\psi(\sigma(\omega)), \qquad \omega\in\Omega \text{ a.e.},$$

for all $\psi \in L_1(0,1)$.

(iii) [**7**] *If T is a sign-preserving operator from $L_1(0,1)$ into Z then T is an isomorphism on some subspace of $L_1(0,1)$ which is isomorphic to $L_1(0,1)$.*

Additional details on operators from L_1-spaces into Banach lattices can be found in [**10**, **11**, **24**, **25**, **7**, **4** and **5**].

1. Spaces with an unconditional basis (atomic lattices). We start this section by presenting an alternative proof of P. Wojtaszczyk's decomposition theorem from [**28**]. The original proof relies on Theorem A while ours uses instead Theorem B whose proof is considerably simpler than that of Theorem A.

THEOREM 1.1. *Let X and Y be two Banach spaces such that every bounded linear operator from Y into X is compact. Then every unconditional basis $\{z_n\}_{n=1}^\infty$ of $X \oplus Y$ splits into two disjoint parts $\{z_n\}_{n\in C_1}$ and $\{z_n\}_{n\in C_2}$ that are respectively equivalent to bases of X and Y. In particular, X and Y have unconditional bases.*

PROOF. We can assume without loss of generality that $\|z_n\| = 1$, for all n, and that the direct sum $X \oplus Y$ is taken in the sense of l_1. In this case, the corrresponding projections P_X and P_Y from $X \oplus Y$ onto X, respectively Y, are of norm one. For each subset C of the integers, let Q_C denote the projection defined by

$$Q_C\left(\sum_{n=1}^\infty a_n z_n\right) = \sum_{n\in C} a_n z_n,$$

for any choice of $\{a_n\}_{n=1}^\infty$. Put $K = \sup \|Q_C\|$.

Split now the integers into two disjoint subsets A_1 and A_2 such that $n \in A_1 \Rightarrow x_n^* x_n \geq \frac{1}{2}$ and $n \in A_2 \Rightarrow y_n^* y_n \geq \frac{1}{2}$, where $x_n = P_X z_n$, $y_n = P_Y z_n$, $x_n^* = P_n^* z_n^*$, $y_n^* = P_Y^* z_n^*$ and $\{z_n^*\}_{n=1}$ are the biorthogonal functionals associated to $\{z_n\}_{n=1}^\infty$.

By Theorem B, we conclude the existence of a constant $M = M(K)$ so that

$$M^{-1}\left\|\sum_{i\in A_1} a_i z_i\right\| \leq \int_0^1 \left\|\sum_{i\in A_1} a_i r_i(t) x_i\right\| dt \leq M \left\|\sum_{i\in A_1} a_i z_i\right\|$$

and

$$M^{-1}\left\|\sum_{i\in A_2} a_i z_i\right\| \leq \int_0^1 \left\|\sum_{i\in A_2} a_i r_i(t) y_i\right\| dt \leq M \left\|\sum_{i\in A_2} a_i y_i\right\|,$$

for any choice of scalars $\{a_i\}_{i=1}^\infty$. Similar inequalities hold in the dual.

LEMMA 1.2. *There exists a subset B_1 of A_1 so that $\{z_n\}_{n\in B_1}$ is equivalent to $\{x_n\}_{n\in B_1}$ and $|A_1 \sim B_1|$ is finite.*

PROOF. Suppose the assertion of Lemma 1.2 is false. Since

$$\left\|\sum_{i\in A_1} a_i x_i\right\| \leq \left\|\sum_{i\in A_1} a_i z_i\right\|,$$

for any choice of $\{a_i\}$, we can easily construct a sequence $\{C_j\}_{j=1}^\infty$ of mutually disjoint finite subsets of A_1 and vectors $u_j = \sum_{i\in C_j} b_i z_i$ such that $\|u_j\| = 1$ and

$\|P_X u_j\| < 2^{-(j+1)}$, for all j. The inequalities mentioned above show the existence of signs $\varepsilon_i = \pm 1$, $i \in C_j$, $j = 1, 2, \ldots$, such that

$$\left\| \sum_{i \in C_j} \varepsilon_i b_i x_i \right\| \geq M^{-1}.$$

Consider now the automorphism T_ε of $X \oplus Y$ defined by $T_\varepsilon z_i = \varepsilon_i z_i$, for all i, where ε_i is chosen as above if $i \in C_j$, for some j, or $\varepsilon_i = +1$, otherwise. Since the operator $P_X T_{\varepsilon|Y}: Y \to X$ must be compact it follows by passing to a subsequence if necessary that $P_X T_\varepsilon P_Y u_j \to x$, as $j \to \infty$, for some $x \in X$. Since $\|u_j - P_Y u_j\| < 2^{-(j+1)}$ we also get that $P_X T_\varepsilon u_j \to x$, as $j \to \infty$. On the other hand, $P_X T_\varepsilon u_j = \sum_{i \in C_j} \varepsilon_i b_i x_i$ and therefore $\|x\| \geq M^{-1}$. This shows that the sequence $\{u_j\}_{j=1}^\infty$ does not tend weakly to zero. Hence, there is a functional u^* on $X \oplus Y$ and a subsequence $\{u_{j(k)}\}_{k=1}^\infty$ of $\{u_j\}_{j=1}$ so that $\inf_k u^* u_{j(k)} > 0$. It follows that $\{u_{j(k)}\}_{k=1}^\infty$ is equivalent to the unit vector basis of l_1, and, moreover, that $[u_{j(k)}]_{k=1}^\infty$ is complemented in $X \oplus Y$. A standard perturbation argument shows that there is a l_0 so that $[P_Y u_{j(k)}]_{k \geq l_0}$ is complemented in Y and isomorphic to l_1. This means that there are quotient maps from Y onto X which, of course, is a contradiction. □

LEMMA 1.3. *There exists a subset B_2 of A_2 so that $\{z_n\}_{n \in B_2}$ is equivalent to $\{y_n\}_{n \in B_2}$ and $|A_2 \sim B_2|$ is finite.*

PROOF. If the assertion of Lemma 1.3 is false then one can construct exactly as in the proof of Lemma 1.2 a sequence of mutually disjoint finite subsets $\{D_j\}_{j=1}^\infty$ of A_2 and vectors $v_j = \sum_{i \in D_j} d_i z_i$ such that $\|v_j\| = 1$ and $\|P_Y v_j\| < 2^{-(j+1)}$, for all j.

If $\{v_j\}_{j=1}^\infty$ does not tend weakly to zero then it contains a subsequence which is equivalent to the unit vector basis of l_1. By Theorem B, the subspace $[r_n(t) y_n]_{n \in A_2}$ of $L_2(Y)$ also contains a subspace isomorphic to l_1. Hence, by G. Pisier [**20**], we conclude that Y contains an isomorphic copy of l_1. A standard gliding hump argument shows that one can find in Y a sequence of elements which, on one hand, is equivalent to the unit vector basis of l_1 and, on the other hand, is equivalent to a block basis of $\{z_n\}_{n=1}^\infty$. Therefore, Y contains a complemented copy of l_1 and we can easily construct noncompact operators from Y into X. This contradiction shows that $v_j \xrightarrow{w} 0$, as $j \to \infty$.

Now for each j, let $v_j^* \in X^*$ be so that $\|v_j^*\| = 1$ and $v_j^*(P_X v_j) \geq 1 - 2^{-(j+1)}$. It follows that $v_j^*(v_j) \geq 1 - 2^{-j}$, for all j. By passing to a subsequence if needed, we can assume without loss of generality that $v_j^* \xrightarrow{w^*} v^*$, as $j \to \infty$, for some $v^* \in X^*$. Since $v^*(v_j) \to 0$, as $j \to \infty$, we can choose an integer j_1 so that $|v^*(v_j)_1| < 1/8$. Then we can find an integer i_1 so that $|v_{i_1}^*(v_{j_1})| < 1/8$. Put $w_1^* = v_{j_1}^* - v_{i_1}^*$ and notice that $w_1^*(v_{j_1}) \geq 1 - 1/2^j - 1/8 > 1/4$. Repeating this procedure, we construct a subsequence $\{v_{j_k}\}_{k=1}^\infty$ of $\{v_j\}_{j=1}^\infty$ and a sequence $\{w_k^*\}_{k=1}^\infty$, which are differences of the v_j^*'s and therefore $w^*\text{-}\lim_{k \to \infty} w_k^* = 0$, such that $w_k^*(v_{j_k}) > 1/4$, for all k. By a gliding hump argument, we can assume without loss of generality that

$$w_k^* = \sum_{i \in \tilde{D}_k} c_i z_i^* + \tilde{w}_k^*,$$

where $\{\tilde{D}_k\}_{k=1}^{\infty}$ are mutually disjoint finite subsets of the integers, $\{c_i\}$ suitable scalars and $\{\tilde{w}_k^*\}_{k=1}^{\infty}$ a sequence which tends in norm to zero. Therefore, we will assume that $\|\tilde{w}_k^*\| < 1/8$, for all k. It follows that

$$\frac{1}{4} < w_k^*(v_{j_k}) \leq \left(\sum_{i\in\tilde{D}_k} c_i z_i^*\right)\left(\sum_{i\in D_{j_k}} d_i z_i\right) + \frac{1}{8}$$

which implies that

$$\left\|\sum_{i\in\tilde{D}_k\cap D_{j_k}} c_i z_i^*\right\| \geq \frac{1}{8},$$

for all k. By Theorem B, applied to the dual of $X \oplus Y$, we find signs $\varepsilon_i = \pm 1$, $i \in \tilde{D}_k \cap D_{j_k}$, $k = 1, 2, \ldots$ so that

$$\left\|\sum_{i\in\tilde{D}_k\cap D_{j_k}} \varepsilon_i c_i y_i^*\right\| > \frac{1}{8M}$$

for all k.

Let now $\tilde{T}_\varepsilon$ be the linear operator on $X \oplus Y$ which is defined by $\tilde{T}_\varepsilon z_i = \varepsilon_i z_i$, if $i \in \tilde{D}_k \cap D_{j_k}$ for some k, or by $\tilde{T}_\varepsilon z_i = 0$, otherwise. Since $P_X \tilde{T}_\varepsilon P_Y$ is compact it follows that its adjoint $P_Y^* \tilde{T}_\varepsilon^* P_X^*$ is also compact and thus, we can assume without loss of generality that $P_Y^* \tilde{T}_\varepsilon^* P_X^* w_k^* \to 0$, as $k \to \infty$. Since $w_k^* \in X^*$, for all k, we get that

$$\sum_{i\in\tilde{D}_k\cap D_{j_k}} \varepsilon_i c_i y_i^* = P_Y^* \tilde{T}_\varepsilon^* \left(\sum_{i\in\tilde{D}_k} c_i z_i^*\right) \to 0,$$

as $k \to \infty$, and this, of course, is a contradiction.

LEMMA 1.4. *There exists a subset D of the integers such that*

(i) $Y = [y_n]_{n\in D}$.

(ii) $\{z_n\}_{n\in D}$ *is equivalent to* $\{y_n\}_{n\in D}$.

PROOF. Suppose that the complement B_2^c of B_2 in $\mathbb{N}$ consists of the integers $\{n_1 < n_2 < \cdots < n_j < \cdots\}$. Put $E_0 = B_2$. If $y_{n_1} \in [y_n]_{n\in E_0}$ then omit this vector; if $y_{n_1} \notin [y_n]_{n\in E_0}$ then set $E_1 = B_2 \cup \{n_1\}$. A simple verification shows that $\{z_n\}_{n\in E_1}$ is equivalent to $\{y_n\}_{n\in E_1}$. Indeed, suppose that there exists a vector $\sum_{n\in E_1} a_n z_n \neq 0$ so that $\sum_{n\in E_1} a_n y_n = 0$. By our assumption on y_{n_1}, we conclude that the coefficient of y_{n_1} must be zero. Hence, $\sum_{n\in E_0} a_n y_n = 0$ which, by Lemma 3, implies that $\sum_{n\in E_0} a_n z_n = 0$, a contradiction.

We continue this procedure with $n_2, n_3, \ldots$ and construct subsets $E_2, E_3, \ldots$ of the integers as above. If the argument stops after m steps then we put $D = E_m$ and the proof is completed since $[y_n]_{n\in E_m}$ must be equal to Y. On the other hand, if this procedure can be repeated as many times as we like, then there exists a sequence $\{E_j\}_{j=1}^{\infty}$ of subsets of the integers so that

(a) $B_2 = E_0 \subset E_1 \subset \cdots \subset E_j \subset \cdots$.

