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CONSISTENCY THEOREMS FOR ALMOST CONVERGENCE

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ABSTRACT. The concept of almost convergence of a sequence of real or complex numbers was introduced by Lorentz, who developed a very elegant theory. The purpose of the present paper is to continue Lorentz's investigations and obtain consistency theorems for almost convergence; this is achieved by studying certain locally convex topological vector spaces.

1. **Introduction** The concept of almost convergence of a sequence of real or complex numbers was introduced, after an idea of Banach, by Lorentz [13] who developed a very elegant theory. Further studies of almost convergence and its relationship with general summability methods have since been carried out in [12], [17] and [19]. The purpose of the present paper is to obtain consistency theorems for almost convergence by studying certain locally convex topological vector spaces.

We adopt the following notation:

ω denotes the space of all scalar (real or complex) sequences;

$e, e^{(k)} \in \omega$ are given by

$$e = (1, 1, \ldots),$$

$$e^{(k)} = (0, \ldots, 0, 1, 0, \ldots) \quad \text{with the one in the } k\text{th position;}$$

φ is the linear span of $\{e^{(k)}: k = 1, 2, \ldots\}$;

$m = \{x \in \omega: \|x\|_\infty = \sup_j |x_j| < \infty\}$;

$c = \{x \in \omega: \lim x = \lim_{j\to\infty} x_j \text{ exists}\}$;

$c_0 = \{x \in \omega: \lim x = 0\}$;

$l = \{x \in \omega: \|x\|_1 = \sum_{j=1}^\infty |x_j| < \infty\}$;

$bv = \{x \in \omega: \|x\|_{bv} = \sum_{j=1}^\infty |x_j - x_{j+1}| + \lim_{j\to\infty} |x_j| < \infty\}$;

$bv_0 = bv \cap c_0$;

$bs = \{x \in \omega: \|x\|_{bs} = \sup_n |\sum_{j=1}^n x_j| < \infty\}$.

A vector subspace of ω is called a *sequence space*. If E is a sequence space with a locally convex topology τ then (E, τ) is a *K-space* provided that the linear functionals

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$$x \to x_j \qquad (j = 1, 2, \ldots)$$

are continuous on E. If, in addition, (E, τ) is complete and metrizable (respectively normable) then (E, τ) is called an *FK-space* (respectively *BK-space*). For $x \in \omega$ we write

$$P_n x = (x_1, x_2, \ldots, x_n, 0, \ldots).$$

(E, τ) is an *AK-space* if $P_n x$ converges to x for every $x \in E$.

If E and F are sequence spaces containing φ such that the bilinear form $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j$ converges whenever $x \in E$ and $y \in F$, then topologies of the dual pairing $\langle E, F\rangle$ provide examples of K-space topologies. In particular, we shall be interested in the weak topology $\sigma(E, F)$, the Mackey topology $\tau(E, F)$ and the strong topology $\beta(E, F)$ (following the notation of Schaefer [18]).

We shall also consider matrix maps and matrix methods of limitation. Let $A = (a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix with scalar entries; we denote by ω_A the set of $x \in \omega$ such that $\sum_{j=1}^{\infty} a_{ij} x_j$ converges for each i. For $x \in \omega_A$ we write

$$(Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j$$

so that $A\colon \omega_A \to \omega$ is a linear map. If E is a sequence space,

$$E_A = \{x \in \omega_A : Ax \in E\}.$$

If E is an FK-space then Zeller [24, Theorem 4.10(a)] has shown that E_A is also an FK-space when topologized by means of the seminorms:

$$x \to x_j \qquad (j = 1, 2, \ldots),$$

$$x \to \sup_n \left| \sum_{j=1}^{n} a_{ij} x_j \right| \qquad (i = 1, 2, \ldots),$$

and

$$x \to q(Ax),$$

where q runs through the continuous seminorms on E. A matrix A defines a method of limitation, viz: if $x \in c_A$, we write $\lim_A x = \lim(Ax)$. A is called *conservative* if $c \subseteq c_A$ or, equivalently (see [26]),

(1) $$\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty,$$

(2) $$\lim_{i\to\infty} a_{ij} = a_j \quad \text{exists} \qquad (j = 1, 2, \ldots),$$

and

(3) $$\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} \quad \text{exists.}$$

We then write

$$\chi(A) = \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} - \sum_{j=1}^{\infty} a_j,$$

and say that A is *conull* when $\chi(A) = 0$. A is called *regular* if $\lim_A x = \lim x$ whenever $x \in c$; for regularity it is necessary and sufficient (see [26]) to have (1), (2) and (3) with $a_j = 0$ $(j = 1, 2, \dots)$ and $\chi(A) = 1$.

2. **Properties of almost convergence.** In this section we develop the theory of almost convergence, deriving the original characterization of almost convergent sequences given by Lorentz [13], as well as several other useful properties of the space ac_0 (to be defined below). Since our approach is from the viewpoint of functional analysis, and therefore differs slightly from Lorentz's, we shall give a complete development of the subject.

The linear functional lim on c has norm one, i.e.

$$|\lim x| \leq \|x\|_\infty \qquad (x \in c)$$

and so by the Hahn-Banach theorem possesses extensions L, of norm one, defined on all of m. We call such a functional L an *extended limit*. If $x \in \omega$, we write

$$Tx = \{x_{n+1}\}_{n=1}^{\infty}$$

and say that an extended limit L is a *Banach limit* if

$$L(Tx) = L(x) \qquad (x \in m).$$

(Some authors insist that a Banach limit should also satisfy $L(x) \geq 0$ whenever $x_n \geqslant 0$ for all n, or even $\lim_{n\to\infty} \sup x_n \geq L(x) \geq \lim_{n\to\infty} \inf x_n$. It is clear, however, that any extended limit has these properties.)

The existence of Banach limits was proved by Banach [2]; another proof can be found in Theorem 1 below. If $x \in m$ is such that for every Banach limit L, $L(x)$ assumes a common value, then we write Lim x for this value, and say that *x is almost convergent to* Lim x. The set of almost convergent sequences is denoted by ac, and the subset $\{x \in ac: \lim x = 0\}$ is denoted by ac_0. ac_0 is a hyperplane in ac and $ac = ac_0 + \{e\}$; it is also easy to show that ac and ac_0 are closed supspaces of m. Our first result (Theorem 1) characterizes these spaces.

Lemma 1. *If L is a continuous linear functional on m with*
(i) $\|L\| = 1$,
(ii) $L(e) = 1$, *and*
(iii) $L(bs) = 0$,
then L is a Banach limit.

Proof. Since $\varphi \subseteq bs$, it follows from (iii) that $L(\varphi) = 0$, and by continuity that

$L(c_0) = 0$; therefore L is an extended limit. Moreover, for $x \in m$, $x - Tx \in bs$ and so $L(x) = L(Tx)$.

