Symmetric linear functionals on function spaces

*Tadeusz Figiel and Nigel Kalton**

To Professor Jaak Peetre on his 65th birthday

Abstract. We discuss the properties of linear functionals defined on spaces of measurable functions which are invariant under rearrangements.

2000 Mathematics Subject Classification: 42A50, 46E30

1. Introduction

Suppose (Ω, Σ, μ) is any nonatomic σ-finite measure space. We let $\mathcal{M}(\Omega)$ be the space of all Σ-measurable complex-valued functions f such that

$$d_f(x) := \mu\{|f| > x\} < \infty, \quad x > 0.$$

We define $\mathcal{M}_{\mathbb{R}}(\Omega)$ and $\mathcal{M}_+(\Omega)$ to be the subsets of all real-valued or all non-negative real-valued functions respectively. If $f \in \mathcal{M}$, then the decreasing rearrangement of $|f|$ is denoted by f^*; this is defined by

$$f^*(t) = \inf_{\mu E = t} \sup_{\omega \notin E} |f(\omega)|.$$

From now we assume that either $\mu(\Omega) = \infty$ or $\mu(\Omega) = 1$. If Ω is a standard measure space we may assume that either $\Omega = (0, \infty)$ or $\Omega = (0, 1]$ with Lebesgue measure. We define a *symmetric ideal* $\mathcal{X}$ to be a linear subspace of $\mathcal{M}$ such that:

- If $f \in \mathcal{X}$ and $|g| \le |f|$ then $g \in \mathcal{X}$,
- If $f \in \mathcal{X}$ and $d_g \le d_f$ then $g \in \mathcal{X}$,
- $\mathcal{X} \ne \{0\}$.

---

*Research supported by NSF grant DMS-9870027

Of course if $\mathcal{X}$ is any symmetric ideal on Ω where $\mu(\Omega) = 1$ (respectively, $\mu(\Omega) = \infty$) then we can define the associated symmetric ideal $\mathcal{X}(0, 1]$ (respectively $\mathcal{X}(0, \infty)$ by $f \in \mathcal{X}(0, 1]$ (respectively $\mathcal{X}(0, \infty)$) if and only there exists $g \in \mathcal{X}$ with $d_g = d_f$.

Note that $\mathcal{X} \neq \{0\}$ implies that $\mathcal{X}$ contains all bounded measurable functions with support of finite measure. It follows from this remark that if $f \in \mathcal{X}$ and $g \in \mathcal{M}$ satisfies $g^*(t) \leq Cf^*(t/C)$ for some constant C then $g \in \mathcal{X}$.

By a symmetric quasi-Banach ideal we mean a symmetric ideal $\mathcal{X}$ equipped with a quasi-norm $f \to \|f\|_{\mathcal{X}}$ such that:

- $\|f\|_{\mathcal{X}} \leq \|g\|_{\mathcal{X}}$ whenever $f^* \leq g^*$.
- $\mathcal{X}$ is complete for the quasi-norm $\|\cdot\|_{\mathcal{X}}$.

In this case we define the dilation operators $D_s : \mathcal{X}(0, 1] \to \mathcal{X}(0, 1]$ by

$$D_s f(t) = \begin{cases} f(t/s) & \text{if } t \leq s \\ 0 & \text{if } t > s, \end{cases}$$

or $D_s : \mathcal{X}(0, \infty) \to \mathcal{X}(0, \infty)$ by

$$D_s f(t) = f(t/s), \quad 0 < t < \infty.$$

The Boyd indices $p_{\mathcal{X}}$ and $q_{\mathcal{X}}$ are defined by

$$\frac{1}{p_{\mathcal{X}}} = \lim_{s\to\infty} \frac{\log \|D_s\|}{\log s}.$$

and

$$\frac{1}{q_{\mathcal{X}}} = \lim_{s\to 0} \frac{\log \|D_s\|}{\log s}.$$

Suppose $\mathcal{X}$ is a symmetric ideal on (Ω, Σ, μ). Let us say that $f \sim g$ if f, g have the same distributions i.e. $\mu \circ f^{-1} = \mu \circ g^{-1}$ coincide as Borel measures on $\mathbb{C} \setminus \{0\}$. We shall say a linear functional $\varphi : \mathcal{X} \to \mathbb{C}$ is *symmetric* if $\varphi(f) = \varphi(g)$ whenever $f \sim g$. We say φ is a *regular symmetric functional* if additionally $\varphi(\chi_A) = 1$ whenever $\mu(A) = 1$. We emphasize here that we have no topological assumptions in general: $\mathcal{X}$ need not be a quasi-Banach ideal, and φ need not have any continuity properties.

The study of such symmetric functionals goes back around twenty years. In fact the corresponding notion for sequence spaces is closely related to the problem of defining unusual traces on certain ideals of operators on a Hilbert space; this subject can be traced back to the work of Dixmier [5] in the 1960's. In the 1980's Pietsch, motivated by the study of traces on ideals of operators on Banach spaces again drew attention to the problem of characterizing symmetric functionals on sequence spaces (see for example [14]). For more recent work in this direction see [11], [6] and [12].

In this paper, we want to draw a neat and perhaps unexpected connection between analytic functions and symmetric functionals. Before doing this in Section 2 we develop the basic theory of symmetric functionals on an arbitrary symmetric ideal of functions. Our main result Theorem 2.8 here characterizes the set $Z_{\mathcal{X}}$ (which we call the center) of all functions f so that $\varphi(f) = 0$ for all symmetric functionals φ. These results were essentially known to the first author in the 1980's; see for example [13] and [14]. It is interesting that $Z_{\mathcal{X}}$ can be characterized purely in terms of a generalized K-functional.

In Section 3 we prove our main results. For the case of the unit circle $\mathbb{T}$, this says that for a symmetric ideal $\mathcal{X}$ which satisfies a very mild condition, (in particular for any quasi-Banach symmetric ideal), that if f is analytic function on $\mathbb{D}$ with $f(0) = 0$, belonging to the Smirnov class such that its boundary values, also denoted by f, belong to $\mathcal{X}(\mathbb{T})$ then f actually belongs to the center $Z_{\mathcal{X}}$. For the special case of $\mathcal{X} = L_1$ this result is proved in two different ways in [10]; the center of L_1 is there denoted by $H^{\text{sym}}_{1,0}$. It is also equivalent (again for the case $\mathcal{X} = L_1$) to an older result of Ceretelli [3], later independently found by Davis [4]. This result is also discussed in [10] and [9]; in [10] it is connected to the theory of commutators in interpolation. Theorems 3.3 and 3.4 are inspired by the analogous results for traces in [12].

Finally in Section 4 we show that for quasi-Banach ideals a real function belongs to $\mathcal{X}(\mathbb{R})$ if and only if it is the real part of a function f in the corresponding Hardy class $H_{\mathcal{X}}$.

2. Symmetric functionals

In order to treat the case $\mu(\Omega) = 1$ let us introduce the notion of a symmetric ideal *of finite type*. If $\mu(\Omega) = \infty$ we say that $\mathcal{X}$ is a symmetric ideal of finite type if $f \in \mathcal{X}$ implies that $\mu(\operatorname{supp} f) < \infty$. If $\mathcal{Y}$ is a symmetric ideal on Ω where $\mu(\Omega) = 1$ we can embed (Ω, μ) in a larger nonatomic measure space (Ω', μ'), by taking a countable disjoint union of copies of (Ω, μ). We can then define $\mathcal{X}(\Omega')$ of finite type by the property that $f \in \mathcal{X}$ if and only if $\mu'(\operatorname{supp} f) < \infty$ and $f\chi_E \in \mathcal{Y}(E)$ whenever $\mu'(E) = 1$. We call $\mathcal{X}$ the *infinite extension* of $\mathcal{Y}$. It is then easy to see that a symmetric linear functional φ on $\mathcal{Y}$ extends to a symmetric linear functional on $\mathcal{X}$. Using this idea of an infinite extension, we shall restrict our attention unless otherwise stated to the case when $\mu(\Omega) = \infty$.

For $f \in \mathcal{M}_+$ we define the *generalized K-functional* of f for $0 < u < v < \infty$ by

$$K(u, v; f) = \int_u^v f^*(t)dt. \tag{2.1}$$

We extend the definitions of K to general complex functions by linearity. To be precise if $f = g + ih \in \mathcal{M}$ where g and h are real we define

$$K(u, v; f) = K(u, v; g_+) - K(u, v; g_-) + iK(u, v; h_+) - iK(u, v; h_-)$$

and where, as usual $g_+ = \max(g, 0)$ and $g_- = \max(-g, 0)$ etc.

