CONSISTENCY THEOREMS FOR ALMOST CONVERGENCE

BY

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ABSTRACT. The concept of almost convergence of a sequence of real or complex numbers was introduced by Lorentz, who developed a very elegant theory. The purpose of the present paper is to continue Lorentz's investigations and obtain consistency theorems for almost convergence; this is achieved by studying certain locally convex topological vector spaces.

1. Introduction The concept of almost convergence of a sequence of real or complex numbers was introduced, after an idea of Banach, by Lorentz [13] who developed a very elegant theory. Further studies of almost convergence and its relationship with general summability methods have since been carried out in [12], [17] and [19]. The purpose of the present paper is to obtain consistency theorems for almost convergence by studying certain locally convex topological vector spaces.

We adopt the following notation:

- \( \omega \) denotes the space of all scalar (real or complex) sequences;
- \( e, e^{(k)} \in \omega \) are given by
  \[
e = (1, 1, \ldots), \\
e^{(k)} = (0, \ldots, 0, 1, 0, \ldots) \text{ with the one in the } k\text{th position};
\]
- \( \varphi \) is the linear span of \( \{e^{(k)} : k = 1, 2, \ldots\} \);
- \( m = \{x \in \omega : \|x\|_\infty = \sup_j |x_j| < \infty\} \);
- \( c = \{x \in \omega : \lim x = \lim_{j \to \infty} x_j \text{ exists}\} \);
- \( c_0 = \{x \in \omega : \lim x = 0\} \);
- \( l = \{x \in \omega : \|x\|_l = \sum_{j=1}^\infty |x_j| < \infty\} \);
- \( bv = \{x \in \omega : \|x\|_{bv} = \sum_{j=1}^\infty |x_j - x_{j+1}| + \lim_{j \to \infty} |x_j| < \infty\} \);
- \( bv_0 = bv \cap c_0 \);
- \( bs = \{x \in \omega : \|x\|_{bs} = \sup_n |\sum_{j=1}^n x_j| < \infty\} \).

A vector subspace of \( \omega \) is called a sequence space. If \( E \) is a sequence space with a locally convex topology \( \tau \) then \( (E, \tau) \) is a K-space provided that the linear functionals
\[ x \rightarrow x_j \quad (j = 1, 2, \ldots) \]

are continuous on \( E \). If, in addition, \((E, \tau)\) is complete and metrizable (respectively normable) then \((E, \tau)\) is called an \textit{FK-space} (respectively \textit{BK-space}). For \( x \in \omega \) we write

\[ P_n x = (x_1, x_2, \ldots, x_n, 0, \ldots). \]

\((E, \tau)\) is an \textit{AK-space} if \( P_n x \) converges to \( x \) for every \( x \in E \).

If \( E \) and \( F \) are sequence spaces containing \( \varphi \) such that the bilinear form \( \langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j \) converges whenever \( x \in E \) and \( y \in F \), then topologies of the dual pairing \( \langle E, F \rangle \) provide examples of \textit{K-space} topologies. In particular, we shall be interested in the weak topology \( \sigma(E, F) \), the Mackey topology \( \tau(E, F) \) and the strong topology \( \beta(E, F) \) (following the notation of Schaefer [18]).

We shall also consider matrix maps and matrix methods of limitation. Let \( A = (a_{ij})_{i,j=1}^{\infty} \) be an infinite matrix with scalar entries; we denote by \( \omega_A \) the set of \( x \in \omega \) such that \( \sum_{j=1}^{\infty} a_{ij} x_j \) converges for each \( i \). For \( x \in \omega_A \) we write

\[ (Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j \]

so that \( A : \omega_A \rightarrow \omega \) is a linear map. If \( E \) is a sequence space,

\[ E_A = \{ x \in \omega_A : Ax \in E \}. \]

If \( E \) is an \textit{FK-space} then Zeller [24, Theorem 4.10(a)] has shown that \( E_A \) is also an \textit{FK-space} when topologized by means of the seminorms:

\[ x \rightarrow x_j \quad (j = 1, 2, \ldots), \]

\[ x \rightarrow \sup_{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \quad (i = 1, 2, \ldots), \]

and

\[ x \rightarrow q(Ax), \]

where \( q \) runs through the continuous seminorms on \( E \). A matrix \( A \) defines a method of limitation, viz: if \( x \in c_A \), we write \( \lim_A x = \lim(Ax) \). \( A \) is called \textit{conservative} if \( c \subseteq c_A \) or, equivalently (see [26]),

1. \( \sup_{i} \sum_{j=1}^{\infty} |a_{ij}| < \infty, \)
2. \( \lim_{l \rightarrow \infty} a_{ij} = a_j \) exists \( (j = 1, 2, \ldots) \),

and

3. \( \lim_{l \rightarrow \infty} \sum_{j=1}^{\infty} a_{ij} \) exists.
We then write
\[ \chi(A) = \lim_{i \to \infty} \sum_{j=1}^{\infty} a_{ij} - \sum_{j=1}^{\infty} a_j, \]
and say that \( A \) is conull when \( \chi(A) = 0 \). \( A \) is called regular if \( \lim_{A} x = \lim_{\chi} x \) whenever \( x \in c \); for regularity it is necessary and sufficient (see [26]) to have (1), (2) and (3) with \( a_j = 0 \) (\( j = 1, 2, \ldots \)) and \( \chi(A) = 1 \).

2. Properties of almost convergence. In this section we develop the theory of almost convergence, deriving the original characterization of almost convergent sequences given by Lorentz [13], as well as several other useful properties of the space \( aco \) (to be defined below). Since our approach is from the viewpoint of functional analysis, and therefore differs slightly from Lorentz's, we shall give a complete development of the subject.

The linear functional \( \lim \) on \( c \) has norm one, i.e.
\[ |\lim x| \leq ||x||_{\infty} \quad (x \in c) \]
and so by the Hahn-Banach theorem possesses extensions \( L \), of norm one, defined on all of \( m \). We call such a functional \( L \) an extended limit. If \( x \in m \), we write
\[ Tx = \{x_{n+1}\}_{n=1}^{\infty} \]
and say that an extended limit \( L \) is a Banach limit if
\[ L(Tx) = L(x) \quad (x \in m). \]
(Some authors insist that a Banach limit should also satisfy \( L(x) \geq 0 \) whenever \( x_n \geq 0 \) for all \( n \), or even \( \lim_{n \to \infty} \sup x_n \geq \lim_{n \to \infty} \inf x_n \). It is clear, however, that any extended limit has these properties.)

The existence of Banach limits was proved by Banach [2]; another proof can be found in Theorem 1 below. If \( x \in m \) is such that for every Banach limit \( L \), \( L(x) \) assumes a common value, then we write \( \text{Lim} \ x \) for this value, and say that \( x \) is almost convergent to \( \text{Lim} \ x \). The set of almost convergent sequences is denoted by \( ac \), and the subset \( \{x \in ac: \lim x = 0\} \) is denoted by \( ac_0 \). \( ac_0 \) is a hyperplane in \( ac \) and \( ac = ac_0 + \{e\} \); it is also easy to show that \( ac \) and \( ac_0 \) are closed subspaces of \( m \). Our first result (Theorem 1) characterizes these spaces.

