Symmetric linear functionals on function spaces

Tadeusz Figiel and Nigel Kalton

To Professor Jaak Peetre on his 65th birthday

Abstract. We discuss the properties of linear functionals defined on spaces of measurable functions which are invariant under rearrangements.

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1. Introduction

Suppose $(\Omega, \Sigma, \mu)$ is any nonatomic $\sigma$-finite measure space. We let $\mathcal{M}(\Omega)$ be the space of all $\Sigma$-measurable complex-valued functions $f$ such that

$$d_f(x) := \mu(|f| > x) < \infty, \quad x > 0.$$ 

We define $\mathcal{M}_R(\Omega)$ and $\mathcal{M}_+(\Omega)$ to be the subsets of all real-valued or all non-negative real-valued functions respectively. If $f \in \mathcal{M}$, then the decreasing rearrangement of $|f|$ is denoted by $f^*$; this is defined by

$$f^*(t) = \inf_{E \supseteq \omega E} \sup_{\omega \in E} |f(\omega)|.$$ 

From now we assume that either $\mu(\Omega) = \infty$ or $\mu(\Omega) = 1$. If $\Omega$ is a standard measure space we may assume that either $\Omega = (0, \infty)$ or $\Omega = (0, 1]$ with Lebesgue measure. We define a symmetric ideal $\mathcal{K}$ to be a linear subspace of $\mathcal{M}$ such that:

- If $f \in \mathcal{K}$ and $|g| \leq |f|$ then $g \in \mathcal{K}$,
- If $f \in \mathcal{K}$ and $d_f \leq d_g$ then $g \in \mathcal{K}$,
- $\mathcal{K} \neq \{0\}$.

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functions. Our main result Theorem 2.8 here characterizes the set $Z_X$ (which we call the center) of all functions $f$ so that $\varphi(f) = 0$ for all symmetric functionals $\varphi$. These results were essentially known to the first author in the 1980’s; see for example [13] and [14]. It is interesting that $Z_X$ can be characterized purely in terms of a generalized $K$-functional.

In Section 3 we prove our main results. For the case of the unit circle $T$, this says that for a symmetric ideal $X$ which satisfies a very mild condition, (in particular for any quasi-Banach symmetric ideal), that if $f$ is analytic function on $D$ with $f(0) = 0$, belonging to the Smirnov class such that its boundary values, also denoted by $f$, belong to $X(T)$ then $f$ actually belongs to the center $Z_X$. For the special case of $X = L_1$ this result is proved in two different ways in [10]; the center of $L_1$ is there denoted by $H^p_{S^0}$. It is also equivalent (again for the case $X = L_1$) to an older result of Ceretelli [3], later independently found by Davis [4]. This result is also discussed in [10] and [9]; in [10] it is connected to the theory of commutators in interpolation. Theorems 3.3 and 3.4 are inspired by the analogous results for traces in [12].

Finally in Section 4 we show that for quasi-Banach ideals a real function belongs to $X(S)$ if and only if it is the real part of a function $f$ in the corresponding Hardy class $H_X$.

## 2. Symmetric functionals

In order to treat the case $\mu(\Omega) = 1$ let us introduce the notion of a symmetric ideal of finite type. If $\mu(\Omega) = \infty$ we say that $X$ is a symmetric ideal of finite type if $f \in X$ implies that $\mu(\text{supp } f) < \infty$. If $f$ is a symmetric ideal on $\Omega$ where $\mu(\Omega) = 1$ we can embed $(\Omega, \mu)$ in a larger nonatomic measure space $(\mathcal{G}, \mu')$, by taking a countable disjoint union of copies of $(\Omega, \mu)$. We can then define $X(\mathcal{G})$ of finite type by the property that $f \in X$ if and only if $\mu'(\text{supp } f) < \infty$ and $f_{\mathcal{G}} \in X(\mathcal{G})$ whenever $\mu'(E) = 1$. We call $X$ the infinite extension of $Y$. It is then easy to see that a symmetric linear functional $\varphi$ on $Y$ extends to a symmetric linear functional on $X$. Using this idea of an infinite extension, we shall restrict our attention unless otherwise stated to the case when $\mu(\Omega) = \infty$.

For $f \in \mathcal{M}$ we define the generalized $K$-functional of $f$ for $0 < u < v < \infty$ by

$$K(u, v; f) = \int_w^v f^3(t)dt.$$  \hspace{1cm} (2.1)

We extend the definitions of $K$ to general complex functions by linearity. To be precise if $f = g + ih \in \mathcal{M}$ where $g$ and $h$ are real we define

$$K(u, v; f) = K(u, v, g_+) - K(u, v, g_-) + iK(u, v, h_+) - iK(u, v, h_-)$$

and where, as usual $g_+ = \max(g, 0)$ and $g_- = \max(-g, 0)$ etc.
We will also need a dual notion which we term the $K^*$-functional which we define for all $f \in \mathcal{M}$:

$$K^*(r, s; f) = \int_{r \leq |f| < s} f \, d\mu. \quad (2.2)$$

If $f \in \mathcal{M}$ we can define measurable sets $E_u(f)$ for $u > 0$ so that $\mu E_u(f) = u$, $E_{u^*}(f) \subset E_u(f)$ if $u < v$ and $E_u \subset \{ |f| \geq f^*(u) \}$. Let $E_{u,v}(f) = E_u(f) \setminus E_{v}(f)$ when $u < v$.

**Lemma 2.1.** Suppose $f, g \in \mathcal{M}$ and $|f| \leq |g|$.  

1. Suppose $0 < u < v < \infty$. Then

$$\left| \int_{E_{u,v}(f)} f \, d\mu - \int_{E_{u,v}(g)} f \, d\mu \right| \leq 2(u g^*(u) + v g^*(v)). \quad (2.3)$$

2. Suppose $0 < u < v < \infty$. Then

$$\left| \int_{r \leq |f| < s} f \, d\mu - \int_{r \leq |g| < s} f \, d\mu \right| \leq 2(r d_g(r) + s g^*(s)). \quad (2.4)$$

**Proof.** (1) Let $h = \chi_{E_u(f)} - \chi_{E_v(g)} - \chi_{E_u(f)} + \chi_{E_v(g)}$. If $\in E_{u,v}(f)$ and $h(\omega) = 1$ then either $\omega \in E_{u,v}(g)$ or $|g(\omega)| > g^*(g)$. Thus

$$\left| \int_{E_{u,v}(f)} h f \, d\mu \right| \leq uf^*(u) + (v - u)g^*(v).$$

If $\in E_u(f)$ then $h(\omega) = -1$ if $\in E_{u,v}(g)$;

$$\left| \int_{E_u(f)} h f \, d\mu \right| \leq ug^*(u).$$

If $\in \Omega \setminus E_u(f)$ then $f(\omega) \leq f^*(u)$ and $h(\omega) = 1$ if $\in E_{u,v}(g)$.

$$\left| \int_{\Omega \setminus E_u(f)} h f \, d\mu \right| \leq (v - u) f^*(v).$$

Combining gives (1).

For (2) note that

$$\int_{r \leq |f| < s} f \, d\mu \leq r d_g(r), \quad \int_{r \leq |f| < s} f \, d\mu \leq s g^*(s),$$

$$\int_{r \leq |f| < s} f \, d\mu \leq r d_g(r), \quad \int_{r \leq |f| < s} f \, d\mu \leq s g^*(s).$$

This concludes the proof.