We will also need a dual notion which we term the K^*-*functional* which we define for all $f \in \mathcal{M}$:

$$K^*(r, s; f) = \int_{r \le |f| < s} f \, d\mu. \tag{2.2}$$

If $f \in \mathcal{M}$ we can define measurable sets $E_u(f)$ for $u > 0$ so that $\mu E_u(f) = u$, $E_u(f) \subset E_v(f)$ if $u < v$ and $E_u \subset \{|f| \ge f^*(u)\}$. Let $E_{u,v}(f) = E_v(f) \setminus E_u(f)$ when $u < v$.

Lemma 2.1. *Suppose $f, g \in \mathcal{M}$ and $|f| \le |g|$.*

(1) *Suppose* $0 < u < v < \infty$. *Then*

$$\left| \int_{E_{u,v}(f)} f \, d\mu - \int_{E_{u,v}(g)} f \, d\mu \right| \le 2(ug^*(u) + vg^*(v)). \tag{2.3}$$

(2) *Suppose* $0 < r < s < \infty$. *Then*

$$\left| \int_{r \le |f| < s} f \, d\mu - \int_{r \le |g| < s} f d\mu \right| \le 2(rd_g(r) + sd_g(s)). \tag{2.4}$$

Proof. (1) Let $h = \chi_{E_v(f)} - \chi_{E_u(f)} + \chi_{E_u(g)} - \chi_{E_v(g)}$. If $\omega \in E_{u,v}(f)$ and $h(\omega) = 1$ then either $\omega \in E_u(g)$ or $|g(\omega)| \le g^*(v)$. Thus

$$\left| \int_{E_{u,v}(f)} hf \, d\mu \right| \le uf^*(u) + (v - u)g^*(v).$$

If $\omega \in E_u(f)$ then $h(\omega) = -1$ if $\omega \in E_{u,v}(g)$;

$$\left| \int_{E_u(f)} hf \, d\mu \right| \le ug^*(u).$$

If $\omega \in \Omega \setminus E_v(f)$ then $|f(\omega)| \le f^*(v)$ and $h(\omega) = 1$ if $\omega \in E_{u,v}(g)$.

$$\left| \int_{\Omega \setminus E_v(f)} hf \, d\mu \right| \le (v - u)f^*(v).$$

Combining gives (1).

For (2) note that

$$\left| \int_{\substack{r \le |f| < s \\ |g| < r}} f \, d\mu \right| \le rd_f(r), \quad \left| \int_{\substack{r \le |f| < s \\ |g| \ge s}} f \, d\mu \right| \le sd_g(s),$$

$$\left| \int_{\substack{r \le |g| < s \\ |f| < r}} f \, d\mu \right| \le rd_g(r), \quad \text{and} \left| \int_{\substack{r \le |g| < s \\ |f| \ge s}} f \, d\mu \right| \le sd_f(s).$$

This concludes the proof. □

The following is an immediate consequence of Lemma 2.1:

Lemma 2.2. *Suppose $f, g \in \mathcal{M}$ and $|f| \le |g|$. Then:*

$$\left| K(u, v; f) - \int_{E_{u,v}(g)} f \, d\mu \right| \le 8(ug^*(u) + vg^*(v)) \tag{2.5}$$

and

$$\left| K^*(r, s; f) - \int_{r \le |g| < s} f \, d\mu \right| \le 2(rd_g(r) + sd_g(s)). \tag{2.6}$$

Proof. Note that (2.6) simply restates (2.4). To prove (2.5) simply note that (2.3) gives this with constant 2 if f is positive. We can apply it in turn to $(\Re f)_+$, $(\Re f)_-$, $(\Im f)_+$ and $(\Im f)_-$. □

Our next lemma shows that the K-functional has approximate linearity properties:

Lemma 2.3. *Suppose $f, g \in \mathcal{M}$ and suppose $h = f + g$ and $\psi = |f| + |g|$. Then if* $0 < u < v < \infty$, $0 < r < s < \infty$,

$$|K(u, v; h) - K(u, v; f) - K(u, v; g)| \le 24(u\psi^*(u) + v\psi^*(v)) \tag{2.7}$$

and

$$|K^*(r, s; h) - K^*(r, s; f) - K^*(r, s; g)| \le 6(rd_\psi(r) + sd_\psi(s)). \tag{2.8}$$

Proof. For (2.7) it is enough to note that

$$\left| K(u, v; \phi) - \int_{E_{u,v}(\psi)} \phi \, d\mu \right| \le 8(u\psi^*(u) + v\psi^*(v))$$

when $\phi = f$, g and h. (2.8) is similar. □

Next we define the notion of the *center* $Z_{\mathcal{X}}$ of a symmetric ideal $\mathcal{X}$ supported on a measure space Ω with $\mu(\Omega) = 1$ or $\mu(\Omega) = \infty$. We shall say that $f \in Z_{\mathcal{X}}$ if there exists $h \in \mathcal{X}_+$ such that whenever $0 < u < v < \infty$ we have:

$$|K(u, v; f)| \le uh^*(u) + vh^*(v). \tag{2.9}$$

Let us make the remark that if $\mathcal{X}$ is of finite type and $\mu(\text{supp } f) = 1$ then by taking v large enough (2.9) will imply that

$$|K(u, 1; f)| \le uh^*(u).$$

Hence it suffices to consider in this case h with $\mu(\text{supp } h) = 1$. This implies that if $\mathcal{Y}$ is a symmetric ideal on a probability space and $\mathcal{X}$ is the infinite extension of $\mathcal{Y}$ then $Z_{\mathcal{Y}}$ coincides with the restriction of $Z_{\mathcal{X}}$ to Ω.

Lemma 2.4. *$f \in Z_{\mathcal{X}}$ if and only if there exists $h \in \mathcal{X}_+$ and $C > 0$ with*

$$|K^*(r, s; f)| \le C(rd_h(r) + sd_h(s)), \quad 0 < r < s < \infty. \tag{2.10}$$

Proof. Assume first $f \in Z_{\mathcal{X}}$. Choose $u = \mu(|f| > r)$ and $v = \mu(|f| > s)$. Then we can assume $\{r \le |f| < s\} = E_{u,v}(f)$ and so

$$K^*(r, s; f) = \int_{E_{u,v}(f)} f \, d\mu$$

and (2.10) follows from (2.5) with $f = g$.

Conversely suppose (2.10) holds and $0 < u < v < \infty$; we can assume $h^* \ge f^*$. Let $r = 2h^*(u)$ and $s = 2h^*(v)$. Then $d_h(r) \le u$ and $d_h(s) \le v$ and so

$$\int_{r \le |f| < s} f \, d\mu \le 2C(uh^*(u) + vh^*(v)).$$

Now

$$\int_{\substack{E_{u,v}(f) \\ |h| \le s}} f \, d\mu \le 2vh^*(v)$$

and

$$\int_{\substack{E_{u,v}(f) \\ |h| > r}} d\mu \le 2uh^*(u).$$

Combining again with (2.5) gives that

$$|K(u, v; f)| \le (2C + 10)(uh^*(u) + vh^*(v)). \qquad \square$$

Proposition 2.5. *The center $Z_{\mathcal{X}}$ is a linear subspace of $\mathcal{X}$ with the property that $f \in Z_{\mathcal{X}}$ if and only if $\Re f, \Im f \in \mathcal{X}$.*

Proof. The only statement that requires proof is linearity; this follows directly from (2.7). $\square$

We will need a result of Kwapień from 1983:

Theorem 2.6 ([13]). *Let $f \in \mathcal{M}$ be a bounded real function with support of finite measure. If $\int f d\mu = 0$ then there exist $g_1, g_2 \in \mathcal{M}$ with $\operatorname{supp} g_j \subset \operatorname{supp} f$ for $j = 1, 2$, $\|g_j\|_\infty \le 6\|f\|_\infty$, $g_1 \sim g_2$ and $f = g_1 - g_2$ (μ-a.e).*

Proof. In effect Kwapień proves this for a standard measure space with $g_2 = g_1 \circ \sigma$ where σ is a measure preserving transformation. For the general case we apply Kwapień's theorem to $(\mathbb{R}, \nu)$ where $\nu(B) = \mu(f^{-1}B \cap \operatorname{supp} f)$. We can then write $\phi(x) = x$ in the form $\phi = \psi_1 - \psi_2$ where $\psi_1 \sim \psi_2$ and $\|\psi_j\|_\infty \le 6\|f\|_\infty$. Let $g_j = \psi_j \circ f$. $\square$

If $\mathcal{X}$ is any symmetric ideal let us introduce a corresponding sequence space $\mathcal{S}\mathcal{X}$. Let $(A_n)_{n\in\mathbb{Z}}$ be a disjoint sequence of measurable sets in Ω such that $\mu A_n = 2^{-n}$. We define $\mathcal{S}\mathcal{X}$ as the space of sequences $\xi = (\xi_n)$ such that $L\xi := \sum_{n\in\mathbb{Z}} 2^n \xi_n \chi_{A_n} \in \mathcal{X}$.