Lemma 2. *If $x \in m \backslash c_0$, then there exists an extended limit L with $L(x) \neq 0$.*

Proof. Since $x \in m \backslash c_0$, we may choose an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that

$$\lim_{k \to \infty} x_{n_k} = \alpha \neq 0.$$

Define L by

$$Ly = \lim_{k \to \infty} y_{n_k}$$

where this limit exists, and extend L to m by the Hahn-Banach theorem.

Theorem 1 (Lorentz) [13]. *$x \in \omega$ is almost convergent (to α) if and only if*

$$\lim_{p \to \infty} \frac{1}{p}(x_n + \cdots + x_{n+p-1}) = \alpha \tag{4}$$

uniformly in n.

Proof. Without loss of generality we may assume that $\alpha = 0$. Let $\{n_p\}_{p=1}^{\infty}$ be any increasing sequence of positive integers, and define the matrix map A: $m \to m$ by

$$(Ax)_p = \frac{1}{p}(x_{n_p} + \cdots + x_{n_p+p-1}) \qquad (x \in m).$$

Then we have $Ae = e$, $A(bs) \subset c_0$, $\|A\|_\infty = 1$.

If L is an extended limit, then, by Lemma 1,

$$LA \text{ is a Banach limit} \tag{5}$$

and so, for $x \in ac_0$, we have

$$L(Ax) = 0.$$

By Lemma 2 we have $Ax \in c_0$ so that

$$\lim_{p \to \infty} \frac{1}{p}(x_{n_p} + \cdots + x_{n_p+p-1}) = 0.$$

Since this is true for any sequence $\{n_p\}_{p=1}^{\infty}$, we conclude that

$$\lim_{p \to \infty} \sup_n \left| \frac{1}{p}(x_n + \cdots + x_{n+p-1}) \right| = 0,$$

which is (4).

Conversely, (4) implies that

$$\lim_{p\to\infty}\left\|\frac{1}{p}(Tx+\cdots+T^p x)\right\|_\infty = 0.$$

Thus, for any Banach limit L, we have $L(x) = 0$, so that $x \in ac_0$.

We remark that (5) gives what is perhaps the easiest proof of the existence of Banach limits. Banach's original proof [2] also uses the Hahn-Banach theorem, but involves a rather sophisticated sublinear functional; Day's elegant proof [9, p.83], using fixed point theory, requires considerably more machinery.

Our next result, which follows at once from Theorem 1, shows that ac_0 and ac are "large" subspaces of m.

Corollary. *$(ac_0, \|\cdot\|_\infty)$ is a nonseparable BK-space.*

We now come to a series of results which relate various properties of ac_0 to those of more familiar sequence spaces.

Theorem 2. *If $\{x^{(n)}\}_{n=1}^\infty$ is a sequence of points in l, and $x \in l$, then the following conditions are equivalent*:

(i) *$\{x^{(n)}\}_{n=1}^\infty$ is $\sigma(l, ac_0)$-convergent to x*:

(ii) *$\{x^{(n)}\}_{n=1}^\infty$ is $\sigma(l, bs + c_0)$-convergent to x*;

(iii) *$\sup_n \|x^{(n)}\|_1 < \infty$ and $\lim_{n\to\infty}\|x^{(n)} - x\|_{bv} = 0$.*

Proof. Without loss of generality we may assume that $x = 0$.

(i) $\Rightarrow$ (ii) follows since $bs + c_0 \subset ac_0$.

(ii) $\Rightarrow$ (iii). If $x^{(n)} \to 0$ $\sigma(l, bs + c_0)$, then $x^{(n)} \to 0$ $\sigma(l, c_0)$ so that

$$\sup_n \|x^{(n)}\|_1 < \infty.$$

Also, $x^{(n)} \to 0$ $\sigma(l, bs)$ so that $x^{(n)} \to 0$ $\sigma(bv_0, bs)$; this is the weak topology on bv_0, and, since bv_0 is isomorphic to l, we may use Schur's theorem [2, p. 137] to deduce that

$$\lim_{n\to\infty}\|x^{(n)}\|_{bv} = 0.$$

(iii) $\Rightarrow$ (i). Let $f \in ac_0$ and $\varepsilon > 0$ be fixed. By Theorem 1 we may choose a positive integer p so that

$$\left\|\frac{1}{p}(Tf+\cdots+T^p f)\right\|_\infty < \varepsilon\Big/2\Big(1+\sup_n\|x^{(n)}\|_1\Big).$$

We then have, for every n,

$$\left|\sum_{k=1}^\infty \frac{1}{p}(f_{k+1}+\cdots+f_{k+p})x_k^{(n)}\right| \le \|x^{(n)}\|_1\left\|\frac{1}{p}(Tf+\cdots+T^p f)\right\|_\infty < \frac{\varepsilon}{2}.$$

Furthermore, fixing p, we may choose a positive integer N so that

$$\|x^{(n)}\|_{bv} < \frac{\varepsilon}{2(p+1)(1+\|f\|_\infty)}$$

whenever $n \geq N$.

Now

$$\left|\sum_{k=1}^{\infty}(f_{k+s}-f_k)x_k^{(n)}\right| = \left|\sum_{k=1}^{\infty} f_k(x_{k-s}^{(n)} - x_k^{(n)})\right| \qquad (\text{putting } x_m^{(n)} = 0 \text{ if } m \leq 0)$$

$$\leq \sum_{k=1}^{\infty} |f_k| \sum_{r=1}^{s} |x_{k-r+1}^{(n)} - x_{k-r}^{(n)}|$$

$$\leq s\|f\|_\infty(\|x^{(n)}\|_{bv} + |x_1^{(n)}|)$$

$$\leq 2s\|f\|_\infty\|x^{(n)}\|_{bv}.$$

Therefore

$$\left|\sum_{k=1}^{\infty}\frac{1}{p}(f_{k+1}+\cdots+f_{k+p})x_k^{(n)} - \sum_{k=1}^{\infty} f_k x_k^{(n)}\right| \leq \frac{1}{p}\frac{p(p+1)}{2}2\|f\|_\infty\|x^{(n)}\|_{bv}$$

$$< \frac{\varepsilon}{2} \quad \text{whenever } n \geq N.$$

Thus, for $n \geq N$, we have

$$\left|\sum_{k=1}^{\infty} f_k x_k^{(n)}\right| < \varepsilon,$$

i.e., $x^{(n)} \longrightarrow 0 \ \sigma(l, ac_0)$.

We remark that condition (iii) of Theorem 2 identifies sequential convergence in $\sigma(l, ac_0)$ with a two-norm topology. For details concerning this type of topology we refer the reader to [1], [6], [22] and [23].

Corollary 1. *l is sequentially complete under both the topologies $\sigma(l, ac_0)$ and $\sigma(l, bs + c_0)$.*

Proof. If $\{x^{(n)}\}_{n=1}^{\infty}$ is a $\sigma(l, bs + c_0)$-Cauchy sequence, the proof of Theorem 2 shows that $\{x^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in bv_0 and bounded in l. Since bv_0 is complete, there exists $x \in bv_0$ such that

$$\lim_{n\to\infty}\|x^{(n)} - x\|_{bv} = 0.$$

But this implies that

$$x_k = \lim_{n\to\infty} x_k^{(n)} \qquad (k = 1, 2, \ldots)$$

so that

$$\sum_{k=1}^{\infty}|x_k| \leq \sup_n \sum_{k=1}^{\infty}|x_k^{(n)}| < \infty,$$

and $x \in l$. It then follows from Theorem 2, (iii) $\Rightarrow$ (i), that $x^{(n)} \to 0$ $\sigma(l, ac_0)$, giving the desired result.

