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Departament de Matemàtica i Informàtica
Universitat de Vic
E-08500 Vic, Spain
E-mail: jlgarcia@uvic.es

Departament de Matemàtica
Aplicada i Anàlisi
Universitat de Barcelona
E-08071 Barcelona, Spain
E-mail: soria@cerber.mat.ub.es

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Uniqueness of unconditional bases in c_0-products

by

P. G. CASAZZA and N. J. KALTON (Columbia, Mo.)

Abstract. We give counterexamples to a conjecture of Bourgain, Casazza, Lindenstrauss and Tzafriri that if X has a unique unconditional basis (up to permutation) then so does $c_0(X)$. We also give some positive results including a simpler proof that $c_0(\ell_1)$ has a unique unconditional basis and a proof that $c_0(\ell_{p_n}^{N_n})$ has a unique unconditional basis when $p_n \downarrow 1$, $N_{n+1} \geq 2N_n$ and $(p_n - p_{n+1}) \log N_n$ remains bounded.

1. Introduction. A Banach space X is said to have a *unique unconditional basis* (or more precisely, a unique unconditional basis up to permutation) if it has an unconditional basis and if whenever (u_n) and (v_n) are two normalized unconditional bases of X, then there is a permutation π of $\mathbb{N}$ such that (v_n) and $(u_{\pi(n)})$ are equivalent. Since unconditional bases correspond to discrete or atomic order-continuous lattice structures on X, this can be reworded as a statement that such a lattice-structure is essentially unique.

The earliest examples of Banach spaces with unique unconditional bases are c_0, ℓ_1 ([10]) and ℓ_2 ([9]). It was shown by Lindenstrauss and Zippin [12] that amongst spaces with symmetric bases this is the complete list. Later Edelstein and Wojtaszczyk showed that direct sums of these spaces also have unique unconditional bases. All these results can be found in [11]. In [3] the authors attempted a complete classification and showed that the spaces $c_0(\ell_1), c_0(\ell_2), \ell_1(c_0)$ and $\ell_1(\ell_2)$ all have unique unconditional bases while $\ell_2(\ell_1)$ does not. They also found an unexpected additional space, 2-convexified Tsirelson (see [5] for the definition), with a unique unconditional basis. Recently, the authors found a new approach to this type of problem and were able to add some more spaces, including Tsirelson space (see [5]) itself and certain Nakano spaces [4] (as pointed out in [4], some spaces considered by Gowers [8] provide further examples); we also showed

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that uniqueness of the unconditional basis need not be inherited by a complemented subspace.

This note is motivated by a question raised in [3]: does $c_0(X)$ have a unique unconditional basis whenever X does? The idea here is that if this and the corresponding dual result for ℓ_1-products holds then one could iterate the results in [3] to produce examples such as $c_0(\ell_1(c_0(\ell_1)))$ and so on.

Unfortunately, as we show below in Section 4, the answer to this question is negative and Tsirelson space T or its 2-convexified version both produce counterexamples. However, we show how our approach in [4] can be used for c_0-products. We give a much shorter proof (Theorem 3.3) of the fact that $c_0(\ell_1)$ has a unique unconditional basis; the original proof of this result in [3] is extremely technical. We show by the same techniques (Theorem 3.4) that examples of the type $c_0(\ell_{p_n}^{N_n})$ where $p_n \downarrow 1$, $N_{n+1} \geq 2N_n$ and $(p_n - p_{n+1}) \log N_n$ remains bounded, must also have unique unconditional bases.

In Section 4, we also use the same techniques to show that for certain right-dominant spaces X, as introduced in [4], such as Tsirelson space T, any unconditional basis of $c_0(X)$ must be equivalent to a *subset* of the canonical basis (Theorem 4.1). Nevertheless we show that the unconditional basis of $c_0(T)$ is *not* unique as already remarked.

We conclude this section with a few remarks on terminology and assumptions. We will frequently index unconditional bases and basic sequences by an unordered countable index set $\mathcal{N}$ which need not be the natural numbers $\mathbb{N}$. We will assume that any unconditional basic sequence $(u_n)_{n\in\mathcal{N}}$ is semi-normalized, i.e. $0 < \inf_{n\in\mathcal{N}} \|u_n\| \leq \sup_{n\in\mathcal{N}} \|u_n\| < \infty$. We will say that two unconditional basic sequences $(u_n)_{n\in\mathcal{N}}$ and $(v_n)_{n\in\mathcal{N}'}$ are *equivalent* if there is a bijection $\pi : \mathcal{N} \to \mathcal{N}'$ so that $(u_n)_{n\in\mathcal{N}}$ and $(v_{\pi(n)})_{n\in\mathcal{N}}$ are equivalent.

An unconditional basic sequence $(u_n)_{n\in\mathcal{N}}$ in X is *complemented* if there is a bounded projection $P : X \to [u_n]_{n\in\mathcal{N}}$. If $(u_n)_{n\in\mathcal{N}}$ is an unconditional basis of X and $(v_n)_{n\in\mathcal{N}'}$ is an unconditional basic sequence of the form $v_n = \sum_{k\in\mathcal{A}_n} a_k u_k$ where the sets $(\mathcal{A}_n)_{n\in\mathcal{N}'}$ are disjoint subsets of $\mathcal{N}$, we say that $(v_n)_{n\in\mathcal{N}'}$ is *disjoint* with respect to $(u_n)_{n\in\mathcal{N}}$. If (v_n) is a complemented disjoint sequence then it may be shown that there is a projection $Px = \sum_{n\in\mathcal{N}'} v_n^*(x) v_n$ where each $v_n^* \in X^*$ is of the form $v_n^* = \sum_{k\in\mathcal{A}_n} b_k u_k^*$ and $(u_k^*)_{k\in\mathcal{N}}$ is the sequence of biorthogonal functions for $(u_k)_{k\in\mathcal{N}}$.

It will be convenient to represent a space X with unconditional basis $(u_n)_{n\in\mathcal{N}}$ as a sequence space modelled on the index set $\mathcal{N}$, identifying $\sum_{k\in\mathcal{N}} a_k u_k$ with the function $f : \mathcal{N} \to \mathbb{R}$ given by $f(k) = a_k$. This identifies X as a discrete Banach lattice and allows us to use functional notation. The canonical basis of a sequence space X modelled on $\mathcal{N}$ is denoted by $(e_n)_{n\in\mathcal{N}}$.

If $(u_n)_{n\in\mathcal{N}}$ is an unconditional basis for X and N is a natural number, we denote by $(u_n)_{n\in\mathcal{N}}^N$ the naturally induced unconditional basis of X^N (the direct sum of N copies of X).

For future reference we note here that our techniques depend critically on the following result, proved in Theorem 3.5 of [4]:

THEOREM 1.1. *Suppose X is a Banach space with an unconditional basis $(u_n)_{n\in\mathcal{N}}$ which does not contain uniformly complemented copies of ℓ_2^n (i.e. is not sufficiently Euclidean). Suppose $(v_n)_{n\in\mathcal{B}}$ is a complemented unconditional basic sequence in X. Then there is a integer N and a complemented disjoint sequence $(w_n)_{n\in\mathcal{B}}$ in the basis $(u_n)_{n\in\mathcal{N}}^N$ such that $(v_n)_{n\in\mathcal{B}}$ is equivalent to $(w_n)_{n\in\mathcal{B}}$.*

2. A criterion for an ℓ_1- or c_0-product to be sufficiently Euclidean. The aim of this section is to establish criteria for a c_0-product to contain uniformly complemented copies of ℓ_2^n so that we can apply Theorem 1.1.