(b) $[y_n]_{n\in E_{j-1}} \neq [y_n]_{n\in E_j} \neq Y$, for all j.

(c) $[y_n]_{n\in\cup_{j=1}^{\infty} E_j} = Y$.

Hence, there exist vectors $y'_j \in Y$ such that $\|y'_j\| = 1$ and $d(y'_j, [y_n]_{n\in E_{j-1}}) \geq \frac{1}{2}$, for all $j \geq 1$. Since, by our assumption, the restriction of the operator $P_X Q_{B_1}$ to Y is compact it follows, by passing to a subsequence if necessary, that $P_X Q_{B_1} y'_j \to x$, as $j \to \infty$, for some $x \in X$. Suppose now that

$$y'_j = \sum_{n=1}^{\infty} a_n^j z_n = \sum_{n=1}^{\infty} a_n^j y_n, \qquad j = 1, 2, \dots .$$

Then

$$\lim_{j\to\infty} \sum_{n\in B_1} a_n^j x_n = x.$$

But, by Lemma 2, $\{z_n\}_{n\in B_1}$ is equivalent to $\{x_n\}_{n\in B_1}$. Hence,

$$\lim_{j\to\infty} \sum_{n\in B_1} a_n^j y_n = y,$$

for some $y \in Y$. Since

$$B_1^c \sim B_2 = (A_1 \sim B_1) \cup (A_2 \sim B_2)$$

there exists an integer l so that $[y_n]_{n\in B_1^c} \subset [y_n]_{n\in E_k}$, whenever $k \geq l$. It follows that, for $j > k \geq l$, we have

$$d\left(\sum_{n\in B_1} a_n^j y_n, [y_n]_{n\in E_k}\right) = d(y'_j, [y_n]_{n\in E_k}) \geq d(y'_j, [y_n]_{n\in E_{j-1}}) \geq \frac{1}{2}.$$

This implies that $d(y, [y_n]_{n\in E_k}) \geq 1/2$ for all k. Hence, $d(y, Y) \geq 1/2$ which, of course, is contradictory. $\square$

We return now to the Proof of Theorem 1.1. Let D be the set constructed in Lemma 1.4 and notice that if $x \in X \cap [z_n]_{n\in D}$ then $x = \sum_{n\in D} d_n z_n = \sum_{n\in D} d_n x_n$ for a suitable sequence of scalars $\{d_n\}_{n\in D}$. Hence, $\sum_{n\in D} a_n y_n = 0$ and, by Lemma 1.4, also $x = 0$. Thus, $X \cap [z_n]_{n\in D} = \{0\}$. Actually, $X + [z_n]_{n\in D}$ is a closed subspace of $X \oplus Y$. Indeed, if there exist sequences $\{u_k\}_{k=1}^{\infty}$ in X and $\{v_k\}_{k=1}^{\infty}$ in $[z_n]_{n\in D}$ with $\|u_k\| = \|v_k\| = 1$, for all k, and $\lim_{k\to\infty} \|u_k + v_k\| = 0$ then $\lim_{k\to\infty} \|u_k + P_X v_k\| = 0$. Let S be the isomorphism from Y onto $[z_n]_{n\in D}$ given by Lemma 1.4. Then $P_X S$ is a compact operator which maps $S^{-1}v_k$ into $P_X v_k$, for all k. Hence, $\{P_X v_k\}_{k=1}^{\infty}$ and thus also $\{u_k\}_{k=1}^{\infty}$ contain convergent subsequences and this contradicts the fact that $X \cap [z_n]_{n\in D} = \{0\}$.

Observe now that Y is contained in the internal direct sum $X \oplus [z_n]_{n\in D}$ since $Y = [y_n]_{n\in D}$ and, for $m \in D$, $y_m = z_m - x_m \in X \oplus [z_n]_{n\in D}$. Therefore,

$$[z_n]_{n=1}^{\infty} = X \oplus Y = X \oplus [z_n]_{n\in D}$$

which readily implies that X is isomorphic to $[z_n]_{n\in D^c}$. This completes the proof since we already know that Y is isomorphic to $[z_n]_{n\in D}$, by Lemma 1.4. $\square$

P. Wojtaszczyk raised in [**28**] the question whether Theorem 1 remains valid when the assumption that all the operators from Y into X are compact is replaced by the weaker condition that all the operators from Y into X are strictly singular. This question seems to be still open. As far as we know, it is still unknown if the above compactness condition can be replaced by the total incomparability of X and Y. We prove such a result under some additional assumptions.

THEOREM 1.5. *Let X and Y be two totally incomparable Banach spaces having each an unconditional basis. Then every unconditional basis $\{z_n\}_{n=1}^\infty$ of $X \oplus Y$ splits into two disjoint parts which are respectively equivalent to bases of X and Y.*

PROOF. We shall maintain the notation used in the proof of Theorem 1.1 as well as the basic assumptions made there. We need first a lemma.

LEMMA 1.6. *If there exists an $\alpha > 0$ so that $x_n^*(x_n) \geq \alpha$, for all n, then*
(i) *$\{z_n\}_{n=m}^\infty$ is equivalent to $\{x_n\}_{n=m}^\infty$, for some integer m.*
(ii) *Y is finite dimensional.*

PROOF. By Theorem B, there is a $M = M(K, \alpha)$ so that

$$M^{-1}\left\|\sum_{n=1}^\infty c_n z_n\right\| \leq \int_0^1 \left\|\sum_{n=1}^\infty c_n r_n(t) x_n\right\| dt \leq M\left\|\sum_{n=1}^\infty c_n z_n\right\|$$

for any choice of $\{c_n\}_{n=1}^\infty$. A similar inequality holds in the dual situation for $\{x_n^*\}_{n=1}^\infty$ and $\{z_n^*\}_{i=1}^\infty$.

Let now $\{\xi_i\}_{i=1}^\infty$ be a normalized unconditional basis for X and let $\{\xi_i^*\}_{i=1}^\infty$ be the corresponding biorthogonal functionals. For $x = \sum_{i=1}^\infty b_i \xi_i$ put

$$|x| = \sum_{i=1}^\infty |b_i| \xi_i \quad \text{and} \quad |x_i|^2 = \sum_{i=1}^\infty |b_i|^2 \xi_i.$$

Then, for any choice of $\{c_n\}_{n=1}^\infty$, we have

$$M\left\|\sum_{n=1}^\infty c_n z_n\right\| \geq \left\|\int_0^1 \left|\sum_{n=1}^\infty c_n r_n(t) x_n\right| dt\right\| \geq \frac{1}{\sqrt{2}}\left\|\left(\sum_{n=1}^\infty |c_n x_n|^2\right)^{1/2}\right\|.$$

Since

$$Pf = \sum_{n=1}^\infty \left(\int_0^1 x_n^*(f(s)) r_n(s) ds\right) \frac{r_n(t) x_n}{x_n^*(x_n)}, \qquad f \in L_2(X),$$

defines a projection of norm $\leq M$ from $L_2(X)$ onto $[r_n(t)x_n]_{n=1}^\infty$ and since, as above,

$$M\sqrt{2}\left\|\sum_{n=1}^\infty d_n z_n^*\right\| \geq \left\|\left(\sum_{n=1}^\infty |d_n x_n^*|^2\right)^{1/2}\right\|,$$

for any choice of $\{d_n\}_{n=1}^\infty$, we can apply a standard duality argument and conclude that

$$\left\|\sum_{n=1}^\infty c_n z_n\right\| \leq M^4\sqrt{2}\left\|\left(\sum_{n=1}^\infty |c_n x_n|^2\right)^{1/2}\right\|,$$

for every choice of scalars $\{c_n\}_{n=1}^\infty$.

Suppose now that either (i) or (ii) do not hold. In this case, it is easily verified that there are integers $p_0 = 1 < p_1 < p_2 < \cdots < p_j < \cdots$ and blocks $u_j = \sum_{i=p_{j-1}+1}^{p_j} a_i z_i$ so that $\|u_j\| = 1$ and $\|P_X u_j\| < 2^{-(j+1)}$, for all j.

Notice that $u_j \xrightarrow{w} 0$, as $j \to \infty$. Indeed, if this is false then we can assume without loss of generality that $\{u_j\}_{j=1}^\infty$ is equivalent to the unit vector basis of l_1. Moreover, $[u_j]_{j=1}^\infty$ is complemented in $X \oplus Y$. In view of the condition satisfied by

$\{P_X u_j\}_{j=1}^{\infty}$, it follows that Y contains a complemented copy of l_1. Since $\{z_n\}_{n=1}^{\infty}$ is equivalent to $\{r_n(t)x_n\}_{n=1}^{\infty}$, considered e.g. as a sequence in $L_2(X)$, we conclude that also $L_2(X)$ contains a complemented subspace isomorphic to l_1. By [**20**], X contains a copy of l_1, too, and this contradicts the assumption that X and Y are totally incomparable.

Now, for each j, put

$$s_j = \left(\sum_{n=p_{j-1}+1}^{p_j} |a_n x_n|^2 \right)^{1/2} \in X.$$

Then, for every i, we have

$$\lim_{j\to\infty} \xi_i^*(s_j) = 0.$$

Indeed, if for some fixed i there exists a $\beta > 0$ so that $\xi_i^*(s_j) \geq \beta$, for all j, then

$$\left(\int_0^1 \left| \sum_{n=p_{j-1}+1}^{p_j} a_n r_n(t) \xi_i^*(x_n) \right|^2 dt \right)^{1/2} = \left(\sum_{n=p_{j-1}+1}^{p_j} |a_n \xi_i^*(x_n)|^2 \right)^{1/2} \geq \beta.$$

Hence, there are signs $\varepsilon_n = \pm 1$, $p_{j-1} < n \leq p_j$, such that

$$(P_X^* \xi_i^*) \left(\sum_{n=p_{j-1}+1}^{p_j} a_n \varepsilon_n z_n \right) = \xi_i^* \left(\sum_{n=p_{j-1}+1}^{p_j} a_n \varepsilon_n x_n \right) \geq \beta,$$

for all j. This contradicts the fact that $\text{w-lim}_{j\to\infty} u_j = 0$. The condition satisfied by $\{s_j\}_{j=1}^{\infty}$ shows that there is no loss of generality in assuming that $\{s_j\}_{j=1}^{\infty}$ is actually a block basis of $\{\xi_i\}_{i=1}^{\infty}$ (note that, by the inequalities established above, $1/M^4\sqrt{2} \leq \|s_j\| \leq M\sqrt{2}$, for all j). These inequalities also yield that, for any choice of $\{b_j\}_{j=1}^{\infty}$, we have

$$M\sqrt{2} \left\| \sum_{j=1}^{\infty} b_j u_j \right\| \geq \left\| \left(\sum_{j=1}^{\infty} |b_j|^2 \sum_{n=p_{j-1}+1}^{p_j} |a_n x_n|^2 \right)^{1/2} \right\| = \left\| \sum_{j=1}^{\infty} |b_j| s_j \right\|$$

and, also

$$\left\| \sum_{j=1}^{\infty} b_j u_j \right\| \leq M^4\sqrt{2} \left\| \sum_{j=1}^{\infty} |b_j| s_j \right\|.$$

It follows that $[u_j]_{j=1}^{\infty}$ is isomorphic to the subspace $[s_j]_{j=1}^{\infty}$ of X. However, the fact that $\|P_X u_j\| < 2^{-(j+1)}$, for all j, implies that $[u_j]_{j=1}^{\infty}$ is isomorphic to a subspace of Y. This, of course, contradicts the total incomparability of X and Y. □

We return to the Proof of Theorem 1.5. Let A_1 and A_2 be defined as in the proof of Theorem 1.1. By Theorem A, there exists an automorphism $\psi\colon X \oplus Y \to X \oplus Y$ such that

$$\psi([z_n]_{n\in A_1}) = X_1 \oplus Y_1,$$

where $X_1 = \psi([z_n]_{n\in A_1}) \cap X$ and $Y_1 = \psi([z_n]_{n\in A_1}) \cap Y$ are complemented subspaces of X, respectively Y.