The following is an immediate consequence of Lemma 2.1:

**Lemma 2.2.** Suppose $f, g \in \mathcal{M}$ and $|f| \leq |g|$. Then:

$$|K(u, v; f) - \int_{E_{u,v}(f)} f \, d\mu| \leq 8(u g^*(u) + v g^*(v)) \quad (2.5)$$

and

$$|K^*(r, s; f) - \int_{r \leq |f| < s} f \, d\mu| \leq 2(r d_g(r) + s g^*(s)). \quad (2.6)$$

**Proof.** Note that (2.6) simply restates (2.4). To prove (2.5) simply note that (2.3) gives this with constant 2 if $f$ is positive. We can apply it in turn to $(\Phi(f))_+$, $(\Phi(f))_-$, $(\Phi(f))_+$ and $(\Phi(f))_-$. □

Our next lemma shows that the $K$-functional has approximate linearity properties:

**Lemma 2.3.** Suppose $f, g \in \mathcal{M}$ and suppose $h = f + g$ and $\psi = |f| + |g|$. Then if $0 < u < v < \infty$, $0 < r < s < \infty$, 

$$|K(u, v; h) - K(u, u; f) - K(u, v; g) - 24(u \psi^*(u) + v \psi^*(v))| \leq 24(u \psi^*(u) + v \psi^*(v)) \quad (2.7)$$

and

$$|K^*(r, s; h) - K^*(r, r; f) - K^*(r, s; g)| \leq 6(r d_g(r) + s g^*(s)). \quad (2.8)$$

**Proof.** For (2.7) it is enough to note that

$$\left| \int_{E_{u,v}(f)} f \, d\mu \right| \leq 8(u \psi^*(u) + v \psi^*(v))$$

when $\phi = f, g$. (2.8) is similar. □

Next we define the notion of the center $Z_X$ of a symmetric ideal $\mathcal{I}$ supported on a measure space $\Omega$ with $\mu(\Omega) = 1$ or $\mu(\Omega) = \infty$. We shall say that $f \in Z_X$ if there exists $h \in \mathcal{X}$ such that whenever $0 < u < v < \infty$ we have:

$$|K(u, v; f)| \leq u h^*(u) + v h^*(v). \quad (2.9)$$

Let us make the remark that if $\mathcal{I}$ is of finite type and $\mu(\text{supp } f) = 1$ then by taking $v$ large enough (2.9) will imply that

$$|K(u, 1; f)| \leq u h^*(u).$$

Hence it suffices to consider in this case $h$ with $\mu(\text{supp } h) = 1$. This implies that if $\mathcal{X}$ is a symmetric ideal on a probability space and $\mathcal{X}$ is the infinite extension of $\mathcal{I}$ then $Z_\mathcal{X}$ coincides with the restriction of $Z_\mathcal{X}$ to $\Omega$.

**Lemma 2.4.** Suppose $f \in Z_\mathcal{X}$ if and only if there exists $h \in \mathcal{X}_+$ and $C > 0$ with

$$|K^*(r, s; f)| \leq C(r d_h(r) + s d_h(s)), \quad 0 < r < s < \infty. \quad (2.10)$$
Proof. Assume first $f \in Z_X$. Choose $u = \mu(|f| > r)$ and $v = \mu(|f| > s)$. Then we can assume $r \leq |f| < s = E_{u,v}(f)$ and so

$$K^*(r, s; f) = \int_{E_{u,v}(f)} f \, d\mu$$

and (2.10) follows from (2.5) with $f = g$.

Conversely suppose (2.10) holds and $0 < u < v < \infty$; we can assume $h^* > f^*$. Let $r = 2h^*(u)$ and $s = 2h^*(v)$. Then $d_0(r) \leq u$ and $d_0(s) \leq v$ and so

$$\int_{r \leq |f| < s} f \, d\mu \leq 2C(uh^*(u) + vh^*(v)).$$

Now

$$\int_{E_{u,v}(f)} f \, d\mu \leq 2v h^*(v)$$

and

$$\int_{E_{u,v}(f)} d\mu \leq 2u h^*(u).$$

Combining again with (2.5) gives that

$$|K(u, v; f)| \leq (2C + 10)(uh^*(u) + vh^*(v)).$$

$\square$

**Proposition 2.5.** The center $Z_X$ is a linear subspace of $X$ with the property that $f \in Z_X$ if and only if $\Re f, \Im f \in X$.

**Proof.** The only statement that requires proof is linearity; this follows directly from (2.7). $\square$

We will need a result of Kwapien from 1983:

**Theorem 2.6** ([13]). Let $f \in M$ be a bounded real function with support of finite measure. If $\int f \, d\mu = 0$ then there exist $g_1, g_2 \in M$ with support $g_j \subset \text{supp } f$ for $j = 1, 2$, $\|g_j\|_\infty \leq 6\|f\|_\infty$, $g_1 \sim g_2$ and $f = g_1 - g_2$ ($\mu$-a.e.).

**Proof.** In effect Kwapien's proves this for a standard measure space with $g_2 = g_1 \circ \sigma$ where $\sigma$ is a measure preserving transformation. For the general case we apply Kwapien's theorem to $(\mathbb{R}, v)$ where $v(B) = \mu(f^{-1}B \cap \text{supp } f)$. We can then write $\phi(x) = x$ in the form $\phi = \psi_1 - \psi_2$ where $\psi_1 \sim \psi_2$ and $\|\psi_j\|_\infty \leq 6\|f\|_\infty$. Let $g_j = \psi_j \circ f$.

If $X$ is any symmetric ideal let us introduce a corresponding sequence $\delta_X$. Let $(A_n)_{n \in \mathbb{Z}}$ be a disjoint sequence of measurable sets in $\Omega$ such that $\mu(A_n) = 2^{-n}$. We define $\delta_X$ as the space of sequences $\xi = (\xi_n)$ such that $L\xi := \sum_{n \in \mathbb{Z}} 2^n \xi_n \chi_{A_n} \in X$.

It is clear that this definition is independent of the choice of $(A_n)$; indeed we may take $\Omega = (0, \infty)$ and $A_{n} = (2^{-n}, 2^{-(n+1)})$.

It is easy to see from the symmetry condition on $X$ that if $\xi \in \delta_X$ then we have $\sum_{n \in \mathbb{Z}} \xi_n \chi_{A_n} \in X$. On $\delta_X$ we may therefore define the translation operator $T((\xi_n)_{n \in \mathbb{Z}}) = (\xi_{n-1})_{n \in \mathbb{Z}}$. We say that a linear functional $\psi$ on $\delta_X$ is translation-invariant if $\psi(T(\xi)) = \psi(\xi)$ for every $\xi \in \delta_X$. It is trivial that $\psi(\xi) = 0$ for every translation-invariant $\psi$ if and only if $\xi$ is in the range $R$ of $I - T$ where $I$ is the identity on $\delta_X$.

**Lemma 2.7.** (1) If $\psi$ is a symmetric linear functional on $X$ then $\psi \circ L$ is translation-invariant functional on $\delta_X$.