If X is a Banach space we say that X has property $P(k, M)$, where $k \in \mathbb{N}$ and $M \geq 1$, if whenever $S : \ell_2^k \to X$ and $T : X \to \ell_2^k$ are operators satisfying $TS = I_{\ell_2^k}$ then $\|S\|\|T\| \geq M$. We say that a sequence of Banach spaces $(X_j)_{j=1}^\infty$ has property $P(k, M)$ if each X_j has property $P(k, M)$. A Banach space X (respectively a sequence of Banach spaces $(X_j)_{j=1}^\infty$) is *sufficiently Euclidean* if there exists M so that X (respectively $(X_j)_{j=1}^\infty$) fails $P(k, M)$ for every $k \in \mathbb{N}$.

We recall that if H is a finite-dimensional Hilbert space and $A : H \to X$ is any linear map then the ℓ-norm of A is given by

$$\ell(A) = \mathbf{E}\Big(\Big\|\sum_{i=1}^m g_i A e_i\Big\|^2\Big)^{1/2}$$

where $(e_1, \ldots, e_m)$ is any orthonormal basis of H and $(g_1, \ldots, g_m)$ is a sequence of independent normalized Gaussian random variables. See [15]. If S is an operator on a Banach space X and E is a closed subspace of X we denote by S_E the restriction of S to E.

LEMMA 2.1. *There exists a universal constant $c > 0$ with the following property: Suppose H is an n-dimensional Hilbert space and X is any Banach space. Suppose $S : H \to X$ is any operator with $\|S\| \leq 1$. Then there is a subspace E of H with $\dim E \geq c\ell(S)^2$ so that $\|S_E\| \leq 3\ell(S)n^{-1/2}$.*

Proof. It will suffice to prove this for S one-to-one, since the result then follows by a simple perturbation argument. Let μ be a normalized invariant measure on the surface of the sphere in ℓ_2^n. Consider the norm $\xi \to \|S\xi\|$; this satisfies $\|S\xi\| \leq \|\xi\|$ for all ξ. We use Theorem 4.2 of [13] (p. 12). If M_r

is a median value of the norm $\|S\xi\|$ then

$$M_r \leq \sqrt{2}\Big(\int \|S\xi\|^2\,d\mu\Big)^{1/2} = \sqrt{2/n}\,\ell(S). \ \blacksquare$$

LEMMA 2.2. *Suppose X is a Banach space with property $P(k,M)$. Suppose H is an n-dimensional Hilbert space and $S: H\to X$ and $T: X\to H$ are bounded operators with $\|T\|\leq 1$. Then*

$$|\mathrm{tr}(TS)| \leq Cn^{1/2}\max(M^{-1}\ell(S), k^{1/2}\|S\|)$$

for some universal constant C.

Proof. Suppose that $1\leq j\leq n$ and s_j is the jth singular value of TS. We can restrict to a subspace H_j of dimension j so that $\|TS\xi\|\geq s_j\|\xi\|$ for all $\xi\in H_j$.

Assume $s_j>0$. Then by Lemma 2.1 there is a subspace E of H_j so that $\dim E\geq c\ell(S_{H_j})^2$ and $\|S_E\|\leq 3\ell(S_E)j^{-1/2}$.

Suppose $\dim E\geq k$. Then we must have $\|T\|\|S_E\|\geq Ms_j$ so that $\ell(S_E)\geq \frac{1}{3}Ms_j j^{1/2}$. This implies that $s_j\leq 3M\ell(S)j^{-1/2}$. If $\dim E<k$ then

$$\ell(S_{H_j})^2\|S_{H_j}\|^{-2}\leq c^{-1}k$$

and so $\ell(TS_{H_j})\leq c^{-1/2}k^{1/2}\|S\|$. From this we deduce that $j^{1/2}s_j\leq c^{-1/2}k^{1/2}\|S\|$ or $s_j\leq c^{-1/2}k^{1/2}j^{-1/2}\|S\|$. Combining we obtain

$$s_j\leq \max(3M^{-1}\ell(S), c^{-1/2}k^{1/2}\|S\|)j^{-1/2}.$$

Now

$$|\mathrm{tr}(TS)|\leq \sum_{j=1}^n s_j\leq Cn^{1/2}\max(M^{-1}\ell(S), k^{1/2}\|S\|)$$

for some universal constant C. $\blacksquare$

LEMMA 2.3. *There is a universal constant C so that if X is a Banach space with property $P(k,M)$ then whenever H is a Hilbert space of dimension n, and $S: H\to X$ and $T: X\to H$ are bounded operators with $\|T\|\leq 1$, we have*

$$|\mathrm{tr}(TS)|\leq C\ell(S)(1/M+k^{1/2}(\log n)^{-1/2})n^{1/2}.$$

Proof. We first choose an orthonormal basis $(e_i)_{i=1}^n$ of H so that $\|Se_i\| = \|S_{H_i}\|$ where $H_i=[e_i,e_{i+1},\ldots,e_n]$. Pick $m=[n^{1/2}]$. Then if $g_1,\ldots,g_n$ are normalized independent Gaussians,

$$\ell(S)\geq \mathbf{E}\Big(\Big\|\sum_{i=1}^m g_iTe_i\Big\|^2\Big)^{1/2}\geq \|Se_m\|\mathbf{E}(\max_{1\leq i\leq m}|g_i|).$$

Now (cf. [13], p. 23) this implies that

$$\|Se_m\|\leq C(\log m)^{-1/2}\ell(S)$$

for some universal constant C. Our choice of m implies that we can replace this estimate by

$$\|Se_m\|\leq C(\log n)^{-1/2}\ell(S)$$

for some universal constant C.

If $E=[e_1,\ldots,e_m]$ then

$$|\mathrm{tr}(TS_E)|\leq m^{1/2}\ell(S_E)\leq n^{1/4}\ell(S).$$

On the other hand,

$$|\mathrm{tr}(TS_{H_{m+1}})|\leq C\max(M^{-1}, k^{1/2}(\log n)^{-1/2})\ell(S)n^{1/2}$$

by Lemma 2.2. Combining these results gives us our estimate. $\blacksquare$

PROPOSITION 2.4. *Suppose $(X_j)_{j=1}^\infty$ is not sufficiently Euclidean. Then $\ell_1(X_j)$ is not sufficiently Euclidean.*

Proof. Suppose (X_j) satisfies property $P(k,M)$. Suppose $n\in\mathbb{N}$ and $S:\ell_2^n\to\ell_1(X_j)$ and $T:\ell_1(X_j)\to\ell_2^n$ are any operators satisfying $TS=I_{\ell_2^n}$; we assume that $\|T\|=1$. We write $S\xi=(S_i\xi)_{i=1}^\infty$ and $T(x_i)_{i=1}^\infty=\sum_{i=1}^\infty T_ix_i$.

Now $n=\mathrm{tr}(TS)=\sum_{i=1}^\infty \mathrm{tr}(T_iS_i)$. On the other hand, by Lemma 2.3 we have

$$|\mathrm{tr}(T_iS_i)|\leq Cn^{1/2}(1/M+k^{1/2}(\log n)^{-1/2}\ell(S_i)). \tag{2.1}$$

Let $(e_1,\ldots,e_n)$ be any orthonormal basis. Then by the Kahane–Khinchin inequality we have

$$\ell(S_i)\leq C_0\mathbf{E}\Big(\Big\|\sum_{j=1}^n g_jS_ie_j\Big\|\Big) \tag{2.2}$$

where the $(g_i)_{i=1}^n$ are normalized independent Gaussians, and C_0 is a universal constant. Hence

$$\sum_{i=1}^\infty \ell(S_i)\leq C_0\mathbf{E}\Big(\Big\|\sum_{j=1}^n g_jSe_j\Big\|\Big)\leq C\ell(S). \tag{2.3}$$