It is clear that this definition is independent of the choice of (A_n); indeed we may take $\Omega = (0, \infty)$ and $A_n = (2^{-n}, 2^{1-n})$.

It is easy to see from the symmetry condition on $\mathcal{X}$ that if $\xi \in \mathcal{S}\mathcal{X}$ then we have $\sum_{n\in\mathbb{Z}} \xi_{n-1}\chi_{A_n} \in \mathcal{X}$. On $\mathcal{S}\mathcal{X}$ we may therefore define the translation operator $\mathcal{T}((\xi_n)_{n\in\mathbb{Z}}) = (\xi_{n-1})_{n\in\mathbb{Z}}$. We say that a linear functional ψ on $\mathcal{S}\mathcal{X}$ is *translation-invariant* if $\psi(\mathcal{T}(\xi)) = \psi(\xi)$ for every $\xi \in \mathcal{S}\mathcal{X}$. It is trivial that $\psi(\xi) = 0$ for every translation-invariant ψ if and only if ξ is in the range $\mathcal{R}$ of $I - \mathcal{T}$ where I is the identity on $\mathcal{S}\mathcal{X}$.

Lemma 2.7. (1) *If φ is a symmetric linear functional on $\mathcal{X}$ then $\varphi \circ L$ is translation-invariant functional on $\mathcal{S}\mathcal{X}$.*

(2) *$\xi \in \mathcal{R}$ if and only if there exists $\eta \in \mathcal{S}\mathcal{X}_+$ such that*

$$\Big|\sum_{k=m+1}^{n} \xi_k\Big| \le \eta_m + \eta_n, \quad -\infty < m < n < \infty. \tag{2.11}$$

Proof. (1) Let $A_n = B_n \cup C_n$ be a partition of A_n into two measurable sets such that $\mu B_n = \mu A_n = 2^{-(n+1)}$. If $\xi \in \mathcal{S}$ then

$$\begin{aligned}\varphi(L\xi) &= \varphi\Big(\sum_{n\in\mathbb{Z}} 2^n \xi_n \chi_{B_n} + \sum_{n\in\mathbb{Z}} 2^n \xi_n \chi_{C_n}\Big) = 2\varphi\Big(\sum_{n\in\mathbb{Z}} 2^n \xi_n \chi_{B_n}\Big) \\ &= 2\varphi\Big(\sum_{n\in\mathbb{Z}} 2^{n-1} \xi_{n-1} \chi_{A_n}\Big) \\ &= \varphi(L\mathcal{T}\xi).\end{aligned}$$

(2) If $\xi = \eta - \mathcal{T}\eta$ then

$$\Big|\sum_{k=m+1}^{n} \xi_k\Big| = |\eta_n - \eta_m| \le |\eta_m| + |\eta_n|.$$

Conversely assume (2.11). It suffices, by splitting into real and imaginary parts, to treat the case when ξ is real. Let

$$\zeta_n = \begin{cases} \sum_{k=0}^{n} \xi_k & n \ge 0 \\ -\sum_{k=1}^{-n} \xi_{-k} & n < 0. \end{cases}$$

Then by (2.11) we have

$$\sup_n(\zeta_n - \eta_n) \le \inf_m(\zeta_m + \eta_m).$$

Pick $\lambda \in \mathbb{R}$ so that

$$\zeta_n - \eta_n \le \lambda \le \zeta_m + \eta_m, \quad m, n \in \mathbb{Z}.$$

Then $(\zeta_n - \lambda)_{n\in\mathbb{Z}} \in \mathcal{S}\mathcal{X}$ and $(I - \mathcal{T})(\zeta_n - \lambda)_{n\in\mathbb{Z}} = \xi$. $\square$

We now come to the main theorem of the section. As mentioned in the Introduction, results of this type were known to the first author in the mid 1980's. See Pietsch [14] where sequence space analogues are considered. A non-commutative analogue of this result appears in Dykema, Figiel, Weiss and Wodzicki [6]; see also [11] for the earlier special case of the trace-class, analogous to the case $\mathcal{X} = L_1$.

Theorem 2.8. *Let $\mathcal{X}$ be a symmetric ideal on (Ω, Σ, μ). Then $f \in Z_{\mathcal{X}}$ if and only if $\varphi(f) = 0$ for every symmetric linear functional on $\mathcal{X}$.*

Proof. Let us introduce the space $\mathcal{N}$ of $\mathcal{X}$ defined by $f \in \mathcal{N}$ if and only if $\varphi(f) = 0$ for every symmetric linear functional. We will prove that $\mathcal{N} = Z_{\mathcal{X}}$. First suppose $f \in \mathcal{N}$; we will show that $f \in Z_{\mathcal{X}}$. Indeed f can be written in the form $f = \sum_{j=1}^{m}(g_j - h_j)$ where $g_j, h_j \in \mathcal{X}$ and $g_j \sim h_j$. It thus suffices to show that $g - h \in Z_{\mathcal{X}}$ whenever $g \in \mathcal{X}$ and $g \sim h$. Note that for any u, v we have $K(u, v; g) = K(u, v; h) = -K(u, v; -h)$. Hence by Lemma 2.3 (2.7) we have

$$|K(u, v; g - h)| \leq 8(u\psi^*(u) + v\psi^*(v))$$

where $\psi = |g| + |h|$. We conclude that $\mathcal{N} \subset Z_{\mathcal{X}}$.

For the converse we shall prove the following statements:

$$\mathcal{X} = \mathcal{N} + L(\mathcal{S}\mathcal{X}). \tag{2.12}$$

$$Z_{\mathcal{X}} = \mathcal{N} + L(\mathcal{R}). \tag{2.13}$$

Let us suppose $f \in \mathcal{X}$. Then we may find $g \in \mathcal{X}$ so that $g \sim f$ and for every n, $g\chi_{A_n} \sim f\chi_{E_{2^{-n},2^{1-n}}(f)}$. Let

$$\xi_n = \int_{A_n} g\, d\mu.$$

By Theorem 2.6 applied to $(g - \xi_n)\chi_{A_n}$ for each n we can write $g - L\xi = h_1 - h_2$ where $h_1 \sim h_2$ and $|h_j\chi_{A_n}| \leq 12f^*(2^{-n})$ for every n and $j = 1, 2$. Hence $h_1, h_2 \in \mathcal{X}$. Now

$$f - L\xi = (f - g) + (h_1 - h_2) \in \mathcal{N}.$$

This proves (2.12). If $f \in Z_{\mathcal{X}}$ we note that

$$\sum_{k=m+1}^{n} \xi_k = \int_{\cup_{k=m+1}^{n} A_k} g\, d\mu = \int_{E_{2^{-n},2^{-m}}(f)} f\, d\mu.$$

Hence by Lemma 2.2 (2.5)

$$\left|\sum_{k=m+1}^{n} \xi_k\right| \leq |K(2^{-n}, 2^{-m}; f)| + 8(2^{-m}f^*(2^{-m}) + 2^{-n}f^*(2^{-n}).$$

It follows that there exists $\psi \in \mathcal{X}_+$ so that

$$\left|\sum_{k=m+1}^{n} \xi_k\right| \leq 2^{-m}\psi^*(2^{-m}) + 2^{-n}\psi^*(2^{-n}), \quad m < n.$$

Now by (2) of Lemma 2.7 this implies $\xi \in \mathcal{R}$. This proves 2.13.