(2) $\xi \in \mathcal{R}$ if and only if there exists $\eta \in \delta_X$ such that

$$\left| \sum_{k=m+1}^{n} \xi_k \right| \leq |\eta_m + \eta_n|, \quad -\infty < m < n < \infty.$$  

(2.11)

**Proof.** (1) Let $A_n = B_n \cup C_n$ be a partition of $A_n$ into two measurable sets such that $\mu(B_n) = \mu(A_n) = 2^{-(n+1)}$. If $\xi \in \delta_X$ then

$$\psi(L\xi) = \psi(\sum_{n \in \mathbb{Z}} 2^n \xi_n \chi_{B_n} + \sum_{n \in \mathbb{Z}} 2^n \xi_n \chi_{C_n}) = 2\psi(\sum_{n \in \mathbb{Z}} 2^n \xi_n \chi_{B_n})$$

$$= 2\psi\left(\sum_{n \in \mathbb{Z}} 2^{n-1} \xi_{n-1} \chi_{A_n}\right)$$

$$= \psi(\xi).$$

(2) If $\xi = \eta - T\eta$ then

$$\left| \sum_{k=m+1}^{n} \xi_k \right| = |\eta_m - \eta_n| = |\eta_m| + |\eta_n|.$$

Conversely assume (2.11). It suffices, by splitting into real and imaginary parts, to treat the case when $\xi$ is real. Let

$$\xi_n = \left\{ \begin{array}{ll} \sum_{k=0}^{n} \xi_k, & n \geq 0 \\ \sum_{k=-n}^{0} \xi_k, & n < 0 \end{array} \right. \quad n \in \mathbb{N}$$

Then by (2.11) we have

$$\sup_n (\xi_n - \eta_n) \leq \inf_m (\xi_m + \eta_m).$$

Pick $\lambda \in \mathbb{R}$ so that

$$\lambda - \eta_n \leq \xi_n - \eta_n,$$

$m, n \in \mathbb{Z}.$

Then $\xi_n - \lambda \in \mathcal{S}$ and $(1 - T)(\xi_n - \lambda)_{n \in \mathbb{Z}} = \xi$. $\square$
We now come to the main theorem of the section. As mentioned in the Introduction, results of this type were known to the first author in the mid 1980's. See Pietsch [14] where sequence space analogues are considered. A non-commutative analogue of this result appears in Dykema, Figiel, Weiss and Wodzicki [6]; see also [11] for the earlier special case of the trace-class, analogous to the case $\mathcal{X} = L_1$.

**Theorem 2.8.** Let $\mathcal{X}$ be a symmetric ideal on $(\Omega, \Sigma, \mu)$. Then $f \in Z_{\mathcal{X}}$ if and only if $\varphi(f) = 0$ for every symmetric linear functional on $\mathcal{X}$.

**Proof.** Let us introduce the space $\mathcal{N}$ of $\mathcal{X}$ defined by $f \in \mathcal{N}$ if and only if $\varphi(f) = 0$ for every symmetric linear functional. We will prove that $\mathcal{N} = Z_{\mathcal{X}}$. First suppose $f \in \mathcal{N}$; we will show that $f \in Z_{\mathcal{X}}$. Indeed $f$ can be written in the form $f = \sum_{j=1}^{n} (g_j - h_j)$ where $g_j, h_j \in \mathcal{X}$ and $g_j \sim h_j$. It thus suffices to show that $g - h \in Z_{\mathcal{X}}$ whenever $g \in \mathcal{X}$ and $g \sim h$. Note that for any $u, v$ we have $K(u, v; g) = K(u, v; h) = -K(u, v; -h)$. Hence by Lemma 2.3 (2.7) we have

$$|K(u, v; g - h)| \leq 8(u\psi^*(v) + v\psi^*(u))$$

where $\psi = |g| + |h|$. We conclude that $\mathcal{N} \subset Z_{\mathcal{X}}$.

For the converse we shall prove the following statements:

$$\mathcal{X} = \mathcal{N} + L(\delta \mathcal{X}).$$  \hspace{1cm} (2.12)

$$Z_{\mathcal{X}} = \mathcal{N} + L(\mathcal{R}).$$  \hspace{1cm} (2.13)

Let us suppose $f \in \mathcal{X}$. Then we may find $g \in \mathcal{X}$ so that $g \sim f$ and for every $n$, $\mathcal{X} \mathcal{X} \sim f \in L(\delta \mathcal{X})$. Let $g \sim f \in L(\mathcal{R})$. Then $\int_a^b g \, d\mu = \int_a^b f \, d\mu$. Hence by Theorem 2.6 applied to $(g - \xi_n) \mathcal{X} \mathcal{X}$ for each $n$ we can write $g - L \xi = h_1 - h_2$ where $h_1 \sim h_2$ and $|h_1 \mathcal{X} \mathcal{X}| \leq 12 f^*(2^{-m})$ for every $n$ and $j = 1, 2$. Hence $h_1, h_2 \in \mathcal{X}$. Now

$$f - L \xi = (f - g) + (h_1 - h_2) \in \mathcal{N}.$$  \hspace{1cm} (2.12)

This proves (2.12). If $f \in Z_{\mathcal{X}}$ we note that

$$\sum_{k=m+1}^{n} \xi_k = \int_{\mathcal{X} \mathcal{X} \mathcal{X}} g \, d\mu = \int_{E_2^{-m}} f \, d\mu.$$  \hspace{1cm} (2.12)

Hence by Lemma 2.2 (2.5)

$$\left| \sum_{k=m+1}^{n} \xi_k \right| \leq |K(2^{-m}, 2^{-m}; f)| + 8(2^{-m} f^*(2^{-m}) + 2^{-m} f^*(2^{-m})).$$

It follows that there exists $\psi \in \mathcal{X}_+$ so that

$$\sum_{k=m+1}^{n} \xi_k \leq 2^{-m} \psi^*(2^{-m}) + 2^{-m} \psi^*(2^{-m}), \quad m < n.$$  \hspace{1cm} (2.14)

Now by (2) of Lemma 2.7 this implies $\xi \in \mathcal{X}$. This proves 2.13.

By (1) of Lemma 2.7 we have $L(\mathcal{R}) \subset \mathcal{N}$ and so $Z_{\mathcal{X}} \subset \mathcal{N}$. This completes the proof. \hspace{1cm} $\square$

In fact we have also proved the following statement:

**Corollary 2.9.** There is a natural isomorphism between the space of symmetric functionals on $\mathcal{X}$ and the space of translation-invariant functionals on $(\delta \mathcal{X}) \subset \mathcal{X}$ implemented by $\varphi \mapsto \varphi \circ L$.