Corollary 2. *For a subset C of l, the following conditions are equivalent:*
(i) *C is $\sigma(l, ac_0)$-relatively compact;*
(ii) *C is $\sigma(l, bs + c_0)$-relatively compact;*
(iii) *C is $\|\cdot\|_1$-bounded and* $\lim_{n\to\infty} \sup_{x\in C} \|x - P_n x\|_{bv} = 0$.

Proof. A subset of a K-space is relatively compact if and only if it is relatively sequentially compact (see [10]) and hence Theorem 2 shows that (i) and (ii) are equivalent. Using the sequential completeness of l in the two-norm convergence defined in (iii) of Theorem 2, it is clear that (i) and (ii) are equivalent to "C is $\|\cdot\|_l$-bounded and $\|\cdot\|_{bv_0}$-relatively compact." However, by a general theorem on bases (see [16]) this is equivalent to (iii).

We note from Corollary 2 that the closed convex hull of a $\sigma(l, ac_0)$-compact set is also $\sigma(l, ac_0)$-compact (using (iii)); hence the Mackey topology, $\tau(ac_0, l)$, is the topology of uniform convergence on $\sigma(l, ac_0)$-compact sets.

We now turn to the relationship between ac_0 and bs.

Theorem 3. (i) $ac_0 = \overline{bs}$, *the closure of bs in m.*
(ii) *If $x \in bs + c_0$, then* $\sup_p \lim \sup_{n\to\infty} |x_{n+1} + \cdots + x_{n+p}| < \infty$.
(iii) $ac_0 \neq bs + c_0$.

Proof. (i) Clearly $\overline{bs} \subseteq ac_0$. Conversely, if $x \in ac_0$ and $\varepsilon > 0$ are given, we may choose a positive integer p so that

$$|x_{n+1} + \cdots + x_{n+p}| < p\varepsilon \qquad (n = 1, 2, \ldots).$$

In particular,

$$x_{mp+1} + \cdots + x_{(m+1)p} = p\delta_m \qquad (m = 0, 1, 2, \ldots) \tag{6}$$

where $|\delta_m| \leq \varepsilon$. Letting y be defined by

$$y_{mp+k} = x_{mp+k} - \delta_m \qquad (k = 1, 2, \ldots, p;\ m = 0, 1, 2, \ldots),$$

it is clear that $\|x - y\|_\infty \leq \varepsilon$; we complete the proof of (i) by showing that $y \in bs$.

Now

$$\sum_{i=1}^{mp+k} y_i = \sum_{n=0}^{m-1} \sum_{j=1}^{p} (x_{np+j} - \delta_n) + \sum_{i=1}^{k} x_{mp+i} - k\delta_m$$
$$= \sum_{i=1}^{k} x_{mp+i} - k\delta_m \quad \text{by (6)}.$$

Consequently

$$\left|\sum_{i=1}^{q} y_i\right| \leq p(\|x\|_\infty + \varepsilon)$$

for every q, and $y \in bs$.

(ii) If $x \in bs + c_0$, then $x = y + z$ for some $y \in bs$ and $z \in c_0$. Then

$$|x_{n+1} + \cdots + x_{n+p}| \leq |y_{n+1} + \cdots + y_{n+p}| + |z_{n+1} + \cdots + z_{n+p}|$$

so that

$$\limsup_{n\to\infty} |x_{n+1} + \cdots + x_{n+p}| = \limsup_{n\to\infty} |y_{n+1} + \cdots + y_{n+p}| \leq 2\|y\|_{bs},$$

giving the desired result.

(iii) By (ii) we may construct $x \in ac_0 \backslash (bs + c_0)$ directly; let

$$\begin{aligned} x_k &= 1 \qquad \text{if } k = 2^n + 2^m \text{ for } n \geq m \geq 1, \\ &= 0 \qquad \text{otherwise}. \end{aligned}$$

Then x does not satisfy (ii), yet it is easy to check that $x \in ac_0$.

It is interesting to note that $bs + c_0$ is a BK-space which is B-invariant in the sense of Garling [10], and $c_0 \subset bs + c_0 \subset m$, yet $bs + c_0$ is not closed in m.

Theorem 4. (i) *$(ac_0, \tau(ac_0, l))$ is a complete AK-space.*

(ii) *$\tau(bs + c_0, l)$ is the restriction of $\tau(ac_0, l)$ to $bs + c_0$ [so that $(ac_0, \tau(ac_0, l))$ is the completion of $(bs + c_0, \tau(bs + c_0, l))$].*

Proof. (i) If C is $\sigma(l, ac_0)$-relatively compact, then by (iii) of Corollary 2 to Theorem 2, the set $P(C) = \{P_n f: f \in C\}$ is $\sigma(l, ac_0)$-relatively compact. It follows that the operators $\{P_n: n = 1, 2, \ldots\}$ are $\tau(ac_0, l) \to \tau(ac_0, l)$-equicontinuous, so that the set

$$S = \{x \in ac_0: P_n x \to x \ \tau(ac_0, l)\}$$

is $\tau(ac_0, l)$-closed. However, $S \supset \varphi$ and φ is $\tau(ac_0, l)$-dense in ac_0 (since φ is $\sigma(ac_0, l)$-dense); hence $S = ac_0$, showing that $(ac_0, \tau(ac_0, l))$ is an AK-space.

To show that $(ac_0, \tau(ac_0, l))$ is complete we use Grothendieck's criterion [6, Proposition 1]. Let θ be a linear functional on l which is $\sigma(l, ac_0)$-continuous on each $\sigma(l, ac_0)$-compact set. Then $\theta(x^{(n)}) \to 0$ whenever $x^{(n)} \to 0 \ \sigma(l, ac_0)$. Consequently, from Theorem 2, θ is continuous in the two-norm topology. Using the standard characterization of the dual of a two-norm space [1, Theorem 4.2], it follows that θ lies in the closure of bs (the dual of $(l, \|\cdot\|_{bv})$) in m (the dual of $(l, \|\cdot\|_\infty)$). Hence, by Theorem 3(i), θ takes the form

$$\theta(x) = \sum_{k=1}^{\infty} f_k x_k,$$

where f is a fixed element from ac_0. It follows from Grothendieck's criterion that $(ac_0, \tau(ac_0, l))$ is complete.

(ii) This follows from Corollary 2 to Theorem 2.