**Lemma 1.** If \( L \) is a continuous linear functional on \( m \) with
(i) \( ||L|| = 1 \),
(ii) \( L(e) = 1 \), and
(iii) \( L(bs) = 0 \),
then \( L \) is a Banach limit.

**Proof.** Since \( \varphi \subseteq bs \), it follows from (iii) that \( L(\varphi) = 0 \), and by continuity that
$L(c_0) = 0$; therefore $L$ is an extended limit. Moreover, for $x \in m$, $x - Tx \in bs$ and so $L(x) = L(Tx)$.

**Lemma 2.** If $x \in m \setminus c_0$, then there exists an extended limit $L$ with $L(x) \neq 0$.

**Proof.** Since $x \in m \setminus c_0$, we may choose an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that

$$\lim_{k \to \infty} x_{n_k} = \alpha \neq 0.$$ 

Define $L$ by

$$Ly = \lim_{k \to \infty} y_{n_k}$$

where this limit exists, and extend $L$ to $m$ by the Hahn-Banach theorem.

**Theorem 1 (Lorentz) [13].** $x \in \omega$ is almost convergent (to $\alpha$) if and only if

$$\lim_{p \to \infty} \frac{1}{p} (x_n + \cdots + x_{n+p-1}) = \alpha$$

uniformly in $n$.

**Proof.** Without loss of generality we may assume that $\alpha = 0$. Let $\{n_p\}_{p=1}^{\infty}$ be any increasing sequence of positive integers, and define the matrix map $A: m \to m$ by

$$(Ax)_p = \frac{1}{p} (x_{n_p} + \cdots + x_{n_p+p-1}) \quad (x \in m).$$

Then we have $Ae = e$, $A(bs) \subset c_0$, $\|A\|_\infty = 1$.

If $L$ is an extended limit, then, by Lemma 1,

$$LA$$

is a Banach limit

and so, for $x \in ac_0$, we have

$$L(Ax) = 0.$$ 

By Lemma 2 we have $Ax \in c_0$ so that

$$\lim_{p \to \infty} \frac{1}{p} (x_{n_p} + \cdots + x_{n_p+p-1}) = 0.$$ 

Since this is true for any sequence $\{n_p\}_{p=1}^{\infty}$, we conclude that

$$\lim_{p \to \infty} \sup_n \left| \frac{1}{p} (x_n + \cdots + x_{n+p-1}) \right| = 0,$$

which is (4).

Conversely, (4) implies that
\[
\lim_{p \to \infty} \left\| \frac{1}{p} (T x + \cdots + T^p x) \right\|_\infty = 0.
\]

Thus, for any Banach limit \( L \), we have \( L(x) = 0 \), so that \( x \in ac^0 \).

We remark that (5) gives what is perhaps the easiest proof of the existence of Banach limits. Banach's original proof [2] also uses the Hahn-Banach theorem, but involves a rather sophisticated sublinear functional; Day's elegant proof [9, p.83], using fixed point theory, requires considerably more machinery.

Our next result, which follows at once from Theorem 1, shows that \( ac_0 \) and \( ac \) are "large" subspaces of \( m \).

**Corollary.** \((ac_0, \| \cdot \|_\infty)\) is a nonseparable BK-space.

We now come to a series of results which relate various properties of \( ac_0 \) to those of more familiar sequence spaces.

**Theorem 2.** If \( \{x^{(n)}\}_{n=1}^{\infty} \) is a sequence of points in \( l \), and \( x \in l \), then the following conditions are equivalent:

(i) \( \{x^{(n)}\}_{n=1}^{\infty} \) is \( \sigma(l, ac_0) \)-convergent to \( x \):

(ii) \( \{x^{(n)}\}_{n=1}^{\infty} \) is \( \sigma(l, bs + c_0) \)-convergent to \( x \);

(iii) \( \sup_n \|x^{(n)}\|_l < \infty \) and \( \lim_{n \to \infty} \|x^{(n)} - x\|_{bv} = 0 \).

**Proof.** Without loss of generality we may assume that \( x = 0 \).

(i) \( \Rightarrow \) (ii) follows since \( bs + c_0 \subset ac_0 \).

(ii) \( \Rightarrow \) (iii). If \( x^{(n)} \to 0 \sigma(l, bs + c_0) \), then \( x^{(n)} \to 0 \sigma(l, c_0) \) so that

\[ \sup_n \|x^{(n)}\|_l < \infty. \]

Also, \( x^{(n)} \to 0 \sigma(l, bs) \) so that \( x^{(n)} \to 0 \sigma(bv_0, bs) \); this is the weak topology on \( bv_0 \), and, since \( bv_0 \) is isomorphic to \( l \), we may use Schur's theorem [2, p. 137] to deduce that

\[ \lim_{n \to \infty} \|x^{(n)}\|_{bv} = 0. \]

(iii) \( \Rightarrow \) (i). Let \( f \in ac_0 \) and \( \varepsilon > 0 \) be fixed. By Theorem 1 we may choose a positive integer \( p \) so that

\[ \left\| \frac{1}{p} (Tf + \cdots + T^p f) \right\|_\infty < \frac{\varepsilon}{2} \left(1 + \sup_n \|x^{(n)}\|_l\right). \]

We then have, for every \( n \),

\[ \left| \sum_{k=1}^{\infty} \frac{1}{p} (f_{k+1} + \cdots + f_{k+p}) x^{(n)}_k \right| \leq \|x^{(n)}\|_l \left\| \frac{1}{p} (Tf + \cdots + T^p f) \right\|_\infty < \frac{\varepsilon}{2}. \]

Furthermore, fixing \( p \), we may choose a positive integer \( N \) so that
whenever \( n \geq N \).

Now

\[
\left| \sum_{k=1}^\infty (f_{k+s} - f_k)x_k^{(n)} \right| = \left| \sum_{k=1}^\infty f_k(x_k^{(n)} - x_k^{(s)}) \right| \quad \text{(putting } x_m^{(n)} = 0 \text{ if } m \leq 0) \\
\leq \sum_{k=1}^\infty |f_k| \sum_{r=1}^s |x_{k-r+1}^{(n)} - x_{k-r}^{(n)}| \\
\leq s\|f\|_\infty (\|x^{(n)}\|_{bv} + |x_1^{(n)}|) \\
\leq 2s\|f\|_\infty \|x^{(n)}\|_{bv}.
\]

Therefore

\[
\left| \sum_{k=1}^\infty \frac{1}{p}(f_{k+1} + \cdots + f_{k+p})x_k^{(n)} - \sum_{k=1}^\infty f_kx_k^{(n)} \right| \leq \frac{1}{p} \left( \frac{p(p+1)}{2} \right) 2\|f\|_\infty \|x^{(n)}\|_{bv} \\
< \frac{\varepsilon}{2} \quad \text{whenever } n \geq N.
\]

Thus, for \( n \geq N \), we have

\[
\left| \sum_{k=1}^\infty f_kx_k^{(n)} \right| < \varepsilon,
\]

i.e., \( x^{(n)} \to 0 \sigma(l, ac_0) \).