By (1) of Lemma 2.7 we have $L(\mathcal{R}) \subset \mathcal{N}$ and so $Z_{\mathcal{X}} \subset \mathcal{N}$. This completes the proof. □

In fact we have also proved the following statement:

Corollary 2.9. *There is a natural isomorphism between the space of symmetric functionals on $\mathcal{X}$ and the space of translation-invariant functionals on $\mathcal{S}\mathcal{X}$ implemented by $\varphi \to \varphi \circ L$.*

Proof. We first show that $L(\mathcal{R}) = \mathcal{N} \cap L(\mathcal{X})$. Indeed by Lemma 2.7 (1) we know $L(\mathcal{R}) \subset \mathcal{N}$. Now suppose $L\xi = f \in \mathcal{N}$. Then $|\xi_n| \leq f^*(2^{-n})$. If $g = \sum_{n\in\mathbb{Z}} f^*(2^{-n})\chi_{A_n}$ then $g \in \mathcal{X}$ and $|f| \leq g$. Hence by Lemma 2.2, (2.5),

$$\left|\sum_{k=m+1}^{n} \xi_k\right| \leq |K(2^{-n}, 2^{-m}; f)| + 8(2^{-m}g^*(2^{-m}) + 2^{-n}g^*(2^{-n})).$$

Now since $f \in Z_{\mathcal{X}}$ this inequality implies that $\xi \in \mathcal{R}$ by Lemma 2.7 (2). Now by (2.12) and the fact that $Z_{\mathcal{X}} = \mathcal{N}$ it follows that L induces an isomorphism between $\mathcal{S}\mathcal{X}/\mathcal{R}$ and $\mathcal{X}/\mathcal{N}$. This implies the corollary. □

We have stated all these results for infinite measure spaces. As noted at the beginning of the section we can quickly derive from this the corresponding results for probability spaces. Indeed it is not difficult to show that for ideals on probability spaces one may consider the sequence space $\mathcal{S}\mathcal{X}(\mathbb{N})$ modelled on the natural numbers, and, in this case, we define translation to be the right shift, $\mathcal{T}(\xi) = (\xi_{n-1})_{n\in\mathbb{N}}$ with the understanding that $\xi_0 = 0$. The result corresponding to Corollary 2.9 then holds.

Let us consider a few examples.

Suppose $\int_0^1 f^*(s)ds < \infty$ for all $f \in \mathcal{X}$. Then we may define $K(0, v; f) = \lim_{u\to 0} K(u, v; f)$. Conversely if $\int_1^\infty f^*(s)ds < \infty$ for all $f \in \mathcal{X}$, then we may define $K(u, \infty; f) = \lim_{v\to\infty} K(u, v; f)$. The latter situation arises if $\mathcal{X}$ is of finite type, or if $\mu(\Omega) = 1$.

Proposition 2.10. (1) *If $\int_0^1 f^*(s)ds < \infty$ for all $f \in \mathcal{X}$ then $f \in Z_{\mathcal{X}}$ and only if*

$$|K(0, u; f)| \leq uh^*(u), \quad 0 < u < \infty, \tag{2.14}$$

for some $h \in \mathcal{X}$.

(2) *If $\mathcal{X}$ has the property that $\int_1^\infty f^*(s)ds < \infty$ for every $f \in \mathcal{X}$ then*

$$|K(u,\infty;f)| \le uh^*(u), \quad 0 < u < \infty, \tag{2.15}$$

for some $h \in \mathcal{X}$.

Proof. In case (1) note that the hypothesis implies $\lim_{u\to 0} uh^*(u) = 0$ for all $h \in \mathcal{X}$; it is then easy to see that (2.9) reduces to (2.14). Similarly if (2) holds then $\lim_{u\to\infty} uh^*(u) = 0$ for all $h \in \mathcal{X}$ and the argument is similar. □

Let us use these remarks to relate the property $Z_{\mathcal{X}} = \mathcal{X}$ to the Boyd indices. We first summarize some properties of the Boyd indices in the next lemma.

Lemma 2.11. *Suppose $p < p_{\mathcal{X}} \le q_{\mathcal{X}} < q$. Suppose $h \in \mathcal{X}(\mathbb{R})$. Then there exists a constant C such that*

$$\Big\| \sum_{k\in\mathbb{Z}} 2^{\min(k/p,k/q)} D_{2^{-k}} h \Big\|_{\mathcal{X}} \le C\|h\|_{\mathcal{X}} \tag{2.16}$$

$$C^{-1}\|h\|_{L_p+L_q} \le \|h\|_{\mathcal{X}} \le C\|h\|_{L_p\cap L_q}. \tag{2.17}$$

It is easy then to see that the following proposition holds.

Proposition 2.12. *Suppose $\mathcal{X}$ is a quasi-Banach ideal.*
(1) *If $q_{\mathcal{X}} < 1$ then $Z_{\mathcal{X}} = \mathcal{X}$.*
(2) *If $p_{\mathcal{X}} > 1$ and $\mu(\Omega) = \infty$ then $Z_{\mathcal{X}} = \mathcal{X}$.*
(3) *If $p_{\mathcal{X}} > 1$ and $\mu(\Omega) = 1$ then $Z_{\mathcal{X}} = \{f \in \mathcal{X} : \int f\,d\mu = 0\}$.*

Proof. Just note that if $f \in \mathcal{X}_+$ then, where applicable,

$$K(0,u;f) = \int_0^1 f^*(su)\,ds \le \sum_{k\le 0} 2^k D_{2^{-k}} f(u)$$

and

$$K(u,\infty;f) = \int_1^\infty f^*(su)ds \le \sum_{k\ge 1} 2^k D_{2^{-k}} f(u).$$

Then (1) and (2) follow from Lemma 2.11 (2.16) and the remarks preceding it. (3) follows from the fact that if $f \in L_1$, is real-valued and $\int f\,d\mu = 0$ then $K(0,u;f) = -K(u,\infty;f)$ This implies that $f \in Z_{\mathcal{X}}$. □

Thus for quasi-Banach ideals the only case where one gets a non-trivial symmetric linear functional is when $p_{\mathcal{X}} \le 1 \le q_{\mathcal{X}}$. We now discuss the cases of L_1.

If f is a real function in $\mathcal{M}$ we introduce the function $f_d : (-\infty,\infty) \to \mathbb{R}$ defined by the conditions $f_d \sim f$, f_d is decreasing and non-positive on $(-\infty, 0)$ and f_d is

decreasing and non-negative on $(0,\infty)$. Then it is clear that

$$K(u,v;f) = \int_{u\le|t|\le v} f_d(t)dt.$$

In the case of L_1 the center was introduced as the *symmetric Hardy class* H_1^{sym} in [10]. A real function $f \in H_1^{\mathrm{sym}}(\Omega)$ if and only if

$$\int_0^\infty |M(t)|\frac{dt}{t} < \infty$$

where

$$M(t) = \int_{-t}^t f_d(s)ds.$$

It follows that there are many discontinuous symmetric functionals on L_1 in both the probability and infinite measure cases.

An interesting case is to take $\mathcal{X}$ to be the space of all f so that $\lim_{t\to 0} tf^*(t) = \lim_{t\to\infty} tf^*(t)dt = 0$. Then $f \in Z_{\mathcal{X}}$ if and only if

$$\lim_{\substack{u\to 0\\ v\to\infty}} \int_{|u|<t<|v|} f_d(s)ds = 0$$

or equivalently

$$\lim_{\substack{r\to 0\\ s\to\infty}} \int_{r<|f|<s} f(s)ds = 0.$$

Symmetric continuous linear functionals on weak L_1 were used in [8] (and are implicit in the work of Cwikel and Fefferman [2] on the dual of weak L_1).

3. Analytic functions

Let $\mathbb{D}$ be the unit disk and $\mathbb{T}$ be the unit circle. We recall that the Nevanlinna class $N(\mathbb{D})$ consists of all functions f which are analytic in $\mathbb{D}$ and satisfy

$$\sup_{r<1} \int_0^{2\pi} \log_+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty.$$

If $f \in N$ then the boundary values $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exist almost everywhere. The Smirnov class $N^+(\mathbb{D})$ consists of all functions $f \in N$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log_+ |f(re^{i\theta}|\frac{d\theta}{2\pi} = \int_0^{2\pi} \log_+ |f(e^{i\theta}|\frac{d\theta}{2\pi}.$$

We can regard N^+ as both a space of analytic functions on $\mathbb{D}$ and as a space of measurable functions on $\mathbb{T}$. We shall use the notation $f_r(e^{i\theta}) = f(re^{i\theta})$ so that $f_r : \mathbb{T} \to \mathbb{C}$. We will use μ to denote the normalized measure $d\theta/2\pi$.