Theorem 5. *Let E be a separable FK-space containing c_0 and bs. Then*
(i) *E contains ac_0;*
(ii) *$x \in ac_0$ implies that $P_n x \to x$ in E;*
(iii) *$e^{(n)} \to 0$ in E.*

Proof. (i) and (ii). The space $(bs + c_0, \tau(bs + c_0, l))$ is a Mackey space whose dual, l, is $\sigma(l, bs + c_0)$-sequentially complete by Corollary 1 to Theorem 2. Thus, by the main result of [11] (see also [7, Theorem 5]), the natural inclusion mapping: $bs + c_0 \to E$, which clearly has closed graph, must be continuous. If $x \in ac_0$, then by Theorem 4, $\{P_n x\}_{n=1}^{\infty}$ is Cauchy in $(bs + c_0, \tau(bs + c_0, l))$ and hence in E. Since E is complete, $\{P_n x\}_{n=1}^{\infty}$ converges in E, and its limit must be x since E is a K-space. This completes the proof of (i) and (ii).

For (iii), we note that if C is a $\sigma(l, bs + c_0)$-compact subset of l, then

$$\sup_{f \in C} |f_n| \leq \sup_{f \in C} \|f - P_{n-1} f\|_{bv} \to 0 \quad \text{as } n \to \infty$$

by Corollary 2 to Theorem 2. Consequently $e^{(n)} \to 0$ $\tau(bs + c_0, l)$ and hence in E.

We note that (iii) is true if we only assume that E contains bs (see [5, Theorem 5]).

Our next result answers a question left open in [6].

Corollary 1. *There exists a BK-space E which is not the intersection of the separable FK-spaces containing it.*

Proof. Using Theorems 3 and 5 we may take E to be $bs + c_0$.

Corollary 2. *If A is a conservative matrix such that $bs \subset c_A$, then*
(i) *$ac \subset c_A$;*
(ii) $\lim_{n\to\infty} \sup_m |a_{mn}| = 0$;
(iii) $\lim_{m\to\infty} \sup_n |a_{mn}| = 0$;
(iv) *for $x \in ac$, we have* $\lim_A x = \sum_{j=1}^{\infty} a_j x_j + \chi(A) \operatorname{Lim} x$.

Proof. (i) c_A is a separable FK-space ([14, 1.4.1]; [4, Corollary 1 to Theorem 4]) and so, by Theorem 5(i), $ac_0 \subset c_A$. Since $e \in c_A$, it follows that $ac \subset c_A$.

(ii) $e^{(n)} \to 0$ in c_A by Theorem 5(iii), so that $Ae^{(n)} \to 0$ in c by Theorem 4.4(c) of [24]. It follows that

$$\lim_{n\to\infty} \sup_m |a_{mn}| = 0.$$

(iii) follows from (ii) as in the proof of Proposition 8 of [5].

(iv) If $x \in ac$, then $x - (\operatorname{Lim} x)e \in ac_0$ and, by Theorem 5(ii),

$$x - (\operatorname{Lim} x)e = \sum_{k=1}^{\infty} (x_k - \operatorname{Lim} x)e^{(k)} \quad \text{in } c_A.$$

Now $\lim_A$ is continuous on c_A so that

$$\lim_A x - (\operatorname{Lim} x)\lim_A e = \sum_{k=1}^{\infty} (x_k - \operatorname{Lim} x)a_k.$$

Since $x \in m$, $\sum_{k=1}^{\infty} a_k x_k$ converges and so

$$\lim_A x = \sum_{k=1}^{\infty} a_k x_k + \operatorname{Lim} x\left(\lim_A e - \sum_{k=1}^{\infty} a_k\right)$$

$$= \sum_{k=1}^{\infty} a_k x_k + \chi(A)\operatorname{Lim} x.$$

3. Consistency theorems. In [6] we used a technique involving the Orlicz-Pettis theorem on unconditional convergence of series to obtain a new proof of the Mazur-Orlicz-Brudno consistency theorem. In this section we apply the same basic technique to derive similar consistency theorems for almost convergence; the details, however, are much more difficult than those in [6] and we shall need considerable preparation before coming to our main results (Theorems 6 and 8).

We begin by introducing an idea which may be of some interest in a more general setting; we say that a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ is *superconvergent to* x (in a locally convex space E) if $\{x^{(n)}\}_{n=1}^{\infty}$ converges to x and

$$\sum_{k=1}^{\infty} (x_{n_k} - x_{n_k-1})$$

converges in E for every increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers.

Our first result is elementary and its proof is omitted.

Lemma 3. *Every subsequence of a superconvergent sequence is superconvergent.*

The validity of the next result is one of the main reasons for studying superconvergence.

Lemma 4. *Let E be a locally convex space with dual E'. If a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ superconverges in the weak topology $\sigma(E, E')$, then $\{x^{(n)}\}_{n=1}^{\infty}$ converges in the topology $\lambda(E, E')$ of uniform convergence on $\sigma(E', E)$-compact sets.*

Proof. Direct application of the general Orlicz-Pettis theorem (see [6], [15] or [21]).

Lemma 5. *Let E be a Fréchet space and suppose that $x^{(n)} \to x$ in E. Then there is a subsequence $\{z^{(n)}\}_{n=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ that superconverges to x.*

Proof. Suppose that $\{p_k\}_{k=1}^{\infty}$ is an increasing sequence of seminorms defining the topology on E. Choose an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers so that

$$p_k(x - x^{(n)}) \le 1/2^k$$

whenever $n \ge n_k$. Putting $z^{(k)} = x^{(n_k)}$, $k = 1, 2, \ldots$, it is easy to see that

$$\sum_{k=1}^{\infty} (z^{(k)} - z^{(k-1)})$$

converges absolutely, so that $\{z^{(k)}\}_{k=1}^{\infty}$ superconverges to x in E.

Lemma 6. *$x^{(n)} \to x$ $\sigma(m, l)$ if and only if $x^{(n)} \to x$ $\sigma(m, \varphi)$ and* $\sup_n \|x^{(n)}\|_\infty < \infty$.

Proof. A simple compactness argument (cf. [6, Lemma 3]). Alternatively, a neat proof may be given by using Lebesgue's dominated convergence theorem.

Lemma 7. *If $x^{(n)} \to 0$ $\sigma(c_0, l)$, then there exists a subsequence $\{z^{(n)}\}_{n=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ such that*

$$\left\| \frac{1}{n}(z^{(1)} + z^{(2)} + \cdots + z^{(n)}) \right\|_\infty \to 0.$$

Proof. In view of Lemma 6 the hypotheses are equivalent to

$$\lim_{j\to\infty} x_j^{(n)} = 0 \qquad (n = 1, 2, \ldots), \tag{7}$$

$$\lim_{n\to\infty} x_j^{(n)} = 0 \qquad (j = 1, 2, \ldots), \tag{8}$$

and

$$\sup_{n,j} |x_j^{(n)}| = M < \infty. \tag{9}$$

We choose increasing sequences $\{s_m\}_{m=1}^{\infty}$ and $\{t_m\}_{m=1}^{\infty}$ of positive integers as follows. Let $s_1 = 1$, $t_0 = 0$, and suppose that $s_1, \ldots, s_m$ and $t_1, \ldots, t_{m-1}$ have been chosen. Using (7), choose $t_m > t_{m-1}$ so that

$$\max_{1 \le n \le s_m} |x_j^{(n)}| \le 2^{-m} \tag{10}$$

whenever $j > t_m$. Next, using (8), choose $s_{m+1} > s_m$ so that

$$|x_j^{(s_{m+1})}| \le 2^{-m} \tag{11}$$

whenever $1 \le j \le t_n$.