Combining (2.1)–(2.3), and taking into account $n=\sum_{i=1}^\infty \mathrm{tr}(T_iS_i)$, we get

$$n\leq C_0n^{1/2}(1/M+k^{1/2}(\log n)^{-1/2})\ell(S)\leq C_1n\|S\|(1/M+k^{1/2}(\log n)^{-1/2})$$

for some universal constant C_1. We thus obtain an estimate

$$\|S\|\geq C^{-1}\min(M, k^{-1/2}(\log n)^{1/2})$$

for some absolute constant C. $\blacksquare$

REMARK. In the case when $X=c_0$ we see that X satisfies $P(ck^{1/2},k)$ for $c>0$ and all k. Thus $\ell_1(X)$ satisfies $P(c(\log k)^{1/4},k)$ for some $c>0$ and all k. On the other hand, Figiel, Lindenstrauss and Milman [7] established the upper estimate that $\ell_1(c_0)$ contains a subspace 2-isomorphic to ℓ_2^k which is $(\log k)^{1/2}$-complemented; this estimate is best possible (see [2]). This suggests that our method, while not optimal, cannot be improved significantly. ∎

COROLLARY 2.5. *Suppose $(X_j)_{j=1}^{\infty}$ is not sufficiently Euclidean. Then $c_0(X_j)$ is not sufficiently Euclidean.*

Proof. This follows by simple duality. ∎

3. Unconditional bases in c_0-products. For convenience we define a sequence space X as a Köthe space of real-valued functions on a countable set I (with counting measure) so that the canonical basis vectors $(e_i)_{i\in I}$ form a 1-unconditional basis. Usually, of course, we take $I=\mathbb{N}$, but for our purposes it is convenient also to allow $I=\mathbb{N}\times\mathbb{N}$ and certain other alternatives. A typical element x of X is of the form $x=(x(i))_{i\in I}$.

Let $(u_n)_{n\in\mathcal{N}}$ be a set of disjointly supported vectors in X. Then $(u_n)_{n\in\mathcal{N}}$ is an unconditional basic sequence which is complemented if and only if there exists a biorthogonal sequence $(u_n^*)_{n\in\mathcal{N}}\in X^*$ with $\operatorname{supp} u_n^*\subset \operatorname{supp} u_n$, $u_n u_n^*\ge 0$, $\langle u_n,u_n^*\rangle=1$ and such that the projection

$$Px=\sum_{n\in\mathcal{N}}\langle x,u_n^*\rangle u_n$$

is well-defined and bounded. If we define $f_n=u_nu_n^*$ then $f_n\ge 0$, $f_n\in\ell_1(I)$ and $\|f_n\|_1=1$ for all $n\in\mathbb{N}$. Under these circumstances we say that (u_n) is a *complemented disjoint sequence* and we assume that (u_n^*) and (f_n) are associated with (u_n). Note that we can always replace u_n and u_n^* by $|u_n|$ and $|u_n^*|$ and hence also assume them positive.

We start with an observation which we will use repeatedly.

LEMMA 3.1. *Suppose X is a sequence space (modelled on an index set I) and $(u_n)_{n\in\mathcal{N}}$ is a complemented disjoint sequence. Let $(A_n)_{n\in\mathcal{N}}$ be any sequence of disjoint sets such that for some $\delta>0$ we have $\|f_n\chi_{A_n}\|_1\ge\delta>0$ for all $n\in\mathcal{N}$. Let $v_n=u_n\chi_{A_n}$. Then $(v_n)_{n\in\mathcal{N}}$ is a complemented disjoint sequence equivalent to $(u_n)_{n\in\mathcal{N}}$. Furthermore, the biorthogonal vectors v_n^* may be chosen so that $v_nv_n^*\le\delta^{-1}f_n$.*

Proof. Let P be the projection onto $[u_n]_{n\in\mathcal{N}}$ as defined above. Let

$$v_n^*=\|f_n\chi_{A_n}\|_1^{-1}u_n^*\chi_{A_n}$$

and define

$$Qx=\sum_{n\in\mathcal{N}}\langle x,v_n^*\rangle v_n.$$

Then it is easy to verify that Q is a bounded projection, $Qu_n=v_n$ and $P(v_n)=\|f_n\chi_{A_n}\|_1u_n$. This quickly establishes the equivalence of $(u_n)_{n\in\mathcal{N}}$ and $(v_n)_{n\in\mathcal{N}}$. Furthermore, $v_n^*v_n\le\delta^{-1}f_n$. ∎

Next suppose $(X_i)_{i=1}^{\infty}$ is a sequence of sequence spaces modelled on index sets J_i (either finite sets or $\mathbb{N}$). We suppose that for some $q<\infty$ the spaces (X_i) satisfy a lower q-estimate uniformly, i.e. there exists $c>0$ so that if $i\in\mathbb{N}$ and $x_1\ldots,x_n$ are disjoint in X_i then

$$\Big\|\sum_{k=1}^n x_k\Big\|_{X_i}\ge c\Big(\sum_{k=1}^n\|x_k\|_{X_i}^q\Big)^{1/q}.$$

Let $Y=c_0(X_j)$ be the sequence space on $J=\{(i,j):j\in J_i,\ i\in\mathbb{N}\}$ of all $x=(x[i])_{i=1}^{\infty}$, where $x[i]\in X_i$ are so that $\lim_{i\to\infty}\|x[i]\|_{X_i}=0$. We define

$$\|x\|_Y=\max_{i\in\mathbb{N}}\|x[i]\|_{X_i}.$$

Now suppose $(u_n)_{n\in\mathcal{N}}$ is a complemented disjoint sequence in Y, with biorthogonal sequence (u_n^*). As above let $f_n=u_n^*u_n$. Then define $F_n\in\ell_1(\mathbb{N})$ by

$$F_n(i)=\sum_{j\in J_i}f_n(i,j).$$

We will say that (u_n) is *C-tempered* if

$$\sum_{i=1}^{\infty}\sup_{n\in\mathcal{N}}F_n(i)\le C.$$

THEOREM 3.2. *Suppose $(X_i)_{i=1}^{\infty}$ is a sequence of sequence spaces satisfying a uniform lower q-estimate for some $q<\infty$. Suppose $(u_n)_{n\in\mathcal{N}}$ is a normalized complemented disjoint sequence in $c_0(X_i)$. Then there is a complemented disjoint sequence $(v_n)_{n\in\mathcal{N}}$ equivalent to $(u_n)_{n\in\mathcal{N}}$ and a partition $\mathcal{N}=\bigcup_{n\in\mathcal{A}}\mathcal{B}_n$ of $\mathcal{N}$ with the following properties:*

(1) For each $i\in\mathbb{N}$ we have either $\|v_k[i]\|_{X_i}=0$ or 1.

(2) For some C each $(v_k)_{k\in\mathcal{B}_n}$ is C-tempered.

(3) There exists an integer N and subsets $(S_n)_{n\in\mathcal{A}}$ of $\mathbb{N}$ such that $\sum_{n\in\mathcal{A}}\chi_{S_k}\le N-1$ and $v_k[i]=0$ whenever $k\in\mathcal{B}_n$ and $i\notin S_n$. Hence for any finitely non-zero sequence $(a_n)_{n\in\mathcal{N}}$,

$$\max_{n\in\mathcal{A}}\Big\|\sum_{k\in\mathcal{B}_n}a_kv_k\Big\|_Y\le\Big\|\sum_{k\in\mathcal{N}}a_kv_k\Big\|_Y\le(N-1)\max_{n\in\mathcal{A}}\Big\|\sum_{k\in\mathcal{B}_n}a_kv_k\Big\|_Y.$$

Proof. As usual, we let P be the induced projection on $[u_n]_{n\in\mathcal{N}}$. First let $A_n=\{(i,j):\|u_n[i]\|_{X_i}\ge(2\|P\|)^{-1}\}$. Notice that

$$\sum_{i\notin A_n}\langle u_n[i],u_n[i]^*\rangle\le\frac{1}{2\|P\|}\sum_{i=1}^{\infty}\|u_n[i]\|_{X_i^*}\le\frac{1}{2\|P\|}\|u_n\|_{Y^*}\le\frac12.$$

Hence $(u_n\chi_{A_n})_{n\in\mathcal{N}}$ is a complemented disjoint sequence equivalent to $(u_n)_{n\in\mathcal{N}}$. It follows after some appropriate renormalization that we can replace $(u_n)_{n\in\mathcal{N}}$ by an equivalent sequence with the additional property that $\|u_n[i]\|_{X_i} = 1$ or $u_n[i] = 0$ for every $i \in \mathbb{N}$. For each $n \in \mathcal{N}$ let $S_n = \{i : \|u_n[i]\|_{X_i} = 1\}$.