**Proof.** We first show that $L(\mathcal{R}) = \mathcal{N} \cap L(\mathcal{X})$. Indeed by Lemma 2.7 (1) we know $L(\mathcal{R}) \subset \mathcal{N}$. Now suppose $L \xi \in \mathcal{N}$. Then $|\xi| \leq f^*(2^{-m})$. If $g = \sum_{m \leq 2^{-m} \mathcal{X} \mathcal{X} \mathcal{X}} f^*(2^{-m})$, then $g \in \mathcal{X}$ and $|f| \leq g$. Hence by Lemma 2.2 (2.5),

$$\left| \sum_{k=m+1}^{n} \xi_k \right| \leq |K(2^{-m}, 2^{-m}; f)| + 8(2^{-m} g^*(2^{-m}) + 2^{-m} g^*(2^{-m})).$$

Now since $f \in Z_{\mathcal{X}}$, this inequality implies that $\xi \in \mathcal{X}$. Now by (2.12) and the fact that $Z_{\mathcal{X}} \subset \mathcal{N}$, it follows that $L$ induces an isomorphism between $(\delta \mathcal{X}) / \mathcal{X} \cap \mathcal{N}$. This implies the corollary. \hspace{1cm} $\square$

We have stated all these results for infinite measure spaces. As noted at the beginning of the section we can quickly derive from this the corresponding results for probability spaces. Indeed it is not difficult to show that for ideals on probability spaces one may consider the sequence space $(\delta \mathcal{X}(\Omega))$ modelled on the natural numbers, and, in this case, we define translation to be the right shift, $T \xi = (\xi_{n-1})_{n \in \mathbb{N}}$ with the understanding that $\xi_0 = 0$. The result corresponding to Corollary 2.9 then holds.

Let us consider a few examples.

Suppose $\int_0^1 f^*(s) \, ds < \infty$ for all $f \in \mathcal{X}$. Then we may define $K(0, v; f) = \lim_{u \to 0} K(u, v; f)$. Conversely if $\int_0^1 f^*(s) \, ds < \infty$ for all $f \in \mathcal{X}$, then we may define $K(u, v; f) = \lim_{u \to 0} K(u, v; f)$. The latter situation arises if $\mathcal{X}$ is of finite type, or if $\mu(\Omega) = 1$.

**Proposition 2.10.** (1) If $\int_0^1 f^*(s) \, ds < \infty$ for all $f \in \mathcal{X}$ then $f \in Z_{\mathcal{X}}$ and only if

$$|K(0, u; f)| \leq ah^*(u), \quad 0 < u < \infty,$$

for some $h \in \mathcal{X}$.  \hspace{1cm} (2.14)
(2) If $X$ has the property that \( \int_1^\infty f^*(s) ds < \infty \) for every \( f \in X \) then
\[
|K(u, \infty; f)| \leq uh^*(u), \quad 0 < u < \infty,
\] (2.15)
for some \( h \in X \).

**Proof.** In case (1) note that the hypothesis implies \( \lim_{u \to 0} uh^*(u) = 0 \) for all \( h \in X \); it is then easy to see that (2.9) reduces to (2.14). Similarly if (2) holds then \( \lim_{u \to \infty} uh^*(u) = 0 \) for all \( h \in X \) and the argument is similar. \( \Box \)

Let us use these remarks to relate the property \( Z_X = X \) to the Boyd indices. We first summarize some properties of the Boyd indices in the next lemma.

**Lemma 2.11.** Suppose \( p < p_X \leq q \leq q_X < q \). Suppose \( h \in X(R) \). Then there exists a constant \( C \) such that
\[
\sum_{k \geq 2} 2^{\min(k/p, k/q)} D_{2^k+h} \|h\|_X \leq C \|h\|_X
\] (2.16)
and
\[
C^{-1} \|h\|_{L_p \cap L_q} \leq \|h\|_X \leq C \|h\|_{L_p \cap L_q}.
\] (2.17)

It is easy then to see that the following proposition holds.

**Proposition 2.12.** Suppose \( X \) is a quasi-Banach ideal.

1. If \( q_X < 1 \) then \( Z_X = X \).
2. If \( q_X > 1 \) and \( \mu(\Omega) = \infty \) then \( Z_X = X \).
3. If \( p_X > 1 \) and \( \mu(\Omega) = 1 \) then \( Z_X = \{ f \in X : \int f \, d\mu = 0 \} \).

**Proof.** Just note that if \( f \in X_4 \) then, where applicable,
\[
K(0, u; f) = \int_0^1 f^*(su) \, ds \leq \sum_{k \geq 0} 2^k D_{2^k+h} f(u)
\]
and
\[
K(u, \infty; f) = \int_1^\infty f^*(su) \, ds \leq \sum_{k \geq 1} 2^k D_{2^k+h} f(u).
\]
Then (1) and (2) follow from Lemma 2.11 (2.16) and the remarks preceding it. (3) follows from the fact that if \( f \in L_{1,15} \) real-valued and \( \int f \, d\mu = 0 \) then \( K(0, u; f) = -K(u, \infty; f) \) This implies that \( f \in Z_X \). \( \Box \)

Thus for quasi-Banach ideals the only case where one gets a non-trivial symmetric linear functional is when \( p_X \leq 1 \). We now discuss the cases of \( L_1 \).

If \( f \) is a real function in \( X \) we introduce the function \( f_d : (-\infty, \infty) \to \mathbb{R} \) defined by the conditions \( f_d \sim f \), \( f_d \) is decreasing and non-positive on \( (-\infty, 0) \) and \( f_d \) is decreasing and non-negative on \( (0, \infty) \). Then it is clear that
\[
K(u, 0; f_d) = \int_{|t| \leq 0} f_d(t) \, dt.
\]

In the case of \( L_1 \) the center was introduced as the symmetric Hardy class \( H^{sym}_1 \) in [10]. A real function \( f \in H^{sym}_1(\Omega) \) if and only if
\[
\int_0^\infty |M(t)| \frac{dt}{T} < \infty
\]
where
\[
M(t) = \int_{-t}^{t} f_d(s) \, ds.
\]

It follows that there are many discontinuous symmetric functionals on \( L_1 \) in both the probability and infinite measure cases.

An interesting case is to take \( X \) to be the space of all \( f \) so that \( \lim_{t \to 0} tf^*(t) = \lim_{t \to \infty} tf^*(t) \, dt = 0 \). Then \( f \in Z_X \) if and only if
\[
\lim_{u \to \infty} \int_{|u| > r < u} f_d(s) \, ds = 0
\]
or equivalently
\[
\lim_{r \to \infty} \int_{|f| > r} f(s) \, ds = 0.
\]

Symmetric continuous linear functionals on weak \( L_1 \) were used in [8] (and are implicit in the work of Cwikel and Fefferman [2] on the dual of weak \( L_1 \)).

### 3. Analytic functions

Let \( D \) be the unit disk and \( T \) be the unit circle. We recall that the Nevanlinna class \( N(D) \) consists of all functions \( f \) which are analytic in \( D \) and satisfy
\[
\sup_{r \in (0,1)} \int_0^{2\pi} \log_+ |f(re^{i\theta})| \, d\theta < \infty.
\]

If \( f \in N \) then the boundary values \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) exist almost everywhere. The Smirnov class \( N^+(D) \) consists of all functions \( f \in N \) such that
\[
\lim_{r \to 1} \int_0^{2\pi} \log_+ |f(re^{i\theta})| \, d\theta = \int_0^{2\pi} \log_+ |f(e^{i\theta})| \, d\theta.
\]
We can regard \( N^+ \) as both a space of analytic functions on \( \mathbb{D} \) and as a space of measurable functions on \( T \). We shall use the notation \( f_r(e^{\theta}) = f_r(e^{\theta_0}) \) so that \( f_r : T \rightarrow \mathbb{C} \). We will use \( \mu \) to denote the normalized measure \( d\theta/2\pi \).