If $t_m < j \le t_{m+1}$ and $m \ge 1$, then

$$\begin{aligned} |x_j^{(s_1)} + x_j^{(s_2)} + \cdots + x_j^{(s_n)}| &\le n \cdot 2^{-m} \quad \text{if } n \le m \text{ by (10)}, \\ &\le m \cdot 2^{-m} + |x_j^{(s_{m+1})}| + \sum_{k=m+1}^{n-1} 2^{-k} \\ &\qquad\qquad \text{if } n > m \text{ by (10) and (11)}, \\ &\le M + 1 \quad \text{by (9)}. \end{aligned}$$

Consequently, putting $z^{(n)} = x^{(s_n)}$, $n = 1, 2, \ldots$, we have

$$\begin{aligned} \left\| \frac{z^{(1)} + z^{(2)} + \cdots + z^{(n)}}{n} \right\|_\infty &= \sup_m \sup_{t_m < j \le t_{m+1}} \left| \frac{x_j^{(s_1)} + x_j^{(s_2)} + \cdots + x_j^{(s_n)}}{n} \right| \\ &\le (M + 1)/n \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Lemma 7 says that in the Banach space c_0 every weakly convergent sequence has a subsequence whose arithmetic means converge in norm. This property, the so-called *Banach-Saks property*, is also known to hold for the spaces l^p and $L^p(0, 1)$ (see [3]). We remark here that not every Banach space has this property.

Lemma 8. *Let $x^{(n)} \in c_0$, $n = 1, 2, \ldots$, and suppose that $x^{(n)} \to x$ $\sigma(m, l)$. Then there exists a subsequence $\{z^{(n)}\}_{n=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ that is superconvergent to x in $\sigma(m, l)$.*

Proof. Without loss of generality we may assume that $x = 0$. The hypotheses are then the same as in Lemma 7 and we may choose $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ as before so that (9) and (10) are satisfied. It is easily seen that

$$\sup_j \sum_{n=1}^{\infty} |x_j^{(s_n)} - x_j^{(s_{n-1})}| < \infty,$$

and so, in view of Lemma 6, we may take $z^{(n)} = x^{(s_n)}$, $n = 1, 2, \ldots$.

Lemma 9. *If $x \in ac_0$ and $\{s_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers then the sequence*

$$y^{(n)} = \frac{1}{n}(P_{s_1}x + P_{s_2}x + \cdots + P_{s_n}x)$$

is superconvergent to x in $\sigma(ac_0, l)$.

Proof. We define

$$Q_0 = 0; \qquad Q_n = \frac{1}{n}(P_{s_1} + P_{s_2} + \cdots + P_{s_n}), \qquad n = 1, 2, \ldots;$$
$$R_n = Q_n - Q_{n-1}.$$

For any finite subset M of the positive integers $\mathbf{Z}$ we write

$$S_M = \sum_{k \in M} R_k = \sum_{k=1}^{\infty} \delta_M(k) R_k$$

where δ_M is the characteristic function of M. We show that the collection $\{S_M: M \text{ is a finite subset of } \mathbf{Z}\}$ is $\tau(ac_0, l) \to \tau(ac_0, l)$-equicontinuous. Since

$$\sum_{k=1}^{\infty} (S_M f)_k x_k = \sum_{k=1}^{\infty} f_k (S_M x)_k,$$

we must show that if C is $\sigma(l, ac_0)$-compact then $S(C) = \{S_M f: f \in C, M \subset \mathbf{Z}\}$ is $\sigma(l, ac_0)$-relatively compact.

For $x \in l$, we have

$$\begin{aligned}(Q_n x)_p &= (1 - k/n)x_p \qquad &&\text{if } s_k < p \leq s_{k+1},\ 1 \leq k \leq n,\\ &= 0 &&\text{if } p > s_n,\end{aligned}$$

and

$$\begin{aligned}(R_n x)_p &= \frac{k}{n(n-1)} x_p \qquad \text{if } s_k < p < s_{k+1},\ 1 \le k \le n-1,\\ &= 0 \qquad\qquad\qquad \text{if } p > s_n,\end{aligned}$$

so that

$$(S_M x)_p = \left\{ \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(n) \right\} x_p \quad \text{if } s_k < p \le s_{k+1}.$$

Now if C is $\sigma(l, ac_0)$-relatively compact, then by Corollary 2 to Theorem 2, we have

$$\sup_{x \in C} \|x\|_1 = K < \infty,$$

and

$$\sup_{x \in C} \sum_{p=n}^{\infty} |x_p - x_{p+1}| = \varepsilon_n \to 0.$$

Now

$$|(S_M x)_p| \le |x_p| \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \le |x_p|$$

so that $\|S_M x\|_1 \le K$. If $s_k < p < s_{k+1}$,

$$(S_M x)_p - (S_M x)_{p+1} = \left(\sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(k) \right) (x_p - x_{p+1})$$

so that

$$|(S_M x)_p - (S_M x)_{p+1}| \le |x_p - x_{p+1}|;$$

while, if $p = s_k$,

$$\begin{aligned}(S_M x)_p - (S_M x)_{p+1} &= \left\{ \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)} \delta_M(n) \right\} x_p - \left\{ \sum_{n=k+1}^{\infty} \frac{k}{n(n-1)} \delta_M(n) \right\} x_{p+1}\\ &= \left\{ \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)} \delta_M(n) \right\} (x_p - x_{p+1}) - \left\{ \sum_{n=k+1}^{\infty} \frac{1}{n(n-1)} \delta_M(n) \right\} x_{p+1}\\ &\qquad + \frac{1}{k} \delta_M(k) x_{p+1},\end{aligned}$$

so that

$$|(S_M x)_p - (S_M x)_{p+1}| \le |x_p - x_{p+1}| + \frac{1}{k} |x_{p+1}|.$$

Consequently, if $s_k \le n < s_{k+1}$,

$$\sum_{p=n+1}^{\infty} |(S_M x)_p - (S_M x)_{p+1}| \leq \frac{K}{k} + \sum_{p=n+1}^{\infty} |x_p - x_{p+1}|$$
$$\leq \frac{K}{k} + \varepsilon_n \to 0 \quad \text{as } n \to \infty.$$

It follows from Corollary 2 to Theorem 2 that $S(C)$ is indeed $\sigma(l, ac_0)$-compact and so the collection $\{S_M\colon M \subseteq \mathbf{Z}\}$ is equicontinuous on $(ac_0, \tau(ac_0, l))$. In particular, if N is an infinite subset of $\mathbf{Z}$, the operators

$$\sum_{k=1}^{n} \delta_N(k) R_k \qquad (n = 1, 2, \ldots)$$

are equicontinuous, and so, since ac_0 is $\tau(ac_0, l)$-complete (Theorem 4(i)) the set of $x \in ac_0$ for which $\sum_{k=1}^{\infty} \delta_N(k) R_k x$ converges is closed. However if $x \in \varphi$ this is clearly so, and so we conclude for all $x \in ac_0$ and all $N \subseteq \mathbf{Z}$ that $\sum_{k=1}^{\infty} \delta_N(k) R_k x$ converges. Hence the sequence $\{Q_n x\}_{n=1}^{\infty}$ superconverges in $(ac_0, \tau(ac_0, l))$.