We remark that condition (iii) of Theorem 2 identifies sequential convergence in \( \sigma(l, ac_0) \) with a two-norm topology. For details concerning this type of topology we refer the reader to [1], [6], [22] and [23].

**Corollary 1.** \( l \) is sequentially complete under both the topologies \( \sigma(l, ac_0) \) and \( \sigma(l, bs + c_0) \).

**Proof.** If \( \{x^{(n)}\}_{n=1}^\infty \) is a \( \sigma(l, bs + c_0) \)-Cauchy sequence, the proof of Theorem 2 shows that \( \{x^{(n)}\}_{n=1}^\infty \) is a Cauchy sequence in \( bv_0 \) and bounded in \( l \). Since \( bv_0 \) is complete, there exists \( x \in bv_0 \) such that

\[
\lim_{n \to \infty} \|x^{(n)} - x\|_{bv} = 0.
\]

But this implies that

\[
x_k = \lim_{n \to \infty} x_k^{(n)} \quad (k = 1, 2, \ldots)
\]

so that

\[
\sum_{k=1}^\infty |x_k| \leq \sup_n \sum_{k=1}^\infty |x_k^{(n)}| < \infty,
\]
and $x \in l$. It then follows from Theorem 2, (iii) $\Rightarrow$ (i), that $x^{(n)} \to 0$ $\sigma(l, \alpha_0)$, giving the desired result.

**Corollary 2.** For a subset $C$ of $l$, the following conditions are equivalent:

(i) $C$ is $\sigma(l, \alpha_0)$-relatively compact;
(ii) $C$ is $\sigma(l, bs + c_0)$-relatively compact;
(iii) $C$ is $\| \cdot \|_1$-bounded and $\lim_{n \to \infty} \sup_{x \in C} \| x - P_n x \|_{bs_0} = 0$.

**Proof.** A subset of a $K$-space is relatively compact if and only if it is relatively sequentially compact (see [10]) and hence Theorem 2 shows that (i) and (ii) are equivalent. Using the sequential completeness of $l$ in the two-norm convergence defined in (iii) of Theorem 2, it is clear that (i) and (ii) are equivalent to "$C$ is $\| \cdot \|_1$-bounded and $\| \cdot \|_{bs_0}$-relatively compact." However, by a general theorem on bases (see [16]) this is equivalent to (iii).

We note from Corollary 2 that the closed convex hull of a $\sigma(l, \alpha_0)$-compact set is also $\sigma(l, \alpha_0)$-compact (using (iii)); hence the Mackey topology, $\tau(\alpha_0, l)$, is the topology of uniform convergence on $\sigma(l, \alpha_0)$-compact sets.

We now turn to the relationship between $\alpha_0$ and $bs$.

**Theorem 3.** (i) $\alpha_0 = \overline{bs}$, the closure of $bs$ in $m$.
(ii) If $x \in bs + c_0$, then $\sup_p \lim_{n \to \infty} |x_{n+1} + \cdots + x_{n+p}| < \infty$.
(iii) $\alpha_0 \neq bs + c_0$.

**Proof.** (i) Clearly $bs \subseteq \alpha_0$. Conversely, if $x \in \alpha_0$ and $\varepsilon > 0$ are given, we may choose a positive integer $p$ so that

$$|x_{n+1} + \cdots + x_{n+p}| < p\varepsilon \quad (n = 1, 2, \ldots).$$

In particular,

$$x_{mp+1} + \cdots + x_{(m+1)p} = p\delta_m \quad (m = 0, 1, 2, \ldots)$$

where $|\delta_m| \leq \varepsilon$. Letting $y$ be defined by

$$y_{mp+k} = x_{mp+k} - \delta_m \quad (k = 1, 2, \ldots, p; \ m = 0, 1, 2, \ldots),$$

it is clear that $\|x - y\|_\infty \leq \varepsilon$; we complete the proof of (i) by showing that $y \in bs$.

Now

$$\sum_{i=1}^{mp+k} y_i = \sum_{n=0}^{m-1} \sum_{j=1}^{p} (x_{np+j} - \delta_n) + \sum_{i=1}^{k} x_{mp+i} - k\delta_m$$

$$= \sum_{i=1}^{k} x_{mp+i} - k\delta_m \quad \text{by (6)}.$$

Consequently
\[ \left| \sum_{i=1}^{q} y_i \right| \leq p(\|x\|_{\infty} + \varepsilon) \]

for every \( q \), and \( y \in bs \).

(ii) If \( x \in bs + c_0 \), then \( x = y + z \) for some \( y \in bs \) and \( z \in c_0 \). Then

\[ |x_{n+1} + \cdots + x_{n+p}| \leq |y_{n+1} + \cdots + y_{n+p}| + |z_{n+1} + \cdots + z_{n+p}| \]

so that

\[ \limsup_{n \to \infty} |x_{n+1} + \cdots + x_{n+p}| = \limsup_{n \to \infty} |y_{n+1} + \cdots + y_{n+p}| \leq 2\|y\|_{bs}, \]

giving the desired result.

(iii) By (ii) we may construct \( x \in ac_0 \setminus (bs + c_0) \) directly; let

\[ x_k = 1 \quad \text{if } k = 2^n + 2^m \text{ for } n \geq m \geq 1, \]

\[ = 0 \quad \text{otherwise}. \]

Then \( x \) does not satisfy (ii), yet it is easy to check that \( x \in ac_0 \).

It is interesting to note that \( bs + c_0 \) is a BK-space which is B-invariant in the sense of Garling [10], and \( c_0 \subset bs + c_0 \subset m \), yet \( bs + c_0 \) is not closed in \( m \).

**Theorem 4.** (i) \((ac_0, \tau(ac_0, l))\) is a complete AK-space.

(ii) \( \tau(bs + c_0, l) \) is the restriction of \( \tau(ac_0, l) \) to \( bs + c_0 \) [so that \((ac_0, \tau(ac_0, l))\) is the completion of \((bs + c_0, \tau(bs + c_0, l))\)].

**Proof.** (i) If \( C \) is \( \sigma(l, ac_0) \)-relatively compact, then by (iii) of Corollary 2 to Theorem 2, the set \( P(C) = \{ P_f : f \in C \} \) is \( \sigma(l, ac_0) \)-relatively compact. It follows that the operators \( \{ P_n : n = 1, 2, \ldots \} \) are \( \tau(ac_0, l) \rightarrow \tau(ac_0, l) \)-equicontinuous, so that the set

\[ S = \{ x \in ac_0 : P_n x \to x \ \tau(ac_0, l) \} \]

is \( \tau(ac_0, l) \)-closed. However, \( S \supseteq \varphi \) and \( \varphi \) is \( \tau(ac_0, l) \)-dense in \( ac_0 \) (since \( \varphi \) is \( \sigma(ac_0, l) \)-dense); hence \( S = ac_0 \), showing that \((ac_0, \tau(ac_0, l))\) is an AK-space.