Next fix any $N \in \mathbb{N}$ so that $N > 1 + c^{-q}(1 + \|P\|)^q$, where c is the constant of the uniform lower q-estimate. Let $\delta = N^{-1}$. We pick a maximal subset $\mathcal{A}$ of $\mathcal{N}$ with the property that if $\mathcal{F}$ is a subset of $\mathcal{A}$ with $|\mathcal{F}| \le N$ then

$$\sum_{i=1}^{\infty} \max_{n\in\mathcal{F}} F_n(i) \ge (1-\delta)|\mathcal{F}|.$$

(Here $F_n(i) = \sum_{j\in J_i} f_n(i,j) = \langle u_n[i], u_n^*[i]\rangle$ as usual.)

Now let $\mathcal{F}$ be any subset of $\mathcal{A}$ with $|\mathcal{F}| = N$. We can partition $\mathbb{N}$ into N disjoint sets $(A_n)_{n=1}^N$ so that if $i \in A_n$ then $F_n(i) = \max_{m\in\mathcal{F}} F_m(i)$. Let $v = \sum_{n\in\mathcal{F}} u_n \chi_{A_n'}$ where $A_n' = \{(i,j) : i \in A_n,\ j \in J_i\}$. Clearly, $\|v\|_Y \le 1$. However,

$$Pv = \sum_{n\in\mathcal{F}} \sum_{i\in A_n} F_n(i) u_n.$$

Since

$$\sum_{n\in\mathcal{F}} \sum_{i\in A_n} F_n(i) \ge N(1-\delta) = N - 1$$

we conclude that

$$\Big\| \sum_{n\in\mathcal{F}} u_n \Big\|_Y \le \|P\| + 1.$$

On the other hand,

$$\Big\| \sum_{n\in\mathcal{F}} u_n \Big\|_Y \ge c \max_{i\in\mathbb{N}} \Big(\sum_{n\in\mathcal{F}} \chi_{S_n}(i) \Big)^{1/q}.$$

Hence

$$\max_{i\in\mathbb{N}} \sum_{n\in\mathcal{F}} \chi_{S_n}(i) \le c^{-q}(\|P\| + 1)^q < N. \tag{3.1}$$

Now suppose for some i we have $\sum_{n\in\mathcal{A}} \chi_{S_n}(i) \ge N$. Then we can find a subset $\mathcal{F}$ of $\mathcal{A}$ with $|\mathcal{F}| = N$ so that $\chi_{S_n}(i) = 1$ for $n \in \mathcal{F}$, contradicting (3.1). We therefore conclude that

$$\max_{i\in\mathbb{N}} \sum_{n\in\mathcal{A}} \chi_{S_n}(i) \le N - 1.$$

Let now $k \in \mathcal{N} \setminus \mathcal{A}$. There exists a subset $\mathcal{F}$ of $\mathcal{A}$ with $1 \le |\mathcal{F}| \le N-1$ and such that

$$\sum_{i=1}^{\infty} \max(F_k(i), \max_{n\in\mathcal{F}} F_n(i)) < (|\mathcal{F}| + 1)(1-\delta).$$

Hence

$$\begin{aligned}\sum_{i=1}^{\infty} \min(F_k(i), \max_{n\in\mathcal{F}} F_n(i)) &= \sum_{i=1}^{\infty} (F_k(i) + \max_{n\in\mathcal{F}} F_n(i)) \\ &\quad - \sum_{i=1}^{\infty} \max(F_k(i), \max_{n\in\mathcal{F}} F_n(i)) \\ &> \sum_{i=1}^{\infty} F_k(i) - 1 + \delta > \delta.\end{aligned}$$

Thus there exists $n \in \mathcal{F}$ so that

$$\sum_{i=1}^{\infty} \min(F_k(i), F_n(i)) > \frac{\delta}{N} = \delta^2.$$

Now put $T_k = \{i : F_k(i) < 2N^2 F_n(i)\}$. Then $\sum_{i\in T_k} F_k(i) > \frac{1}{2}\delta^2$. In the case when $k \in \mathcal{A}$ we will take $T_k = S_k$.

We now can partition $\mathcal{N}$ into disjoint sets $(\mathcal{B}_n)_{n\in\mathcal{A}}$ so that $n \in \mathcal{B}_n$ and if $k \in \mathcal{B}_n$ then $\sum_{i\in T_k} F_k(i) > \frac{1}{2}\delta^2$ and $F_k(i) < 2N^2 F_n(i)$ for $i \in T_k \subset S_n$. If we let $T_k' = \{(i,j) : i \in T_k,\ j \in J_i\}$ and $v_k = u_k \chi_{T_k'}$ then by Lemma 3.1 we find that $(v_k)_{k\in\mathcal{N}}$ is a complemented disjoint sequence in Y equivalent to $(u_k)_{k\in\mathcal{N}}$. Furthermore, if (v_k^*) is the biorthogonal sequence then we have an estimate $v_k v_k^* \le M f_k$ for a suitable constant M. It follows that for each $n \in \mathcal{A}$ we have

$$\langle v_k[i], v_k^*[i]\rangle \le 2MN^2 F_n(i)$$

whenever $k \in \mathcal{B}_n$. Thus the sets $(v_k)_{k\in\mathcal{B}_n}$ are each C-tempered where C is a constant depending only on c, q and $\|P\|$.

Finally suppose $(a_n)_{n\in\mathcal{N}}$ is finitely non-zero. Then

$$\begin{aligned}\Big\| \sum_{k\in\mathcal{N}} a_k v_k \Big\|_Y &= \max_{i\in\mathbb{N}} \Big\| \sum_{k\in\mathcal{N}} a_k v_k[i] \Big\|_{X_i} = \max_{i\in\mathbb{N}} \Big\| \sum_{n\in\mathcal{A}, i\in S_n} \sum_{k\in\mathcal{B}_n} a_k v_k[i] \Big\|_{X_i} \\ &\le (N-1) \max_{n\in\mathcal{A}} \Big\| \sum_{k\in\mathcal{B}_n} a_k v_k \Big\|_Y. \ \blacksquare\end{aligned}$$

Let us first use this theorem to give a simpler proof of the result of [3] that $c_0(\ell_1)$ has a unique unconditional basis (up to permutation).

THEOREM 3.3. *The space $c_0(\ell_1)$ has a unique unconditional basis.*

Proof. We start with the remark that $c_0(\ell_1)$ is not sufficiently Euclidean (cf. Bourgain [2] or Corollary 2.5 above). Hence any complemented unconditional basic sequence is equivalent to a complemented positive disjoint sequence in $c_0(\ell_1)^m$ for some m. We can clearly suppose $m = 1$.

We next show that any C-tempered C-complemented disjoint sequence $(u_n)_{n\in\mathcal{N}}$ is K-equivalent to the standard ℓ_1-basis where K depends only on C. Indeed, we may suppose $\|u_n[i]\|_1 = 1$ or $u_n[i] = 0$ for each i, n. Let $G(i) = \max_n \langle u_n[i], u_n^*[i]\rangle$. Then $\|G\|_1 \le C$.

