TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 266, Number 1, July 1981

A RIGID SUBSPACE OF L_0

BY

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ABSTRACT. We construct a closed infinite-dimensional subspace of $L_0(0, 1)$ (or L_p for $0 < p < 1$) which is rigid, i.e. such that every endomorphism in the space is a multiple of the identity.

1. Introduction. In this paper we shall show how to construct a closed infinite-dimensional linear subspace X of $L_0 = L_0(0, 1)$ which is *rigid*, i.e. such that every linear operator from X into itself is a multiple of the identity operator. In fact the space X can be chosen to embed in every L_p for $0 < p < 1$, and to have the property that every quotient space of X is also rigid.

In **[9]** Waelbroeck constructed the first known example of a rigid topological vector space. The space he constructed was metrizable but not complete. Its completion X was an F-space with the property that the algebra of all endomorphisms of X, $\mathfrak{L}(X)$, was commutative; in fact $\mathfrak{L}(X) \cong L_\infty$. Shortly after this, the second author constructed a rigid F-space **[7]** but the details have never been published. In this paper we modify the construction in **[7]**, allowing us to construct such a subspace of L_0.

All vector spaces in this paper will be real. It is not difficult to check the construction also works for complex scalars, with very minor modifications.

Our notation is fairly standard. An *F-norm* on a real vector space X is a map Λ: $X \to \mathbf{R}$ satisfying

(1.0.1) $\Lambda(x) > 0$ if $x \neq 0$,

(1.0.2) $\Lambda(\alpha x) \leqslant \Lambda(x)$, $|\alpha| \leqslant 1$, $x \in X$,

(1.0.3) $\lim_{\alpha \to 0} \Lambda(\alpha x) = \Lambda(0) = 0$, $x \in X$,

(1.0.4) $\Lambda(x + y) \leqslant \Lambda(x) + \Lambda(y)$, $x, y \in X$.

A *quasi-norm* is a map $x \to \|x\|$ $(X \to \mathbf{R})$ satisfying

(1.0.5) $\|x\| > 0$, $x \neq 0$,

(1.0.6) $\|\alpha x\| = |\alpha|\, \|x\|$, $\alpha \in \mathbf{R}$, $x \in X$,

(1.0.7) $\|x + y\| \leqslant C(\|x\| + \|y\|)$, $x, y \in X$,

where C is independent of x and y. The quasi-norm is p-subadditive $(0 < p \leqslant 1)$ if

(1.0.8) $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$, $x, y \in X$.

Received by the editors July 16, 1980.
AMS (MOS) subject classifications (1970). Primary 46A10, 46E30.
Key words and phrases. Rigid F-space, L_p-spaces for $0 \leqslant p < 1$.
[1] Supported in part by NSF grant no. MCS-7903079.
[2] Supported in part by NSF grant no. MCS-7901044.

A complete metrizable topological vector space X is called an F-space and its topology may be induced by an F-norm; if it is locally bounded then its topology may be induced by a quasi-norm, and is then a quasi-Banach space.

The space $L_0 = L_0(0, 1)$ consists of all Lebesgue measurable real functions, where functions differing only on a set of measure zero are identified. Equipped with the topology of convergence in measure, L_0 is an F-space and may be F-normed by

$$f \to \int_0^1 \frac{|f(x)|}{1 + |f(x)|}\, dx.$$

For $0 < p < \infty$, $L_p = L_p(0, 1)$ consists of all $f \in L_0$ such that

$$\|f\|_p = \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{1/p} < \infty,$$

L_p is a locally bounded F-space, and $\|\cdot\|_p$ is a quasi-norm on L_p; of course for $1 \leqslant p < \infty$, $\|\cdot\|_p$ is a norm, while for $0 < p < 1$, $\|\cdot\|_p$ is only p-subadditive.

If X and Y are two quasi-Banach spaces and $T\colon X \to Y$ is a linear operator, then $\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\|$. One easily established fact we shall use in the sequel is that if $T\colon X \to X$ satisfies $\|T\| < 1$ then $I - T$ is invertible on X; the proof is exactly the same as for Banach spaces.