We also need analogous spaces in the upper half-plane \( \mathbb{U} = \{z : \Im z > 0\} \). We define \( N^+(\mathbb{U}) \) to be the class of all functions \( f \) analytic in \( \mathbb{U} \) so that \( f \circ \phi \in N^+ \) where \( \phi : \mathbb{D} \rightarrow \mathbb{U} \) is given by \( \phi(z) = i(1 + z)/(1 - z) \). If \( f \in N^+(\mathbb{U}) \) then

\[
\lim_{y \to 0} f(x + iy) = f(x)
\]

exists a.e. and

\[
\log |f(x + iy)| \leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |f(x)|}{(t - x)^2 + y^2} dt.
\]

We will write \( f_r(x) = f(x + iy) \) and in this case \( \mu \) is Lebesgue measure. As before we can consider \( N^+(\mathbb{U}) \) as a space of analytic functions in the upper half-plane or as a space of measurable functions on the real line.

In the next technical theorem we prove a result on the distribution of boundary values of functions in \( N^+(\mathbb{D}) \) and \( N^+(\mathbb{U}) \).

**Theorem 3.1.** There exists a constant \( C \) so that:

1. If \( f \in N^+(\mathbb{D}) \) and \( f(0) = 0 \) then, whenever \( 0 < r < s < \infty \),

\[
\int_{|z|=r} f(e^{\theta}) \frac{d\theta}{2\pi} \leq C \left( r \mu(|f| > r) + s \mu(|f| > s) + \int_0^{2\pi} r \log \frac{|f(e^{\theta})|}{r} + s \log \frac{|f(e^{\theta})|}{s} \frac{d\theta}{2\pi} \right)
\]

2. If \( f \in N^+(\mathbb{U}) \) then

\[
\int_{|z|=r} f'(z) \frac{dx}{r} \leq C \left( r \mu(|f| > r) + s \mu(|f| > s) + \int_{-\infty}^{\infty} r \log \frac{|f(x)|}{r} + s \log \frac{|f(x)|}{s} \frac{dx}{s} \right)
\]

**Proof.** We begin by fixing a smooth bump function \( b : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \text{supp } b \subset (0, 1/2) \), \( b \geq 0 \), \( \int b(x) dx = 1 \). Let \( \beta(t) = 2|b(t)| + |b'(t)| \).

Note first that the estimates are trivial when \( s \leq 2r \).

Now suppose \( 0 < r < s < \infty \), with \( s \geq 2r \). We define

\[
\varphi(t) = \varphi_{r,s}(t) = \int_{-\infty}^{t} b(t - \log r) - b(t - \log s) dt.
\]

Notice that the two terms in the integrand are never simultaneously positive (since \( \log 2 > \frac{1}{2} \)), and \( \varphi \) is a bump function which satisfies \( \varphi(\tau) = 0 \) if \( \tau < \log r \) or \( \tau > \frac{1}{2} + \log s \) while \( \varphi(\tau) = 1 \) if \( \log r + \frac{1}{2} \leq \tau \leq \log s \). Then let \( \psi = \psi_{r,s} \) be defined to be the function such that \( \psi(\tau) = 0 \) if \( \tau < \log r \) and

\[
\psi''(\tau) = e^\tau (2|\psi''(\tau)| + |\psi'(\tau)|).
\]

In fact, this implies that

\[
\psi''(\tau) = e^\tau (\beta(\tau - \log r) + \beta(\tau - \log s))
\]

and then

\[
\psi(\tau) = \psi_{r,s}(\tau) = \int_{-\infty}^{\tau} e^\tau (\beta(t - \log r) + \beta(t - \log s)) dt,
\]

and

\[
\psi'(\tau) = \int_{-\infty}^{\tau} e^\tau (\beta(t - \log r) + \beta(t - \log s)) dt.
\]

Thus, if we set

\[
C_0 = \int_{-\infty}^{\infty} e^\tau \beta(t) dt
\]

then

\[
\psi'(\tau) \leq C_0 (r \chi(\tau > \log r) + s \chi(\tau > \log s))
\]

and so

\[
\psi_{r,s}(\tau) \leq C_0 (r \chi(\tau > \log r + s \chi(\tau > \log s)).
\]

Now we use the argument of Lemma 2.6 of [12]. If we define

\[
h(z) = h_{r,s}(z) = \psi(\log |z|) - x \varphi(|\log |z||), z = x + iy \neq 0,
\]

and \( h(0) = 0 \) then \( h \) is a \( C^2 \)-subharmonic function which vanishes on a neighborhood of 0. We note the estimates (from 3.4)

\[
\psi(\log |z|) \leq C_0 \left( r \log + \frac{|z|}{r} + s \log \frac{|z|}{s} \right)
\]

and

\[
0 \leq h(z) \leq C_0 \left( r \log + \frac{|z|}{r} + s \log \frac{|z|}{s} \right), \quad |z| \geq 2r.
\]

Note of course that \( C_0 \) is independent of \( r, s \).

Let us first treat the case \( \psi \) when \( f \in N^+(\mathbb{D}) \) and \( f(0) = 0 \). Then \( h \circ f \) is subharmonic on \( \mathbb{D} \). Hence

\[
0 \leq \int_0^{2\pi} h \circ f_r \frac{d\theta}{2\pi}
\]
for $0 \leq r < 1$. Now $h_+(z) := \max(h(z), 0) \leq M \log_+ |z| + 1$ for a suitable constant $M$ and $h_+ - h$ is bounded. From the definition of the Smirnov class we have:

$$\lim_{r \to 1} \int_{|f| > 2f} \log_+ |f| \, d\theta = 0.$$ 

Hence

$$\lim_{r \to 1} \int_{|f| > 2f} h_+ \circ f \, d\theta = 0.$$ 

Now by the Dominated Convergence Theorem we have

$$\lim_{r \to 1} \int_{|f| \leq 2|f|} h_+ \circ f \, d\theta = \int_0^{2\pi} h_+ \circ f \, d\theta.$$ 

Thus

$$\lim_{r \to 1} \int_0^{2\pi} h_+ \circ f \, d\theta = \int_0^{2\pi} h_+ \circ f \, d\theta.$$ 

By the Bounded Convergence Theorem,

$$\lim_{r \to 1} \int_0^{2\pi} (h - h_+) \circ f \, d\theta = \int_0^{2\pi} (h - h_+) \circ f \, d\theta.$$ 

We thus conclude that

$$\int_0^{2\pi} h \circ f \, d\theta \geq 0$$

and thus

$$\forall \int_0^{2\pi} f \varphi(\log |f|) \, d\theta \leq \int_0^{2\pi} \psi(\log |f|) \, d\theta.$$ 

Applying this inequality to $\alpha f$ for every $\alpha$ with $|\alpha| = 1$ gives

$$\int_0^{2\pi} f \varphi(\log |f|) \, d\theta \leq \int_0^{2\pi} \psi(\log |f|) \, d\theta.$$ 

(3.7)

Now note that

$$\int_0^{2\pi} f (\varphi(\log |f|) - \chi_{(c \leq |f| < c^2)}) \, d\theta \leq \int_0^{2\pi} |f|(\chi_{(c \leq |f| < c^2)} + \chi_{(c < |f| < 2c)}) \, d\theta \leq 2\mu(|f| > r) + 2s\mu(|f| > s).$$