Now

$$\Big\| \sum_{n\in\mathcal{N}} a_n u_n \Big\|_{c_0(\ell_1)} \ge \frac{1}{C} \sum_{i=1}^{\infty} G(i) \sum_{u_n[i]\ne 0} |a_n| = \frac{1}{C} \sum_{n\in\mathcal{N}} |a_n| \sum_{u_n[i]\ne 0} G(i)$$

$$\ge \frac{1}{C} \sum_{n\in\mathcal{N}} |a_n| \sum_{i=1}^{\infty} \langle u_n[i], u_n^*[i]\rangle = \frac{1}{C} \sum_{n\in\mathcal{N}} |a_n|.$$

It follows that any unconditional basis of $c_0(\ell_1)$ is equivalent to the canonical unconditional basis of $c_0(X_n)$ where each X_n is either ℓ_1 or ℓ_1^m for some $m = m(n)$. However, there must be infinitely many indices n for which $X_n = \ell_1$ (since $c_0(\ell_1)$ cannot be decomposed as $\ell_1 \oplus Z$ where Z contains no copy of ℓ_1.) It then easily follows that $c_0(\ell_1)$ has a unique unconditional basis. ■

THEOREM 3.4. *Suppose $1 < p_n < \infty$ and $p_n \downarrow 1$. Let (N_n) be an increasing sequence of natural numbers such that $(p_n - p_{n+1}) \log N_n$ is bounded and $N_{n+1} \ge 2N_n$. Then $c_0(\ell_{p_n}^{N_n})$ has a unique unconditional basis.*

Proof. We suppose the sequence (p_n) fixed and first consider $c_0(\ell_{p_n}^{M_n})$ for any sequence of integers (M_n). It is easy to see by considering the ultraproduct $\prod_{\mathcal{U}} \ell_{p_n}^{M_n}$ (which is an L_1-space) that the sequence $(\ell_{p_n}^{M_n})$ is not sufficiently Euclidean. By Corollary 2.5 also the space $Y = c_0(\ell_{p_n}^{M_n})$ is not sufficiently Euclidean and therefore every complemented unconditional basic sequence in Y is equivalent to a complemented disjoint sequence in $Y^r = c_0(\ell_{p_n}^{rM_n})$ for some $r \in \mathbb{N}$.

In this case our spaces X_n are modelled on the sets $J_n = \{1, 2, \ldots, M_n\}$. Thus $c_0(\ell_{p_n}^{M_n})$ is modelled on the set $\{(n, k) : n \in \mathbb{N},\ 1 \le k \le M_n\}$.

Now suppose $(u_n)_{n\in\mathcal{N}}$ is a C-tempered C-complemented disjoint sequence in $c_0(\ell_{p_n}^{M_n})$ with the property that $\|u_n[i]\|_{\ell_{p_i}^{M_i}} = 1$ or 0 for each i, and let $A = \{i : \exists n,\ u_n[i] \ne 0\}$.

CLAIM. *There exists a constant $K = K(C)$, an integer r depending only on C, a subset B of A with $|B| \le r$ and $P_i \le rM_i$ for $i \in' B$ so that $(u_n)_{n\in\mathcal{N}}$ is K-equivalent to the canonical basis of $(\sum_{i\in B} \oplus \ell_{p_i}^{P_i})_{c_0}$.*

To show this let $G(i) = \sup\langle u_n[i], u_n^*[i]\rangle$ as usual. We have $\sum_{i=1}^{\infty} G(i) \le C$. We first consider the case when $G(i) \le 1/20$ for every i. Then it is possible to find a finite increasing sequence of integers $(k_j)_{j=0}^{N}$ where $N = [10C]$ depends on C such that

$$\sum_{i<k_j} G(i) \le \frac{j}{10} \quad\text{and}\quad \sum_{i\le k_j} G(i) \ge \frac{j}{10}.$$

Notice that

$$\sum_{i>k_N} G(i) \le \frac{1}{10}.$$

It follows that for each n there exist at least three values of $1 \le j \le N$ so that

$$\sum_{i=k_{j-1}+1}^{k_j} \langle u_n[i], u_n^*[i]\rangle > \frac{1}{4N}.$$

We can then assign to each n a value of j which is neither the largest nor smallest with this property. In this way we partition $\mathcal{N}$ into sets $(\mathcal{N}_j)_{2\le j\le N-1}$.

Consider $(u_n)_{n\in\mathcal{N}_j}$. We note that this is equivalent (with constants depending only on C) to each of $(v_n)_{n\in\mathcal{N}_j}$ and $(w_n)_{n\in\mathcal{N}_j}$, where $v_n[i] = u_n[i]$ if $i \le k_{j-1}$ and 0 otherwise while $w_n[i] = u_n[i]$ if $i > k_j$ and 0 otherwise.

Now for any finitely non-zero sequence $(a_n)_{n\in\mathcal{N}_j}$ we have

$$\Big\| \sum_{n\in\mathcal{N}_j} a_n v_n \Big\|_Y \le \Big(\sum_{n\in\mathcal{N}_j} |a_n|^{q_{j-1}} \Big)^{1/q_{j-1}}$$

where $q_j = p_{k_j}$. On the other hand,

$$\Big\| \sum_{n\in\mathcal{N}_j} a_n w_n \Big\|_Y \ge \max_{i>k_j} \Big(\sum_{u_n[i]\ne 0} |a_n|^{q_j} \Big)^{1/q_j}.$$

Thus

$$\Big\| \sum_{n\in\mathcal{N}_j} a_n w_n \Big\|_Y^{q_j} \ge \frac{1}{C} \sum_{i>k_j} \sum_{u_n[i]\ne 0} G(i)|a_n|^{q_j}.$$

Now for fixed n,

$$\sum_{i>k_j,\, u_n[i]\ne 0} G(i) \ge \sum_{i>k_j} \langle u_n[i], u_n^*[i]\rangle \ge \frac{1}{4N}.$$

It follows that $(u_n)_{n\in\mathcal{N}_j}$ satisfies an upper q_{j-1}-estimate and a lower q_j-estimate with constants depending only on C. We next estimate $|\mathcal{N}_j|$. In fact,

$$\frac{1}{4N}|\mathcal{N}_j| \le \sum_{n\in\mathcal{N}_j} \sum_{i=k_{j-1}+1}^{k_j} \langle u_n[i], u_n^*[i]\rangle \le \sum_{i=k_{j-1}+1}^{k_j} G(i) M_i$$

$$\le C \max_{k_{j-1}<i\le k_j} M_i.$$

Hence if we select $k_{j-1} < i \le k_j$ appropriately we have $(u_n)_{n\in\mathcal{N}_j}$ equivalent to a subset of the standard basis of $\ell_{p_i}^{rM_i}$ where r and the constant of equivalence depend only on C.

We must now treat the case when $G(i) > 1/20$ for some i. In this case we split $\mathcal{N}$ into two groups $\mathcal{N}'$ and $\mathcal{N}''$ where $\mathcal{N}' = \{n : \exists i,\ \langle u_n[i], u_n^*[i]\rangle > 1/20\}$ and $\mathcal{N}''$ is the remainder. Then $\mathcal{N}''$ can be treated as before. For $\mathcal{N}'$ we note that $(u_n)_{n\in\mathcal{N}'}$ is equivalent to a sequence (u_n') where $u_n'[i] = u_n[i]$ for precisely one index $i = i_n$ such that $\langle u_n[i_n], u_n^*[i_n]\rangle > 1/20$ and is zero elsewhere. The appropriate representation of $(u_n)_{n\in\mathcal{N}'}$ follows once we observe that the set $\{i_n : n \in \mathcal{N}'\}$ is bounded in cardinality with a bound depending only on C. But this is clear since $G(i_n) > 1/20$ but $\sum G(i) \le C$.

Thus the claim is established.

Returning to our original hypotheses we see that if (u_n) is any unconditional basis of $c_0(\ell_{p_n}^{N_n})$ then (u_n) is equivalent to the canonical basis of $c_0(\ell_{p_n}^{M_n})$ where $M_n \le rN_n$ for all n and some fixed r. By the same token the canonical basis of $c_0(\ell_{p_n}^{N_n})$ is equivalent to a subset of $(u_n)^s$ for some $s \in \mathbb{N}$.

Now the additional hypotheses on N_n ensure that the original basis is equivalent to its square. Hence the s-fold product $(u_n)^s$ is equivalent to a subset of the canonical basis and so it follows from the Cantor–Bernstein principle (apparently first noticed by Mityagin, [14], [16] and [17]) that $(u_n)^s$ and the original basis are equivalent.