If X is a quasi-Banach space and N is a closed subspace of X, then the quotient space X/N is quasi-normed by the quotient quasi-norm

$$\|x + N\| = \inf_{y \in N} \|x + y\|.$$

Then X/N is also a quasi-Banach space.

The plan of the paper is as follows. In §2, we list some basic results. In §3 we construct a simple example of a rigid closed subspace of L_0. This construction is self-contained and fairly elementary. In §4, we use some results from [3] to obtain a stronger example, a rigid closed subspace of L_0 for which every quotient space is also rigid.

2. Some basic results. Our first lemma is a finite-dimensional result due to N. T. Peck, who kindly showed us this improvement of our original estimate (replacing $(\dim X)^{1/p}$ by $(\dim X)^{1/p-1}$).

LEMMA 2.1. *Let X be a finite-dimensional quasi-normed space, and suppose the quasi-norm is p-subadditive. Then for $x_1, \ldots, x_m \in X$*

$$\left\| \sum_{i=1}^m x_i \right\| \leqslant (\dim X)^{1/p-1} \sum_{i=1}^m \|x_i\|.$$

PROOF. Let $B = \{x\colon \|x\| \leqslant 1\}$. Then as $x \to \|x\|$ is certainly continuous (it is p-subadditive), B is compact.

We may suppose $\Sigma\|x_i\| > 0$. Let $u = (\Sigma_{i=1}^m \|x_i\|)^{-1} \Sigma_{i=1}^m x_i$. Then $u \in \operatorname{co} B$. Now by a well-known result of Carathéodory, since B is balanced, we may write $u = \Sigma_{j=1}^N c_j v_j$ where $N = \dim X$, $v_j \in B$ $(1 \leqslant j \leqslant n)$ and $c_j \geqslant 0$ with $\Sigma c_j = 1$.

Hence

$$\|u\| \leqslant \left(\sum_{j=1}^{N} |c_j|^p\right)^{1/p} \leqslant N^{1/p-1}$$

and

$$\left\|\sum_{i=1}^{m} x_i\right\| \leqslant N^{1/p-1} \sum_{i=1}^{m} \|x_i\|.$$

Our remaining results in this section concern the spaces L_p for $0 \leqslant p < \infty$.

LEMMA 2.2. *Suppose $0 < q \leqslant p < \infty$. Then if $f \in L_q$ there exists a linear operator $T: L_p \to L_q$ with $T1 = f$ and $\|T\| = \|f\|_q$.*

REMARK. Here 1 denotes the constant function one.

PROOF. For the case $p = q$ this is essentially proved in Rolewicz [**8**, pp. 253–254]. The general case follows easily by composing with the inclusion map.

Now let C be the one-dimensional subspace of L_p consisting of the constant functions. Let $\rho_0: L_p \to L_p/C$ be the quotient map so that

$$\|\rho_0 f\|_p = \min_{\lambda \in \mathbf{R}} \|f - \lambda\|_p, \qquad f \in L_p.$$

The space L_p/C is (isomorphically) the space $L_p/1$ as defined in [**4**]. The next result shows that $L_p/1$, although not isomorphic to L_p [**4**] nevertheless embeds into L_p; this fact was independently observed by N. T. Peck. The same result is true for $p = 0$ by essentially the same argument.

LEMMA 2.3. *There is a linear operator $S: L_p \to L_p$ such that $\|\rho_0 f\|_p \leqslant \|Sf\|_p \leqslant 2^{1/p}\|\rho_0 f\|_p, f \in L_p$.*

PROOF. Note that L_p is isometric to $L_p((0, 1) \times (0, 1))$. We define $S: L_p \to L_p((0, 1) \times (0, 1))$ by $Sf(x, y) = f(x) - f(y)$. Then

$$\|Sf\|_p^p = \int_0^1 \int_0^1 |f(x) - f(y)|^p \, dx \, dy \geqslant \|\rho_0 f\|_p^p$$

and

$$\|Sf\|_p^p \leqslant \int_0^1 \int_0^1 |f(x)|^p + |f(y)|^p \, dx \, dy = 2\|f\|_p^p.$$

As $S1 = 0$, $\|Sf\|^p \leqslant 2\|\rho_0 f\|_p^p$.

The next lemma is a well-known application of stable processes; see [**5**].