Now combining with (3.4) and (3.7) gives (3.2). The proof of (3.3) is somewhat similar. We only sketch the details. It is clear that we may assume that

$$\int_{-\infty}^{\infty} \log_+ |f| \, dx < \infty.$$

It then follows from (3.1) that

$$\int_{-\infty}^{\infty} \log_+ |f| \, dx \leq \int_{-\infty}^{\infty} \log_+ |f| \, dx$$

whenever $y > 0$ and also that $\lim_{y \to \infty} \sup |f(x + iy)| = 0$. As above we note that $h \circ f$ is subharmonic on $U$; in this case $h \circ f$ vanishes for $y$ large enough. It follows using (3.6) that $\int_{-\infty}^{\infty} h \circ f \, dx$ is a convex function of $y > 0$ and so that

$$\lim_{y \to \infty} \int_{-\infty}^{\infty} h \circ f \, dx \geq 0.$$

In this case we use (3.8) in place of the Smirnov condition to deduce

$$\int_{-\infty}^{\infty} h \circ f \, dx \geq 0.$$

The remaining details are then similar. □

We now define (cf. [12]) a symmetric ideal $\mathcal{X}$ to be geographically stable if given $f \in \mathcal{X}$ there exists $h \in \mathcal{X}$ with

$$\exp \left( \frac{1}{t} \int_0^t \log f^*(s) \, ds \right) \leq h^*(t), \quad 0 < t < \infty.$$ 

(3.9)

Note of course that we require $\log_+ |f| \in L_1$ for all $f \in \mathcal{X}$.

**Proposition 3.2.** Every symmetric quasi-Banach ideal is geometrically stable.

**Proof.** For $0 < p \leq 1$ we note that for an appropriate constant $C_1$,

$$\exp \left( \frac{1}{t} \int_0^t \log f^*(s) \, ds \right) \leq \left( \frac{1}{t} \int_0^t f^*(s)^p \, ds \right)^{\frac{1}{p}} \leq \sum_{n=1}^\infty 2^{\frac{n}{2} - \frac{p}{2}} D_2 f^*(t)^p \leq C_1 \sum_{n=1}^\infty 2^{\frac{n}{2} - \frac{p}{2}} D_2 f^*(t).$$

Now if $\frac{1}{2} + \frac{1}{2p} > \frac{1}{p_0}$ the series

$$C_1 \sum_{n=1}^\infty 2^{\frac{n}{2} - \frac{p}{2}} D_2 f^*$$

converges in $\mathcal{X}(0, 1)$ or $\mathcal{X}(0, \infty)$ to some $h = h^*$ satisfying (3.9). □

If $\mathcal{X}$ is a symmetric ideal on $\mathcal{T}$ let us define $H_{\mathcal{X}}(\mathcal{T})$ to be the space of all $f \in \mathcal{X}(\mathcal{T})$ so that $f \in N^+(\mathcal{D})$. Similarly $H_{\mathcal{X}}(\mathcal{R})$ consists of all $f \in \mathcal{X}(\mathcal{R})$ so that $f \in N^+(\mathcal{U})$. 
Theorem 3.3. Suppose $\mathcal{X}$ is a geometrically stable symmetric ideal on $\mathbb{T}$ (in particular this applies when $\mathcal{X}$ is a quasi-Banach ideal). Suppose $f \in H_{\mathcal{X}}$ with $f(0) = 0$. Then $f \in Z_{\mathcal{X}}(\mathbb{T})$ and so for every symmetric linear functional $\varphi$ on $\mathcal{X}$ we have $\varphi(f) = 0$.

Theorem 3.4. Suppose $\mathcal{X}$ is a geometrically stable symmetric ideal on $\mathbb{R}$ (in particular this applies when $\mathcal{X}$ is a quasi-Banach ideal). Suppose $f \in H_{\mathcal{X}}(\mathbb{R})$. Then $f \in Z_{\mathcal{X}}$ and so for every symmetric linear functional $\varphi$ on $\mathcal{X}$ we have $\varphi(f) = 0$.

Proof. The proofs of both theorems are essentially the same. We prove only Theorem 3.3. Assume first $f$ satisfies the condition that $\mu(\{|f| = r\}) = 0$ for all $r > 0$. Then if $0 < u < v < \infty$ we have $E_{\alpha, r}(f) = |f^*(u)| \leq |f < f^*(v)|$ and so
\[
|K(u, v; f) - \int_{f^*(u) < f} f \frac{d\theta}{2\pi}| \leq 2(u f^*(u) + v f^*(v))
\]
by Lemma 2.1, (2.3). Thus using Theorem 3.1 (3.2),
\[
|K(u, v; f)| \leq (C + 2)(u f^*(u) + v f^*(v)) + \int_0^{2\pi} f^*(u) \log \frac{|f|}{f^*(u)} + f^*(u) \log \frac{|f|}{f^*(u)} + \frac{|f|}{f^*(u)} \frac{d\theta}{2\pi}.
\]
Now by (3.9) we can find $h$ with
\[
\frac{1}{I} \int_0^I \log f^*(s) ds \leq \log h^*(t).
\]
Now
\[
\int_0^{2\pi} f^*(u) \log \frac{|f|}{f^*(u)} \frac{d\theta}{2\pi} = f^*(u) \int_0^u \log f^*(t) - \log f^*(u) dt \leq u f^*(u) \log h^*(u)
\]
\[
\leq u h^*(u)
\]
This together with a similar calculation for $v$ gives the estimate
\[
K(u, v; f) \leq u g^*(u) + v g^*(v)
\]
where $g^* = (C + 2) f^* + h^*$.

Now for the general case we may clearly define some $g \in N^+(\mathbb{D})$ so that $g \in \mathcal{X}(\mathbb{T})$ and $\mu(\{|g| = r\}) = 0$ for every choice of $r$. It then follows that for almost every choice of $(e^{i\theta}, e^{i\phi}) \in \mathbb{T}^2$ we have $|g(e^{i\theta})| \neq |g(e^{i\phi})|$. Let $A$ be the set of $(t, e^{i\theta}, e^{i\phi}) \in [-1, 1] \times \mathbb{T}^2$ so that $|f(e^{i\theta}) + t g(e^{i\phi})| = |f(e^{i\theta}) + t g(e^{i\phi})|$. By expanding as a polynomial in $t$ it follows that for almost every $(e^{i\theta}, e^{i\phi})$ the set of $t$ such that $(t, e^{i\theta}, e^{i\phi}) \in A$ has Lebesgue measure zero. Applying Fubini's theorem $A$ has product measure zero and so we may find $t \neq 0$ so that the set of $(e^{i\theta}, e^{i\phi})$ so that either $|f(e^{i\theta}) + t g(e^{i\phi})| = |f(e^{i\theta}) + t g(e^{i\phi})|$ or $|f(e^{i\theta}) - t g(e^{i\phi})| = |f(e^{i\theta}) - t g(e^{i\phi})|$ has measure zero. From this it follows that for every $r > 0$ we have $\mu(|f + t g| = r) = \mu(|f - t g| = r) = 0$. Now applying the first part $f \pm t g \in Z_{\mathcal{X}}$ and so $f \in Z_{\mathcal{X}}$. □

Theorem 3.3 has some amusing applications of which we give two.