Put $\hat{z}_n = \psi(z_n); n = 1, 2, \ldots$, and let $\{\hat{z}_n^*\}_{n=1}^\infty$ be the functionals biorthogonal to $\{\hat{z}_n\}_{n=1}^\infty$. In general, the set

$$\hat{A}_1 = \{n \in \mathbb{N}; \hat{z}_n^*(P_X \hat{z}_n) \geq \tfrac{1}{2}\}$$

need not coincide with A_1. However, A_1 and $\hat{A}_1$ can differ only by a finite number of elements. Indeed, if e.g. $\hat{A}_1 \cap A_2$ contains an infinite sequence $\{n_i\}_{i=1}^\infty$ and $\{z_{n_i}\}_{i=1}^\infty$ does not tend weakly to zero then, by Theorem B, both $L_2(X)$ and $L_2(Y)$ must contain an isomorphic copy of l_1. By [**20**], so do X and Y and this is a contradiction. If, on the other hand, $z_{n_i} \xrightarrow{w} 0$ then we can assume without loss of generality that $\{y_{n_i}\}_{i=1}^\infty$ and $\{P_X \hat{z}_{n_i}\}_{i=1}^\infty$ are mutually disjoint blocks on $\{z_n\}_{n=1}^\infty$, respectively $\{\hat{z}_n\}_{n=1}^\infty$. Hence, again by Theorem B, $\{y_{n_i}\}_{i=1}^\infty$ is equivalent to $\{z_{n_i}\}_{i=1}^\infty$ and $\{P_X \hat{z}_{n_i}\}_{i=1}^\infty$ to $\{\hat{z}_{n_i}\}_{i=1}^\infty$; i.e. X and Y are not totally incomparable.

It follows from these considerations that $\psi([z_n]_{n \in A_1 \cap \hat{A}_1})$ is a space of finite codimension in $X_1 \oplus Y_1$, and

$$\psi([z_n]_{n \in A_1 \cap \hat{A}_1}) = \hat{X}_1 \oplus \hat{Y}_1,$$

where $\hat{X}_1$ and $\hat{Y}_1$ are spaces of finite codimension in X_1, respectively Y_1. By Lemma 1.6, Y_1 must be finite dimensional and thus $k = \dim Y_1 < \infty$.

Suppose now that $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Then $\psi([z_n]_{n \in A_2})$ is clearly isomorphic to $X_2 \oplus Y_2$. In view of the total symmetry of our assumptions, we can conclude, as above, that $m = \dim X_2 < \infty$. If e.g. $m \geq k$ then we switch $m - k$ vectors from A_2 to A_1 thus obtaining new sets C_1 and C_2. It is easily checked that $\psi([z_n]_{n \in C_1})$ is isomorphic to X and $\psi([z_n]_{n \in C_2})$ to Y. $\square$

REMARK. It is not clear if the assumption made in the statement of Theorem 1.5 that X and Y have unconditional bases is actually needed. As is well known, the problem whether every complemented subspace of a space with an unconditional basis must have itself an unconditional basis is still unsolved. Theorem 1.5 could perhaps provide some means to construct a counterexample since it specifies in a precise manner an unconditional basis for X whenever $X \oplus Y$ has an unconditional basis and X and Y are totally incomparable.

In order to prove results on splitting of bases of a direct sum, one has to make some additional assumptions, otherwise, the result clearly fails. This fact is put in evidence by a simple example pointed out by P. Wojtaszczyk in [**28**]. He noticed that, for $1 < p \neq 2$, $L_p(0,1)$ is isomorphic to $L_p(0,1) \oplus l_2$ but the image under any isomorphism of the normalized Haar basis in $L_p(0,1)$ is an unconditional basis of $L_p(0,1) \oplus l_2$ which does not split as e.g. in Theorem 1.1.

In the simpler case when one of the factors of the direct sum is c_0 or l_p, the situation is considerably clearer, as shown by the next result.

THEOREM 1.7. *Let X and Y be two Banach spaces and suppose that X is isomorphic to c_0 or l_p, for some $p \geq 1$. If $X \oplus Y$ has an unconditional basis then so does Y.*

PROOF. The cases when X is isomorphic either to c_0 or to l_1 can be treated separately. For instance, suppose that X is isomorphic to c_0 and that $X \oplus Y$ has a normalized unconditional basis $\{z_n\}_{n=1}^\infty$. If every operator from X into Y is compact then we conclude that Y has an unconditional basis, by using Theorem 1. On the other hand, if there exists a noncompact operator T from c_0 into Y then we can find a $\gamma > 0$, a normalized block basis $\{u_n\}_{n=1}^\infty$ of the unit vector basis of c_0 and a block basis $\{v_n\}_{n=1}^\infty$ of $\{z_n\}_{n=1}^\infty$ so that $\|v_n\| \geq \gamma$ and $\|Tu_n - v_n\| \leq 2^{-n}$, for all n. It follows easily that $\{v_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of c_0 and therefore Y contains a complemented copy of c_0. Hence, $X \oplus Y$ is isomorphic to Y and this completes the proof of this case.

We pass now to the case when X is isomorphic to l_p, for some $p > 1$. We shall assume that every bounded linear operator from Y into X is strictly singular; otherwise, Y contains a complemented subspace isomorphic to l_p and thus Y is isomorphic to $Y \oplus X$ which has an unconditional basis. We shall keep the notations and the conventions introduced throughout the proof of Theorem 1.1.

We consider first the vectors $\{z_n\}_{n\in A_1}$. By Theorem B, $[z_n]_{n\in A_1}$ is isomorphic to $[r_n(t)x_n]_{n\in A_1}$, considered e.g. as a complemented subspace of $L_p(X)$ which, in turn, is isomorphic to $L_p(0,1)$. By Theorem A, $[z_n]_{n\in A_1}$ is isomorphic to a direct sum of the form $X_1 \oplus Y_1$, where X_1 and Y_1 are complemented subspaces of X, respectively Y. It follows that Y_1 is either isomorphic to l_2 or is an $\mathcal{L}_p$-space. In the case when Y_1 is an infinite dimensional $\mathcal{L}_p$-space it contains a complemented copy of l_p (cf. [**14**]) and thus, there exist nonstrictly singular operators from Y into X. Therefore, either Y_1 is isomorphic to l_2 or it is finite dimensional. In both these cases, Y_1 has an unconditional basis.

We proceed now with $[z_n]_{n\in A_2}$. Let X_2 and Y_2 be such that $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Then, by Theorem A, $[z_n]_{n\in A_2}$ is isomorphic to $X_2 \oplus Y_2$. If $\dim X_2 < \infty$ then Y_2 is of finite codimension in a space with an unconditional basis and therefore it has itself, an unconditional basis. Suppose now that X_2 is isomorphic to l_p. In this case, we can find a block basis $u_j = \sum_{n=q_{j-1}+1}^{q_j} a_n z_n$, $j = 1, 2, \ldots$, of $\{z_n\}_{n\in A_2}$ so that

(a) $\|u_j\| = 1$, for all j.

(b) $\{u_j\}_{j=1}^\infty$ is equivalent to the unit vector basis of l_p.

(c) $\{u_j\}_{j=1}^\infty$ is equivalent to a sequence in X.

(d) $[u_j]_{j=1}^\infty$ is a complemented subspace in $[z_n]_{n\in A_2}$.

Let R be a projection from $[z_n]_{n\in A_2}$ onto $[u_j]_{j=1}^\infty$. We can assume without loss of generality that

$$R(z) = \sum_{j=1}^\infty u_j^*(z)u_j, \qquad z \in [z_n]_{n\in A_2},$$

where $u_j^* = \sum_{n=q_{j-1}+1}^{q_j} b_n z_n^*$, for all j, and $\{b_n\}_{n\in A_2}^\infty$ are suitable scalars so that $b_n a_n \geq 0$, for all n. Notice that $\{u_j^*\}_{j=1}^\infty$ is equivalent to the unit vector basis of l_q, where $q = p/(p-1)$. Since

$$\int_0^1 \left(\sum_{n=q_{j-1}+1}^{q_j} b_n r_n(t) y_n^*\right)\left(\sum_{n=q_{j-1}+1}^{q_j} a_n r_n(t) y_n\right) dt$$
$$= \sum_{n=q_{j-1}+1}^{q_j} a_n b_n y_n^*(y_n) \geq \frac{1}{2} u_j^*(u_j)$$

for all j, we conclude the existence of signs $\varepsilon_n = \pm 1$, $q_{j-1} < n \le q_j$, $j = 1, 2, \ldots$, so that

$$\left(\sum_{n=q_{j-1}+1}^{q_j} b_n \varepsilon_n y_n^*\right)\left(\sum_{n=q_{j-1}+1}^{q_j} a_n \varepsilon_n y_n\right) \ge \frac{1}{2},$$

for all j. We write now

$$v_j = \sum_{n=q_{j-1}+1}^{q_j} a_n \varepsilon_n y_n \quad \text{and} \quad v_j^* = \sum_{n=q_{j-1}+1}^{q_j} b_n \varepsilon_n y_n^*, \qquad j = 1, 2, \ldots.$$

By switching to a subsequence, we may assume with no loss of generality that $\{v_j\}_{j=1}^\infty$ and $\{v_j^*\}_{j=1}^\infty$ are block bases of $\{z_n\}_{n=1}^\infty$, respectively $\{z_n^*\}_{n=1}^\infty$. Hence,

(a^1) $v_j^*(v_j) \ge \frac{1}{2}$ and $v_j^*(v_h) = 0$, for all j and all $h \ne j$.

(b^1) $\{v_j\}_{j=1}^\infty$ is dominated by the unit vector basis of l_p and $\{v_j^*\}_{j=1}^\infty$ by that of l_q.

It follows easily, by a simple duality argument, that $\{v_j\}_{j=1}^\infty$ is equivalent to the unit vector basis of l_p and $\{v_j^*\}_{j=1}^\infty$ to that of l_q. Moreover,

$$R^1(y) = \sum_{j=1}^\infty \frac{v_j^*(y)}{v_j^*(v_j)} v_j, \qquad y \in Y,$$

defines a bounded projection from Y onto $[v_j]_{j=1}^\infty$. This means that Y contains a complemented copy of l_p and thus there are nonstrictly singular operators from Y into X.

Since both Y_1 and Y_2 have unconditional bases it follows that so does Y. □

We conclude this section with a result on block bases of an unconditional basis which span a complemented subspace.

THEOREM 1.8. *Let Z be a Banach space with a normalized unconditional basis $\{z_n\}_{n=1}^\infty$ and let $\{x_j\}_{j=1}^\infty$ be a normalized block basis of $\{z_n\}_{n=1}^\infty$. Suppose that $Z = [x_j]_{j=1}^\infty \oplus Y$, for some subspace Y of Z. Then there is a partition of the integers into two subsets A and B so that*

(i) *$\{x_j\}_{j\in A}$ is equivalent to a subsequence of $\{z_n\}_{n=1}^\infty$.*

(ii) *$[x_j]_{j\in B}$ is isomorphic to a complemented subspace of Y.*

PROOF. Denote by K the unconditionality constant of $\{z_n\}_{n=1}^\infty$ and suppose that

$$x_j = \sum_{n=p_{j-1}+1}^{p_j} a_n z_n,$$

for a suitable sequence of scalars $\{a_n\}_{n=p_{j-1}+1}^{p_j}$, $j = 1, 2, \ldots$, and for integers $p_0 = 1 < p_1 < p_2 < \cdots$. Let P be a bounded projection from Z onto $[x_j]_{j=1}^\infty$. As in the proof of Theorem 1.7, we an assume without loss of generality that

$$P(z) = \sum_{j=1}^\infty x_j^*(z) x_j, \qquad z \in Z,$$

where $x_j^* = \sum_{n=p_{j-1}+1}^{p_j} b_n z_n^*$, $j = 1, 2, \ldots$, are functionals biorthogonal to $\{x_j\}_{j=1}^\infty$ and $\{b_n\}_{n=p_{j-1}+1}^{p_j}$, $j = 1, 2, \ldots$, are suitable scalars so that $a_n b_n \ge 0$, for all

n ($\{z_n^*\}_{n=1}^\infty$ denotes, as usual, the sequence of the biorthogonal functions associated with $\{z_n\}_{n=1}^\infty$).