LEMMA 2.4. *There is a linear embedding (isomorphism into) $S: L_p \to L_0$ such that*

$$\int_0^1 \exp(itSf(x)) \, dx = \exp(-|t|^p \, \|f\|_p^p).$$

3. Elementary construction of a rigid space. Let us suppose that for $\frac{1}{2} \leqslant p < 1$ we are given:

(3.0.1) A closed subspace W_p of L_p such that $1 \in W_p$ and $\phi(1) = 0$ for every continuous linear functional ϕ on W_p. This means that 1 belongs to the convex hull of every neighborhood of zero in W_p.

(3.0.2) A constant $K(p)$ where $0 \leqslant K(p) < \infty$.

For the purposes of this section it will suffice to take $W_p = L_p$ and $K(p) \equiv 0$. The greater generality will however be useful in §4 when we construct a further rigid subspace of L_0 with every quotient also rigid.

Select now any sequence $(c_n\colon n \geqslant 1)$ of positive numbers so that $\sum c_n^{1/2} < \frac{1}{2}$.

With these assumptions we prove:

LEMMA 3.1. *We may select sequences* $(p_n\colon n \geqslant 0)$ *and* $(\varepsilon_n\colon n \geqslant 0)$ *and a sequence* $(V_n\colon n \geqslant 0)$ *of finite-dimensional subspaces of* L_{p_n} *so that*:

(3.1.1) (p_n) *is increasing with* $p_0 = \frac{1}{2}$, $p_n < 1$ *for all* n, *and* $\lim_{n\to\infty} p_n = 1$.

(3.1.2) $\varepsilon_n > 0$ *for all* n.

(3.1.3) $1 \in V_n$ *and* $V_n \subset W_{p_n}$.

(3.1.4) *If* $M_n = \sum_{i=0}^{n-1} \dim V_i$ *then*

$$nM_n\varepsilon_n < c_n, \quad n \geqslant 1, \qquad nK(p_n)\varepsilon_n < c_n, \quad n \geqslant 1.$$

(3.1.5) *For* $n \geqslant 0$, *there exists* $\{v_{n,k}\colon 1 \leqslant k \leqslant l(n)\}$ *in* V_n *with* $\sum_{k=1}^{l(n)} v_{n,k} = 1$ *and*

$$\sum_{k=1}^{l(n)} \|v_{n,k}\| < \varepsilon_n, \qquad n \geqslant 0,$$

$$\sum_{k=1}^{l(n)} \|v_{n,k}\|^{p_{n+1}} < \varepsilon_n^{p_{n+1}}, \qquad n \geqslant 0. \tag{3.1.6}$$

PROOF. We select the sequences by induction. To start the induction take $p_0 = \frac{1}{2}$, $\varepsilon_0 = 2$ and $V_0 = C$, the space of constants in $L_{1/2}$. Then let $l(1) = 1$ and $v_{0,1} = 1$.

Now suppose $(p_0, \ldots, p_{m-1})$, $(\varepsilon_0, \ldots, \varepsilon_{m-1})$ and $(V_0, \ldots, V_{m-1})$ have been chosen so that (3.1.1)–(3.1.5) hold for $n \leqslant m-1$ and (3.1.6) holds for $n \leqslant m-2$. Then since $\sum_{k=1}^{l(m-1)} \|v_{m-1,k}\| < \varepsilon_{m-1}$, we may choose $p_n > p_{n-1}$ sufficiently close to 1 so that (3.1.6) holds for $n = m-1$, and so that $p_n > 1 - 1/m$. Now $\varepsilon_m > 0$ so that (3.1.4) holds for $n = m$. Since 1 is in the convex hull of every neighborhood of 0 in W_{p_m}, there exist $v_{m,1}, \ldots, v_{m,l(n)} \in W_{p_m}$ so that (3.1.5) holds for $n = m$. Finally we may let $V_m = \operatorname{lin}(v_{m,1}, \ldots, v_{m,l(n)})$.