Thus the canonical bases of $c_0(\ell_{p_n}^{sM_n})$ and $c_0(\ell_{p_n}^{sN_n})$ are equivalent. Let $\mathcal{M} = \{(i,j) : 1 \le j \le M_i\}$ and $\mathcal{N} = \{(i,j) : 1 \le j \le N_i\}$. Suppose the former basis is indexed by $\mathcal{M}^s = \{(i,j) : 1 \le j \le sM_i\}$ and the latter by $\mathcal{N}^s = \{(i,j) : 1 \le j \le sN_i\}$. Let $\varphi : \mathcal{M}^s \to \mathcal{N}^s$ be a bijection implementing the claimed equivalence of bases. By elementary considerations concerning c_0-sums it is clear that for each fixed i the set $\{\varphi(i,j) : 1 \le j \le M_i\}$ can have at most t distinct first coordinates where t depends only on the constant of equivalence; similarly, for each fixed i the set of possible first coordinates of $\varphi^{-1}(i,j)$ can be bounded by the same t. For each $(a,b) \in \mathbb{N}^2$, let E_{ab} be the set of (i,j) so that $i = a$ and the first coordinate of $\varphi(i,j)$ is b. Then let F_{ab} be a subset of E_{ab} of size $[|E_{ab}|/s]$. Restricting φ to $\bigcup_{(a,b)} F_{ab}$ produces an equivalence between the bases of $c_0(\ell_{p_n}^{M_n-\alpha_n})$ and $c_0(\ell_{p_n}^{N_n-\beta_n})$ where $0 \le \alpha_n, \beta_n \le (s-1)t$ for each n. This clearly implies the equivalence of (u_n) and the original basis. ■

4. Uniqueness of unconditional bases in c_0-products of right-dominant spaces. We first introduce some standard notation. Let A, B be subsets of $\mathbb{N}$. We write $A < B$ to indicate that $\max\{a : a \in A\} < \min\{b : b \in B\}$.

Let X be a sequence space modelled on $\mathbb{N}$. We say that X is *right-dominant* if there is a constant $\kappa = \kappa(X)$ so that whenever $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are disjointly supported sequences satisfying $\operatorname{supp} u_k < \operatorname{supp} v_k$ and $\|u_k\|_X = \|v_k\|_X$ for $1 \le k \le n$ then $\|\sum_{k=1}^n u_k\|_X \le \kappa\|\sum_{k=1}^n v_k\|_X$. We say that X is *left-dominant* if there is a constant $\varrho = \varrho(X)$ so that whenever $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are disjointly supported sequences satisfying $\operatorname{supp} u_k < \operatorname{supp} v_k$ and $\|u_k\|_X = \|v_k\|_X$ for $1 \le k \le n$ then $\|\sum_{k=1}^n v_k\|_X \le \varrho\|\sum_{k=1}^n u_k\|_X$.

Left- and right-dominant spaces were studied in [4]. It is established (Lemma 5.2 of [4]) that in these spaces there is exactly one $r = r(X)$ (the *index* of X) so that ℓ_r is disjointly finitely representable in X. If X is right-dominant then X satisfies an upper r-estimate and a lower s-estimate for any $s > r$; the corresponding dual statements hold for left-dominant spaces. Clearly, if a space X is both left- and right-dominant then $X = \ell_r$.

THEOREM 4.1. *Let X be a right-dominant sequence space with $r(X) = 1$. Then every complemented unconditional basic sequence in $c_0(X)$ is equivalent to a subsequence of the canonical basis.*

REMARK. In particular, this applies when X is a Nakano space $\ell_{(p_n)}$ where $p_n \downarrow 1$, or when X is Tsirelson space T (see [4]).

Proof (of Theorem 4.1). In this case we note that in the notation of Section 3, $J_i = \mathbb{N}$ for all $i \in \mathbb{N}$. We first note that by Corollary 2.5, $c_0(X)$ is not sufficiently Euclidean. Hence by [4], Theorem 3.5, every complemented unconditional basic sequence is equivalent to a complemented positive disjoint sequence in $c_0(X)^N$ for some N and hence also to a complemented disjoint sequence in $c_0(X)$.

Now by Theorem 3.2 it will suffice to show that if $(u_n)_{n\in\mathcal{N}}$ is a C-tempered C-complemented unconditional basic sequence then $(u_n)_{n\in\mathcal{N}}$ is K-equivalent to a subsequence of the canonical basis of $c_0(X)$ where K depends only on C. In fact we will show that it is K-equivalent to a subsequence of the canonical basis of $\ell_\infty^N(X)$ where N depends only on C.

We may suppose that, as before, $\|u_n[i]\|_X = 1$ or $u_n[i] = 0$. Let $f_n = u_n u_n^*$ and $F_n(i) = \langle u_n[i], u_n^*[i]\rangle$. Let $G(i) = \max_n F_n(i)$ so that $\|G\|_1 \le C$.

Pick any integer $N > 2C$. For each $n \in \mathcal{N}$ we pick natural numbers $r_n \le s_n$ such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{r_n-1} f_n(i,j) < \frac{1}{4}, \quad \sum_{i=1}^{\infty}\sum_{j=1}^{r_n} f_n(i,j) \ge \frac{1}{4},$$
$$\sum_{i=1}^{\infty}\sum_{j=1}^{s_n-1} f_n(i,j) < \frac{3}{4}, \quad \sum_{i=1}^{\infty}\sum_{j=1}^{s_n} f_n(i,j) \ge \frac{3}{4}.$$

We will argue by Hall's Marriage Lemma (see Bollobas [1]) that it is possible to find a map $\varphi : \mathcal{N} \to \mathbb{N}$ such that $\varphi(n) \in [r_n, s_n]$, with $|\varphi^{-1}(k)| \le N$ for all $k \in \mathbb{N}$. Indeed, if not, the Marriage Lemma implies there is a minimal finite subset $\mathcal{M}$ of $\mathcal{N}$ such that $N|\bigcup_{n\in\mathcal{M}}[r_n,s_n]|<|\mathcal{M}|$. It follows easily from the mininality that $\bigcup_{n\in\mathcal{M}}[r_n,s_n]$ is an interval $[a,b]$. From the disjointness of the (f_n) we get

$$\sum_{j=a}^{b}\sup_{n\in\mathcal{M}} f_n(i,j)\le (b-a+1)G(i)$$

so that

$$\sum_{i=1}^{\infty}\sum_{j=a}^{b} f_n(i,j)\le (b-a+1)C.$$

However,

$$\sum_{i=1}^{\infty}\sum_{j=a}^{b} f_n(i,j)\ge \frac{1}{2}|\mathcal{M}|$$

so that $|\mathcal{M}|\le 2(b-a+1)C<N(b-a+1)$, which is a contradiction.

We can now split $\mathcal{N}$ into at most N disjoint subsets $(\mathcal{N}_k)_{k\in\mathcal{M}}$ so that φ is injective on each $\mathcal{N}_k$.

For each $n\in\mathcal{N}$ let $A_n=\{(i,j):j\le\varphi(n)\}$ and $B_n=\{(i,j):j\ge\varphi(n)\}$. Then, by Lemma 3.1, we find that $(u_n)_{n\in\mathcal{N}}$ is equivalent to both $(u_n\chi_{A_n})_{n\in\mathcal{N}}$ and $(u_n\chi_{B_n})_{n\in\mathcal{N}}$ with constants of equivalence depending only on C.