Lemma 10. *Let $x^{(n)} \in c_0$, $n = 1, 2, \ldots$, and suppose that $x^{(n)} \to x$ $\sigma(ac_0, l)$. Then there exists a subsequence $\{z^{(n)}\}_{n=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ such that some subsequence $\{w^{(n)}\}_{n=1}^{\infty}$ of $\{(z^{(1)} + z^{(2)} + \cdots + z^{(n)})/n\}_{n=1}^{\infty}$ superconverges to x in $\sigma(ac_0, l)$.*

Proof. Since $P_n x \to x$ $\sigma(ac_0, l)$, we have

$$x^{(n)} - P_n x \to 0 \ \sigma(c_0, l).$$

By Lemma 7 we may take a subsequence $\{z^{(n)}\}_{n=1}^{\infty} = \{x^{(s_n)}\}_{n=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ such that

$$\left\| \frac{1}{n}(z^{(1)} + \cdots + z^{(n)}) - \frac{1}{n}(P_{s_1} x + \cdots + P_{s_n} x) \right\|_{\infty} \to 0.$$

Taking a subsequence again, we may suppose that, for each integer n,

$$\left\| \frac{1}{m_n}(z^{(1)} + \cdots + z^{(m_n)}) - \frac{1}{m_n}(P_{s_1} x + \cdots + P_{s_{m_n}} x) \right\|_{\infty} \leq \frac{1}{2^n},$$

so that the sequence

$$\left\{ \frac{1}{m_n}(z^{(1)} + \cdots + z^{(m_n)}) - \frac{1}{m_n}(P_{s_1} x + \cdots + P_{s_{m_n}} x) \right\}_{n=1}^{\infty}$$

is superconvergent to 0 in c_0. However, by Lemmas 3 and 9, the sequence

$$\left\{ \frac{1}{m_n}(P_{s_1} x + \cdots + P_{s_{m_n}} x) \right\}_{n=1}^{\infty}$$

is superconvergent to x in $\sigma(ac_0, l)$. Hence, with

$$w^{(n)} = \frac{1}{m_n}(z^{(1)} + \cdots + z^{(m_n)}) \qquad (n = 1,2,\ldots),$$

$\{w^{(n)}\}_{n=1}^{\infty}$ is superconvergent to x in $\sigma(ac_0, l)$.

Before stating our next result we recall the following notation. For an infinite matrix A, ac_A denotes the set

$$ac_A = \{x \in \omega \colon Ax \in ac\}.$$

If $x \in ac_A$, we write $\mathrm{Lim}_A x$ in place of $\mathrm{Lim}(Ax)$, and denote by $(ac_0)_A$ the subspace of $(ac)_A$ on which Lim_Avanishes.

Theorem 6. *Let A be a matrix such that*

(i) $\sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$, *and*

(ii) $\lim_{i\to\infty} a_{ij} = 0$ *for* $j = 1, 2, \ldots$. *Then l is $\sigma(l, (ac_0)_A \cap m)$-sequentially complete.*

Proof. Let $x \in (ac_0)_A \cap m$ be fixed. We construct a sequence $\{z^{(n)}\}_{n=1}^{\infty}$ of elements of φ such that $\{z^{(n)}\}_{n=1}^{\infty}$ superconverges to x in $\sigma((ac_0)_A \cap m, l)$. To do this, we first observe that

$$AP_n x \to Ax \qquad \sigma(\omega, \varphi).$$

Condition (i) ensures that $A\colon m \to m$ is continuous and hence

$$\|AP_n x\|_\infty \leq \|A\|\, \|x\|_\infty.$$

Lemma 6 gives

$$AP_n x \to Ax \qquad \sigma(m, l).$$

Now condition (ii) implies that $AP_n x \in A(\varphi) \subset c_0$ so we may apply Lemma 10 to deduce the existence of a sequence $\{v^{(k)}\}_{k=1}^{\infty}$ such that $\{Av^{(k)}\}_{k=1}^{\infty}$ superconverges to Ax in $\sigma(ac_0, l)$ and $\{v^{(k)}\}_{k=1}^{\infty}$ takes the form

$$v^{(k)} = \frac{1}{m_k}(u^{(1)} + \cdots + u^{(m_k)})$$

where $\{u^{(k)}\}_{k=1}^{\infty}$ is some subsequence of $\{P_n x\}_{n=1}^{\infty}$. Clearly we have

$$\sup_k \|v^{(k)}\|_\infty \leq \|x\|_\infty$$

and $v^{(k)} \to x$ $\sigma(\omega, \varphi)$ so that $v^{(k)} \to x$ $\sigma(m, l)$ by Lemma 6.

Furthermore, since $Av^{(k)} \to Ax$ in ω, we have

$$v^{(k)} \to x \quad \text{in } \omega_A.$$

We now apply Lemmas 3, 5 and 8 to obtain a subsequence $\{z^{(n)}\}_{n=1}^{\infty}$ of $\{v^{(n)}\}_{n=1}^{\infty}$ such that $\{z^{(n)}\}_{n=1}^{\infty}$ superconverges to x in both ω_A and $\sigma(m, l)$; it is also clear that

$\{Az^{(n)}\}_{n=1}^{\infty}$ superconverges in $(ac_0, \sigma(ac_0, l))$. Now suppose that $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence taking only the values 1 and 0; for each k let

$$y_k = \sum_{n=1}^{\infty} \varepsilon_n (z_k^{(n)} - z_k^{(n-1)}) \quad (\text{where } z^{(0)} = 0).$$

Since $\{z^{(n)}\}_{n=1}^{\infty}$ superconverges in both ω_A and $(m, \sigma(m, l))$, the series

$$\sum_{n=1}^{\infty} \varepsilon_n (z^{(n)} - z^{(n-1)})$$

converges to y in both ω_A and $(m, \sigma(m, l))$; therefore $y \in m \cap \omega_A$. Now $A: \omega_A \to \omega$ is continuous [24, Theorem 4.4(c)] and so

$$Ay = \sum_{n=1}^{\infty} \varepsilon_n (Az^{(n)} - Az^{(n-1)}) \qquad \sigma(\omega, \varphi).$$

However, $\{Az^{(n)}\}_{n=1}^{\infty}$ superconverges in $(ac_0, \sigma(ac_0, l))$ so that

$$Ay = \sum_{n=1}^{\infty} \varepsilon_n (Az^{(n)} - Az^{(n-1)}) \qquad \sigma(ac_0, l),$$

and $Ay \in ac_0$, i.e., $y \in (ac_0)_A$. Thus $\{z^{(n)}\}_{n=1}^{\infty}$ superconverges to x in $((ac_0)_A \cap m, \sigma((ac_0)_A \cap m, l))$.