Corollary 3.5. Suppose $f \in N^+(\mathbb{D})$ and $\lim_{r \to \infty} \mu(|f| > r) = 0$. Then
\[
f(0) = \lim_{r \to \infty} \int_{|f| < r} f(e^{i\theta}) \frac{d\theta}{2\pi}.
\]
Proof. If $f(0) = 0$ this follows directly from Theorem 3.3 and the remarks at the end of Section 2. If $f(0) = \alpha \neq 0$ then note that
\[
\lim_{r \to \infty} \left| \int_{|f| < r} f(e^{i\theta}) \frac{d\theta}{2\pi} \right| + \left| \int_{|f| > r} f(e^{i\theta}) \frac{d\theta}{2\pi} \right| = 0
\]
follows from $\lim_{r \to \infty} \mu(|f| > r) = 0$. □

Corollary 3.6. Suppose $\mathcal{X}$ is a geometrically stable symmetric ideal on $\mathbb{T}$ which admits a regular symmetric linear functional $\varphi$. Then if $f \in H_{\mathcal{X}}(\mathbb{T})$ we have $\varphi(f) = f(0)$.

We omit the trivial proof.

4. Conjugate functions

Cerdà-[3] and later (independently) Davis [4] proved that a real function $f \in L_1(\mathbb{T})$ is in $H^\text{syn}(\mathbb{T}) \iff f \sim 9g$ for some $g \in H_1(\mathbb{T})$ with $g(0) = 0$. See also [10] for an alternate proof and [9] for a vector-valued generalization. In [7] there is a general discussion of problems of this type. Let us state our extensions of this result to arbitrary quasi-Banach ideals.

Theorem 4.1. Let $\mathcal{X}$ be a symmetric quasi-Banach ideal on $\mathbb{T}$ with $\frac{1}{2} < p_\mathcal{X} < q_\mathcal{X} < \infty$. If $f \in \mathcal{X}_p$ then in order that there exist $g \in H_{\mathcal{X}}(\mathbb{T})$ with $g(0) = 0$, and $\mathcal{X}g \sim f$ it is necessary and sufficient that $f \in Z_{\mathcal{X}}$.

Theorem 4.2. Let $\mathcal{X}$ be a symmetric quasi-Banach ideal on $\mathbb{R}$ with $q_\mathcal{X} < \infty$. If $f \in \mathcal{X}_p$ then in order that there exist $g \in H_{\mathcal{X}}(\mathbb{R})$, with $\mathcal{X}g \sim f$ it is necessary and sufficient that $f \in Z_{\mathcal{X}}$.

Before proving this result we note that Davis [4] establishes Theorem 4.1 for the case $\mathcal{X} = L_p$ where $\frac{1}{2} < p < 1$. In this case $Z_{L_p} = L_p$. In fact Aleksandrov proves a stronger theorem [1] that the map $f \to \mathcal{H}f$ maps $H(\mathbb{T})$ onto $H(\mathbb{T})$ when $q_\mathcal{X} < 1$. If $1 < p_\mathcal{X} < q_\mathcal{X} < \infty$ then the Hilbert transform is bounded on $\mathcal{X}$ and
Define $f_t$ as before (see after Proposition 2.12). Now let $J_n = (2^n, 2^{n+1}] \cup [-2^{n-1}, -2^n)$. Let $E_0$ be a measurable subset of $J_0$ so that $\mu(E) = \frac{1}{2} \mu(J_0) = 1$ and

$$\int_E t^k dt = \frac{1}{2} \int_{J_0} t^k dt, \quad 0 \leq k \leq N.$$ 

Let $F_0 = J_0 \setminus E_0$ and then let $E_n = 2^n E_0$, $F_n = 2^n F_0$ for $n \in \mathbb{Z}$.

By using Lemma 4.3 on each set $E_n \cup F_{n+1}$ (which has measure $(3/2) \mu J_n$) we can find a function $f_c$ so that $f_c|_{E_n \cup F_{n+1}} \sim f_{c|_{J_n}}$ for $n \in \mathbb{Z}$ and

$$\int_{E_n \cup F_{n+1}} t^k f_c(t) dt = 2^{-n-1} \int_{E_n \cup F_n} t^k dt \int_{J_n} f_c(t) dt, \quad 1 \leq k \leq N, \quad n \in \mathbb{Z}.$$ 

Let us define $K(u, v; f) = -K(v, u; f)$ if $u > v$. Then there exists $h \in X$ so that

$$|K(u, v; f)| \leq uh^*(u) + vh^*(v), \quad u, v > 0.$$ 

Hence

$$K(u, 1; f) - uh^*(u) \leq K(v, 1; f) + vh^*(v), \quad u, v > 0.$$ 

Now (as in Lemma 2.7) this implies the existence of $\lambda \in \mathbb{R}$ so that

$$|K(u, 1; f) - \lambda| \leq uh^*(u), \quad 0 \leq u < \infty.$$ 

We may assume $h^*$ is constant on each interval $(2^n, 2^{n+1}]$ and that $|f_c(t)| \leq h^*(t)$. For $n \in \mathbb{Z}$ we define $\phi_n$ by

$$\phi_n(t) = \begin{cases} 
 f_c(t) - 2^{-n-1}(K(2^n, 1; f) - \lambda) & t \in E_n \\
 -2^{-n}(K(2^n, 1; f) - \lambda) & t \in F_n \\
 2^{-n-2}(K(2^{n+1}, 1; f) - \lambda) & t \in E_{n+1} \\
 f_c(t) + 2^{-n-2}(K(2^{n+1}, 1; f) - \lambda) & t \in F_{n+1} \\
 0 & \text{otherwise} \n \end{cases}$$ 

Then

$$\int \phi_n(t) dt = \int_{E_n \cup F_{n+1}} f_c(t) dt + K(2^{n+1}, 1; f) - K(2^n, 1; f) = 0.$$ 

We thus have

$$\int t^k \phi_n(t) dt = 0, \quad 0 \leq k \leq N. \quad (4.1)$$ 

Note also that we have an estimate

$$|\phi_n(t)| \leq Ch^*(2^n)$$ 

for a suitable constant $C$. 

The theorems follow easily (cf. Proposition 2.12). Thus the only interesting cases are when $px \leq 1 \leq \phi x$. 

Let $L_{log}$ denote the (non-locally convex) Orlicz space of all functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\int \frac{\log(1 + |f(x)|)}{1 + x^2} dx < \infty.$$ 

This is an F-space when it is equipped with the F-norm

$$f \mapsto \int \frac{\log(1 + |f(x)|)}{1 + x^2} dx.$$ 

It is easy to see from (2.17) that $X$ is continuously embedded into $L_{log}$. Now, when identified with its boundary values, $N^1(U)$ is a closed subspace of $L_{log}$ and hence $X$ is a closed subspace of $X$. 