Keeping the notation of the preceding lemma we introduce now a space Z of real-valued measurable functions on $(0, \infty)$. Z consists of all f such that

$$\Lambda(f) = \sum_{n=0}^{\infty} \int_n^{n+1} |f(x)|^{p_n}\, dx < \infty.$$

Then (modulo functions zero almost everywhere), Λ is an F-norm on Z and Z is an F-space. Furthermore Z is locally bounded and may be quasi-normed so that $\|f\| \leqslant 1$ if and only if $\Lambda(f) \leqslant 1$. It is easy to see that the unit ball is the $\frac{1}{2}$-convex, or equivalently

$$\|f + g\|^{1/2} \leqslant \|f\|^{1/2} + \|g\|^{1/2}, \qquad f, g \in Z, \tag{3.1.7}$$

and this implies

$$\|f + g\| \leqslant 2(\|f\| + \|g\|), \qquad f, g \in Z. \tag{3.1.8}$$

We denote by $Z(a, b)$ the subspace of Z of functions supported on the interval (a, b). Let P_n, E_n and Q_n be the natural projections of Z onto $Z(0, n)$, $Z(n, n+1)$ and $Z(n, \infty)$ respectively. Then $\|P_n\| = \|E_n\| = \|Q_n\| = 1$ for $n \in \mathbf{N}$ and $I = P_n + Q_n = P_n + E_n + Q_{n+1}$.

Note that if $f, g \in Z(n, \infty)$ then

$$\|f+g\|^{p_n} \leqslant \|f\|^{p_n} + \|g\|^{p_n}. \tag{3.1.9}$$

There is a natural isometric isomorphism $\tau_n\colon L_{p_n} \to Z(n, n+1)$ given by

$$\begin{aligned} \tau_n f(x) &= f(x-n), \qquad n < x < n+1, \\ &= 0, \qquad x \notin (n, n+1). \end{aligned}$$

Let $U_n = \tau_n V_n$, $e_n = \tau_n 1$ and $e_{n,k} = \tau_n v_{n,k}$, and $1 \leqslant k \leqslant l(n)$. We shall let Y be the closed subspace of Z spanned by $\bigcup_{n=1}^{\infty} U_n$, and let M be the closed linear span of $(e_n\colon n \geqslant 0)$. Let $\rho\colon Z \to Z/M$ be the quotient map so that $\|\rho f\| = \inf_{g \in M} \|f - g\|$. Note that if $f \in Z(n, n+1)$,

$$\|\rho f\| = \inf_{\lambda \in \mathbf{R}} \|f - \lambda e_n\| = \min_{\lambda \in \mathbf{R}} \|f - \lambda e_n\|.$$

LEMMA 3.2. *Suppose $f \in Z(0, n)$ with $\|f\| = 1$. Then there exists a linear operator $A\colon Z(n, n+1) \to Z(0, n)$ with $Ae_n = f$ and $\|A\| = 1$.*

PROOF. Suppose $f = h_0 + \cdots + h_{n-1}$ where $h_i \in Z(i, i+1)$ for $0 \leqslant i \leqslant n-1$. Then $\sum_{i=1}^{n-1} \|h_i\|^{p_i} = 1$. By Lemma 2.2, there exist linear operators $F_i\colon Z(n, n+1) \to Z(i, i+1)$ with $\|F_i\| = \|h_i\|$ and $F_i e_n = h_i$. Let $A = F_0 + \cdots + F_{n-1}$. Then $Ae_n = f$ and if $g \in Z(n, n+1)$ with $\|g\| = 1$ then

$$\Lambda(Ag) = \sum_{i=0}^{n-1} \|F_i g\|^{p_i} \leqslant \sum_{i=0}^{n-1} \|h_i\|^{p_i} \leqslant 1.$$

Hence $\|A\| = 1$.

Now let $(\mathfrak{B}_n)_{n=0}^{\infty}$ be a partitioning of $\mathbf{N}$ into infinite disjoint subsets with the property that, for every $n \geqslant 0$, $n < \min \mathfrak{B}_n$. For each n, l choose $(\gamma_k\colon k \in \mathfrak{B}_n)$ to be a dense subset of $\{f\colon f \in U_0 + \cdots + U_n,\ \|f\| = 1)$ with the property that $\gamma_k = e_n$ infinitely often.