We now repeat the argument used in the proof of Theorem 3 of [6]. Consider the topology $\lambda((ac_0)_A \cap m, l)$ on $(ac_0)_A \cap m$ of uniform convergence on the $\sigma(l, (ac_0)_A \cap m)$-compact subsets of l; by Lemma 4 we have

$$z^{(n)} \to x \qquad \lambda((ac_0)_A \cap m, l).$$

Suppose now that ψ is a linear functional on $(ac_0)_A \cap m$ whose restrictions to $\lambda((ac_0)_A \cap m, l)$-precompact sets are λ-continuous. Then ψ is λ-sequentially continuous and since $\lambda \leq \beta((ac_0)_A \cap m, l) \leq \beta(c_0, l)$, ψ is $\|\cdot\|_\infty$-continuous on c_0 so that

$$\sum_{j=1}^{\infty} |f_j| < \infty$$

where $\psi(e^{(j)}) = f_j$ $(j = 1, 2, \ldots)$. Now

$$\psi(z^{(n)}) = \sum_{j=1}^{\infty} z_j^{(n)} f_j$$

since $z^{(n)} \in \varphi$, and

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} z_j^{(n)} f_j = \sum_{j=1}^{\infty} x_j f_j$$

since $z^{(n)} \to x$ $\sigma(m, l)$. Consequently, for each $x \in (ac_0)_A \cap m$, we have

$$\psi(x) = \sum_{j=1}^{\infty} x_j f_j.$$

It follows (as in the proof of Theorem 3 of [6]) by Grothendieck's completeness theorem that the topology ρ on l, of uniform convergence on λ-precompact subsets of $(ac_0)_A \cap m$, must be complete. Furthermore, ρ defines the same convergent and Cauchy sequences as $\sigma(l, (ac_0)_A \cap m)$ so that l is $\sigma(l, (ac_0)_A \cap m)$-sequentially complete.

We now come to our first consistency theorem.

Theorem 7. *Let A and B be regular matrices and suppose that $ac_A \cap m \subset c_B$. Then* $\lim_B x = \operatorname{Lim}_A x$ *whenever $x \in ac_A \cap m$.*

Proof. Since A is regular, the conditions of Theorem 6 are satisfied. Let $b^{(n)} \in l$, $n = 1, 2, \ldots$, be defined by

$$b_k^{(n)} = b_{nk} \qquad (k = 1, 2, \ldots),$$

(so that $b^{(n)}$ is the nth row of B). Since $(ac_0)_A \cap m \subseteq c_B$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)} x_k \quad \text{exists}$$

whenever $x \in (ac_0)_A \cap m$. Hence $\{b^{(n)}\}_{n=1}^{\infty}$ is $\sigma(l, (ac_0)_A \cap m)$-Cauchy and so converges, say $b^{(n)} \to b$, by Theorem 6. Clearly

$$b_k = \lim_{k\to\infty} b_{nk} = 0,$$

so that $b^{(n)} \to 0$ $\sigma(l, (ac_0)_A \cap m)$.

Now, if $x \in (ac)_A \cap m$, then $x - (\operatorname{Lim}_A x)e \in (ac_0)_A \cap m$, and so

$$\lim_B\left(x - \left(\operatorname*{Lim}_A x\right)e\right) = \lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)}\left(x_k - \operatorname*{Lim}_A x\right) = 0,$$

i.e. $\lim_B x = \operatorname{Lim}_A x$.

When A is the identity matrix, Theorem 7 reduces to the following.

Corollary. *Let B be a regular matrix with $ac \subset c_B$. Then* $\lim_B x = \operatorname{Lim} x$ *whenever $x \in ac$.*

This special result may also be derived from Corollary 2 to Theorem 5 and was first obtained by Lorentz [13].

Before stating our next result let us recall the following notation. If E is an FK-space containing φ, then we write

$$W_E = \{x \in E\colon P_n x \to x \text{ weakly in } E\}$$

and

$$S_E = \{x \in E\colon P_n x \to x \text{ in } E\}.$$

Theorem 8. *Let E be an FK-space containing c_0. Then l is sequentially complete* *under both the topologies* $\sigma(l, W_E \cap ac_0)$ *and* $\sigma(l, S_E \cap ac_0)$.

Proof. As with Theorem 6, the proof hinges on ideas developed in Theorem 3 of [6].

Let $x \in W_E \cap ac_0$ be fixed: by Theorem 2 of [6] (see also [20]) there is a sequence $\{u^{(n)}\}_{n=1}^{\infty}$ of elements of φ with

$$\tau - \lim_{n\to\infty} u^{(n)} = x$$

and

$$\sup_n \|u^{(n)}\|_\infty \leq \|x\|_\infty,$$

where τ denotes the FK-topology on E. By Lemma 6,

$$\lim_{n\to\infty} u^{(n)} = x \qquad \sigma(ac_0, l)$$

and so, by Lemma 10, there exists a sequence $\{v^{(n)}\}_{n=1}^{\infty}$, of arithmetic means of a subsequence of $\{u^{(n)}\}_{n=1}^{\infty}$, such that $\{v^{(n)}\}_{n=1}^{\infty}$ superconverges to x in $\sigma(ac_0, l)$; clearly

$$\tau - \lim_{n\to\infty} v^{(n)} = x.$$

By using Lemmas 3 and 5 we may select a subsequence $\{z^{(n)}\}_{n=1}^{\infty}$ which superconverges to x in both τ and $\sigma(ac_0, l)$. Thus every subseries of $\Sigma_{n=1}^{\infty} (z^{(n)} - z^{(n-1)})$ converges in $E \cap ac_0$; i.e., if $\epsilon_n = 0$ or 1 for all n and

$$y = \sum_{n=1}^{\infty} \varepsilon_n (z^{(n)} - z^{(n-1)}) \quad (\text{where } z^{(0)} = 0),$$

then $y \in E \cap ac_0$. Since this series converges in $\sigma(m, l)$, we have

$$\sup_k \left\| \sum_{n=1}^{k} \varepsilon_n (z^{(n)} - z^{(n-1)}) \right\|_\infty < \infty,$$

and since the series converges in τ we have $y \in W_E$ by Theorem 2 of [6]. Thus $\{z^{(n)}\}_{n=1}^{\infty}$ superconverges to x in $\sigma(W_E \cap ac_0, l)$, and the remaining details follow those of Theorem 6 (or Theorem 3 of [6]).

For the second half of the theorem we observe that [10, pp. 1015–1016] $S_E = W_F$, where F is the FK-space defined as follows.

$$F = \{x \in E \colon \{P_n x\}_{n=1}^{\infty} \text{ is } \tau\text{-bounded}\}$$

with the topology given by the seminorms

$$\nu(x) = \sup_n \nu(P_n x) \qquad (x \in F)$$

for each τ-continuous seminorm ν.