We also need analogous spaces in the upper half-plane $\mathbb{U} = \{z : \Im z > 0\}$. We define $N^+(\mathbb{U})$ to be the class of all functions f analytic in $\mathbb{U}$ so that $f \circ \varphi \in N^+$ where $\varphi : \mathbb{D} \to \mathbb{U}$ is given by $\varphi(z) = i(1+z)/(1-z)$. If $f \in N^+(\mathbb{U})$ then

$$\lim_{y\to 0} f(x+iy) = f(x)$$

exists a.e. and

$$\log|f(x+iy)| \le \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(x)|}{(t-x)^2+y^2}dt. \tag{3.1}$$

We will write $f_y(x) = f(x+iy)$ and in this case μ is Lebesgue measure. As before we can consider $N^+(\mathbb{U})$ as a space of analytic functions in the upper half-plane or as a space of measurable functions on the real line.

In the next technical theorem we prove a result on the distribution of boundary values of functions in $N^+(\mathbb{D})$ and $N^+(\mathbb{U})$.

Theorem 3.1. *There exists a constant C so that:*

(1) *If $f \in N^+(\mathbb{D})$ and $f(0) = 0$ then, whenever $0 < r < s < \infty$,*

$$\begin{aligned}\left|\int_{r\le|f|<s} f(e^{i\theta})\frac{d\theta}{2\pi}\right| &\le C\Bigg(r\mu(|f|>r) + s\mu(|f|>s)\\ &\quad + \int_0^{2\pi} r\log_+\frac{|f(e^{i\theta})|}{r} + s\log_+\frac{|f(e^{i\theta})|}{s}\frac{d\theta}{2\pi}\Bigg)\end{aligned} \tag{3.2}$$

(2) *If $f \in N^+(\mathbb{U})$ then*

$$\begin{aligned}\left|\int_{r\le|f|<s} f(x)dx\right| &\le C\Bigg(r\mu(|f|>r) + s\mu(|f|>s)\\ &\quad + \int_{-\infty}^{\infty} r\log_+\frac{|f(x)|}{r} + s\log_+\frac{|f(x)|}{s}dx\Bigg)\end{aligned} \tag{3.3}$$

Proof. We begin by fixing a smooth bump function $b : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} b \subset (0, 1/2)$, $b \ge 0$, $\int b(x)dx = 1$. Let $\beta(t) = 2|b(t)| + |b'(t)|$.

Note first that the estimates are trivial when $s \le 2r$.

Now suppose $0 < r < s < \infty$, with $s \ge 2r$. We define

$$\varphi(\tau) = \varphi_{r,s}(\tau) = \int_{-\infty}^{\tau} b(t-\log r) - b(t-\log s)dt.$$

Notice that the two terms in the integrand are never simultaneously positive (since $\log 2 > \frac12$), and φ is a bump function which satisfies $\varphi(\tau) = 0$ if $\tau < \log r$ or $\tau > \frac12 + \log s$ while $\varphi(\tau) = 1$ if $\log r + \frac12 \le \tau \le \log s$. Then let $\psi = \psi_{r,s}$ be defined

to be the function such that $\psi(\tau) = 0$ if $\tau < \log r$ and

$$\psi''(\tau) = e^{\tau}(2|\varphi''(\tau)| + |\varphi'(\tau)|).$$

In fact, this implies that

$$\psi''(\tau) = e^{\tau}(\beta(\tau-\log r) + \beta(\tau-\log s))$$

and then

$$\psi(\tau) = \psi_{r,s}(\tau) = \int_{-\infty}^{\tau} (\tau-t)e^t(\beta(t-\log r) + \beta(t-\log s))\,dt,$$

and

$$\psi'(\tau) = \int_{-\infty}^{\tau} e^t(\beta(t-\log r) + \beta(t-\log s))dt.$$

Thus, if we set

$$C_0 = \int_{-\infty}^{\infty} e^t\beta(t)dt$$

then,

$$\psi'(\tau) \le C_0(r\chi_{(\tau>\log r)} + s\chi_{(\tau>\log s)})$$

and so

$$\psi_{r,s}(\tau) \le C_0(r(\tau-\log r)_+ + s(\tau-\log s)_+). \tag{3.4}$$

Now we use the argument of Lemma 2.6 of [12]. If we define

$$h(z) = h_{r,s}(z) = \psi(\log|z|) - x\varphi(\log|z|), \quad z = x+iy \ne 0,$$

and $h(0) = 0$ then h is a C^2-subharmonic function which vanishes on a neighborhood of 0. We note the estimates (from (3.4))

$$\psi(\log|z|) \le C_0\left(r\log_+\frac{|z|}{r} + s\log_+\frac{|z|}{r}\right) \tag{3.5}$$

and

$$0 \le h(z) \le C_0\left(r\log_+\frac{|z|}{r} + s\log_+\frac{|z|}{r}\right), \quad |z| \ge 2s. \tag{3.6}$$

Note of course that C_0 is independent of r, s.

Let us first treat the case when $f \in N^+(\mathbb{D})$ and $f(0) = 0$. Then $h \circ f$ is subharmonic on $\mathbb{D}$. Hence

$$0 \le \int_0^{2\pi} h\circ f_r\frac{d\theta}{2\pi}$$

for $0 \le r < 1$. Now $h_+(z) := \max(h(z), 0) \le M(\log_+ |z| + 1)$ for a suitable constant M and $h_+ - h$ is bounded. From the definition of the Smirnov class we have:

$$\lim_{r\to 1} \int_{|f_r|>2|f|} \log_+ |f_r| \frac{d\theta}{2\pi} = 0.$$

Hence

$$\lim_{r\to 1} \int_{|f_r|>2|f|} h_+ \circ f_r \frac{d\theta}{2\pi} = 0.$$

Now by the Dominated Convergence Theorem we have

$$\lim_{r\to 1} \int_{|f_r|\le 2|f|} h_+ \circ f_r \frac{d\theta}{2\pi} = \int_0^{2\pi} h_+ \circ f \frac{d\theta}{2\pi}.$$

Thus

$$\lim_{r\to 1} \int_0^{2\pi} h_+ \circ f_r \frac{d\theta}{2\pi} = \int_0^{2\pi} h_+ \circ f \frac{d\theta}{2\pi}.$$

By the Bounded Convergence Theorem,

$$\lim_{r\to 1} \int_0^{2\pi} (h - h_+) \circ f_r \frac{d\theta}{2\pi} = \int_0^{2\pi} (h - h_+) \circ f \frac{d\theta}{2\pi}.$$

We thus conclude that

$$\int_0^{2\pi} h \circ f \frac{d\theta}{2\pi} \ge 0$$

and thus

$$\Re \int_0^{2\pi} f\varphi(\log|f|) \frac{d\theta}{2\pi} \le \int_0^{2\pi} \psi(\log|f|) \frac{d\theta}{2\pi}.$$

Applying this inequality to αf for every α with $|\alpha| = 1$ gives

$$\left| \int_0^{2\pi} f\varphi(\log|f|) \frac{d\theta}{2\pi} \right| \le \int_0^{2\pi} \psi(\log|f|) \frac{d\theta}{2\pi}. \tag{3.7}$$

Now note that

$$\begin{aligned} \left| \int_0^{2\pi} f(\varphi(\log|f|) - \chi_{(r\le|f|<s)}) \frac{d\theta}{2\pi} \right| &\le \int_0^{2\pi} |f|(\chi_{(r<f<2r)} + \chi_{(s<|f|<2s)}) \frac{d\theta}{2\pi} \\ &\le 2r\mu(|f| > r) + 2s\mu(|f| > s). \end{aligned}$$

Now combining with (3.4) and (3.7) gives (3.2).

The proof of (3.3) is somewhat similar. We only sketch the details. It is clear that we may assume that

$$\int_{-\infty}^{\infty} \log_+ |f| dx < \infty.$$

It then follows from (3.1) that

$$\int_{-\infty}^{\infty} \log_+ |f_y| dx \le \int_{-\infty}^{\infty} \log_+ |f| dx \tag{3.8}$$

whenever $y > 0$ and also that $\lim_{y\to\infty} \sup |f(x + iy)| = 0$. As above we note that $h \circ f$ is subharmonic on $\mathbb{U}$; in this case $h \circ f$ vanishes for y large enough. It follows using (3.6) that $\int_{-\infty}^{\infty} h \circ f_y dx$ is a convex function of $y > 0$ and so that

$$\lim_{y\to 0} \int_{-\infty}^{\infty} h \circ f_y dx \ge 0.$$

In this case we use (3.8) in place of the Smirnov condition to deduce

$$\int_{-\infty}^{\infty} h \circ f \, dx \ge 0.$$

The remaining details are then similar. □

We now define (cf. [12]) a symmetric ideal $\mathcal{X}$ to be *geometrically stable* if given $f \in \mathcal{X}$ there exists $h \in \mathcal{X}$ with

$$\exp\left(\frac{1}{t}\int_0^t \log f^*(s)\, ds\right) \le h^*(t), \quad 0 < t < \infty. \tag{3.9}$$

Note of course that we require $\log_+ |f| \in L_1$ for all $f \in \mathcal{X}$.