Put

$$A = \left\{ j \in \mathbb{N};\ \sup_{p_{j-1} < n \leq p_j} a_n b_n > 1/2 \right\}$$

and $B = \mathbb{N} \sim A$. We prove first that $\{x_j\}_{j \in A}$ is equivalent to $\{z_{n(j)}\}_{j \in A}$, where, for each $j \in A$, $n(j)$ is selected to be one of the indices satisfying $p_{j-1} < n(j) \leq p(j)$ and $a_{n(j)} b_{n(j)} > 1/2$. Notice that $|a_{n(j)}| \leq K$ and $|b_{n(j)}| \leq \|P\|$, for all j. Hence, for any choice of scalars $\{c_j\}_{j \in A}$, we have

$$\begin{aligned} \|P\| \cdot \left\| \sum_{j \in A} c_j z_{n(j)} \right\| &\geq \left\| \sum_{j \in A} c_j P(z_{n(j)}) \right\| = \left\| \sum_{j \in A} c_j b_{n(j)} x_j \right\| \geq \frac{1}{4K^2} \left\| \sum_{j \in A} c_j x_j \right\| \\ &\geq \frac{1}{4K^3} \left\| \sum_{j \in A} c_j a_{n(j)} z_{n(j)} \right\| \geq \frac{1}{16K^5} \left\| \sum_{j \in A} c_j z_{n(j)} \right\|, \end{aligned}$$

and this completes the proof of (i).

We consider now the set B. For $j \in B$ and $p_{j-1} < n \leq p_j$, we have $0 \leq a_n b_n \leq 1/2$. Thus, we can choose signs $\varepsilon_n = \pm 1$ so that

$$\omega_j = \sum_{n=p_{j-1}+1}^{p_j} \varepsilon_n a_n b_n$$

satisfies $|\omega_j| \leq 1/2$, for all j. Put

$$y_j = \sum_{n=p_{j-1}+1}^{p_j} \varepsilon_n a_n (I - P) z_n \in Y$$

and observe that

$$y_j = \sum_{n=p_{j-1}+1}^{p_j} \varepsilon_n a_n z_n - \omega_j x_j = \sum_{n=p_{j-1}+1}^{p_j} (\varepsilon_n - \omega_j) a_n z_n,$$

for all j. Hence, for every choice of scalars $\{d_j\}_{j \in B}$, we get

$$\begin{aligned} K \left\| \sum_{j \in B} d_j x_j \right\| &\geq \left\| \sum_{j \in B} d_j \sum_{n=p_{j-1}+1}^{p_j} \varepsilon_n a_n z_n \right\| \geq \frac{1}{\|I - P\|} \left\| \sum_{j \in B} d_j y_j \right\| \\ &= \frac{1}{\|I - P\|} \left\| \sum_{j \in B} d_j \sum_{n=p_{j-1}+1}^{p_j} (\varepsilon_n - \omega_j) a_n z_n \right\| \\ &\geq \frac{1}{4K\|I - P\|} \left\| \sum_{j \in B} d_j x_j \right\| \end{aligned}$$

since $|\varepsilon_n - \omega_j| \geq 1/2$, for all $p_{j-1} < n \leq p_j$, $j = 1, 2, \ldots$.

Finally, in order to prove that $[y_j]_{j=1}^{\infty}$ is complemented in Y, we notice that the operator Q, defined by,

$$Q(z) = \sum_{j=1}^{\infty} \left(\sum_{n=p_{j-1}+1}^{p_j} l_n b_n c_n \right) y_j, \qquad z = \sum_{i=1}^{\infty} c_i z_i \in Z,$$

where $l_n = 1/(\varepsilon_n - \omega_j)$, $p_{j-1} < n \leq p_j$, $j = 1, 2, \ldots$, is a bounded linear projection from Z onto $[y_j]_{j=1}^{\infty}$. $\square$

REMARK. The assumption made in Theorem 1.8 that $[x_j]_{j=1}^{\infty}$ is a block basis of $\{z_n\}_{n=1}^{\infty}$ is not redundant. Indeed, take as Z the direct sum $l_p \oplus l_1, 1 < p \neq 2$, and as $\{z_n\}_{n=1}^{\infty}$ the union of the unit vector bases of l_p and l_1. Let $\{v_j\}_{j=1}^{\infty}$ denote the unit vector basis of the space $(\sum_{n=1}^{\infty} \bigoplus l_2^n)p$ and T_p an isomorphism from this space onto l_p. Then clearly the sequence $x_j = T_p(v_j)$, $j = 1, 2, \ldots$, cannot be split as in the statement of Theorem 1.8.

2. Nonatomic lattices. In this section we shall consider some continuous versions of the results of section one. We show that if Z is a nonatomic Banach lattice with some nontrivial cotype and $Z = X \oplus Y$ where X and Y are totally incomparable then either $\dim X < \infty$ or $\dim Y < \infty$. Some similar but rather less complete results are obtained for the situation when every operator $T: X \to Y$ is strictly singular. We give an example to show that the cotype assumption is necessary and prove a lattice analogue of Theorem 1.7.

Let us state first our main result on totally incomparable subspaces.

THEOREM 2.1. *Let Z be a nonatomic Banach lattice with nontrivial cotype and suppose $Z = X \oplus Y$ where X and Y are totally incomparable subspaces of Z. Then either* $\dim X < \infty$ *or* $\dim Y < \infty$.

Theorem 2.1 is an immediate consequence of Theorem 2.2(a).

THEOREM 2.2. *Let Z be a nonatomic Banach lattice with nontrivial cotype, and suppose $Z = X \oplus Y$ where every bounded operator $T: X \to Y$ is strictly singular. Then each of the following conditions implies* $\dim X < \infty$:

(a) *X contains no subspace isomorphic to l_2.*

(b) *Z has nontrivial type and X contains no complemented subspace isomorphic to l_2.*

(c) *Z has nontrivial type and Y contains a complemented infinite dimensional subspace with an unconditional basis.*

The proof of Theorem 2.2 involves a series of lemmas. The consequence of Lemma 2.3 is that we need only consider the case when Z is a Banach function space as described in §0.

LEMMA 2.3. *Let Z be an order-continuous Banach lattice and suppose $Z = X \oplus Y$ where X and Y are infinite dimensional subspaces of Z such that every bounded operator $T: X \to Y$ is strictly singular. Then there is a band Z_0 in Z with a weak order-unit such that $Z_0 = X_0 \oplus Y_0$, where X_0 and Y_0 are, respectively, infinite-dimensional complemented subspaces of X and Y.*

If, in addition, Y contains a complemented subspace with an unconditional basis, then we can further assume that Y_0 has a complemented subspace with unconditional basis.

PROOF. Let X' and Y' be separable infinite dimensional subspaces of X and Y, respectively. Then there is a band Z_0 with a weak order-unit containing X' and Y'. As Z_0 is complemented by a band projection, P_0 say, we can use Theorem A to deduce that $Z_0 = X_0 \oplus Y_0$, where X_0 and Y_0 are complemented subspaces of X and Y, respectively. If P_Y is the projection of Z onto Y then $P_{Y|Z_0}$ is not strictly singular so that $\dim Y_0 = \infty$ (since $P_{Y|Z_0}$ is identity on Y'). Similarly $P_{X|Z_0}$ is not strictly singular so that $\dim X_0 = \infty$.

For the last part we may assume Y' is complemented and has an unconditional basis. Then Y' is complemented in Z_0 and so $Y' = X_1 \oplus Y_1$, where X_1 is complemented in X_0 and Y_1 is complemented in Y_0. Clearly we must have $\dim X_1 < \infty$ and so Y_1 has an unconditional basis. □

LEMMA 2.4. *Let Z be an order-continuous Banach function space on a nonatomic probability measure space (Ω, Σ, μ). Suppose $f \in Z$ with $f \neq 0$.*

(a) *There is a sequence $\{f_n\} \in Z$ with $|f_n| = |f|$, for all n and $f_n \xrightarrow{w} 0$, as $n \to \infty$.*

(b) *If Z has nontrivial cotype then $\{f_n\}_{n=1}^\infty$ may be chosen in* (a) *to be a basic sequence equivalent to the standard l_2-basis.*

(c) *If Z has nontrivial type then $\{f_n\}_{n=1}^\infty$ may be chosen in* (b) *so that $[f_n]_{n=1}^\infty$ is, in addition, complemented in Z.*

PROOF. We first prove (b). If Z has nontrivial cotype then the map

$$A\colon L_\infty(\Omega, \Sigma, \mu) \to Z$$

given by $A(g) = |f|g$ is q-absolutely summing for some $2 < q < \infty$ (cf. [**18**]). By the Pietsch Factorization Theorem (cf. [**15**, 2.b.2]) there is a positive linear functional F on L_∞ with $F(\chi_\Omega) = 1$ and a constant C so that

$$\| |f|g \| \leq C(F(|g|^q))^{1/q}.$$

Clearly we can assume F is μ-continuous so that for some $h \in L_1(\mu), h \geq 0$ and $\int_\Omega h\, d\mu = 1$, we have

$$\| |f|g \| \leq C \left(\int_\Omega |g|^q h \, d\mu \right)^{1/q}.$$

Now, by Liapunoff's theorem [**16**, 2.e.8], we can find $\Omega_{n,j} \in \Sigma$ for $n = 0, 1, 2, \ldots,$ $j = 1, 2, \ldots, 2^n$ with $\Omega_{0,1} = \Omega$ so that

(i) $\Omega_{n,2j-1} \cup \Omega_{n,2j} = \Omega_{n-1,j}$.

(ii) $\int_{\Omega_{n,j}} |f|\, d\mu = 2^{-n} \int_\Omega |f|\, d\mu$.

(iii) $\int_{\Omega_{n,j}} h\, d\mu = 2^{-n}$.

Let $\phi_n = \sum_{j=1}^{2^n} (-1)^j \chi_{\Omega_{n,j}}$ and let $f_n = |f|\phi_n$. Then, by Khintchine's inequality, $\{f_n\}_{n=1}^\infty$ is equivalent in $L_1(\mu)$ to the l_2-basis, and so

$$\begin{aligned} \left\| \sum_{n=1}^\infty a_n f_n \right\| &\geq \int_\Omega \left| \sum_{n=1}^\infty a_n f_n \right| d\mu = \int_\Omega \left| \sum_{n=1}^\infty a_n \phi_n \right| |f|\, d\mu \\ &\geq \frac{1}{\sqrt{2}} \left(\sum_{n=1}^\infty |a_n|^2 \right)^{1/2}. \end{aligned}$$

On the other hand,

$$\left\| \sum_{n=1}^{\infty} a_n f_n \right\| \leq C \left(\int_\Omega \left| \sum_{n=1}^{\infty} a_n \phi_n \right|^q h\, d\mu \right)^{1/q} \leq C' \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2},$$

by Khintchine's inequality in $L_q(h\, d\mu)$, when C' is a suitable constant. Thus (b) is established.

Note that (a) follows by constructing $\Omega_{n,j}$ to verify only (i) and (ii). Then $f_n \to 0$ weakly in $L_1(\mu)$ as $n \to \infty$ and since $\{f_n\}_{n=1}^{\infty}$ is relatively weakly compact in Z we have also $f_n \to 0$ weakly in Z, as $n \to \infty$.

For (c) note that Z is a p-convex lattice for some $p > 1$, and the inclusion $Z \to L_1(\mu)$ is thus a p-convex operator. Thus, by [**12**] (cf. [**16**, 1.d.12]), the inclusion can be factored through an L_p-space. Then there exist operators $A_1 \colon Z \to L_p(\nu)$, $A_2 \colon L_p(\nu) \to L_1(\mu)$ so that $A_2 A_1(f) = f$. In particular, $\{A_1(f_n)\}_{n=1}^{\infty}$ is a basic sequence equivalent to the l_2-basis in $L_p(\nu)$ and so has a subsequence $\{A_1(f_{n_k})\}_{k=1}^{\infty}$, whose span is complemented. Then $[f_{n_k}]_{k=1}^{\infty}$ is also complemented in Z. □

Before proceeding with the main part of the proof we introduce a concept which will be useful and make some preliminary observations. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be any two basic sequences in Z. We say $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are *strongly equivalent* if there exist operators $S \colon Z \to Z$ and $T \colon Z \to Z$ with $Su_n = v_n$, $Tv_n = u_n$. Note that strong equivalence implies equivalence. Note also that if X and Y are complemented subspaces of Z and $\{u_n\}_{n=1}^{\infty} \subset X, \{v_n\}_{n=1}^{\infty} \subset Y$ are strongly equivalent then there exist nonstrictly singular operators from X into Y and from Y into X.