Now suppose $k\in\mathcal{M}$, and let $(a_n)_{n\in\mathcal{N}_k}$ be a finitely non-zero sequence. Then, by the right-dominance property, for each i we have

$$\Big\|\sum_{n\in\mathcal{N}_k}a_nu_n\chi_{A_n}[i]\Big\|_X\le\kappa\Big\|\sum_{n\in\mathcal{N}_k}a_n\|u_n\chi_{A_n}[i]\|_Xe_{\varphi(n)}\Big\|_X.$$

Hence

$$\Big\|\sum_{n\in\mathcal{N}_k}a_nu_n\chi_{A_n}[i]\Big\|_Y\le\kappa\Big\|\sum_{n\in\mathcal{N}}a_ne_{\varphi(n)}\Big\|_X. \tag{4.1}$$

In the opposite direction, again by the right-dominance property, we have

$$\Big\|\sum_{n\in\mathcal{N}_k}a_n\|u_n\chi_{B_n}[i]\|_Xe_{\varphi(n)}\Big\|_X\le\kappa\Big\|\sum_{n\in\mathcal{N}_k}a_nu_n\chi_{B_n}[i]\Big\|_X. \tag{4.2}$$

Combining (4.1) and (4.2) gives

$$\Big\|\sum_{n\in\mathcal{N}_k}a_n\sum_{i=1}^{\infty}G(i)\|u_n\chi_{B_n}[i]\|_Xe_{\varphi(n)}\Big\|_X\le C\kappa\Big\|\sum_{n\in\mathcal{N}_k}a_nu_n\Big\|_Y. \tag{4.3}$$

For each n let $I_n=\{i:8CG(i)\ge\|u_n^*[i]\|_{X^*}\}$. Then

$$\sum_{i\notin I_n}G(i)\le\frac{1}{8C}\sum_{i=1}^{\infty}\|u_n^*[i]\|_{X^*}\le\frac{1}{8}.$$

Hence

$$\sum_{i\notin I_n}\langle u_n[i]\chi_{B_n},u_n^*[i]\rangle\le\frac{1}{8}.$$

However, by choice of $\varphi(n)$ we have

$$\sum_{i=1}^{\infty}\langle u_n[i]\chi_{B_n},u_n^*[i]\rangle\ge\frac{1}{4}.$$

Thus

$$\begin{aligned}\frac{1}{8}&\le\sum_{i\in I_n}\|u_n[i]\chi_{B_n}\|_X\|u_n^*[i]\|_{X^*}\le 8C\sum_{i\in I_n}G(i)\|u_n[i]\chi_{B_n}\|_X\\&\le 8C\sum_{i=1}^{\infty}G(i)\|u_n[i]\chi_{B_n}\|_X.\end{aligned}$$

The estimate above combined with (4.3) yields the inequality

$$\Big\|\sum_{n\in\mathcal{N}_k}a_ne_{\varphi(n)}\Big\|_X\le 64C^2\kappa\Big\|\sum_{n\in\mathcal{N}_k}a_nu_n\Big\|_Y.$$

Thus each $(u_n)_{n\in\mathcal{N}_k}$ is equivalent to $(e_{\varphi(n)})_{n\in\mathcal{N}_k}$ in X with constant of equivalence depending only on C. Since $|\mathcal{M}|\le N$ where N depends only on C, the result is proved. ∎

Let us say that an unconditional basis $(u_n)_{n\in\mathcal{N}}$ is *molecular* if there exists a constant C and a natural number N so that if $\mathcal{N}$ is partitioned into N disjoint sets $(\mathcal{N}_k)_{k=1}^N$ then there exists a proper subset $\mathcal{M}$ of $\{1,\dots,N\}$ such that $(u_n)_{n\in\mathcal{N}}$ is C-equivalent to a subset of $\bigcup_{k\in\mathcal{M}}(u_n)_{n\in\mathcal{N}_k}$. Otherwise we will say that $(u_n)_{n\in\mathcal{N}}$ is *non-molecular*. It follows from the quantitative form of the Cantor–Bernstein principle [14, 16, 17] that $(u_n)_{n\in\mathcal{N}}$ is molecular if and only if there is a constant C so that if $\mathcal{N}$ is partitioned into N disjoint sets $(\mathcal{N}_k)_{k=1}^N$ then there is a proper subset $\mathcal{M}$ of $\{1,\dots,N\}$ so that $(u_n)_{n\in\mathcal{N}_k,k\in\mathcal{M}}$ is C-equivalent to $(u_n)_{n\in\mathcal{N}}$. Let us note that any subsymmetric basis is molecular with $N=2$ as is the usual basis of $(\sum_{n=1}^{\infty}\ell_q^n)_{\ell_p}$ when $1\le p,q<\infty$. The canonical basis of $\ell_p\oplus\ell_q$ for $p\ne q$ is molecular with $N=3$.

LEMMA 4.2. *Suppose $(u_n)_{n=1}^{\infty}$ is a non-molecular unconditional basis. Then for any $\varepsilon>0$, $N\in\mathbb{N}$ and constant C there exists $M>N$ and subsets $(A_k)_{k=1}^M$ of $\mathbb{N}$ so that:*

(1) *if $\mathcal{M}$ is a subset of $\{1,\dots,M\}$ with $|\mathcal{M}|<N$ then $(u_n)_{n=1}^{\infty}$ is not C-equivalent to any subset of $\{u_n:n\in\bigcup_{k\in\mathcal{M}}A_k\}$, and*

(2) $M^{-1}\sum_{k=1}^{M}\chi_{A_k}\ge(1-\varepsilon)\chi_{\mathbb{N}}$.

Proof. It suffices to consider the case when $\varepsilon = r/s$ is rational. We then may pick an integer m so large that $mr > N$, and so that if $L = \binom{ms}{mr}$ then we can partition $\mathbb{N}$ into L sets so that (u_n) is not C-equivalent to a subset of $(u_n)_{n\in\mathcal{N}}$ where $\mathcal{N}$ is the union of any $L-1$ sets.

Let Ω be the collection of all $m(s-r)$ subsets of $\{1,\ldots,ms\}$. We can partition $\mathbb{N} = \bigcup_{\omega\in\Omega} B_\omega$ so that $(u_n)_{n\in\mathbb{N}}$ is not C-equivalent to a subset of $(u_n)_{n\in\mathcal{N}}$ where $\mathcal{N} = \bigcup_{\omega\in D} B_\omega$ for some proper subset D of Ω.

Now let $A_k = \bigcup_{\omega:k\in\omega} B_\omega$ for $1 \le k \le M = ms$. It is clear that

$$\sum_{k=1}^{ms} \chi_{A_k} = m(s-r)\chi_{\mathbb{N}}$$

so that (2) holds. Suppose $(u_n)_{n\in\mathbb{N}}$ is C-equivalent to a subset of $(u_n)_{n\in\mathcal{N}}$ where $\mathcal{N} = \bigcup_{k\in\mathcal{M}} A_k$. Then we have $\bigcup_{k\in\mathcal{M}} A_k = \Omega$ whence $|\mathcal{M}| > mr$. ∎

THEOREM 4.3. *Let X be a space with non-trivial cotype and an unconditional basis (u_n). If $c_0(X)$ has a unique unconditional basis then (u_n) is molecular.*

Proof. We will assume, on the contrary, that the basis (u_n) is not molecular. Let us regard X as a sequence space so that the given unconditional basis is identified with $(e_n)_{n\in\mathbb{N}}$. We start by using Lemma 4.2 repeatedly to generate, for each $r\in\mathbb{N}$, subsets $(A_{rk})_{k=1}^{M_r}$ of $\mathbb{N}$ so that:

(1) for any subset $\mathcal{M}$ of $\{1,\ldots,M_r\}$ with $|\mathcal{M}| < r$ the basis $(e_n)_{n=1}^\infty$ is not r-equivalent to any subset of $\{e_n : n \in \bigcup_{k\in\mathcal{M}} A_{nk}\}$, and

(2) $M_r^{-1}\sum_{k=1}^{M_r}\chi_{A_{rk}} \ge (1-2^{-(r+1)})\chi_{\mathbb{N}}$.