To show that \((ac_0, \tau(ac_0, l))\) is complete we use Grothendieck's criterion [6, Proposition 1]. Let \( \theta \) be a linear functional on \( l \) which is \( \sigma(l, ac_0) \)-continuous on each \( \sigma(l, ac_0) \)-compact set. Then \( \theta(x^{(n)}) \to 0 \) whenever \( x^{(n)} \to 0 \ \sigma(l, ac_0) \). Consequently, from Theorem 2, \( \theta \) is continuous in the two-norm topology. Using the standard characterization of the dual of a two-norm space [1, Theorem 4.2], it follows that \( \theta \) lies in the closure of \( bs \) (the dual of \((l, \| \cdot \|_{bs})\)) in \( m \) (the dual of \((l, \| \cdot \|_{\infty})\)). Hence, by Theorem 3(i), \( \theta \) takes the form

\[ \theta(x) = \sum_{k=1}^{\infty} f_k x_k, \]

where \( f \) is a fixed element from \( ac_0 \). It follows from Grothendieck's criterion that \((ac_0, \tau(ac_0, l))\) is complete.
(ii) This follows from Corollary 2 to Theorem 2.

**Theorem 5.** Let $E$ be a separable FK-space containing $c_0$ and $bs$. Then

(i) $E$ contains $ac_0$;
(ii) $x \in ac_0$ implies that $P_n x \to x$ in $E$;
(iii) $e^{(n)} \to 0$ in $E$.

**Proof.** (i) and (ii). The space $(bs + c_0, \tau(bs + c_0, l))$ is a Mackey space whose dual, $l$, is $\sigma(l, bs + c_0)$-sequentially complete by Corollary 1 to Theorem 2. Thus, by the main result of [11] (see also [7, Theorem 5]), the natural inclusion mapping: $bs + c_0 \to E$, which clearly has closed graph, must be continuous. If $x \in ac_0$, then by Theorem 4, $(P_n x)_{n=1}^\infty$ is Cauchy in $(bs + c_0, \tau(bs + c_0, l))$ and hence in $E$. Since $E$ is complete, $(P_n x)_{n=1}^\infty$ converges in $E$, and its limit must be $x$ since $E$ is a $K$-space. This completes the proof of (i) and (ii).

For (iii), we note that if $C$ is a $\sigma(l, bs + c_0)$-compact subset of $l$, then

$$\sup_{f \in C} |f_n| \leq \sup_{f \in C} \|f - P_{n-1} f\|_{bv} \to 0 \text{ as } n \to \infty$$

by Corollary 2 to Theorem 2. Consequently $e^{(n)} \to 0$ in $(bs + c_0, \tau(bs + c_0, l))$ and hence in $E$.

We note that (iii) is true if we only assume that $E$ contains $bs$ (see [5, Theorem 5]).

Our next result answers a question left open in [6].

**Corollary 1.** There exists a BK-space $E$ which is not the intersection of the separable FK-spaces containing it.

**Proof.** Using Theorems 3 and 5 we may take $E$ to be $bs + c_0$.

**Corollary 2.** If $A$ is a conservative matrix such that $bs \subseteq c_A$, then

(i) $ac \subseteq c_A$;
(ii) $\lim_{n \to \infty} \sup_{m} |a_{mn}| = 0$;
(iii) $\lim_{m \to \infty} \sup_{n} |a_{mn}| = 0$;
(iv) for $x \in ac$, we have $\lim_{A} x = \sum_{j=1}^{\infty} a_j x_j + \chi(A) \text{Lim } x$.

**Proof.** (i) $c_A$ is a separable FK-space ([14, 1.4.1]; [4, Corollary 1 to Theorem 4]) and so, by Theorem 5(i), $ac_0 \subseteq c_A$. Since $e \in c_A$, it follows that $ac \subseteq c_A$.

(ii) $e^{(n)} \to 0$ in $c_A$ by Theorem 5(iii), so that $A e^{(n)} \to 0$ in $c$ by Theorem 4.4(c) of [24]. It follows that

$$\lim_{n \to \infty} \sup_{m} |a_{mn}| = 0.$$

(iii) follows from (ii) as in the proof of Proposition 8 of [5].

(iv) If $x \in ac$, then $x - (\text{Lim } x)e \in ac_0$ and, by Theorem 5(ii),

$$x - (\text{Lim } x)e = \sum_{k=1}^{\infty} (x_k - \text{Lim } x)e^{(k)} \in c_A.$$

Now $\lim_{A}$ is continuous on $c_A$ so that
lim x - (Lim x) lim e = \sum_{k=1}^{\infty} (x_k - \text{Lim}_k x) a_k.

Since x \in m, \sum_{k=1}^{\infty} a_k x_k converges and so
\[
\lim x = \sum_{k=1}^{\infty} a_k x_k + \text{Lim}_k x \left( \lim e - \sum_{k=1}^{\infty} a_k \right)
= \sum_{k=1}^{\infty} a_k x_k + \chi(A) \text{Lim}_k x.
\]

3. Consistency theorems. In [6] we used a technique involving the Orlicz-Pettis theorem on unconditional convergence of series to obtain a new proof of the Mazur-Orlicz-Brudno consistency theorem. In this section we apply the same basic technique to derive similar consistency theorems for almost convergence; the details, however, are much more difficult than those in [6] and we shall need considerable preparation before coming to our main results (Theorems 6 and 8).

We begin by introducing an idea which may be of some interest in a more general setting; we say that a sequence \( \{x^{(n)}\}_{n=1}^{\infty} \) is superconvergent to \( x \) (in a locally convex space \( E \)) if \( \{x^{(n)}\}_{n=1}^{\infty} \) converges to \( x \) and
\[
\sum_{k=1}^{\infty} (x_{n_k} - x_{n_k-1})
\]
converges in \( E \) for every increasing sequence \( \{n_k\}_{k=1}^{\infty} \) of positive integers.

Our first result is elementary and its proof is omitted.

**Lemma 3.** Every subsequence of a superconvergent sequence is superconvergent.

The validity of the next result is one of the main reasons for studying superconvergence.

**Lemma 4.** Let \( E \) be a locally convex space with dual \( E' \). If a sequence \( \{x^{(n)}\}_{n=1}^{\infty} \) superconverges in the weak topology \( \sigma(E, E') \), then \( \{x^{(n)}\}_{n=1}^{\infty} \) converges in the topology \( \lambda(E, E') \) of uniform convergence on \( \sigma(E', E) \)-compact sets.