We shall use an easy perturbation argument repeatedly. Let us suppose $\{u_n\}_{n=1}^{\infty}$ is a basic sequence so that $0 < \inf_n \|u_n\| \leq \sup_n \|u_n\| < \infty$. Suppose $\{v_n\}_{n=1}^{\infty}$ is a sequence in Z and $A \colon Z \to Z$, $B \colon Z \to Z$ are operators such that $\|u_n - B(v_n)\| \to 0$ and $\|v_n - A(u_n)\| \to 0$, as $n \to \infty$. Then there are subsequences $\{u_{n_k}\}_{k=1}^{\infty}$ and $\{v_{n_k}\}_{k=1}^{\infty}$ which are strongly equivalent basic sequences. In particular this holds when $A = B = I$.

LEMMA 2.5. *Let Z be a Banach function space on (Ω, Σ, μ) with nontrivial type. Let $\{f_n\}_{n=1}^{\infty} \subset Z$ be a normalized unconditional basic sequence such that $[f_n]_{n=1}^{\infty}$ is complemented in Z. Then either $\{f_n\}_{n=1}^{\infty}$ is equivalent to the standard l_2-basis or $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which is strongly equivalent to a disjoint sequence $\{h_k\}_{k=1}^{\infty}$ in Z.*

REMARK. Indeed $[h_k]_{k=1}^{\infty}$ is complemented.

PROOF. Let $\{f_n^*\}_{n=1}^{\infty}$ be the dual basis in Z^*, so that the projection $P \colon Z \to [f_n]_{n=1}^{\infty}$ is given by

$$P(g) = \sum_{n=1}^{\infty} \left(\int_\Omega f_n^* \cdot g\, d\mu \right) f_n.$$

Note that Z is reflexive so that Z^* is also order-continuous. We distinguish three cases.

(1) Suppose
$$\inf_n \int_\Omega |f_n|\,d\mu = 0.$$
Then, by passing to a subsequence, we may suppose
$$\lim_{n\to\infty} \int_\Omega |f_n|\,d\mu = 0$$
and, for each m,
$$\lim_{n\to\infty} \| \, |f_m| \wedge |f_n| \, \| = 0.$$
Now, by a standard disjointification argument, we may pass to a further subsequence so that, for some disjoint sequence $\{h_n\}_{n=1}^\infty$,
$$\lim_{n\to\infty} \|f_n - h_n\| = 0.$$
The conclusion follows on passing to yet another subsequence.

(2) Suppose
$$\inf_n \int |f_n^*|\,d\mu = 0.$$
Arguing as in (1) we may pass to a subsequence and assume that $\{f_n^*\}_{n=1}^\infty$ is strongly equivalent to a disjoint sequence $\{h_n^*\}_{n=1}^\infty$. Further $[h_n^*]_{n=1}^\infty$ is complemented in Z^*. If $A_n = \operatorname{supp} h_n^*$ and $Q\colon Z^* \to [h_n^*]$ is a projection then set
$$Q_1(g^*) = \sum_{n=1}^\infty \chi_{A_n} Q(g^* \chi_{A_n}).$$
Then $\|Q_1\| \leq \|Q\|$ and Q_1 is also a projection (cf. [**15**, 1.c.8] for the discrete version). Now
$$Q_1(g^*) = \sum_{n=1}^\infty \left(\int_\Omega g^* h_n \,d\mu \right) h_n^*$$
for a disjoint sequence $\{h_n\}_{n=1}^\infty$ in Z. Then $\{h_n\}_{n=1}^\infty$ is equivalent to $\{f_n\}_{n=1}^\infty$ and, as $[h_n]_{n=1}^\infty$ is complemented, it is strongly equivalent.

(3) Suppose $\delta > 0$ and
$$\int_\Omega |f_n|\,d\mu \geq \delta, \quad \int_\Omega |f_n^*|\,d\mu \geq \delta, \qquad n = 1, 2, \ldots .$$
Then, by Khintchine's inequality, for $a_1, \ldots, a_n \in \mathbb{R}$, $n = 1, 2, \ldots$,
$$\sup_{\varepsilon_i = \pm 1} \int_\Omega \left| \sum_{i=1}^n a_i \varepsilon_i f_i \right| d\mu \geq \frac{\delta}{\sqrt{2}} \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}$$
and so, for some $\gamma > 0$ independent of n,
$$\left\| \sum_{i=1}^n a_i f_i \right\| \geq \gamma \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}$$
and similarly we can assume
$$\left\| \sum_{i=1}^n a_i f_i^* \right\| \geq \gamma \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}.$$

Now there exists $g^* \in Z^*$ with $\|g^*\| = 1$ and

$$\sum_{i=1}^{n} a_i \int_\Omega g^* f_i \, d\mu = \left\| \sum_{i=1}^{n} a_i f_i \right\|.$$

If P is the projection of Z onto $[f_n]_{n=1}^\infty$ then

$$\|P^*\| \geq \|P^*(g^*)\| \geq \gamma \left(\sum_{i=1}^{n} \left| \int_\Omega g^* f_i \, d\mu \right|^2 \right)^{1/2}$$

and hence

$$\left\| \sum_{i=1}^{n} a_i f_i \right\| \leq \gamma^{-1} \|P\| \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2}$$

so that $\{f_i\}_{i=1}^\infty$ is equivalent to the l_2-basis. □

From now on we assume that Z is a Banach function space over a nonatomic probability measure space (Ω, Σ, μ) with nontrivial cotype. We suppose $Z = X \oplus Y$ and P_X and P_Y are the respective projections.

LEMMA 2.6. *If X contains no subspace isomorphic to l_2, or if Z has nontrivial type and X contains no complemented copy of l_2 then X satisfies*:

(+) *For every $f \in Z$ and $\varepsilon > 0$ there exists $g \in Z$ and $A \in \Sigma$ so that*

(a) $|g| = |f|$.

(b) $\|g \cdot \chi_A\| < \varepsilon$.

(c) $\|(P_X(g))\chi_{\Omega \sim A}\| < \varepsilon$.

(d) $\int_\Omega |P_X(g)| \, d\mu < \varepsilon$.

PROOF. Let us consider the first case. If $f \in Z$ and $f \neq 0$ then, by Lemma 2.4, we can find $f_n \in Z$, $n = 1, 2, \ldots$ with $|f_n| = |f|$, for all n, and $\{f_n\}_{n=1}^\infty$ equivalent to the l_2-basis. We show

$$\inf_n \int_\Omega |P_X(f_n)| \, d\mu = 0.$$

Indeed, if not then, for some $\delta > 0$,

$$\int_\Omega |P_X(f_n)| \, d\mu \geq \delta, \qquad n = 1, 2, \ldots .$$

By passing to a subsequence, we may suppose $\{P_X(f_n)\}_{n=1}^\infty$ is basic in X (since $\|P_X(f_n)\| \geq \delta$ but $P_X(f_n) \xrightarrow{w} 0$ as $n \to \infty$). Now, by a theorem of D. J. Aldous and D. H. Fremlin [**1**], we can pass to a further subsequence and assume that, for some $c > 0$,

$$\int_\Omega \left| \sum_{i=1}^{n} a_i P_X(f_i) \right| \, d\mu \geq c \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2},$$

for $a_1, \ldots, a_n \in \mathbb{R}$ and all n. Thus

$$\left\| \sum_{i=1}^{n} a_i P_X(f_i) \right\| \geq c \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2}.$$

However,

$$\left\| \sum_{i=1}^{n} a_i P_X(f_i) \right\| \leq \|P_X\| \left\| \sum_{i=1}^{n} a_i f_i \right\| \leq K\|P_X\| \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2},$$

for some constant K. Thus $\{P_X f_n\}_{n=1}^{\infty}$ is equivalent to the l_2-basis which is a contradiction.

If, in addition Z has nontrivial type, then we can repeat the argument at the end of Lemma 2.5 since $\{P_X(f_n)\}_{n=1}^{\infty}$ is equivalent to l_2-basis in both Z and $L_1(\mu)$. We then can pass to a further subsequence and obtain that $[P_X f_n]_{n=1}^{\infty}$ is complemented which is also a contradiction.

In either case we may now assume $\lim_{n\to\infty} \int_\Omega |P_X(f_n)|\, d\mu = 0$. Let $A_n = \{\omega \in \Omega\colon |P_X(f_n)| \geq \varepsilon/3\}$. Then $\mu(A_n) \to 0$ and so $\|f\chi_{A_n}\| \to 0$, as $n \to \infty$. Also

$$\|P_X(f_n) \cdot \chi_{\Omega\sim A_n}\| \leq 2\|P_X(f_n) \cdot \chi_{\Omega\sim A_n}\|_\infty < \varepsilon.$$

Hence, for large enough n, we may take $g = f_n$ and $A = A_n$ and (+) will hold. □

LEMMA 2.7. *Suppose X satisfies (+) and $\{f_n\}_{n=1}^{\infty}$ is a normalized disjoint sequence in Z. Then $\{f_n\}_{n=1}^{\infty}$ has a subsequence strongly equivalent to a basic sequence in Y.*

PROOF. Use (+) to pick $h_n \in Z$, $A_n \in \Sigma$ so that $\|h_n\chi_{A_n}\| \leq 1/n$, $\|P_X(h_n) \cdot \chi_{\Omega\sim A_n}\| \leq 1/n$ and $\int_\Omega |P_X(h_n)|\, d\mu \leq 1/n$, for all n. Let us put $g_n = h_n\chi_{\Omega\sim A_n} - P_X(h_n)\chi_{A_n}$. Then $\{g_n\}_{n=1}^{\infty}$ is a bounded sequence in Z. If $m \leq n$

$$\int_\Omega |g_m| \wedge |P_X(h_n)|\chi_{A_n}\, d\mu \leq \frac{1}{n}$$

so that

$$\lim_{n\to\infty} \|\, |g_m| \wedge |g_n|\, \| = 0.$$

Now there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ and a disjoint sequence $\{B_k\}_{k=1}^{\infty}$ of sets in Σ so that

$$\lim_{k\to\infty} \|g_{n_k} - g_{n_k}\chi_{B_k}\| = 0.$$

Let $C_k = (\operatorname{supp} f_{n_k}) \cap B_k \cap (\Omega \sim A_{n_k})$, and define $\phi \in L_\infty$ by

$$\psi(\omega) = \begin{cases} h_{n_k}(\omega)/f_{n_k}(\omega), & \omega \in C_k, \\ 0, & \omega \in \Omega \sim \bigcup_{k=1}^{\infty} C_k. \end{cases}$$

Define $A\colon Z \to Z$ by $A(f) = \phi f$. First we note that $A(f_{n_k}) = h_{n_k}\chi_{C_k}$ and $\|h_{n_k} - h_{n_k}\chi_{C_k}\| \leq \|g_{n_k} - g_{n_k}\chi_{B_k}\| \to 0$, as $k \to \infty$.