Our next result may be thought of as a generalized consistency theorem.

Theorem 9. *Let E be an FK-space containing c_0 and let F be an FK-space containing no (closed) subspace isomorphic to m. If $W_E \cap ac_0 \subset F$, then $W_E \cap ac_0 \subset W_F$.*

Proof. As in the proof of Theorem 8 we can show that each $x \in W_E \cap ac_0$ can be written in the form $x = \sum_{n=1}^{\infty} x^{(n)}$ where $x^{(n)} \in \varphi$, $n = 1, 2, \ldots$, and the convergence is $\sigma(W_E \cap ac_0, l)$-subseries. This observation enables us to replace $W_E \cap m$ in the statement of Proposition 1 of [8] by $W_E \cap ac_0$; but then the present result follows just as in the proof of Theorem 2 of [8].

In particular, it should be noted that Theorem 9 remains valid when F is a separable FK-space.

Theorem 10. *Let A and B be regular matrices and suppose that $ac \cap c_A \subseteq c_B$. Then there exists a constant α such that*

$$\lim_B x = \alpha \lim_A x + (1 - \alpha) \operatorname{Lim} x$$

whenever $x \in ac \cap c_A$.

Proof. Since A is regular we have, by Theorem 3.6 of [25], $(c_0)_A \cap ac_0 = W_{c_A} \cap ac_0$. Now c_B is separable ([14], [4]) so that

$$(c_0)_A \cap ac_0 \subset W_{c_B} \cap ac_0 = (c_0)_B \cap ac_0$$

by Theorem 9. Hence $\lim_A x = \operatorname{Lim} x = 0$ implies that $\lim_B x = 0$, and so

$$\lim_B x = \alpha \lim_A x + \beta \operatorname{Lim} x$$

whenever $x \in ac \cap c_A$. However $1 = \lim_B e = \alpha + \beta$ and the desired conclusion follows.

Theorem 11. *Let A and B be conservative matrices and suppose that $ac \cap c_A \subseteq c_B$. Then there exist constants α, β such that*

(i) $\lim_B x - \sum_{j=1}^{\infty} b_j x_j = \alpha(\lim_A x - \sum_{j=1}^{\infty} a_j x_j) + \beta \operatorname{Lim} x$ *whenever* $x \in ac \cap c_A$, *and*

(ii) $\chi(B) = \alpha\chi(A) + \beta$.

Proof. This is a simple extension of Theorem 10; we observe that

$$W_{c_A} \cap ac_0 = \left\{x: \lim_A x = \sum_{j=1}^{\infty} a_j x_j\right\} \cap ac_0$$

and apply the same method.

Corollary 1. *Let A be conull and B be regular and suppose that $ac \cap c_A \subseteq c_B$. Then there exists a constant α such that*

$$\lim_B x = \operatorname{Lim} x + \alpha\left(\lim_A x - \sum_{j=1}^{\infty} a_j x_j\right)$$

whenever $x \in ac \cap c_A$.

Corollary 2. *Let A be regular and B be conull and suppose that* $ac \cap c_A \subseteq c_B$. *Then there exists a constant* α *such that*

$$\lim_B x = \alpha\left(\lim_A x - \operatorname{Lim} x\right) + \sum_{j=1}^{\infty} b_j x_j$$

whenever $x \in ac \cap c_A$.

REFERENCES

1. A. Alexiewicz and Z. Semadeni, *Linear functionals on two-norm spaces,* Studia Math. **17** (1958), 121–140. MR **20** #6644.

2. S. Banach, *Théorie des operations linéaires*, Monografie Mat., PWN, Warsaw, 1932.

3. S. Banach and S. Saks, *Sur la convergence forte dans les champs* L^p, Studia Math. **2** (1930), 51–57.

4. G. Bennett, *A representation theorem for summability domains*, Proc. Lond. Math. Soc. (3) **24** (1972), 193–203. MR **45** #776.

5.———, *A new class of sequence spaces with applications in summability theory*, J. Reine Angew. Math. 266 (1974), 49–75.

6. G. Bennett and N.J. Kalton, *FK-spaces containing* c_0, Duke Math. J. **39** (1972), 561–582.

7.———, *Inclusion theorems for K-spaces*, Canad. J. Math. (to appear).

8.———, *Addendum to FK-spaces containing* c_0, Duke Math. J. 39 (1972), 819–821.

9. M.M. Day, *Normed linear spaces*, 2nd rev. ed., Academic Press, New York; Springer-Verlag, Berlin, 1962. MR **26** #2847.

10. D.J.H. Garling, *On topological sequence spaces*, Proc. Cambridge Philos. Soc. **63** (1967), 997–1019. MR **36** #1964.

11. N.J. Kalton, *Some forms of the closed graph theorem*, Proc. Cambridge Philos. Soc. **70** (1971), 401–408.

12. J.P. King, *Almost summable sequences*, Proc. Amer. Math. Soc. **17** (1966), 1219–1225. MR **34** #1752.

13. G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190. MR **10**, 367.

14. S. Mazur and W. Orlicz, *On linear methods of summability*, Studia Math. **14** (1954), 129–160. MR **16**, 814.

15. C.W. McArthur, *On a theorem of Orlicz and Pettis*, Pacific J. Math. **22** (1967), 297–302. MR **35** #4702.

16. C.W. McArthur and J.R. Retherford, *Uniform and equicontinuous Schauder bases of subspaces*, Canad. J. Math. **17** (1965), 207–212. MR **30** #4141.

17. G.M. Petersen, *Almost convergence and the Buck-Pollard property*, Proc. Amer. Math. Soc. **11** (1960), 469–477. MR **22** #2819.

18. H.H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966. MR **33** #1689.

19. P. Schaefer, *Almost convergent and almost summable sequences*, Proc. Amer. Math. Soc. **20** (1969), 51–54. MR **38** #3649.

20. A.K. Snyder, *Conull and coregular FK-spaces*, Math. Z. **90** (1965), 376–381. MR **32** #2783.

21. I. Tweddle, *Vector-valued measures*, Proc. London Math. Soc. (3) **20** (1970), 469–489. MR **41** #3707.

22. A. Wilansky, *Topics in functional analysis*, Lecture Notes in Math., no. 45, Springer-Verlag, Berlin and New York, 1967. MR **36** #6901.

23. A. Wiweger, *Linear spaces with mixed topology*, Studia Math. **20** (1961), 47–68. MR **24** #A3490.

24. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. **53** (1951), 463–487. MR **12**, 604.

25.———, *Abschnittskonvergenz in FK-Räumen*, Math. Z. **55** (1951), 55–70. MR **13**, 934.

26. K. Zeller and W. Beekman, *Theorie der Limitierungsverfahren. Zweite, erweiterte und verbesserte Auflage*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 15, Springer-Verlag, Berlin and New York, 1970. MR **41** #8863.

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