By Lemma 3.2, we find linear operators $A_k\colon Z(k, k+1) \to Z(0, k)$ with $\|A_k\| = 1$ so that $A_k e_k = \gamma_k$. Define $T\colon Z \to Z$ by $T = \sum_{k=1}^{\infty} c_k A_k E_k$. Then since Z is $\frac{1}{2}$-convex,

$$\|T\|^{1/2} \leqslant \sum_{k=1}^{\infty} c_k^{1/2}$$

i.e.

$$\|T\| \leqslant \tfrac{1}{4} \tag{3.2.1}$$

and

$$\begin{aligned} T(Z(0, k+1)) &\subset Z(0, k), \qquad k \in \mathbf{N}, \\ T(Z(0, 1)) &= \{0\}. \end{aligned} \tag{3.2.2}$$

Now let $S = I - T$. Then $S: Z \to Z$ is invertible. If we let $M_1 = S(M)$ then M_1 is closed and $M_1 \subset Y$. Let $\pi: Z \to Z/M_1$ be the quotient mapping and let $X = \pi(Y) \cong Y/M_1$. We shall show that X is a rigid space.

First we prove

LEMMA 3.3. *Suppose $f \in Z(0, n+1)$. Then*

$$\|\rho E_n f\| \leqslant 4\|\pi f\|. \tag{3.3.1}$$

PROOF. For $\delta > 1$, choose $g \in M$ with $\|f - Sg\| \leqslant \delta\|\pi f\|$. Then

$$\|\rho E_n f - \rho E_n Sg\| \leqslant \delta\|\pi f\|.$$

Since $\rho E_n g = 0$, this implies $\|\rho E_n f + \rho E_n Tg\| \leqslant \delta\|\pi f\|$. Now $E_n Tg = E_n TQ_{n+1} g$ (since $T(Z(0, n+1)) \subset Z(0, n)$) and so

$$\begin{aligned}\|\rho E_n f\| &\leqslant 2(\delta\|\pi f\| + \|T\|\,\|Q_{n+1} g\|)\\ &\leqslant 2\delta\|\pi f\| + \tfrac{1}{2}\|Q_{n+1} g\|.\end{aligned} \tag{3.3.2}$$

However, since $Q_{n+1} f = 0$, $\|Q_{n+1} Sg\| \leqslant \delta\|\pi f\|$ and so

$$\begin{aligned}\|Q_{n+1} g\| &\leqslant 2(\delta\|\pi f\| + \|Q_{n+1} Tg\|)\\ &= 2(\delta\|\pi f\| + \|Q_{n+1} TQ_{n+1} g\|)\\ &\leqslant 2\delta\|\pi f\| + \tfrac{1}{2}\|Q_{n+1} g\|\end{aligned}$$

so that $\|Q_{n+1} g\| \leqslant 4\delta\|\pi f\|$.

Returning to (3.3.2) we obtain $\|\rho E_n f\| \leqslant 4\delta\|\pi f\|$. As $\delta > 1$ is arbitrary, the lemma follows.

LEMMA 3.4. *X is infinite-dimensional.*

PROOF. It follows from condition (3.1.4) of Lemma 3.1 that $\varepsilon_n \to 0$, and from (3.1.5) combined with Lemma 2.1 (note all spaces have $\frac{1}{2}$-subadditive quasi-norm) that $\dim(V_n) \to \infty$. Hence $\dim(U_n) \to \infty$. For $f \in U_n$, by Lemma 3.3

$$\|\pi f\| \geqslant \tfrac{1}{4}\|\rho f\| = \tfrac{1}{4}\min_{\lambda \in \mathbf{R}} \|f - \lambda e_n\|.$$

Hence $\dim \pi(U_n) \geqslant \dim U_n - 1$, and so $\dim X = \infty$.

LEMMA 3.5. *The set $(\lambda\pi(e_n)$: $n \in \mathbf{N}$, $\lambda \in \mathbf{R})$ is dense in X.*

PROOF. Suppose $f \in U_0 + \cdots + U_n$ and $\|f\| = 1$. Then there is an infinite subsequence $\mathcal{C}$ of $\mathbf{N}$ with $\lim_{j \in \mathcal{C}} \gamma_j = f$. Now $T(c_j^{-1} e_j) = \gamma_j$ and so

$$\pi\left(c_j^{-1} e_j\right) = \pi(\gamma_j) \to \pi(f).$$

Since multiples of such f are dense in Y, the lemma follows immediately.