Thus

$$\lim_{k\to\infty} \|Af_{n_k} - h_{n_k}\| = 0.$$

Now $P_X(g_n) = P_X(h_n - h_n \cdot \chi_{A_n}) - P_X(P_X h_n - (P_X h_n) \cdot \chi_{\Omega - A_n})$ so that

$$\|P_X(g_n)\| \leq 2\|P_X\|/n.$$

Thus $\|P_Y(g_n) - g_n\| \to 0$. Also

$$\|P_Y((P_X h_n)\cdot \chi_{A_n})\| = \|P_Y((P_X h_n)\cdot \chi_{\Omega - A_n})\| \le \|P_Y\|/n \to 0$$

so that

$$\|P_Y(g_n) - P_Y(h_n \cdot \chi_{\Omega - A_n})\| \to 0 \quad \text{and} \quad \|P_Y(g_n) - P_Y(h_n)\| \to 0, \quad \text{as } n \to \infty.$$

We conclude that

$$\lim_{k\to\infty} \|g_{n_k} - P_Y(h_{n_k})\| = 0.$$

Next note $A(g_{n_k}) = f_{n_k} \cdot \chi_{(\Omega - A_{n_k})\cap B_k}$ and so

$$\begin{aligned}\|A(g_{n_k}) - f_{n_k}\| &\le 1/n_k + \|h_{n_k}\chi_{(\Omega - A_{n_k})\sim B_k}\| \\ &\le 1/n_k + \|g_{n_k} - g_{n_k}\chi_{B_k}\| \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Combining these remarks we have

$$\lim_{k\to\infty} \|P_Y A f_{n_k} - g_{n_k}\| = 0, \qquad \lim_{k\to\infty} \|A g_{n_k} - f_{n_k}\| = 0$$

so that

$$\lim_{k\to\infty} \|A P_Y A f_{n_k} - f_{n_k}\| = 0.$$

Thus, for some further subsequence $\{f'_n\}_{n=1}^\infty, \{f'_n\}_{n=1}^\infty$ is strongly equivalent to $\{P_Y A(f'_n)\}_{n=1}^\infty$ which is a basic sequence in Y. □

LEMMA 2.8. *Suppose X has the property that no basic sequence in X is strongly equivalent to a disjoint sequence in Z. Then Z embeds into $Y^n = Y \oplus \cdots \oplus Y$ (n times), for some $n \in \mathbb{N}$.*

PROOF. First we note that the norm on X must be equivalent to the L_1-norm i.e., for some C, we have

$$\|f\| \le C \int_\Omega |f|\, d\mu, \qquad f \in X.$$

Otherwise, we can find a sequence $\{f_n\}_{n=1}^\infty \subset X$ with $\|f_n\| = 1$ and

$$\lim_{n\to\infty} \int_\Omega |f_n|\, d\mu = 0.$$

Then $\lim_{n\to\infty} \| \, |f_m| \wedge |f_n| \, \| = 0$ and so $\{f_n\}_{n=1}^\infty$ has a subsequence strongly equivalent to a disjoint sequence.

Next we note that X must be reflexive. For otherwise l_1 must embed complementably into X, and Z must contain a disjoint sequence equivalent to the l_1-basis whose closed linear span is complemented.

Thus the unit ball of X is weakly compact and so, for any $\varepsilon > 0$, there exists $\delta > 0$ so that if $f \in X$, $\|f\| \le 1$ and $A \in \Sigma$ with $\mu(A) < \delta$ then $\int_A |f|\, d\mu < \varepsilon$. Choose δ corresponding to $\varepsilon = 1/2C$ and partition Ω into n disjoint sets Ω_k, $1 \le k \le n$, with $\mu(\Omega_k) < \delta$. Let $Z_k = Z|\Omega_k$.

If $f \in Z_k$ then

$$\begin{aligned}\frac{1}{2C}\|P_X(f)\| &\geq \int_{\Omega_k} |P_X(f)|\,d\mu \\ &= \int_{\Omega} |P_X(f)|\,d\mu - \int_{\Omega\sim\Omega_k} |P_X(f)|\,d\mu \\ &\geq \frac{1}{C}\|P_X(f)\| - \int_{\Omega\sim\Omega_k} |P_Y(f)|\,d\mu \\ &\geq \frac{1}{C}\|P_X(f)\| - \|P_Y(f)\|.\end{aligned}$$

Hence,

$$\|P_Y(f)\| \geq \frac{1}{2C}\|P_X(f)\| \geq \frac{1}{2C}(\|f\| - \|P_Y(f)\|)$$

and

$$\|P_Y(f)\| \geq \frac{1}{1+2C}\|f\|.$$

Hence, Z_k embeds in Y and $Z \approx Z_1 \oplus \cdots \oplus Z_n$ embeds into Y^n. □

Now we complete the proof of Theorem 2.2. We may suppose by Lemma 2.3 that Z is a Banach function space.

PROOF OF 2.2(a). By Lemma 2.6, X satisfies (+) and so by Lemma 2.7, X satisfies the hypotheses of Lemma 2.8. Thus X embeds into Y^n and hence $\dim X < \infty$. □

PROOF OF 2.2(b). Similar. □

PROOF OF 2.2(c). If Y contains a complemented copy of l_2 this reduces to case (b). Otherwise, we can assume Y contains no complemented copy of l_2. Thus Y satisfies (+), but by Lemma 2.5 there is a basic sequence in Y strongly equivalent to a disjoint sequence in Z. Lemma 2.7 then completes the proof. □

The statement of Theorem 2.1 requires that Z be a Banach lattice with some nontrivial cotype. The purpose of the following example is to show that this assumption is not redundant.

EXAMPLE 2.9. There exists a nonatomic order continuous Banach lattice Z which decomposes as a direct sum $Z = X \oplus Y$ and X and Y are infinite dimensional nontotally incomparable spaces.

The idea is to construct an order continuous Banach function space X on $[0,1]$ which contains no isomorphic copy of l_2. Then $Z = X \oplus L_2(1,2)$ provides the desired counter-example.

As X we shall take a separable Orlicz function space $H_M(0,1)$ which is considerably "smaller" than the space $H_N(0,1)$ with $N(x) = (e^{x^2}-1)/(e-1)$. The space $H_N(0,1)$ is, by [**21**], the smallest rearrangement invariant space in which the Rademacher functions span l_2. More precisely, let e.g. $M(x) = (e^{x^4}-1)/(e-1)$ and consider the space $H_M(0,1)$ of all measurable functions f on $[0,1]$ so that

$$\int_0^1 M\left(\frac{|f(t)|}{\lambda}\right)\,dt < \infty,$$

for every $\lambda > 0$. The norm in $H_M(0,1)$ is defined, as usual, by

$$\|f\|_M = \inf\left\{\lambda > 0; \int_0^1 M\left(\frac{|f(t)|}{\lambda}\right)\,dt \leq 1\right\}.$$

It is well known that the simple functions are dense in $H_M(0,1)$ and therefore this space is a separable order continuous lattice. Suppose now that $H_M(0,1)$ contains a subspace V isomorphic to l_2. Then, by [**6**], either there exists a constant $\alpha > 0$ so that $\mu\{t \in [0,1]; |f(t)| \geq \alpha\|f\|_M\} \geq \alpha$, for all $f \in V$, or V contains a normalized sequence which is equivalent to a sequence of mutually disjoint norm one functions in $H_M(0,1)$.

In the first case,

$$\|f\|_2 \geq \alpha^{3/2}\|f\|_M, \qquad f \in V.$$

On the other hand, a simple calculation shows that, for $f \geq 1$ and $x \geq 0$, $x^p \leq (p/4e)^{p/4}e^{x^r}$. Hence, there exists a constant C independent of p so that

$$\|f\|_p \leq Cp^{1/4}\|f\|_M, \qquad f \in H_M(0,1),$$

i.e. on V the $\|\cdot\|_2$ and $\|\cdot\|_M$-norms are equivalent. Let R be the orthogonal projection from $L_2(0,1)$ onto V. Then, for any $p > 2$ and $g \in L_p(0,1)$, we have

$$\|R(g)\|_p \leq Cp^{1/4}\|R(g)\|_M \leq Cp^{1/4}\alpha^{-3/2}\|R(g)\|_2 \leq Cp^{1/4}\alpha^{-3/2}\|g\|_p,$$

i.e. the norm of R as a projection in $L_p(0,1)$ is $\leq C'p^{1/4}$, for all $p > 2$ and some constant C' independent of p. This implies that the factorization constant $\gamma_p(l_2)$ is $\leq C''p^{1/4}$, for some C'', while in fact, by [**8**], $\gamma_p(l_2)$ behaves like $p^{1/2}$, when $p \to \infty$. This contradiction completes the proof in the first case.

We consider now the case when V contains a normalized sequence equivalent to a sequence of mutually disjoint functions in $H_M(0,1)$. This situation, however, is again contradictory since any such sequence of norm one disjoint functions in $H_M(0,1)$ contains a subsequence equivalent to the unit vector basis of c_0. This fact is known and can be deduced easily in the following way. Let $\{h_n\}_{n=1}^{\infty}$ be a normalized sequence of mutually disjoint elements of $H_M(0,1)$ and assume, as we clearly may by passing to a subsequence, that there are sets $\{B_n\}_{n=1}^{\infty}$ such that $\|h_n - h_n \cdot \chi_{B_n}\| \leq 2^{-(n+1)}$ and $|h_n(t)| \geq 2^{n+1}$, for $t \in B_n$ and all n. Since $e^{ax} \leq 2e^x/A$, for $0 < a < 1/2$ and $x \geq A$, it follows that, whenever $0 < |a_n| < 1/2$, for all n,

$$\int_0^1 M\left(\left|\sum_{n=1}^{\infty} a_n h_n \cdot \chi_{B_n}\right|\right) dt = \sum_{n=1}^{\infty} \int_{B_n} \frac{e^{|a_n h_n|^4} - 1}{e-1}\, dt \leq 1,$$

i.e.

$$\left\|\sum_{n=1}^{\infty} a_n h_n \cdot \chi_{B_n}\right\|_M \leq 1 \quad \text{and} \quad \left\|\sum_{n=1}^{\infty} a_n h_n\right\|_M \leq 2.$$

It is highly likely that Theorem 2.2 can be improved in the case when Z has nontrivial type. Note however that the example $L_1(0,1) \oplus L_2(1,2)$ shows that Theorem 2.2(a) requires some condition on X.

We conclude this section with an analogue of Theorem 1.7.

THEOREM 2.10. *Let X be a Banach space with nontrivial type and assume that $1 < p < \infty$. Suppose $X \oplus l_p$ is isomorphic to a Banach lattice Z. Then X is isomorphic to a Banach lattice.*

REMARK. The case $p = 1$ can be obtained from results in the next section and the case of c_0 then would follow by duality. We leave the reader to fill in the details.

PROOF. If X contains a complemented copy of l_p, then $X \approx Z$ and the theorem is immediate. Assume therefore that every bounded operator $T: X \to l_p$ is strictly singular. Z can then be decomposed into two bands, Z_1 and Z_2 so that Z_1 is atomic and Z_2 is nonatomic. By Theorem A, $Z_1 \approx X_1 \oplus Y_1$ and $Z_2 \approx X_2 \oplus Y_2$, where X_1 and X_2 are complemented subspaces of X and Y_1, Y_2 are complemented subspaces of l_p.

Assume first $\dim Y_2 = \infty$, so that $Y_2 \approx l_p$. Then Theorem 2.2(c) implies $\dim X_2 < \infty$ and so $Z_2 \approx l_p$ (which implies $p = 2$). Then $X \oplus l_p$ has a possibly uncountable unconditional basis. However, the copy of l_p can be supported only on countably many basis elements and so we can write $X = W_1 \oplus W_2$, where W_1 has an unconditional basis, W_2 is separable and $W_2 \oplus l_p$ has an unconditional basis. Thus, by Theorem 1.7, X is isomorphic to a Banach lattice.

Assume then $\dim Y_2 < \infty$. Thus l_p embeds complementably in Z_1 and so $Y_1 \approx l_p$. Note that if $\dim Z_2 = 0$, then arguing as above the result follows from Theorem 1.7. Assume therefore $\dim Z_2 > 0$ so that $Z_2 \approx l_2 \oplus Z_2$.

Now $Z \approx (Z_2 \oplus X_1) \oplus l_p \approx X \oplus l_p$ and so by Theorem A, $Z_2 \oplus X_1 \approx U_1 \oplus V_1$, $l_p \approx U_2 \oplus V_2$, where $X_1 \approx U_1 \oplus U_2$ and $l_p \approx V_1 \oplus V_2$. Clearly $\dim V_1 < \infty$ and $\dim U_2 < \infty$. However, $(Z_2 \oplus X_1) \oplus U_2 \approx X \oplus V_1$ and $(Z_2 \oplus X_1) \oplus U_2 \approx Z_2 \oplus X_1$ since $Z_2 \approx Z_2 \oplus l_2$. Thus $Z_2 \oplus X_1 \approx X$.

Again from Theorem 1.7, X_1 is isomorphic to a Banach lattice and so the theorem is proved. □

3. Banach lattices containing complemented copies of L_1-spaces.

The object of this section is to study different situations in which an L_1-space embeds complementably into a Banach lattice. We consider first lattices which contain complemented subspaces isomorphic to l_1, in the spirit of Theorem 2.10.

THEOREM 3.1. *Let Z be a nonatomic order continuous Banach lattice and suppose that $Z = X \oplus Y$, with Y being isomorphic to l_1. Then X is isomorphic to Z.*

PROOF. It clearly suffices to show that X contains a complemented subspace which is isomorphic to l_1.