**Proof.** Direct application of the general Orlicz-Pettis theorem (see [6], [15] or [21]).

**Lemma 5.** Let \( E \) be a Fréchet space and suppose that \( x^{(n)} \to x \) in \( E \). Then there is a subsequence \( \{z^{(n)}\}_{n=1}^{\infty} \) of \( \{x^{(n)}\}_{n=1}^{\infty} \) that superconverges to \( x \).

**Proof.** Suppose that \( \{p_k\}_{k=1}^{\infty} \) is an increasing sequence of seminorms defining the topology on \( E \). Choose an increasing sequence \( \{n_k\}_{k=1}^{\infty} \) of positive integers so that
\[
p_k(x - x^{(n)}) \leq 1/2^k
\]
whenever \( n \geq n_k \). Putting \( z^{(k)} = x^{(n_k)}, k = 1, 2, \ldots, \) it is easy to see that
\[
\sum_{k=1}^{\infty} (z^{(k)} - z^{(k-1)})
\]
converges absolutely, so that \( \{z^{(k)}\}_{k=1}^\infty \) superconverges to \( x \) in \( E \).

**Lemma 6.** \( x^{(n)} \to x \sigma(m, l) \) if and only if \( x^{(n)} \to x \sigma(m, q) \) and \( \sup_n \|x^{(n)}\|_\infty < \infty \).

**Proof.** A simple compactness argument (cf. [6, Lemma 3]). Alternatively, a neat proof may be given by using Lebesgue's dominated convergence theorem.

**Lemma 7.** If \( x^{(n)} \to 0 \sigma(c_0, l) \), then there exists a subsequence \( \{z^{(n)}\}_{n=1}^\infty \) of \( \{x^{(n)}\}_{n=1}^\infty \) such that

\[
\left\| \frac{1}{n} (z^{(1)} + z^{(2)} + \cdots + z^{(n)}) \right\|_\infty \to 0.
\]

**Proof.** In view of Lemma 6 the hypotheses are equivalent to

\[
\begin{align*}
\lim_{j \to \infty} x^{(n)}_j &= 0 \quad (n = 1, 2, \ldots), \\
\lim_{n \to \infty} x^{(n)}_j &= 0 \quad (j = 1, 2, \ldots),
\end{align*}
\]
and

\[
\sup_{n, j} |x^{(n)}_j| = M < \infty.
\]

We choose increasing sequences \( \{s_m\}_{m=1}^\infty \) and \( \{t_m\}_{m=1}^\infty \) of positive integers as follows. Let \( s_1 = 1, \ t_0 = 0 \), and suppose that \( s_1, \ldots, s_m \) and \( t_1, \ldots, t_m-1 \) have been chosen. Using (7), choose \( t_m > t_{m-1} \) so that

\[
\max_{1 \leq n \leq s_m} |x^{(n)}_j| \leq 2^{-m}
\]
whenever \( j > t_m \). Next, using (8), choose \( s_{m+1} > s_m \) so that

\[
|x^{(n+1)}_{s_{m+1}}| \leq 2^{-m}
\]
whenever \( 1 \leq j \leq t_n \).

If \( t_m < j \leq t_{m+1} \) and \( m \geq 1 \), then

\[
|x^{(n)}_j + x^{(n+1)} + \cdots + x^{(n+1)}_j| \leq n \cdot 2^{-m - 1} + 2^{-m} + \sum_{k=m+1}^{n-1} 2^{-k}
\]

if \( n \leq m \) by (10),

\[
\leq m \cdot 2^{-m} + |x^{(n+1)}| + \sum_{k=m+1}^{n-1} 2^{-k}
\]

if \( n > m \) by (10) and (11),

\[
\leq M + 1 \quad \text{by (9)}.
\]

Consequently, putting \( z^{(n)} = x^{(n)}_j, \ n = 1, 2, \ldots, \) we have

\[
\left\| \frac{z^{(1)} + z^{(2)} + \cdots + z^{(n)}}{n} \right\|_\infty \leq \frac{(M + 1)/n \to 0 \quad \text{as} \quad n \to \infty.}
\]
Lemma 7 says that in the Banach space $c_0$ every weakly convergent sequence has a subsequence whose arithmetic means converge in norm. This property, the so-called Banach-Saks property, is also known to hold for the spaces $l^p$ and $L^p(0, 1)$ (see [3]). We remark here that not every Banach space has this property.

Lemma 8. Let $x^{(n)} \in c_0$, $n = 1, 2, \ldots$, and suppose that $x^{(n)} \rightarrow x$ in $\sigma(m, l)$. Then there exists a subsequence $\{z^{(n)}\}_{n=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ that is superconvergent to $x$ in $\sigma(m, l)$.

Proof. Without loss of generality we may assume that $x = 0$. The hypotheses are then the same as in Lemma 7 and we may choose $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ as before so that (9) and (10) are satisfied. It is easily seen that

$$\sup_j \sum_{n=1}^{\infty} |x_j^{(s_n)} - x_j^{(t_n - 1)}| < \infty,$$

and so, in view of Lemma 6, we may take $z^{(n)} = x^{(s_n)}$, $n = 1, 2, \ldots$.

Lemma 9. If $x \in ac_0$ and $\{s_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers then the sequence

$$y^{(n)} = \frac{1}{n} (p_1 x + p_2 x + \cdots + p_n x)$$

is superconvergent to $x$ in $\sigma(ac_0, l)$.

Proof. We define

$$Q_0 = 0; \quad Q_n = \frac{1}{n} (p_1 + p_2 + \cdots + p_n), \quad n = 1, 2, \ldots;$$

$$R_n = Q_n - Q_{n-1}.$$

For any finite subset $M$ of the positive integers $\mathbb{Z}$ we write

$$S_M = \sum_{k \in M} R_k = \sum_{k=1}^{\infty} \delta_M(k) R_k$$

where $\delta_M$ is the characteristic function of $M$. We show that the collection $\{S_M: M$ is a finite subset of $\mathbb{Z}\}$ is $\tau(ac_0, l) \rightarrow \tau(ac_0, l)$-equicontinuous. Since

$$\sum_{k=1}^{\infty} (S_M f)_k x_k = \sum_{k=1}^{\infty} f_k(S_M x)_k,$$

we must show that if $C$ is $\sigma(l, ac_0)$-compact then $S(C) = \{S_M f: f \in C, M \subset \mathbb{Z}\}$ is $\sigma(l, ac_0)$-relatively compact.