Proposition 3.2. *Every symmetric quasi-Banach ideal is geometrically stable.*

Proof. For $0 < p \le 1$ we note that for an appropriate constant C_1,

$$\begin{aligned} \exp\left(\frac{1}{t}\int_0^t \log f^*(s)\, ds\right) &\le \left(\frac{1}{t}\int_0^t f^*(s)^p\, ds\right)^{\frac{1}{p}} \\ &\le \left(\sum_{n=1}^{\infty} 2^{-n} D_{2^n} f^*(t)^p\right)^{\frac{1}{p}} \\ &\le C_1 \sum_{n=1}^{\infty} 2^{-\frac{n}{2}-\frac{n}{2p}} D_{2^n} f^*(t). \end{aligned}$$

Now if $\frac{1}{2} + \frac{1}{2p} > \frac{1}{p\mathcal{X}}$ the series

$$C_1 \sum_{n=1}^{\infty} 2^{-\frac{n}{2}-\frac{n}{2p}} D_{2^n} f^*$$

converges in $\mathcal{X}(0, 1)$ or $\mathcal{X}(0, \infty)$ to some $h = h^*$ satisfying (3.9). □

If $\mathcal{X}$ is a symmetric ideal on $\mathbb{T}$ let us define $H_{\mathcal{X}}(\mathbb{T})$ to be the space of all $f \in \mathcal{X}(\mathbb{T})$ so that $f \in N^+(\mathbb{D})$. Similarly $H_{\mathcal{X}}(\mathbb{R})$ consists of all $f \in \mathcal{X}(\mathbb{R})$ so that $f \in N^+(\mathbb{U})$.

Theorem 3.3. *Suppose $\mathcal{X}$ is a geometrically stable symmetric ideal on $\mathbb{T}$ (in particular this applies when $\mathcal{X}$ is a quasi-Banach ideal). Suppose $f \in H_{\mathcal{X}}$ with $f(0) = 0$. Then $f \in Z_{\mathcal{X}}(\mathbb{T})$ and so for every symmetric linear functional φ on $\mathcal{X}$ we have $\varphi(f) = 0$.*

Theorem 3.4. *Suppose $\mathcal{X}$ is a geometrically stable symmetric ideal on $\mathbb{R}$ (in particular this applies when $\mathcal{X}$ is a quasi-Banach ideal). Suppose $f \in H_{\mathcal{X}}(\mathbb{R})$. Then $f \in Z_{\mathcal{X}}$ and so for every symmetric linear functional φ on $\mathcal{X}$ we have $\varphi(f) = 0$.*

Proof. The proofs of both theorems are essentially the same. We prove only Theorem 3.3. Assume first f satisfies the condition that $\mu(|f| = r) = 0$ for all $r > 0$. Then if $0 < u < v < \infty$ we have $E_{u,v}(|f|) = \{f^*(u) \le |f| < f^*(v)\}$ and so

$$\left| K(u, v; f) - \int_{f^*(u) \le |f| < f^*(v)} f \frac{d\theta}{2\pi} \right| \le 2(uf^*(u) + vf^*(v))$$

by Lemma 2.1, (2.3). Thus using Theorem 3.1 (3.2),

$$|K(u, v; f)|$$
$$\le (C+2)(uf^*(u) + vf^*(v)) + \int_0^{2\pi} f^*(u) \log_+ \frac{|f|}{f^*(u)} + f^*(v) \log_+ \frac{|f|}{f^*(v)} \frac{d\theta}{2\pi}.$$

Now by (3.9) we can find h with

$$\frac{1}{t} \int_0^t \log f^*(s)\, ds \le \log h^*(t).$$

Now

$$\begin{aligned} \int_0^{2\pi} f^*(u) \log_+ \frac{|f|}{f^*(u)} \frac{d\theta}{2\pi} &= f^*(u) \int_0^u (\log f^*(t) - \log f^*(u)) dt \\ &\le uf^*(u) \log \frac{h^*(u)}{f^*(u)} \\ &\le uh^*(u). \end{aligned}$$

This together with a similar calculation for v gives the estimate

$$K(u, v; f) \le ug^*(u) + vg^*(v)$$

where $g^* = (C+2)f^* + h^*$.

Now for the general case we may clearly define some $g \in N^+(\mathbb{D})$ so that $g \in \mathcal{X}(\mathbb{T})$ and $\mu(|g| = r) = 0$ for every choice of r. It then follows that for almost every choice of $(e^{i\theta}, e^{i\phi}) \in \mathbb{T}^2$ we have $|g(e^{i\theta})| \ne |g(e^{i\phi})|$. Let A be the set of $(t, e^{i\theta}, e^{i\phi}) \in [-1, 1] \times \mathbb{T}^2$ so that $|f(e^{i\theta}) + tg(e^{i\theta})| = |f(e^{i\phi}) + tg(e^{i\phi})|$. By expanding as a polynomial in t it follows that for almost every $(e^{i\theta}, e^{i\phi})$ the set of t such that $(t, e^{i\theta}, e^{i\phi}) \in A$ has Lebesgue measure zero. Applying Fubini's theorem A has product measure zero and so we may find $t \ne 0$ so that the set of $(e^{i\theta}, e^{i\phi})$ so that either $|f(e^{i\theta}) + tg(e^{i\theta})| = |f(e^{i\phi}) + tg(e^{i\phi})|$ or $|f(e^{i\theta}) - tg(e^{i\theta})| = |f(e^{i\phi}) - tg(e^{i\phi})|$ has measure zero. From this it follows that for every $r > 0$ we have $\mu(|f + tg| = r) = \mu(|f - tg| = r) = 0$. Now applying the first part $f \pm tg \in Z_{\mathcal{X}}$ and so $f \in Z_{\mathcal{X}}$. □

Theorem 3.3 has some amusing applications of which we give two.

Corollary 3.5. *Suppose $f \in N^+(\mathbb{D})$ and $\lim_{r\to\infty} r\mu(|f| > r) = 0$. Then*

$$f(0) = \lim_{r\to\infty} \int_{|f|<r} f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Proof. If $f(0) = 0$ this follows directly from Theorem 3.3 and the remarks at the end of Section 2. If $f(0) = \alpha \ne 0$ then note that

$$\lim_{r\to\infty} \left| \int_{\substack{|f|<r \\ |f-\alpha|\le r}} f \frac{d\theta}{2\pi} \right| + \left| \int_{\substack{|f|\ge r \\ |f-\alpha|<r}} f \frac{d\theta}{2\pi} \right| = 0$$

follows from $\lim_{r\to\infty} r\mu(|f| > r) = 0$. □

Corollary 3.6. *Suppose $\mathcal{X}$ is a geometrically stable symmetric ideal on $\mathbb{T}$ which admits a regular symmetric linear functional φ. Then if $f \in H_{\mathcal{X}}(\mathbb{T})$ we have $\varphi(f) = f(0)$.*

We omit the trivial proof.

4. Conjugate functions

Ceretelli [3] and later (independently) Davis [4] proved that a real function $f \in L_1(\mathbb{T})$ is in $H_{1,0}^{\mathrm{sym}} = Z_{L_1}$ if and only if $f \sim \Re g$ for some $g \in H_1$ with $g(0) = 0$. See also [10] for an alternate proof and [9] for a vector-valued generalization. In [7] there is a general discussion of problems of this type. Let us state our extensions of this result to arbitrary quasi-Banach ideals.

Theorem 4.1. *Let $\mathcal{X}$ be a symmetric quasi-Banach ideal on $\mathbb{T}$ with $\frac{1}{2} < p_{\mathcal{X}} \le q_{\mathcal{X}} < \infty$. If $f \in \mathcal{X}_{\mathbb{R}}$ then in order that there exist $g \in H_{\mathcal{X}}(\mathbb{T})$ with $g(0) = 0$, and $\Re g \sim f$ it is necessary and sufficient that $f \in Z_{\mathcal{X}}$.*

Theorem 4.2. *Let $\mathcal{X}$ be a symmetric quasi-Banach ideal on $\mathbb{R}$ with $q_{\mathcal{X}} < \infty$. If $f \in \mathcal{X}_{\mathbb{R}}$ then in order that there exist $g \in H_{\mathcal{X}}(\mathbb{R})$, with $\Re g \sim f$ it is necessary and sufficient that $f \in Z_{\mathcal{X}}$.*

Before proving this result we note that Davis [4] establishes Theorem 4.1 for the case $\mathcal{X} = L_p$ where $\frac{1}{2} < p < 1$. In this case $Z_{L_p} = L_p$. In fact Aleksandrov proves a stronger theorem [1] that the map $f \to \Re f$ maps $H_{\mathcal{X}}(\mathbb{T})$ *onto* $\Re\mathcal{X}(\mathbb{T})$ when $q_{\mathcal{X}} < 1$. If $1 < p_{\mathcal{X}} < q_{\mathcal{X}} < \infty$ then the Hilbert transform is bounded on $\mathcal{X}$ and the theorems follow easily (cf. Proposition 2.12). Thus the only interesting cases are when $p_{\mathcal{X}} \leq 1 \leq q_{\mathcal{X}}$.