Since Z is order continuous and contains an isomorphic copy of l_1 there exists a sequence $\{u_k\}_{k=1}^\infty$ of mutually disjoint elements in Z which is equivalent to the unit vector basis of l_1 (this fact is well known; cf. [**26**]). Then, for each k, we use Lemma 2.3 in order to construct a sequence $\{u_{k,n}\}_{n=1}^\infty$ which converges weakly to zero and so that $|u_{k,n}| = |u_k|$, for all n. If P_X and P_Y denote the corresponding projections from Z onto X, respectively Y, then $P_Y(u_{k,n}) \xrightarrow{w} 0$, as $n \to \infty$. However, since l_1 has the Schur property it follows that $\|P_Y(u_{k,n})\| \to 0$, as $n \to \infty$. Choose now an integer $n(k)$ such that $\|P_Y(u_{k,n(k)})\| < 2^{-(k+3)}$, for all k, and notice that $\{u_{k,n(i)}\}_{k=1}^\infty$ is equivalent to the unit vector basis of l_1 and its span is complemented in Z. The above choice of $n(k)$ shows that so is $\{P_X(u_{k,n(k)})\}_{k=1}^\infty$ and this completes the proof.

It was shown in [**11**, Theorem 3.1] that if a Banach lattice Z contains no isomorphic copy of c_0 and has a subspace isomorphic to $L_1(0,1)$ then Z also has a sublattice which is order isomorphic to $L_1(0,1)$. In the next theorems, we consider this situation from different points of view.

THEOREM 3.2. *Let Z be a Banach function space over a probability space (Ω, Σ, μ) which contains no subspace isomorphic to c_0. Suppose that $Z = X \oplus Y$, where X does not contain isomorphic copies of $L_1(0,1)$ while Y is an $\mathcal{L}_1$-space. Then Z has a band Z_0 which is order isomorphic to an L_1-space such that its orthogonal complement $Z_0^\perp$ contains no isomorphic copies of $L_1(0,1)$.*

PROOF. *Step* I. Our first aim is to construct the band Z_0. To this end, we call a set $A \in \Sigma$ with $\mu(A) > 0$ *acceptable* provided there exists a lattice homomorphism $S: L_1(0,1) \to Z$ having the form

$$S(\psi)(\omega) = a(\omega)\psi(\sigma(\omega)), \qquad \psi \in L_1(0,1),$$

where $a: \Omega \to \mathbb{R}$ and $\sigma: \Omega \to [0,1]$ are measurable functions and $a(\omega) > 0$ for $\omega \in A$. Notice that a subset of positive measure of an acceptable set is acceptable and also the countable union $A = \bigcup_{j=1}^\infty A_j$ of a sequence $\{A_j\}_{j=1}^\infty$ of acceptable sets is acceptable. Indeed, if $S_j(\psi)(\omega) = a_j(\omega)\psi(\sigma_j(\omega))$ is the lattice homomorphism corresponding to A_j then we set

$$\left.\begin{aligned} a(\omega) &= a_j(\omega)/2^j \|S_j\|, \\ \sigma(\omega) &= \sigma_j(\omega), \end{aligned}\right\} \qquad \omega \in A_j, \ j = 1, 2, \ldots,$$

and $a(\omega) = \sigma(\omega) = 0$, for $\omega \notin A$. It follows that $S(\psi)(\omega) = a(\omega)\psi(\sigma(\omega))$; $\omega \in \Omega$, $\psi \in L_1(0,1)$ is a lattice homomorphism from $L_1(0,1)$ into Z and $a(\omega) > 0$ for $\omega \in A$. The above observations show that there exists a maximal acceptable set Ω_0 i.e. a subset Ω_0 of Ω having the following properties:

(i) Ω_0 is acceptable.

(ii) $\Omega \sim \Omega_0$ contains no acceptable subset.

Let Z_0 be the band of Z generated by Ω_0 i.e. $Z_0 = \{f \cdot \chi_{\Omega_0}; f \in Z\}$ and $Z_0^\perp$ its orthogonal complement.

Suppose now that there exists an isomorphism T from $L_1(0,1)$ onto a subspace of $Z_0^\perp$. Then, by Theorem D(i), there exists also an order isomorphism S from $L_1(0,1)$ into $Z_0^\perp$. Let J denote the formal identity mapping from Z into $L_1(\Omega, \Sigma, \mu)$, given by Theorem C. Then JS is a lattice homomorphism from $L_1(0,1)$ into $L_1(\Omega \sim \Omega_0, \Sigma_{|\Omega\sim\Omega_0}, \mu_{|\Omega\sim\Omega_0})$ and thus by Theorem D(ii), $JS(\psi)(\omega) = a(\omega)\psi(\sigma(\omega))$; $\omega \in \Omega \sim \Omega_0$, $\psi \in L_1(0,1)$, for suitable a and σ. Since $S \neq 0$ it follows that $a(\omega) > 0$ on a subset A_0 of positive measure of $\Omega \sim \Omega_0$. This contradiction to the maximality of Ω_0 shows that $Z_0^\perp$ contains no subspaces isomorphic to $L_1(0,1)$.

Step II. It remains to prove that Z_0 is order isomorphic to an L_1-space. To this end, we shall prove first that, for every $\varepsilon > 0$ and $z \in Z_0$, there exists a function ψ so that $|\psi| = 1$ and $\|P_X(\psi z)\| < \varepsilon$.

Since Ω_0 is an acceptable set there exists a lattice homomorphism $S_0: L_1(0,1) \to Z$ having the form

$$S_0(\psi)(\omega) = a_0(\omega)\psi(\sigma_0(\omega)), \qquad \omega \in \Omega, \ \psi \in L_1(0,1),$$

with a_0 and σ_0 being measurable and $a_0(\omega) > 0$, $\omega \in \Omega_0$. Fix now $\varepsilon > 0$ and $z \in Z_0$, and choose a bounded measurable function g on Ω_0 such that $\|z - ag\| < \varepsilon/2\|P_X\|$, where P_X and P_Y have the usual meaning. Then define the operator $T: L_1(0,1) \to Z$, by setting

$$T(\psi)(\omega) = g(\omega)S_0(\psi)(\omega), \qquad \omega \in \Omega, \ \psi \in L_1(0,1).$$

Since $\|T\| \leq \|g\|_\infty \|S_0\|$ it follows that $P_X T$ is a bounded operator from $L_1(0,1)$ into X. By our assumption, X contains no isomorphic copy of $L_1(0,1)$ and therefore, by Theorem D(iii), $P_X T$ is not sign preserving. Hence, there exists a mean zero function $\rho \in L_1(0,1)$ with $|\rho| = 1$ so that $\|P_X T(\rho)\| < \varepsilon/2$. Then, with the notation $\psi(\omega) = \rho(\sigma_0(\omega))$, $\omega \in \Omega$, we get

$$\|P_X(\psi z)\| < \varepsilon/2 + \|P_X T(\rho)\| < \varepsilon,$$

which completes the proof of Step II.

Step III. We shall prove now that Z_0 is order isomorphic to an L_1-space. Let $\{z_j\}_{j=1}^m$ be a sequence of mutually disjoint elements in Z_0. Then, by the assertion proved in Step II with $\varepsilon = \|\sum_{j=1}^m z_j\|/m$, we find functions $\{\psi_j\}_{j=1}^m$ for which $|\psi_j| = 1$ and $\|P_X(\psi_j z_j)\| < \|\sum_{j=1}^m z_j\|/m$, for all $1 \leq j \leq m$. It follows that

$$\sum_{j=1}^m \|z_j\| = \sum_{j=1}^m \|\psi_j z_j\| < \sum_{j=1}^m \|P_Y(\psi_j z_j)\| + \left\|\sum_{j=1}^m z_j\right\|.$$

For each $1 \leq j \leq m$, let $z_j^* \in Z^*$ be so that $z_j^*(P_Y(\psi_j z_j)) = \|P_Y(\psi_j z_j)\|$ and $\|z_j^*\| = 1$, and consider the operator $W: Y \to l_2^m$, defined by

$$W(f) = \{z_j^*(P_Y(f \cdot \chi_{B_j}))\}_{j=1}^m, \qquad f \in Y,$$

where B_j denotes the support of z_j. In order to verify that W is a bounded operator, suppose that Y is an $\mathcal{L}_{1,\lambda}$-space, for some $\lambda > 1$. Then, since l_1 is of cotype 2 with constant $\sqrt{2}$, we get

$$\begin{aligned}\|W(f)\|^2 &= \sum_{j=1}^m |z_j^*(P_Y(f \cdot \chi_{B_j}))|^2 \leq \sum_{j=1}^m \|P_Y(f \cdot \chi_{B_j})\|^2 \\ &\leq 2\lambda^2 \left(\int_0^1 \left\|\sum_{j=1}^m r_j(t) P_Y(f \cdot \chi_{B_j})\right\|^2 dt\right),\end{aligned}$$

for all $f \in Y$. Hence, $\|W\| \leq \sqrt{2}\lambda \|P_Y\|$.

However, by a famous result due to A. Grothendieck (see e.g. [**15**, 2.b.6]), every operator from an $\mathcal{L}_1$-space to a Hilbert space is absolutely summing. It follows that the absolutely summing norm $\pi_1(W)$ of W satisfies

$$\pi_1(W) \leq \sqrt{2} K_G \lambda^2 \|P_Y\|,$$

where K_G stands for Grothendieck's constant. Hence,

$$\begin{aligned}\sum_{j=1}^m \|P_Y(\psi_j z_j)\| &= \sum_{j=1}^m z_j^*(P_Y(\psi_j z_j)) = \sum_{j=1}^m \|W(\psi_j z_j)\| \\ &\leq \pi_1(W) \max_{\varepsilon_j = \pm 1} \left\|\sum_{j=1}^m \varepsilon_j \psi_j z_j\right\| \leq \sqrt{2} K_G \lambda^2 \|P_Y\| \cdot \left\|\sum_{j=1}^m z_j\right\|.\end{aligned}$$

Therefore,

$$\sum_{j=1}^m \|z_j\| \leq (1 + \sqrt{2} K_G \lambda^2 \|P_Y\|) \left\|\sum_{j=1}^m z_j\right\|,$$

which clearly implies that Z_0 is order isomorphic to an L_1-space. □

One cannot expect to prove in Theorem 3.2 that $Z_0^\perp$ is isomorphic to X. For instance, if $X = l_1 \oplus X_1$ with X_1 being a reflexive band of Z and $Y = L_1(0,1)$ then we shall get that $Z_0 = L_1(0,1)$ and $Z_0^\perp = X_1$. As we will see later, it is precisely the assumption that X contains no isomorphic copy of l_1 that is needed in order to conclude that, essentially speaking, X is isomorphic to $Z_0^\perp$. Before proving this fact, we need a preliminary result.

THEOREM 3.3. *Let Z be a Banach function space over a probability space (Ω, Σ, μ). If Z contains no isomorphic copy of c_0 and Z^{**}/Z is an $\mathcal{L}_1$-space then either Z has a band Z_0 which is order isomorphic to an L_1-space or Z is a dual space.*

PROOF. Since Z contains no subspaces isomorphic to c_0 it is a band of Z^{**} and, moreover,

$$P(z^{**}) = \bigvee\{z \in Z; 0 \leq z \leq z^{**}\}, \qquad 0 \leq z^{**} \in Z^{**},$$

extends to a norm one positive projection from Z^{**} onto the canonical embedding of Z into Z^{**}. Thus, $Z^{**} = Z \oplus Z^\perp$, where $Z^\perp$ denotes the orthogonal complement of Z in Z^{**}. By our assumption, $Z^\perp$ is an $\mathcal{L}_1$-space. However, it is well known that a lattice, which is an $\mathcal{L}_1$-space, is already order isomorphic to an L_1-space. Therefore, there exists an L_1-norm $\|\cdot\|_L$ on $Z^\perp$ which satisfies $C^{-1}\|z^{**}\| \leq \|z^{**}\|_L \leq C\|z^{**}\|$, for some $0 < C < \infty$ and any $z^{**} \in Z^\perp$. Define now a function F on Z^*, by setting

$$F(z^*) = \sup\{|z^{**}(z^*)| : z^{**} \in Z^\perp, \|z^{**}\|_L \leq 1\},$$

for all $z^* \in Z^*$. We clearly have

$$|F(z^*)| \leq C\|z^*\|, \qquad z^* \in Z^*,$$

and, moreover, the duality between L_1 and M-norms show that F is a seminorm on Z^* so that

$$F(z_1^* + z_2^*) = \max(F(z_1^*), F(z_2^*)),$$

whenever z_1^* and z_2^* in Z^* satisfy $z_1^* \wedge z_2^* = 0$.