For $x \in l$, we have

$$(Q_n x)_p = (1 - k/n)x_p \quad \text{if } s_k < p \leq s_{k+1}, \ 1 \leq k \leq n,$$

$$= 0 \quad \text{if } p > s_n,$$

and
\[(R_n x)_p = \frac{k}{n(n-1)} x_p \quad \text{if } s_k < p < s_{k+1}, \ 1 \leq k \leq n - 1,\]
\[= 0 \quad \text{if } p > s_n,\]
so that
\[(S_M x)_p = \left\{ \sum_{s_{k+1}}^{s_k} \frac{k}{n(n-1)} \delta_M(n) \right\} x_p \quad \text{if } s_k < p \leq s_{k+1}.\]

Now if \(C\) is \(o(l, aco)\)-relatively compact, then by Corollary 2 to Theorem 2, we have
\[
\sup_{x \in C} \|x\| = K < \infty,
\]
and
\[
\sup_{x \in C} \sum_{p=n}^{\infty} |x_p - x_{p+1}| = \varepsilon_n \to 0.
\]

Now
\[
|(S_M x)_p| \leq |x_p| \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \leq |x_p|
\]
so that \(\|S_M x\| \leq K\). If \(s_k < p < s_{k+1}\),
\[
(S_M x)_p - (S_M x)_{p+1} = \left( \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(k) \right) (x_p - x_{p+1})
\]
so that
\[
|(S_M x)_p - (S_M x)_{p+1}| \leq |x_p - x_{p+1}|;
\]
while, if \(p = s_k\),
\[
(S_M x)_p - (S_M x)_{p+1} = \left\{ \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)} \delta_M(n) \right\} x_p - \left\{ \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(n) \right\} x_{p+1}
\]
\[= \left\{ \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)} \delta_M(n) \right\} (x_p - x_{p+1}) - \left\{ \sum_{n=k+1}^{\infty} \frac{1}{n(n-1)} \delta_M(n) \right\} x_{p+1}
\[+ \frac{1}{k} \delta_M(k) x_{p+1},
\]
so that
\[
|(S_M x)_p - (S_M x)_{p+1}| \leq |x_p - x_{p+1}| + \frac{1}{k} |x_{p+1}|.
\]
Consequently, if \(s_k \leq n < s_{k+1}\),
It follows from Corollary 2 to Theorem 2 that $S(C)$ is indeed $\sigma(l, a_0)$-compact and so the collection $\{S_M : M \subseteq \mathbb{Z}\}$ is equicontinuous on $(a_0, \tau(a_0, 1))$. In particular, if $N$ is an infinite subset of $\mathbb{Z}$, the operators

$$
\sum_{k=1}^{n} \delta_N(k) R_k \quad (n = 1, 2, \ldots)
$$

are equicontinuous, and so, since $a_0$ is $\tau(a_0, 1)$-complete (Theorem 4(i)) the set of $x \in a_0$ for which $\sum_{k=1}^{\infty} \delta_N(k) R_k x$ converges is closed. However if $x \notin a_0$ this is clearly so, and so we conclude for all $x \in a_0$ and all $N \subseteq \mathbb{Z}$ that $\sum_{k=1}^{\infty} \delta_N(k) R_k x$ converges. Hence the sequence $\{Q_n x\}_{n=1}^{\infty}$ superconverges in $(a_0, \tau(a_0, 1))$.

**Lemma 10.** Let $x(n) \in c_0$, $n = 1, 2, \ldots$, and suppose that $x(n) \to x$ $\sigma(a_0, l)$. Then there exists a subsequence $\{z(n)\}_{n=1}^{\infty}$ of $\{x(n)\}_{n=1}^{\infty}$ such that some subsequence $\{w(n)\}_{n=1}^{\infty}$ of $\{(z(1) + z(2) + \cdots + z(n))/n\}_{n=1}^{\infty}$ superconverges to $x$ in $\sigma(a_0, l)$.

**Proof.** Since $P_n x \to x$ $\sigma(a_0, l)$, we have

$$
x(n) - P_n x \to 0 \quad \sigma(c_0, l).
$$

By Lemma 7 we may take a subsequence $\{x(n)\}_{n=1}^{\infty} = \{w(n)\}_{n=1}^{\infty}$ of $\{x(n)\}_{n=1}^{\infty}$ such that

$$
\left\| \frac{1}{n} (z(1) + \cdots + z(n)) - \frac{1}{n} (P_n x + \cdots + P_n x) \right\|_\infty \to 0.
$$

Taking a subsequence again, we may suppose that, for each integer $n$,

$$
\left\| \frac{1}{m_n} (z(1) + \cdots + z(m_n)) - \frac{1}{m_n} (P_n x + \cdots + P_n x) \right\|_\infty \leq \frac{1}{2^n},
$$

so that the sequence

$$
\left\{ \frac{1}{m_n} (z(1) + \cdots + z(m_n)) - \frac{1}{m_n} (P_n x + \cdots + P_n x) \right\}_{n=1}^{\infty}
$$

is superconvergent to 0 in $c_0$. However, by Lemmas 3 and 9, the sequence

$$
\left\{ \frac{1}{m_n} (P_n x + \cdots + P_n x) \right\}_{n=1}^{\infty}
$$

is superconvergent to $x$ in $\sigma(a_0, l)$. Hence, with
\[ w(n) = \frac{1}{m_n} (z^{(1)} + \cdots + z^{(m_n)}) \quad (n = 1, 2, \ldots), \]

\( \{w(n)\}_{n=1}^{\infty} \) is superconvergent to \( x \) in \( \sigma(ac_0, l) \).

Before stating our next result we recall the following notation. For an infinite matrix \( A \), \( ac_A \) denotes the set

\[ ac_A = \{ x \in \omega : Ax \in ac \}. \]

If \( x \in ac_A \), we write \( \text{Lim}_A x \) in place of \( \text{Lim}(Ax) \), and denote by \( (ac_0)_A \) the subspace of \( (ac)_A \) on which \( \text{Lim}_A \) vanishes.

Theorem 6. Let \( A \) be a matrix such that

(i) \( \sup \sum_{j=1}^{\infty} |a_{ij}| < \infty \), and

(ii) \( \lim_{i \to \infty} a_{ij} = 0 \) for \( j = 1, 2, \ldots \) Then \( l \) is \( \sigma(l, (ac_0)_A \cap m) \)-sequentially complete.

Proof. Let \( x \in (ac_0)_A \cap m \) be fixed. We construct a sequence \( \{x^{(n)}\}_{n=1}^{\infty} \) of elements of \( \varphi \) such that \( \{x^{(n)}\}_{n=1}^{\infty} \) superconverges to \( x \) in \( \sigma((ac_0)_A \cap m, l) \). To do this, we first observe that

\[ A P_n x \to Ax \quad \sigma(\omega, \varphi). \]

Condition (i) ensures that \( A : m \to m \) is continuous and hence

\[ \|A P_n x\|_\infty \leq \|A\| \|x\|_\infty. \]

Lemma 6 gives

\[ A P_n x \to Ax \quad \sigma(m, l). \]

Now condition (ii) implies that \( A P_n x \in A(\varphi) \subset c_0 \) so we may apply Lemma 10 to deduce the existence of a sequence \( \{v^{(k)}\}_{k=1}^{\infty} \) such that \( \{Av^{(k)}\}_{k=1}^{\infty} \) superconverges to \( Ax \) in \( \sigma(ac_0, l) \) and \( \{v^{(k)}\}_{k=1}^{\infty} \) takes the form

\[ v^{(k)} = \frac{1}{m_k} (u^{(l)} + \cdots + u^{(m_k)}) \]

where \( \{u^{(k)}\}_{k=1}^{\infty} \) is some subsequence of \( \{P_n x\}_{n=1}^{\infty} \). Clearly we have

\[ \sup_k \|v^{(k)}\|_\infty \leq \|x\|_\infty \]

and \( v^{(k)} \to x \quad \sigma(\omega, \varphi) \) so that \( v^{(k)} \to x \quad \sigma(m, l) \) by Lemma 6.