THEOREM 3.6. *The space X is rigid.*

PROOF. Suppose $A: X \to X$ and $\|A\| < 1$. We shall show that, for each $n \in \mathbf{N}$, $\pi(e_n)$ is an eigenvector of A. In view of Lemma 3.5, this will show that each $x \in X$ is an eigenvector of A, and this will imply by easy algebraic arguments that $A = \lambda I$ for some $\lambda \in \mathbf{R}$.

Fix $n \in \mathbf{N}$ and let $\mathfrak{B}_n' = \{j \in \mathfrak{B}_n;\ \gamma_j = e_n\}$. For $j \in \mathfrak{B}_n'$, we have $Te_j = c_j e_n$ and hence $\pi(e_j) = c_j \pi(e_n)$. Now

$$e_j = \sum_{k=1}^{l(j)} e_{j,k} \quad \text{and} \quad \sum_{k=1}^{l(j)} \|e_{j,k}\| < \varepsilon_j.$$

As $\|A\| < 1$, there exists $g_{j,k} \in Y$ with $\pi g_{j,k} = A\pi e_{j,k}$ and

$$\|g_{j,k}\| \leqslant \|e_{j,k}\|, \qquad 1 \leqslant k \leqslant l(j).$$

Let $h_j = \sum_{k=1}^{l(j)} g_{j,k}$. Then $\pi h_j = A\pi e_j$. Now $P_j g_{j,k} \in U_0 + \cdots + U_{j-1}$, and so by Lemma 2.1

$$\|P_j h_j\| \leqslant M_j \sum_{k=1}^{l(j)} \|P_j g_{j,k}\| \leqslant M_j \varepsilon_j \leqslant c_j/j.$$

Similarly $Q_{j+1} g_{j,k} \in Z(j+1, \infty)$ and so applying (3.1.9)

$$\|Q_{j+1} h_j\| \leqslant \left(\sum_{k=1}^{l(j)} \|g_{j,k}\|^{p_{j+1}} \right)^{1/p_{j+1}} < \varepsilon_j.$$

Thus

$$\|P_j h_j + Q_{j+1} h_j\| \leqslant 4c_j/j, \quad \|h_j - E_j h_j\| \leqslant 4c_j/j, \tag{3.6.1}$$

and this implies

$$\left\|A\pi e_n - c_j^{-1}\pi E_j h_j\right\| \leqslant 4/j, \qquad j \in \mathfrak{B}_n'. \tag{3.6.2}$$

Now suppose $i, j \in \mathfrak{B}_n'$ and $i < j$. We have from (3.6.2)

$$\left\|c_i^{-1}\pi E_i h_i - c_j^{-1}\pi E_j h_j\right\| \leqslant 8(1/i + 1/j).$$

Now $c_j^{-1}E_j h_j - c_i^{-1}E_i h_i \in Z(0, j+1)$ and applying Lemma 3.3, equation (3.3.1):

$$\left\|c_j^{-1}\rho E_j h_j\right\| \leqslant 32(1/i + 1/j)$$

i.e. there exists $\lambda = \lambda(i,j) \in \mathbf{R}$ so that $\|c_j^{-1}(E_j h_j - \lambda e_j)\| \leqslant 32(1/i + 1/j)$. Thus

$$\left\|c_j^{-1}\big(\pi E_j h_j - \lambda\pi(e_j)\big)\right\| \leqslant 32(1/i + 1/j).$$

Now by (3.6.2) and since $c_j^{-1}\pi(e_j) = \pi(e_n)$

$$\|A\pi(e_n) - \lambda\pi(e_n)\| \leqslant 64/i + 72/j.$$

As $i, j \in \mathfrak{B}_n'$ can be chosen arbitrarily large we deduce that, for some $\mu \in \mathbf{R}$, $A\pi(e_n) = \mu\pi(e_n)$ and this completes the proof.