Let $L_{\log}$ denote the (non-locally convex) Orlicz space of all functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\int \frac{\log_+ |f(x)|}{1+x^2} dx < \infty.$$

This is an F-space when it is equipped with the F-norm

$$f \to \int \frac{\log(1+|f(x)|)}{1+x^2} dx.$$

It is easy to see from (2.17) that $\mathcal{X}$ is continuously embedded into $L_{\log}$. Now, when identified with its boundary values, $N^+(\mathbb{U})$ is a closed subspace of $L_{\log}$ and hence $H_{\mathcal{X}}$ is a closed subspace of $\mathcal{X}$.

Lemma 4.3. *Suppose E is a bounded measurable subset of $\mathbb{R}$ and $N \in \mathbb{N}$. If $f : E \to \mathbb{R}$ is measurable there exists $g : E \to \mathbb{R}$ with $g \sim f$ and*

$$\int_E t^k g(t)\, dt = \frac{1}{\mu(E)} \left(\int_E t^k dt \right) \left(\int_E g(t)\, dt \right), \quad 0 \leq k \leq N.$$

Proof. By repeated applications of Lyapunov's theorem taking $A_1 = E$ we can find Borel sets A_n with $A_n = A_{2n} \cup A_{2n+1}$, $A_{2n} \cap A_{2n+1} = \emptyset$, $\mu(A_{2n}) = \mu(A_{2n+1}) = \frac{1}{2}\mu(A_n)$ and

$$\int_{A_{2n}} t^k\, dt = \int_{A_{2n+1}} t^k\, dt = \frac{1}{2} \int_{A_n} t^k\, dt, \quad 0 \leq k \leq N.$$

The sets $(A_n)_{n=0}^{\infty}$ generate a non-atomic sub$-\sigma$-algebra Σ_0 of measurable subsets of E so that $B \in \Sigma_0$ implies

$$\int_B t^k\, dt = \mu(B) \int_E t^k\, dt, \quad 0 \leq k \leq N.$$

Take $g \sim f$ to be Σ_0-measurable. □

We now turn to the proofs of the theorems. We first consider Theorem 4.2, and then indicate the modifications which establish Theorem 4.1.

Proof. One direction follows immediately from Theorems 3.3 and 3.4. In Theorem 4.1 if $g \in H_{\mathcal{X}}$ and $g(0) = 0$ then Theorem 3.3 implies that $g \in Z_{\mathcal{X}}(\mathbb{T})$. Thus $\Re g \in Z_{\mathcal{X}}$ by Proposition 2.5. A similar argument applies in Theorem 4.2.

We prove first Theorem 4.2. In the proof we use C to denote a constant which depends only on $\mathcal{X}$ but which can vary from line to line. Let us pick $N \in \mathbb{N}$ so that $N + 2 > \frac{1}{p_{\mathcal{X}}}$.

Define f_d as before (see after Proposition 2.12). Now let $J_n = (2^n, 2^{n+1}] \cup [-2^{-n-1}, -2^n)$. Let E_0 be a measurable subset of J_0 so that $\mu(E) = \frac{1}{2}\mu(J_0) = 1$ and

$$\int_E t^k dt = \frac{1}{2} \int_{J_0} t^k\, dt, \quad 0 \leq k \leq N.$$

Let $F_0 = J_0 \setminus E_0$ and then let $E_n = 2^n E_0$, $F_n = 2^n F_0$ for $n \in \mathbb{Z}$.

By using Lemma 4.3 on each set $E_n \cup F_{n+1}$ (which has measure $(3/2)\mu J_n$) we can find a function f_c so that $f_c \chi_{E_n \cup F_{n+1}} \sim f_d \chi_{J_n}$ for $n \in \mathbb{Z}$ and

$$\int_{E_n \cup F_{n+1}} t^k f_c(t) dt = 2^{-n-1} \int_{E_n \cup F_n} t^k\, dt \int_{J_n} f_c(t)\, dt, \quad 1 \leq k \leq N, \ n \in \mathbb{Z}.$$

Let us define $K(u, v; f) = -K(v, u; f)$ if $u > v$. Then there exists $h \in \mathcal{X}$ so that

$$|K(u, v; f)| \leq u h^*(u) + v h^*(v), \quad u, v > 0.$$

Hence

$$K(u, 1; f) - u h^*(u) \leq K(v, 1; f) + v h^*(v), \quad u, v > 0.$$

Now (as in Lemma 2.7) this implies the existence of $\lambda \in \mathbb{R}$ so that

$$|K(u, 1; f) - \lambda| \leq u h^*(u), \quad 0 < u < \infty.$$

We may assume h^* is constant on each interval $(2^n, 2^{n+1})$ and that $|f_c(t)| \leq h^*(t)$. For $n \in \mathbb{Z}$ we define ϕ_n by

$$\phi_n(t) = \begin{cases} f_c(t) - 2^{-n-1}(K(2^n, 1; f) - \lambda) & t \in E_n \\ -2^{-(n-1)}(K(2^n, 1; f) - \lambda) & t \in F_n \\ 2^{-n-2}(K(2^{n+1}, 1; f) - \lambda) & t \in E_{n+1} \\ f_c(t) + 2^{-n-2}(K(2^{n+1}, 1; f) - \lambda) & t \in F_{n+1} \\ 0 & t \notin J_n \cup J_{n+1}. \end{cases}$$

Then

$$\int \phi_n(t) dt = \int_{E_n \cup F_{n+1}} f_c(t)\, dt + K(2^{n+1}, 1; f) - K(2^n, 1; f) = 0.$$

We thus have

$$\int t^k \phi_n(t)\, dt = 0, \quad 0 \leq k \leq N. \tag{4.1}$$

Note also that we have an estimate

$$|\phi_n(t)| \leq C h^*(2^n)$$

for a suitable constant C.

Define

$$\psi_n(z) := \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\phi_n(t)}{z-t}\,dt$$

for $z \in \mathbb{U}$. Then $\psi_n \in H^\infty$ and has boundary values for $\Im z = 0$ such that $\Re\psi_n(x) = \phi_n(x)$ a.e. Note that

$$\Im\psi_n(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{\phi_n(t)}{x-t}dx.$$

We complete the argument by showing that $\sum_{n\in\mathbb{Z}}\psi_n(x)$ converges a.e. to a function g in $H_{\mathcal{X}}$. To show this we show that $\sum_{n\in\mathbb{Z}}|\psi_n(x)|$ converges a.e. to a function in $\mathcal{X}$. It follows then that the series $\sum_{n\in\mathbb{Z}}\psi_n$ converges also in the topology of $L_p + L_q$ provided $p < p_{\mathcal{X}}$ and $q > q_{\mathcal{X}}$ and hence $g \in N^+ \cap \mathcal{X}$. Once this is done it is clear that $\Re g(x) = f_c(x)$ a.e. and so $\Re g \sim f$.

We first observe that if $|x| \le 2^{n-1}$ then we have an easy estimate that

$$|\psi_n(x)| \le Ch^*(2^n).$$

Next suppose $|x| \ge 2^{n+3}$. Then

$$\frac{1}{(x-t)} = \frac{1}{x} + \cdots + \frac{t^N}{x^{N+1}} + \frac{t^{N+1}}{x^{N+1}(x-t)},$$

and so by (4.1),

$$|\psi_n(x)| \le C|x|^{-(N+2)}2^{(N+2)n}h^*(2^n).$$

Thus if $x \in J_n$, and we have

$$|\psi_{n-k}(x)| \le C2^{-k(N+2)}h^*(2^{-k}|x|)$$

for $k \ge 3$ and

$$|\psi_{n+k}(x)| \le Ch^*(2^k|x|)$$

for $k \ge 2$.