Furthermore, since \( Av^{(k)} \to Ax \) in \( \omega \), we have

\[ v^{(k)} \to x \quad \text{in } \omega_A. \]

We now apply Lemmas 3, 5 and 8 to obtain a subsequence \( \{z^{(n)}\}_{n=1}^{\infty} \) of \( \{v^{(n)}\}_{n=1}^{\infty} \) such that \( \{z^{(n)}\}_{n=1}^{\infty} \) superconverges to \( x \) in both \( \omega_A \) and \( \sigma(m, l) \); it is also clear that
\{Az^{(n)}\}_{n=1}^{\infty} \text{ superconverges in } (ac_0, \sigma(ac_0, l))$. Now suppose that \{\varepsilon_n\}_{n=1}^{\infty} is a sequence taking only the values 1 and 0; for each \(k\) let

\[ y_k = \sum_{n=1}^{\infty} \varepsilon_n (z^{(n)}_k - z^{(n-1)}_k) \quad \text{(where } z^{(0)} = 0). \]

Since \{z^{(n)}\}_{n=1}^{\infty} superconverges in both \(\omega_A\) and \((m, \sigma(m, l))\), the series

\[ \sum_{n=1}^{\infty} \varepsilon_n (z^{(n)} - z^{(n-1)}) \]

converges to \(y\) in both \(\omega_A\) and \((m, \sigma(m, l))\); therefore \(y \in m \cap \omega_A\). Now \(A : \omega_A \to \omega\) is continuous [24, Theorem 4.4(c)] and so

\[ Ay = \sum_{n=1}^{\infty} \varepsilon_n (Az^{(n)} - Az^{(n-1)}) \quad \sigma(\omega, \varphi). \]

However, \(\{Az^{(n)}\}_{n=1}^{\infty} \text{ superconverges in } (ac_0, \sigma(ac_0, l))\) so that

\[ Ay = \sum_{n=1}^{\infty} \varepsilon_n (Az^{(n)} - Az^{(n-1)}) \quad \sigma(ac_0, l), \]

and \(Ay \in ac_0\), i.e., \(y \in (ac_0)_A\). Thus \{z^{(n)}\}_{n=1}^{\infty} superconverges to \(x\) in \(((ac_0)_A \cap m, \sigma((ac_0)_A \cap m, l))\).

We now repeat the argument used in the proof of Theorem 3 of [6]. Consider the topology \(\lambda((ac_0)_A \cap m, l)\) on \((ac_0)_A \cap m\) of uniform convergence on the \(\sigma(l, (ac_0)_A \cap m)-\text{compact subsets of } l\); by Lemma 4 we have

\[ z^{(n)} \to x \quad \lambda((ac_0)_A \cap m, l). \]

Suppose now that \(\psi\) is a linear functional on \((ac_0)_A \cap m\) whose restrictions to \(\lambda((ac_0)_A \cap m, l)-\text{precompact sets are } \lambda\text{-continuous}\). Then \(\psi\) is \(\lambda\)-sequentially continuous and since \(\lambda \leq \beta((ac_0)_A \cap m, l) \leq \beta(c_0, l)\), \(\psi\) is \(\|\cdot\|_{\infty}\)-continuous on \(c_0\) so that

\[ \sum_{j=1}^{\infty} |f_j| < \infty \]

where \(\psi(e^{(j)}) = f_j\) \((j = 1, 2, \ldots)\). Now

\[ \psi(z^{(n)}) = \sum_{j=1}^{\infty} z^{(n)}_j f_j \]

since \(z^{(n)} \in \varphi\), and

\[ \lim_{n \to \infty} \sum_{j=1}^{\infty} z^{(n)}_j f_j = \sum_{j=1}^{\infty} x_j f_j \]

since \(z^{(n)} \to x \quad \sigma(m, l)\). Consequently, for each \(x \in (ac_0)_A \cap m\), we have

\[ \psi(x) = \sum_{j=1}^{\infty} x_j f_j. \]
It follows (as in the proof of Theorem 3 of [6]) by Grothendieck’s completeness theorem that the topology $\rho$ on $l$, of uniform convergence on $\lambda$-precompact subsets of $(ac_0)_A \cap m$, must be complete. Furthermore, $\rho$ defines the same convergent and Cauchy sequences as $\sigma(l, (ac_0)_A \cap m)$ so that $l$ is $\sigma(l, (ac_0)_A \cap m)$-sequentially complete.

We now come to our first consistency theorem.

**Theorem 7.** Let $A$ and $B$ be regular matrices and suppose that $ac_A \cap m \subseteq c_B$. Then $\lim_B x = \lim_A x$ whenever $x \in ac_A \cap m$.

**Proof.** Since $A$ is regular, the conditions of Theorem 6 are satisfied. Let $b^{(n)} \in l$, $n = 1, 2, \ldots$, be defined by

$$b_k^{(n)} = b_{nk} \quad (k = 1, 2, \ldots),$$

(so that $b^{(n)}$ is the $n$th row of $B$). Since $(ac_0)_A \cap m \subseteq c_B$, we have

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} b_k^{(n)} x_k$$

exists whenever $x \in (ac_0)_A \cap m$. Hence $(b^{(n)})_{n=1}^{\infty}$ is $\sigma(l, (ac_0)_A \cap m)$-Cauchy and so converges, say $b^{(n)} \to b$, by Theorem 6. Clearly

$$b_k = \lim_{k \to \infty} b_{nk} = 0,$$

so that $b^{(n)} \to 0 \sigma(l, (ac_0)_A \cap m)$.

Now, if $x \in (ac)_A \cap m$, then $x - (\lim_A x)e \in (ac_0)_A \cap m$, and so

$$\lim_B \left( x - \left( \lim_A x \right) e \right) = \lim_{n \to \infty} \sum_{k=1}^{\infty} b_k^{(n)} \left( x_k - \lim_A x \right) = 0,$$

i.e. $\lim_B x = \lim_A x$.

When $A$ is the identity matrix, Theorem 7 reduces to the following.

**Corollary.** Let $B$ be a regular matrix with $ac \subseteq c_B$. Then $\lim_B x = \lim x$ whenever $x \in ac$.

This special result may also be derived from Corollary 2 to Theorem 5 and was first obtained by Lorentz [13].