THEOREM 3.7. (a) *The space X is isomorphic to a subspace of L_0.*
(b) *X is isomorphic to a subspace of L_p for $0 < p < 1$.*

PROOF. (a) $X \cong Y/M_1$ and embeds into Z/M_1. As $S\colon Z \to Z$ is an invertible operator and $S(M) = M_1$, we have $Z/M \cong Z/M_1$.

The proof will be completed by showing that Z/M is isomorphic to a subspace of Z, and that Z is isomorphic to a subspace of L_0.

For the former statement we note by Lemma 2.3 there exists a linear operator $A_n\colon Z(n, n+1) \to Z(n, n+1)$ with

$$\|\rho f\| \leqslant \|A_n f\| \leqslant 2^{1/p_n}\|\rho f\|.$$

Then if $A = \sum_{n=0}^{\infty} A_n E_n$, and $f \in Z$

$$\Lambda(Af) = \sum_{n=0}^{\infty} \|A_n E_n f\|^{p_n} \leq \sum_{n=0}^{\infty} 2\|\rho E_n f\|^{p_n}$$
$$\leq 2\Lambda(f) \leq \Lambda(4f).$$

Hence $\|A\| \leq 4$, and $A(M) = 0$ so that

$$\|Af\| \leq 4\|\rho f\|, \qquad f \in Z. \tag{3.7.1}$$

Conversely $\Lambda(Af) \geq \sum_{n=0}^{\infty} \|\rho E_n f\|^{p_n}$.

If $f \in Z(0, m)$, then there exists λ_n, $0 \leq n \leq m-1$, so that $\|\rho E_n f\| = \|E_n f - \lambda_n e_n\|$ and hence

$$\sum_{n=0}^{\infty} \|\rho E_n f\|^{p_n} = \sum_{n=0}^{\infty} \|E_n f - \lambda_n e_n\|^{p_n} = \Lambda(f-g)$$

where $g = \sum_{n=0}^{m-1} \lambda_n e_n$. Hence if $\|\rho f\| = 1$, $\Lambda(Af) \geq 1$ and so $\|Af\| \geq 1$. By a density argument we have

$$\|Af\| \geq \|\rho f\|, \qquad f \in Z. \tag{3.7.2}$$

Combining (3.7.1) and (3.7.2), we have Z/M isomorphic to a subspace of Z.

To embed Z in L_0 we first note that $L_0 \cong L_0(\Omega)$ where Ω is the countable product of $(0, 1)$ with the product measure m. Suppose B_n: $Z(n, n+1) \to L_0(0, 1)$ is an isomorphism with

$$\int_0^1 \exp(itB_n f(x))\, dx = \exp\left(-|t|^{p_n} \int_n^{n+1} |f(x)|^{p_n}\, dx\right)$$

(see Lemma 2.4). Then define B: $Z \to L_0(\Omega)$ by

$$Bf(\omega_0, \omega_1, \dots, \omega_n, \dots) = \sum_{n=0}^{\infty} B_n(E_n f)(\omega_n).$$

(Formally this is defined for f of bounded support and then extended by continuity.) Then

$$\int_\Omega \exp(itBf)\, dm = \exp(-\Lambda(tf))$$

and this implies easily that B: $Z \to L_0(\Omega)$ is an isomorphic embedding.

This last step is of course standard. For general results on embeddings of Musielak-Orlicz spaces into L_0 see **[1]**.

(b) First observe that, for each n, Y is a direct sum of a finite-dimensional space and a p_n-convex space. Hence Y is p_n-convex. As $p_n \to 1$, Y is p-convex for every p, $0 < p < 1$, and the same is true of its quotient X. Now by applying Nikišin's theorem **[2]**, **[6]**, if X embeds into L_0 then X embeds into every L_p for $0 < p < 1$.