**Lemma 4.3.** Suppose $E$ is a bounded measurable subset of $\mathbb{R}$ and $N \in \mathbb{N}$. If $f : E \to \mathbb{R}$ is measurable there exists $g : E \to \mathbb{R}$ with $g \sim f$ and

$$\int_E t^k g(t) dt = \frac{1}{\mu(E)} \left( \int_E t^k dt \right) \left( \int_E g(t) dt \right), \quad 0 \leq k \leq N.$$ 

**Proof.** By repeated applications of Lyapunov's theorem taking $A_1 = E$ we can find Borel sets $A_n$ with $A_n = A_{2n} \cup A_{2n+1}, A_{2n} \cap A_{2n+1} = \emptyset, \mu(A_{2n}) = \mu(A_{2n+1}) = \frac{1}{2} \mu(A_n)$ and

$$\int_{A_n} t^k dt = \int_{A_{2n+1}} t^k dt = \frac{1}{2} \int_{A_n} t^k dt, \quad 0 \leq k \leq N.$$ 

The sets $(A_n)_{n=0}^\infty$ generate a non-atomic sub-$\sigma$-algebra $\Sigma_0$ of measurable subsets of $E$ so that $B \in \Sigma_0$ implies

$$\int_B t^k dt = \mu(B) \int_E t^k dt, \quad 0 \leq k \leq N.$$ 

Take $g \sim f$ to be $\Sigma_0$-measurable. \qed

We now turn to the proofs of the theorems. We first consider Theorem 4.2, and then indicate the modifications which establish Theorem 4.1.

**Proof.** One direction follows immediately from Theorems 3.3 and 3.4. In Theorem 4.1 if $g \in H_X$ and $g(0) = 0$ then Theorem 3.3 implies that $g \in Z_X(T)$. Thus $9g \in Z_X$ by Proposition 2.5. A similar argument applies in Theorem 4.2.

We prove first Theorem 4.2. In the proof we use $C$ to denote a constant which depends only on $X$ but which can vary from line to line. Let us pick $N \in \mathbb{N}$ so that

$$N + 2 > \frac{1}{p}.$$ 

Define
\[ \psi_n(z) := \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\phi_n(t)}{z - t} \, dt \]
for \( z \in U \). Then \( \psi_n \in H^\infty \) and has boundary values for \( \Im z = 0 \) such that \( \Re \psi_n(x) = \phi_n(x) \) a.e. Note that
\[ \Im \psi_n(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi_n(t)}{x - t} \, dx. \]

We complete the argument by showing that \( \sum_{n \in \mathbb{Z}} \psi_n(x) \) converges a.e. to a function \( g \) in \( H^\infty \). To show this we show that \( \sum_{n \in \mathbb{Z}} \psi_n(x) \) converges a.e. to a function in \( X \). It follows then that the series \( \sum_{n \in \mathbb{Z}} \psi_n \) converges also in the topology of \( L_p + L_q \) provided \( p < p_X \) and \( q > q_X \) and hence \( g \in N^+ \cap \mathcal{X} \). Once this is done it is clear that \( \Re g(x) = f(x) \) a.e. and so \( \Im g \sim f \).

We first observe that if \( |x| \leq 2^{n-1} \) then we have an easy estimate that
\[ |\psi_n(x)| \leq C^* h^*(2^n). \]

Next suppose \( |x| \geq 2^{n+1} \). Then
\[ \frac{1}{(x-t)} = \frac{1}{x} + \cdots + \frac{t}{x^{N+1}} + \frac{t}{x^{N+1}(x-t)}, \]
and by (4.1),
\[ |\psi_n(x)| \leq C |x|^{-(N+2)} 2^{(N+2)b} h^*(2^n). \]

Thus if \( x \in J_n \), and we have
\[ |\psi_n^{(k)}(x)| \leq C 2^{-k(N+2)} h^*(2^{-k}|x|) \]
for \( k \geq 3 \) and
\[ |\psi_n^{(k)}(x)| \leq C h^*(2^k|x|) \]
for \( k \geq 2 \).

Hence if \( x \in J_n \),
\[ \sum_{k=2}^{n} |\psi_n^{(k)}(x)| \leq C \sum_{k=2}^{n} 2^{-k(N+2)} h^*(2^{-k}|x|) \]
and
\[ \sum_{k=2}^{n} |\psi_n^{(k)}(x)| \leq C \sum_{k=2}^{n} h^*(2^k|x|). \]

Now by Lemma 2.11 we have that
\[ \sum_{k=2}^{n} 2^{2 n(N+2,0)} h^*(2^k|x|) < \infty, \quad x \neq 0 \]
and
\[ \left\| \sum_{k \in \mathbb{Z}} 2^{\min(k(N+2),0)} h^*(2^k|x|) \right\|_X \leq C \|h\|_X. \]

Consider the cases \(-2 \leq k \leq 1 \). Then
\[ \left( \int_{J_n} |\psi_n(x)|^q \, dx \right)^{\frac{1}{q}} \leq C 2^{-n+b} h^*(2^{n+b}). \]

For \( r > 0 \), let \( A_{nr} \) be the subset of \( J_n \) of all \( x \) such that \( \sum_{k=-2}^{1} |\psi_n^{(k)}(x)| \geq 2^r h^*(2^{n-2}). \) Then
\[ \mu(A_{nr}) \leq C 2^{-n+r}. \]

Hence, if \( q_X < q' < q \),
\[ \left\| \sum_{n \geq 2} 2^{-n} h^*(2^{n-2}) \chi_{J_n} \right\|_X \leq C 2^{-r}/q' \|h\|_X. \]

This quickly yields the estimate:
\[ \left\| \sum_{n \geq 2} \sum_{k=-2}^{1} |\psi_n^{(k)}| \chi_{J_n} \right\|_X \leq C \|h\|_X. \]

Combining gives us that \( \sum_{n \in \mathbb{Z}} |\psi_n| \) converges a.e. to a function in \( \mathcal{X}(\mathbb{R}) \). This completes the proof of Theorem 4.2. \( \square \)

**Proof of Theorem 4.1.** Only small modifications are necessary for Theorem 4.1. We note that \( f_d \) is now supported on a set \([a, 1-a]\) where \( 0 \leq a \leq 1 \) and \( h^* \) is supported on \([0, 1] \). In this case with the additional assumption that \( p_X > \frac{1}{2} \) we can take \( N = 0 \). This allows us to simplify the construction by taking
\[ \phi_n(t) = \left\{ \begin{array}{ll} f_d(t) & t \in J_n \\ 2^{-n-1} K(2^n, 1; f) & t \in J_n+1 \\ 0 & t \notin J_n \cup J_n+1. \end{array} \right. \]

The calculations are then very similar. Construct \( g \) as above and then define \( g(0) = 0 \) and
\[ g(0) \left( r e^{i\theta} \right) = \sum_{n \in \mathbb{Z}} g \left( \frac{1}{2\pi} (\theta - i \log r) + n \right). \]

One only really needs to show that \( g \in \mathcal{X}(\mathbb{T}) \) or equivalently that \( \sum_{n \in \mathbb{Z}} g(x + n) \in \mathcal{X}(0, 1] \). Clearly \( g(x, n+1) \in \mathcal{X}(n, n+1) \) for all \( n \in \mathbb{Z} \). If \( |n| \geq 3 \) then one gets an estimate
\[ \|g(x, n+1)\|_\infty \leq C |n|^{-2}. \]
by using (4.2) above. It is then easy to see that $g_0$ solves the problem. 

Let us remark that it should be possible to prove Theorem 4.1 without the restriction $pX > \frac{1}{2}$ as in the case of Theorem 4.2.

References


Tadeusz Figiel, Institute of Mathematics of the Polish Academy of Sciences, 81-825 Sopot, Poland
E-mail: t.figiel@impan.gda.pl

Nigel Kalton, Department of Mathematics, University of Missouri, Columbia, MS 65211, U.S.A.
E-mail: nigel@math.missouri.edu
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