Hence if $x \in J_n$,

$$\sum_{k\ge3}|\psi_{n-k}(x)| \le C\sum_{k\ge2}2^{-k(N+2)}h^*(2^{-k}|x|) \tag{4.2}$$

and

$$\sum_{k\ge2}|\psi_{n+k}(x)| \le C\sum_{k=2}^{\infty}h^*(2^k|x|). \tag{4.3}$$

Now by Lemma 2.11 we have that

$$\sum_{k\in\mathbb{Z}}2^{\min(k(N+2),0)}h^*(2^k|x|) < \infty, \quad x \neq 0$$

and

$$\Big\|\sum_{k\in\mathbb{Z}}2^{\min(k(N+2),0)}h^*(2^k|x|)\Big\|_{\mathcal{X}} \le C\|h\|_{\mathcal{X}}.$$

Consider the cases $-2 \le k \le 1$. Then

$$\left(\int_{J_n}|\psi_{n+k}(x)|^q dx\right)^{\frac{1}{q}} \le C2^{\frac{n+k}{q}}h^*(2^{n+k}).$$

For $r \ge 0$, let A_{nr} be the subset of J_n of all x such that $\sum_{-2}^{1}|\psi_{n+k}(x)| \ge 2^r h^*(2^{n-2})$. Then

$$\mu(A_{nr}) \le C2^{n-rq}.$$

Hence, if $q_{\mathcal{X}} < q' < q$,

$$\Big\|\sum_{n\in\mathbb{Z}}\sum_{k=-2}^{1}h^*(2^{n-2})\chi_{A_{nr}}\Big\|_{\mathcal{X}} \le C2^{-rq/q'}\|h\|_{\mathcal{X}}.$$

This quickly yields the estimate:

$$\Big\|\sum_{n\in\mathbb{Z}}\sum_{k=-2}^{1}|\psi_{n+k}|\chi_{J_n}\Big\|_{\mathcal{X}} \le C\|h\|_{\mathcal{X}}.$$

Combining gives us that $\sum_{n\in\mathbb{Z}}|\psi_n|$ converges a.e. to a function in $\mathcal{X}(\mathbb{R})$. This completes the proof of Theorem 4.2. □

Proof of Theorem 4.1. Only small modifications are necessary for Theorem 4.1. We note that f_d is now supported on a set $[a, 1-a]$ where $0 \le a \le 1$ and h^* is supported on $[0, 1]$. In this case with the additional assumption that $p_{\mathcal{X}} > \frac{1}{2}$ we can take $N = 0$. This allows us to simplify the construction by taking

$$\phi_n(t) = \begin{cases} f_d(t) - 2^{-n-1}K(2^n, 1; f) & t \in J_n \\ 2^{-n-2}K(2^{n+1}, 1; f) & t \in J_{n+1} \\ 0 & t \notin J_n \cup J_{n+1}. \end{cases}$$

The calculations are then very similar. Construct g as above and then define $g_0(0) = 0$ and

$$g_0(re^{i\theta}) = \sum_{n\in\mathbb{Z}}g\Big(\frac{1}{2\pi}(\theta - i\log r) + n\Big).$$

One only really needs to show that $g_0 \in \mathcal{X}(\mathbb{T})$ or equivalently that $\sum_{n\in\mathbb{Z}}g(x+n) \in \mathcal{X}(0, 1]$. Clearly $g\chi_{(n,n+1)} \in \mathcal{X}(n, n+1)$ for all $n \in \mathbb{Z}$. If $|n| \ge 3$ then one gets an estimate

$$\|g\chi_{(n,n+1]}\|_\infty \le C|n|^{-2}$$

by using (4.2) above. It is then easy to see that g_0 solves the problem. □

Let us remark that it should be possible to prove Theorem 4.1 without the restriction $p_X > \frac{1}{2}$ as in the case of Theorem 4.2.

References

[1] A. B. Aleksandrov, Essays on nonlocally convex Hardy classes, in: Complex analysis and spectral theory (Leningrad, 1979/1980), Lecture Notes in Math. 864, Springer-Verlag, Berlin–New York. 1981, 1–89.

[2] M. Cwikel and C. Fefferman, Maximal seminorms on weak L^1, Studia Math. 69 (1980), 149–154.

[3] O. D. Ceretelli, A metric characterization of the set of functions whose conjugate functions are integrable, Bull. Acad. Sci. Georgian S.S.R. 81 (1976), 281–283 (in Russian).

[4] B. Davis, Hardy spaces and rearrangements, Trans. Amer. Math. Soc. 261 (1980), 211–233.

[5] J. Dixmier, Existence de traces non normales, C. R. Acad. Sci. Paris Ser. A-B 262 (1966), A1107–A1108.

[6] K. J. Dykema, T. Figiel, G. Weiss and M. Wodzicki, The commutator structure of operator ideals, preprints 1997, 2001.

[7] A. Gulisashvili, Rearrangement-invariant spaces of functions on LCA groups. J. Funct. Anal. 156 (1998), 384–410.

[8] N. J. Kalton, Banach spaces embedding into L_0, Israel J. Math. 52 (1985), 305–319.

[9] N. J. Kalton, On rearrangements of vector-valued H_1–functions, Ark. Mat. 26 (1988,) 221–229.

[10] N. J. Kalton, Nonlinear commutators in interpolation theory, Mem. Amer. Math. Soc. 85, 1988.

[11] N. J. Kalton, Trace-class operators and commutators, J. Funct. Anal. 86 (1989), 41–74.

[12] N. J. Kalton, Spectral characterization of sums of commutators, I, J. Reine und Angew. Math. 504 (1998), 115–125.

[13] S. Kwapień, Linear functionals invariant under measure preserving transformations. Math. Nachr. 119 (1984), 175–179.

[14] A. Pietsch, Traces and shift invariant functionals, Math. Nachr. 145 (1990), 7–43.

Tadeusz Figiel, Institute of Mathematics of the Polish Academy of Sciences, 81-825 Sopot, Poland

E-mail: t.figiel@impan.gda.pl

Nigel Kalton, Department of Mathematics, University of Missouri, Columbia, MS 65211, U.S.A.

E-mail: nigel@math.missouri.edu

Function Spaces, Interpolation Theory and Related Topics

Proceedings of the International Conference
in honour of Jaak Peetre
on his 65th birthday

Lund, Sweden
August 17–22, 2000

Editors

Michael Cwikel
Miroslav Engliš
Alois Kufner
Lars-Erik Persson
Gunnar Sparr

Walter de Gruyter · Berlin · New York 2002

Editors

Michael Cwikel
Department of Mathematics
Technion – Israel Institute of Technology
Haifa 32000
Israel

Miroslav Engliš
Mathematical Institute
Czech Academy of Sciences
11567 Prague 1
Czech Republic

Alois Kufner
Mathematical Institute
Czech Academy of Sciences
11567 Prague 1
Czech Republic

Lars-Erik Persson
Department of Mathematics
Lulea University of Technology
97187 Lulea
Sweden

Gunnar Sparr
Department of Mathematics
Lund Institute of Technology/Lund University
22100 Lund
Sweden

Mathematics Subject Classification 2000:
00B30; 46-06; 47-06; 46B70; 46EXX, 47BXX

Keywords:
Interpolation theory, function spaces, operator theory

Printed on acid-free paper which falls within the guidelines of the ANSI to ensure permanence and durability.

Library of Congress – Cataloging-in-Publication Data

Function spaces, interpolation theory and related topics : proceedings of the international conference in honour of Jaak Peetre on his 65th birthday : Lund, Sweden, August 17–22, 2000 / editors, Michael Cwikel ... [et al.].
p. cm.
ISBN 3110171171 (acid-free paper)
1. Functions spaces – Congresses. 2. Interpolation spaces – Congresses. I. Peetre, Jaak, 1935– II. Cwikel, M. (Michael), 1948–
QA323 .F8652 2002
515'.73–dc21
2002067478

Die Deutsche Bibliothek – Cataloging-in-Publication Data

Function spaces, interpolation theory and related topics : proceedings of the International Conference in Honour of Jaak Peetre on His 65th Birthday, Lund, Sweden, August, 17–22, 2000 / ed. Michael Cwikel ... . – Berlin ; New York : de Gruyter, 2002
ISBN 3-11-017117-1

Printed in Germany.
Cover design: Thomas Bonnie, Hamburg.
Typeset using the authors' $\mathrm{T_EX}$ files: I. Zimmermann, Freiburg.
Printing and binding: Hubert & Co. GmbH & Co. KG, Göttingen.