4. Modified construction. In this section we use a result proved in **[3]**, i.e. one may choose W_p to satisfy (3.0.1) in such a way that the quotient W_p/C (C = space of constants) is isomorphic to a Banach space. As shown in **[3]** we may arrange $W_p/C \cong l_1$. Thus for each p, $\frac{1}{2} \leq p < 1$, there is a constant $K(p)$ such that

$$\left\| \sum_{i=1}^{n} \rho_0 f_i \right\| \leq K(p) \sum_{i=1}^{n} \|\rho_0 f_i\| \tag{4.0.1}$$

for $f_1, \ldots, f_n \in W_p$. This determines $K(p)$ in (3.0.2). We now repeat the construction in §3, and we shall have the property that if $f_1, \ldots, f_n \in U_m$ then

$$\left\| \sum_{i=1}^{n} \rho f_i \right\| \leqslant K(p_n) \sum_{i=1}^{n} \|\rho f_i\|. \tag{4.0.2}$$

THEOREM 4.1. *Suppose X is constructed as above. Suppose N is a closed subspace of X and q: $X \to X/N$ is the quotiet mapping. Then if A: $X \to X/N$ is any linear operator we have $A = \lambda q$ for some $\lambda \in \mathbf{R}$.*

PROOF. We may suppose $\|A\| < 1$. As in the proof of Theorem 3.6, if $n \in \mathbf{N}$ and $j \in \mathfrak{B}_n'$ we may pick $g_{j,k} \in Y$ with

$$q\pi g_{j,k} = A\pi e_{j,k} \qquad 1 \leqslant k \leqslant l(j),$$

and

$$\|g_{j,k}\| \leqslant \|e_{j,k}\|, \qquad 1 \leqslant k \leqslant l(j).$$

Let $h_j = \sum_{k=1}^{l(j)} g_{j,k}$. As before $\|h_j - E_j h_j\| \leqslant 4c_j/j$.

However in this case we use $E_j g_{j,k} \in U_j$ so that we can use (4.0.2) to deduce

$$\|\rho E_j h_j\| \leqslant K(p_j) \sum_{k=1}^{l(j)} \|\rho g_{j,k}\| \leqslant K(p_j)\varepsilon_j \leqslant c_j/j.$$

Hence there exists $\lambda_j \in \mathbf{R}$ with $\|E_j h_j - \lambda_j e_j\| \leqslant c_j/j$ and

$$\|h_j - \lambda_j e_j\| \leqslant 10c_j/j,$$

$$\left\|q\pi\left(c_j^{-1}h_j\right) - \lambda_j c_j^{-1} q\pi(e_j)\right\| \leqslant 10/j,$$

$$\left\|A\pi e_n - \lambda_j q\pi(e_n)\right\| \leqslant 10/j.$$

Again as $j \in \mathfrak{B}_n'$ can be chosen arbitrarily large we have $A\pi e_n = \mu_n q\pi e_n$ for some μ_n and as before we can deduce that $A = \mu q$ for some $\mu \in \mathbf{R}$.

COROLLARY 4.2. *Every quotient space of X is rigid.*

PROOF. If A: $X/N \to X/N$, then $Aq = \lambda q$ for some $\lambda \in \mathbf{R}$ so that $A = \lambda I$.

COROLLARY 4.3. *If two quotient spaces of X, X/N_1 and X/N_2, are isomorphic then $N_1 = N_2$.*

PROOF. Suppose S: $X/N_1 \to X/N_2$ is an isomorphism, and q_1, q_2 are the respective quotient maps. Then Sq_1: $X \to X/N_2$ and hence $Sq_1 = \lambda q_2$ for some $\lambda \in \mathbf{R}$. Clearly $\lambda \neq 0$ since S is onto; hence if $x \in N_1$, $q_2 x = 0$, i.e. $x \in N_2$. By a symmetric argument $N_1 = N_2$.

COROLLARY 4.4. *There is an uncountable family (X_α: $\alpha \in \mathcal{C}$) of mutually nonisomorphic rigid F-spaces, each of which is isomorphic to a subspace of L_p for $0 < p < 1$.*

PROOF. Let (F_α: $\alpha \in \mathcal{C}$) be the uncountable family of one-dimensional subspaces of X. Each $X_\alpha = X/F_\alpha$ is rigid by 4.2 and the spaces are mutually nonisomorphic by 4.3. Each embeds into $L_p/1$ (see §2) and hence into L_p for $0 < p < 1$ (Lemma 2.3).

5. Concluding remarks. We here mention three problems. First, does L_p $(0 < p < 1)$ have a rigid quotient? It seems that this might be more difficult to achieve. More generally, does every F-space with trivial dual have a rigid quotient? Similarly does every F-space with trivial dual have a closed rigid subspace?

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