Now for each $s\in\mathbb{N}$ let $P_s = \prod_{r=1}^s M_r$ and let $(B_{sk})_{k=1}^{P_s}$ be a listing of all sets of the form $\bigcap_{r=1}^s A_{rk_r}$. We observe that

$$P_s^{-1}\sum_{k=1}^{P_s}\chi_{B_{sk}} \ge \frac{1}{2}\chi_{\mathbb{N}}.$$

Consider the index set $I = \{(s,k) : 1 \le k \le P_s,\ s\in\mathbb{N}\}$. We will treat the space $c_0(X)$ as a sequence space modelled on $I\times\mathbb{N}$.

Consider now the block basic sequence

$$u_{sn} = \sum_{k=1}^{P_s} e_{skn}.$$

If we define the biorthogonal functionals

$$u_{sn}^* = \frac{1}{P_s}\sum_{k=1}^{P_s} e_{skn}$$

then it is clear that $(u_{sn})_{s,n}$ is a complemented disjoint sequence equivalent to the canonical basis of $c_0(X)$.

Now let $D = \{(s,k,n) : n \in B_{sk}\}$. Then $\|u_n u_n^* \chi_D\|_1 \ge 1/2$. It follows from Lemma 3.1 that $(u_{sn}\chi_D)_{s,n}$ is also a complemented disjoint sequence equivalent to the canonical basis of $c_0(X)$.

The basis vectors (e_{skn}) for $(s,k,n)\in D$ span a complemented subspace Y of $c_0(X)$ which by the above remark contains a complemented copy of $c_0(X)$. By the Pełczyński decomposition argument, Y is isomorphic to $c_0(X)$. If we assume that $c_0(X)$ has a unique unconditional basis then it will follow that the whole basis $(e_{skn})_{(s,k)\in I, n\in\mathbb{N}}$ is C-equivalent, for some C, to its subset $(e_{skn})_{(s,k,n)\in D}$.

Thus we can partition D into subsets $(D_t)_{t=1}^\infty$ so that each subset (e_{skn}) for $(s,k,n)\in D_t$ is C-equivalent to the canonical basis (e_n) of X while any subset obtained by picking one element from each D_t is C-equivalent to the standard c_0-basis. From this and the fact that X has a lower estimate it is clear that for fixed (s,k) at most finitely many D_t can intersect the set of all (s,k,n) for $n\in\mathbb{N}$. Note also that the set of (s,k) such that $(s,k,n)\in D_t$ for some n must also be uniformly bounded by some constant K again by the lower estimate on X.

In particular, for any s_0 there exists t so that if $(s,k,n)\in D_t$ then $s > s_0$. Hence, the canonical basis of X is C-equivalent to a subset of $\bigcup_{(s,k)\in\mathcal{M}} B_{sk}$ where $(s,k)\in\mathcal{M}$ implies $s > s_0$ and $|\mathcal{M}| \le K$. Now each B_{sk} is contained in some $A_{s_0,k}$ and so we must have $K \ge s_0$. By choosing s_0 large enough we get a contradiction. ∎

We now state a general theorem which can be proved by exactly the same argument.

THEOREM 4.4. *Suppose $1 \le p < \infty$ and suppose X is a Banach space with a non-molecular unconditional basis $(u_n)_{n=1}^\infty$ with the property that it does not contain subsets uniformly equivalent to the unit vector bases of ℓ_p^m for $m = 1,2,\ldots$ Let $(u_{mn})_{m,n=1}^\infty$ be the induced basis of $\ell_p(X)$. Then there is a subset $\mathcal{A}\subset\mathbb{N}\times\mathbb{N}$ so that $(u_{mn})_{(m,n)\in\mathcal{A}}$ is not equivalent to the full basis (u_{mn}) and spans a subspace isomorphic to $\ell_p(X)$.*

We conclude with a theorem which gives us a large number of examples of right-dominant spaces with non-molecular unconditional bases.

THEOREM 4.5. *Suppose X is a right-dominant sequence space with $r(X) = r$. Suppose the canonical basis is molecular. Then $X = \ell_r$.*

Proof. It is enough to show that X is left-dominant. Let us assume the contrary. Then:

CLAIM. *Given any $a \in \mathbb{N}$ and $C > 0$ there exists $b > a$ so that $(e_k)_{a<k\le b}$ is not C-equivalent to any subset of $(e_k)_{k\le a} \cup (e_k)_{b<k}$.*

To prove the claim let $C_1 > C^2\kappa + a$. Since X is not left-dominant there exist disjoint sequences $(u_n)_{n=1}^N$ and $(v_n)_{n=1}^N$ with finite supports so that $a < \operatorname{supp} u_n < \operatorname{supp} v_n$ for each n, $\|u_n\|_X = \|v_n\|_X$ and

$$\Big\|\sum_{n=1}^N v_n\Big\|_X > C_1\Big\|\sum_{n=1}^N u_n\Big\|_X.$$

Pick b so large that $\operatorname{supp} v_n \le b$ for all n. Suppose $(e_k)_{a<k\le b}$ is C-equivalent to some subset of $(e_k)_{k\le a} \cup (e_k)_{k>b}$. Then there exist $(w_n)_{n=1}^N$, each with finite disjoint support not intersecting $(a,b]$, so that $\|w_n\|_X = \|u_n\|_X$ for $1 \le n \le N$ and

$$\Big\|\sum_{n=1}^N w_n\Big\|_X \le C^2\Big\|\sum_{n=1}^N u_n\Big\|_X.$$

Let $\mathcal{M} = \{n : \operatorname{supp} w_n \cap [1,a] \ne \emptyset\}$. Then $|\mathcal{M}| \le a$. Thus

$$\Big\|\sum_{n\in\mathcal{M}} v_n\Big\|_X \le a\Big\|\sum_{n=1}^N u_n\Big\|_X.$$

On the other hand,

$$\Big\|\sum_{n\notin\mathcal{M}} v_n\Big\|_X \le \kappa\Big\|\sum_{n\notin\mathcal{M}} w_n\Big\|_X \le C^2\kappa\Big\|\sum_{n=1}^N u_n\Big\|_X.$$

It follows that $C_1 < C^2\kappa + a$, contrary to assumption. This establishes the claim.

To prove the theorem we use the claim to find an increasing sequence $(a_n)_{n=1}^\infty$ so that $(e_k)_{a_n<k\le a_{n+1}}$ is not n-equivalent to any subset of $(e_k)_{k\le a_n} \cup (e_k)_{k>a_{n+1}}$. Then fix any $s \in \mathbb{N}$ and consider the sets $A_j = \bigcup\{(a_n, a_{n+1}] : n \equiv j \bmod s\}$ for $0 \le j \le s-1$. Now the sets $(e_n)_{n\in A_j}$ partition the basis into s sets in such a way that no $s-1$ sets contain a subset equivalent to the original basis. This contradicts our assumption that the basis is molecular. ∎

EXAMPLES. We can now give many examples of spaces X with a unique unconditional basis but such that the c_0-product $c_0(X)$ fails to have unique unconditional basis. This will answer negatively a question raised in [3].

In fact, if X is right-dominant and $c_0(X)$ has unique unconditional basis then X must be one of the three spaces c_0, ℓ_1 or ℓ_2. This follows by observing that if it is not in this list then $r(X) < \infty$ and hence X has cotype. Then Theorems 4.3 and 4.5 show that $X = \ell_r$ for some finite r. The uniqueness then forces either $r = 1$ or $r = 2$.

On the other hand, there are many known examples of right-dominant spaces with unique unconditional bases. In [3], 2-convexified Tsirelson space is shown to have unique unconditional basis. In [4], Tsirelson space itself and certain Nakano spaces $\ell_{(p_n)}$ are shown to have unique unconditional bases. These latter examples satisfy $r(X) = 1$ so that we can apply Theorem 4.1. The second non-equivalent basis constructed in Theorem 4.3 is indeed equivalent to a subset of the original basis. ∎

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Department of Mathematics
University of Missouri
Columbia, Missouri 65211
U.S.A.
E-mail: pete@math.missouri.edu
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