By using F, we define, for each partition $\pi = \{B_k\}_{k=1}^m$ of Ω into sets of positive measure, a function $\beta_\pi \in L_\infty(\Omega, \Sigma, \mu)$ by

$$\beta_\pi(\omega) = F(\chi_{B_k}) \quad \text{if } \omega \in B_k,\ 1 \leq k \leq m.$$

It is quite clear that $\beta_\pi \geq \beta_{\pi'}$, whenever the partition π' refines π. Therefore, $\{\beta_\pi\}_\pi$ is a decreasing net of functions in the lattice $L_\infty(\Omega, \Sigma, \mu)$, which is order complete. Consequently, there exists a $\beta \in L_\infty(\Omega, \Sigma, \mu)$ so that $\beta = \bigwedge_\pi \beta_\pi$ and then $\beta_\pi \geq \beta$ a.e. for every π.

We distinguish between two mutually exclusive cases.

Case I. Suppose that $\beta(\omega) > 0$ on a subset of Ω of positive measure and find a $\delta > 0$ and a subset Ω_0 of Ω with $\mu(\Omega_0) > 0$ so that $\beta(\omega) \geq \delta$, $\omega \in \Omega_0$. We shall prove that in this case the band $Z_0 = \{f \cdot \chi_{\Omega_0}; f \in Z\}$ is order isomorphic to an L_1-space. Let $g \in Z^*$ be a simple function of the form $\sum_{i=1}^m a_i \chi_{A_i}$, where $\{A_i\}_{i=1}^m$ are mutually disjoint subsets of Ω_0 each having positive measure. Then, since F is an M-seminorm, we obtain

$$\begin{aligned}F(g) &= \max_{1\le i\le m} |a_i|\cdot F(\chi_{A_i}) \ge \left(\max_{1\le i\le m} |a_i|\right)\cdot\left(\min_{1\le i\le m} F(\chi_{A_i})\right)\\ &= \|g\|_\infty \cdot \min_{\omega\in\bigcup_{i=1}^m A_i} \beta_\pi(\omega),\end{aligned}$$

for any partition π which includes the sets $\{A_i\}_{i=1}^m$. Since $\beta_\pi(\omega) \ge \beta(\omega)$ a.e. it follows that

$$C\|g\| \ge F(g) \ge \|g\|_\infty \cdot \min_{\omega\in\bigcup_{i=1}^m A_i} \beta(\omega) \ge \delta\|g\|_\infty,$$

which clearly implies that $C\|z^*\| \ge \delta\|z^*\|_\infty$, whenever $z^* \in Z^*$ is supported by Ω_0. Hence, for any $z \in Z_0$, we have $\|z\|_1 \le \|z\| \le C\delta^{-1}\|z\|_1$. This completes the proof in Case I.

Case II. $\beta(\omega) = 0$, for a.e. $\omega \in \Omega$. We shall prove that in this case Z is a conjugate space. Choose first an increasing sequence $\{\pi_n\}_{n=1}^\infty$ of partitions of Ω such that $\beta_{\pi_n}(\omega) \to 0$, as $n \to \infty$, for a.e. $\omega \in \Omega$. By Egoroff's theorem, there exists a countable partition $\{\Omega_j\}_{j=1}^\infty$ of Ω so that $\mu(\Omega \sim \bigcup_{j=1}^\infty \Omega_j) = 0$ and $\beta_{\pi_n}(\omega) \to 0$, as $n \to \infty$, uniformly for $\omega \in \Omega_j$, $j = 1, 2, \ldots$. For each j, let V_j be the closure of $L_\infty(\Omega_j, \Sigma_{|\Omega_j}, \mu_{|\Omega_j})$ in Z^* and $Z_j = \{f\chi_{\Omega_j} : f \in Z\}$. Since $Z = \sum_{j=1}^\infty \bigoplus Z_j$ is a boundedly complete unconditional decomposition (for Z contains no copy of c_0) it would follow that Z is a dual space provided we show that Z_j is a conjugate space, for every j. This will be achieved by proving that Z_j is order isometric to V_j^*. To this end, fix j, let $\theta \in V_j^*$ and find a Hahn-Banach extension z^{**} of θ to an element of Z^{**}. If $Z_j^\perp$ denotes the orthogonal complement of Z_j in Z then clearly $Z^{**} = Z_j \oplus Z_j^\perp \oplus Z^\perp$. Hence, $z^{**} = z + z' + z_0^{**}$ with $z \in Z_j$, $z' \in Z_j^\perp$ and $z_0^{**} \in Z^\perp$. Then, for $v^* \in V_j$,

$$\theta(v^*) = z^{**}(v^*) = v^*(z_j) + z_0^{**}(v^*)$$

and the proof will be completed once we show that $z_0^{**}(v^*) = 0$, whenever $z_0^{**} \in Z^\perp$ and $v^* \in V_j$. However, in view of the definitions of F and V_j, it suffices to prove that F vanishes on $L_\infty(\Omega_j, \Sigma_{|\Omega_j}, \mu_{|\Omega_j})$. In order to verify this fact, notice that, for any measurable subset A of Ω_j and each n, we have

$$F(\chi_A) = \max_{B\in\pi_n} F(\chi_{A\cap B}) \le \max_{\substack{B\in\pi_n\\ A\cap B\neq\varnothing}} F(\chi_B) \le \max_{\omega\in\Omega_j} \beta_{\pi_n}(\omega) \to 0,$$

as $n \to \infty$. $\square$

THEOREM 3.4. *Let Z be a Banach function space over a nonatomic probability space (Ω, Σ, μ). Suppose that $Z = X \oplus Y$, where X is a reflexive space and Y is isomorphic to an L_1-space. Then there exists a band Z_0 of Z such that Z_0 is linearly isomorphic to Y and order isomorphic to an L_1-space while its orthogonal complement $Z_0^\perp$ is reflexive and isomorphic to X, up to a finite dimensional space.*

PROOF. By Theorem 3.2, Z has a band Z_0 which is order isomorphic to an L_1-space and such that $Z_0^\perp$ contains no isomorphic copies of $L_1(0,1)$.

We observe now that every operator T from an L_1-space into a reflexive space is strictly singular since T is weakly compact and L_1 has the Dunford-Pettis property (thus, the unit ball of any reflexive subspace of L_1 is mapped by T into a norm-compact set). We may therefore apply Theorem A and find decompositions

$X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ so that Z_0 is isomorphic to $X_1 \oplus Y_1$ and $Z_0^\perp$ to $X_2 \oplus Y_2$. This already implies that $\dim X_1 < \infty$.

Since $(Z_0^\perp)^{**}$ is isomorphic to $X_2 \oplus Y_2^{**}$ it follows that $(Z_0^\perp)^{**}/Z_0^\perp$ is isomorphic to Y_2^{**}/Y_2. On the other hand, Y_2 is clearly an $\mathcal{L}_1$-space and thus so is Y_2^{**}/Y_2. By using Theorem 3.3, we conclude that $Z_0^\perp$ is a dual space (here, we use the fact that (Ω, Σ, μ) is nonatomic which implies that any nontrivial band of $Z_0^\perp$ that is order isomorphic to an L_1-space must contain an isomorphic copy of $L_1(0,1)$, a contradiction). Let X_2' and Y_2' be subspaces of $Z_0^\perp$ so that $Z_0^\perp = X_2' \oplus Y_2'$, X_2' is isomorphic to X_2 and Y_2' to Y_2. Since X_2' is reflexive it would be a w^*-closed subspace of the dual space $Z_0^\perp$. Hence, Y_2', which is isomorphic to $Z_0^\perp/X_2'$, will be a dual space, too.

Notice now that, since the dual Z^* of Z is contained in $L_1(\Omega, \Sigma, \mu)$, the set $\{\chi_A; A \in \Sigma\}$ is weakly compact and its span is dense in Z. Hence, Z is a weakly compactly generated (WCG) space and so are its complemented subspaces Y and Y_2. By using a result of H. P. Rosenthal [**23**, Corollary 2.2], we obtain that Y_2 is isomorphic to a complemented subspace of $L_1(0,1)$. Thus, by D. R. Lewis and C. Stegall [**13**], Y_2 is either isomorphic to l_1 or $\dim Y_2 < \infty$. However, in view of Theorem 3.1, the first possibility cannot take place. Hence, $\dim Y_2 < \infty$. It follows easily that, up to a finite dimensional space, Z_0 is isomorphic to Y and $Z_0^\perp$ to X. However, Z_0 contains a complemented subspace isomorphic to l_1 and therefore Z_0 is precisely isomorphic to Y. □

REMARK. Theorem 3.4 remains true even when (Ω, Σ, μ) is an arbitrary probability space. The proof of this fact uses both Theorems 1.1 and 3.4. Let Z' be the band of Z containing all the atoms of Z and Z'' its orthogonal complement, which is a nonatomic lattice. If $Z = X \oplus Y$ with Y being reflexive and Y isomorphic to an L_1-space then, by Theorem A, there exist decompositions $X = X' \oplus X''$ and $Y' = Y' \oplus Y''$ so that Z' is isomorphic to $X' \oplus Y'$ and Z'' to $X'' \oplus Y''$. The band Z' is actually a space with an unconditional basis. Since every operator from X' into Y' is compact it follows from Theorem 1.1 that $Z' = Z_1' \oplus Z_2'$, where Z_1' and Z_2' are orthogonal bands so that Z_1' is isomorphic to X' and Z_2' to Y'. On the other hand, by Theorem 3.4, $Z'' = Z_1'' \oplus Z_2''$, where again Z_1'' and Z_2'' are orthogonal bands such that Z_1'' is isomorphic to X'', up to a finite dimensional space, and Z_2'' is isomorphic to Y''. Put $Z_1 = Z_1' \oplus Z_1''$ and $Z_2 = Z_2' \oplus Z_2''$. Then $Z = Z_1 \oplus Z_2$, where Z_1 and Z_2 are orthogonal bands so that Z_1 is isomorphic to X, up to a finite dimensional space, and Z_2 to Y.

The difficulties encountered in Theorem 3.4 and the remark following it stem from the fact that we do not know whether a reflexive Banach lattice must be isomorphic to its hyperplanes.

Suppose that we replace the assumption made in the above remark that X is reflexive by the assumption that X has nontrivial type. In this case, Z_1 will be a lattice with nontrivial type and therefore, by Lemma 2.4(c), Z_1 would contain a complemented copy of l_2. Thus, Z_1 will be isomorphic to its hyperplanes which implies the existence of an isomorphism between Z_1 and X. We summarize these conclusions in the following corollary.

COROLLARY 3.5. *Let Z be a Banach lattice with a weak unit and suppose that $Z = X \oplus Y$, where X is a subspace with nontrivial type and Y is isomorphic to an*

L_1-space. Then there exists a band Z_0 of Z which is isomorphic to Y and so that its orthogonal complement $Z_0^{\perp}$ is isomorphic to X.

Corollary 3.5 enables us to solve positively a problem raised by P. Wojtaszczyk in [**28**].

COROLLARY 3.6. *If a nonatomic Banach lattice Z is linearly isomorphic to the direct sum $L_1(0,1) \oplus L_2(0,1)$ then Z is already order isomorphic to $L_1(0,1) \oplus L_2(0,1)$.*

This result means that $L_1(0,1) \oplus L_2(0,1)$ has, up to isomorphism, a unique structure as a Banach function space on $[0,1]$. Further results on uniqueness of structures in Banach function spaces can be found in [**9**].

We remark that we can now list all the Banach lattices isomorphic to $L_1 \oplus L_2$. These are $L_1 \oplus L_2$, $L_1 \oplus l_2$, $L_1 \oplus L_2 \oplus l_2$, $L_1 \oplus l_1 \oplus L_2$, $L_1 \oplus l_1 \oplus l_2$, $L_1 \oplus l_1 \oplus L_2 \oplus l_2$, $L_1 \oplus L_2 \oplus l_2^n$ for $n \in \mathbb{N}$.

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