Before stating our next result let us recall the following notation. If $E$ is an FK-space containing $c_0$, then we write

$$W_E = \{ x \in E : P_n x \to x \text{ weakly in } E \}$$

and

$$S_E = \{ x \in E : P_n x \to x \text{ in } E \}.$$

**Theorem 8.** Let $E$ be an FK-space containing $c_0$. Then $l$ is sequentially complete.
under both the topologies \( \sigma(l, W_E \cap ac_0) \) and \( \sigma(l, S_E \cap ac_0) \).

**Proof.** As with Theorem 6, the proof hinges on ideas developed in Theorem 3 of [6].

Let \( x \in W_E \cap ac_0 \) be fixed: by Theorem 2 of [6] (see also [20]) there is a sequence \( \{u(n)\}_{n=1}^{\infty} \) of elements of \( \varphi \) with

\[
\tau - \lim_{n \to \infty} u(n) = x
\]

and

\[
\sup_n \|u(n)\|_\infty \leq \|x\|_\infty,
\]

where \( \tau \) denotes the FK-topology on \( E \). By Lemma 6,

\[
\lim_{n \to \infty} u(n) = x \quad \sigma(ac_0, l)
\]

and so, by Lemma 10, there exists a sequence \( \{v(n)\}_{n=1}^{\infty} \) of arithmetic means of a subsequence of \( \{u(n)\}_{n=1}^{\infty} \), such that \( \{v(n)\}_{n=1}^{\infty} \) superconverges to \( x \) in \( \sigma(ac_0, l) \); clearly

\[
\tau - \lim_{n \to \infty} v(n) = x.
\]

By using Lemmas 3 and 5 we may select a subsequence \( \{z(n)\}_{n=1}^{\infty} \) which superconverges to \( x \) in both \( \tau \) and \( \sigma(ac_0, l) \). Thus every subseries of

\[
\sum_{n=1}^{\infty} (z^{(n)} - z^{(n-1)})
\]

converges in \( E \cap ac_0 \); i.e., if \( \epsilon_n = 0 \) or 1 for all \( n \) and

\[
y = \sum_{n=1}^{\infty} \epsilon_n (z^{(n)} - z^{(n-1)}) \quad (\text{where } z^{(0)} = 0),
\]

then \( y \in E \cap ac_0 \). Since this series converges in \( \sigma(m, l) \), we have

\[
\sup_k \left\| \sum_{n=1}^{k} \epsilon_n (z^{(n)} - z^{(n-1)}) \right\|_\infty < \infty,
\]

and since the series converges in \( \tau \) we have \( y \in W_E \) by Theorem 2 of [6]. Thus \( \{z^{(n)}\}_{n=1}^{\infty} \) superconverges to \( x \) in \( \sigma(W_E \cap ac_0, l) \), and the remaining details follow those of Theorem 6 (or Theorem 3 of [6]).

For the second half of the theorem we observe that [10, pp. 1015–1016] \( S_E = W_F \), where \( F \) is the FK-space defined as follows.

\[
F = \{ x \in E : \{P_n x\}_{n=1}^{\infty} \text{ is } \tau\text{-bounded} \}
\]

with the topology given by the seminorms

\[
\nu(x) = \sup_n \nu(P_n x) \quad (x \in F)
\]

for each \( \tau \)-continuous seminorm \( \nu \).

Our next result may be thought of as a generalized consistency theorem.
Theorem 9. Let $E$ be an FK-space containing $c_0$ and let $F$ be an FK-space containing no (closed) subspace isomorphic to $m$. If $W_E \cap ac_0 \subseteq F$, then $W_E \cap ac_0 \subseteq W_F$.

Proof. As in the proof of Theorem 8 we can show that each $x \in W_E \cap ac_0$ can be written in the form $x = \sum_{n=1}^{\infty} x^{(n)}$ where $x^{(n)} \in \varphi$, $n = 1, 2, \ldots$, and the convergence is $\sigma(W_E \cap ac_0, l)$-subseries. This observation enables us to replace $W_E \cap m$ in the statement of Proposition 1 of [8] by $W_E \cap ac_0$; but then the present result follows just as in the proof of Theorem 2 of [8].

In particular, it should be noted that Theorem 9 remains valid when $F$ is a separable FK-space.

Theorem 10. Let $A$ and $B$ be regular matrices and suppose that $ac \cap c_A \subseteq c_B$. Then there exists a constant $\alpha$ such that

$$\lim_{B} x = \alpha \lim_{A} x + (1 - \alpha) \lim_{A} x$$

whenever $x \in ac \cap c_A$.

Proof. Since $A$ is regular we have, by Theorem 3.6 of [25], $(c_0)_A \cap ac_0 = W_{c_A} \cap ac_0$. Now $c_B$ is separable ([14], [4]) so that

$$(c_0)_A \cap ac_0 \subseteq W_{c_B} \cap ac_0 = (c_0)_B \cap ac_0$$

by Theorem 9. Hence $\lim_{A} x = \lim_{B} x = 0$ implies that $\lim_{B} x = 0$, and so

$$\lim_{B} x = \alpha \lim_{A} x + \beta \lim_{A} x$$

whenever $x \in ac \cap c_A$. However $1 = \lim_{B} e = \alpha + \beta$ and the desired conclusion follows.

Theorem 11. Let $A$ and $B$ be conservative matrices and suppose that $ac \cap c_A \subseteq c_B$. Then there exist constants $\alpha, \beta$ such that

(i) $\lim_{B} x - \sum_{j=1}^{\infty} b_j x_j = \alpha (\lim_{A} x - \sum_{j=1}^{\infty} a_j x_j) + \beta \lim_{A} x$ whenever $x \in ac \cap c_A$, and

(ii) $\chi(B) = \alpha \chi(A) + \beta$.

Proof. This is a simple extension of Theorem 10; we observe that

$$W_{c_A} \cap ac_0 = \left\{ x : \lim_{A} x = \sum_{j=1}^{\infty} a_j x_j \right\} \cap ac_0$$

and apply the same method.

Corollary 1. Let $A$ be conull and $B$ be regular and suppose that $ac \cap c_A \subseteq c_B$. Then there exists a constant $\alpha$ such that

$$\lim_{B} x = \lim_{A} x + \alpha \left( \lim_{A} x - \sum_{j=1}^{\infty} a_j x_j \right)$$
whenever \( x \in ac \cap c_A \).

**Corollary 2.** Let \( A \) be regular and \( B \) be conull and suppose that \( ac \cap c_A \subseteq c_B \). Then there exists a constant \( \alpha \) such that

\[
\lim_{\substack{\to \vphantom{B} \scriptstyle B \\
\vphantom{B} \scriptstyle A \to}} x = \alpha \left( \lim_{\to \scriptstyle A} x - \lim_{\to \scriptstyle B} x \right) + \sum_{j=1}^{\infty} b_j x_j
\]

whenever \( x \in ac \cap c_A \).

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