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Some Ramsey type theorems for normed and quasinormed spaces

by

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Abstract. We prove that every bounded, uniformly separated sequence in a normed space contains a "uniformly independent" subsequence (see definition); the constants involved do not depend on the sequence or the space. The finite version of this result is true for all quasinormed spaces. We give a counterexample to the infinite version in $L_p[0,1]$ for each $0 < p < 1$. Some consequences for nonstandard topological vector spaces are derived.

0. Introduction. We are concerned with the following problem: given a bounded sequence in a quasinormed space V whose terms are uniformly far apart, can we pass to a subsequence such that each term is uniformly far from the subspace spanned by the remaining terms?

If V is a normed space, it is well known that the answer is "yes". We strengthen this result by showing that the distance of each term from the subspace spanned by the other terms can be determined rather uniformly; in particular, it need not depend on the geometry of the given sequence. The finite version of this result turns out to be true for all quasinormed spaces, and it is tempting to conjecture that the infinite result is also true for all quasinormed spaces. However, we give a counterexample to this conjecture.

Before continuing the discussion we introduce some definitions and notation:

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DEFINITION. Let $(V, \|\cdot\|)$ be a quasinormed space (see [7]) and $(x_i)_{i\in I}$ be a system of elements of V.

(a) Given $\varepsilon > 0$, the system of elements $(x_i)_{i\in I}$ will be called *ε-separated* if for any distinct $i, j \in I$ we have $\|x_i - x_j\| \geq \varepsilon$. The system (x_i) will be called *uniformly separated* if it is ε-separated for some $\varepsilon > 0$.

(b) Given $\delta > 0$, the system of elements $(x_i)_{i\in I}$ is called *δ-independent* if for any $i \in I$ we have

$$\operatorname{dist}(x_i, [x_j]_{i\neq j\in I}) \geq \delta,$$

where for any sets $X, Y \subseteq V$,

$$\operatorname{dist}(X, Y) = \inf\{\|x - y\| : x \in X,\ y \in Y\}$$

denotes their distance with respect to the quasinorm $\|\cdot\|$ and $[X]$ is the closed linear span of X. The system (x_i) will be called *uniformly independent* if it is δ-independent for some $\delta > 0$.

In the first section of this paper we show that in a Banach space, every bounded ε-separated sequence contains a δ-independent subsequence for each positive $\delta < \varepsilon/2$. (As noted in the remark following the statement of Theorem 1.1, this result in the weaker form with "for some $\delta > 0$" replacing "for each positive $\delta < \varepsilon/2$", has an easy, direct proof.) Call a sequence (x_i) in a Banach space X *λ-biorthogonal* if there exists a sequence (f_i) in the dual space X^* with $\|f_i\| \leq \lambda$ and $f_i(x_j) = \delta_{ij}$ for all i and j. Evidently, (x_i) is δ-independent if and only if it is $1/\delta$-biorthogonal. Therefore, the main result of this section can be rephrased as follows: Every bounded ε-separated sequence in a Banach space has, for each $\eta > 0$, a $(2/\varepsilon + \eta)$-biorthogonal subsequence.

In the second section we prove the finite version of this result for an arbitrary p-normed space; our proof is based on the finite version of Ramsey's Theorem. For our other results neither the finite nor the infinite version of the combinatorial Ramsey Principle is sufficient; to prove them one must also use some structural properties of the space in question. Nonetheless, there is a close connection between the results in this paper and Ramsey's Theorem.

In the third section we study various types of irreducible sequences in quasi-Banach spaces; these are sequences in which any infinite subsequence generates the same closed linear span as the original sequence, possibly even when "small" perturbations are allowed. (See the beginning of Section 3 for the definitions.) One of the main results of this section states that in a separable infinite-dimensional quasi-Banach space V, a bounded sequence (x_n) contains either a uniformly independent subsequence or a subsequence some compact perturbation of which is completely irreducible. As a consequence, V contains a bounded uniformly separated sequence with no uniformly independent subsequence if and only if V contains a completely irreducible sequence not identically equal to 0. Note that our results imply that in a Banach space, all completely irreducible sequences are identically 0. Therefore the dichotomy described above has content exactly in quasi-Banach spaces that are nonnormable.

In the fourth section we construct a bounded sequence (f_n) in $L_p[0,1]$, for each $0 < p < 1$, such that (f_n) is uniformly separated and such that for every null sequence (g_n) and for every infinite subset M of $\mathbb{N}$, we have $[f_n + g_n]_{n\in M} = L_p[0,1]$. Obviously, (f_n) has no uniformly independent subsequence.

Finally, in the fifth section we use the results in Section 2 to derive some consequences relating two kinds of compactness properties for nonstandard topological vector spaces.

1. The normed case—infinite version

THEOREM 1.1. *Let $(V, \|\cdot\|)$ be a normed linear space, $\varepsilon > 0$, and $(x_n)_{n=1}^\infty$ be a bounded ε-separated sequence in V. Then for each positive $\delta < \varepsilon/2$ there is a δ-independent subsequence $(y_k)_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$.*

Remark. The weaker statement: "*Every bounded uniformly separated sequence (x_n) in a normed space V contains a uniformly independent subsequence (y_k)*", can be proved rather easily using a Pełczyński style argument. Indeed, by the Banach–Alaoglu Theorem, the bounded set $\{x_1, x_2, \ldots\}$ has a weak* cluster point $y_0 \in V^{**}$. If $y_0 = 0$, then $(x_n)_{n=1}^\infty$ contains a basic subsequence $(y_k)_{k=1}^\infty$, and we are done. If $y_0 \neq 0$, take an $\varepsilon > 0$ such that (x_n) is ε-separated. Then $\|x_n - y_0\| < \varepsilon/2$ for at most one n and (see e.g. [2], Ch. V, Lemma 4) $(x_n)_{n=1}^\infty$ has a subsequence $(y_k)_{k=1}^\infty$ such that $(y_0, y_1 - y_0, y_2 - y_0, \ldots)$ is a basic sequence, hence it is δ-independent for some $\delta > 0$. The δ-independence of $(y_k)_{k=1}^\infty$ is an immediate consequence.

Thus the main point of Theorem 1.1 is the fact that for each given $\varepsilon > 0$, we can choose $\delta > 0$ *uniformly*, regardless of the sequence (x_n), and *each* $\delta < \varepsilon/2$ will do.

Proof of Theorem 1.1. We will be working in the Banach spaces V^* and V^{**}—the first and the second dual of V, respectively. V will be identified with a subspace of V^{**} in the canonical isometric way, and so will be V^* with a subspace of V^{***}. That is, each $f \in V^*$ will be regarded as a bounded linear functional on V^{**} and we will write $f(z)$ instead of $z(f)$ for $z \in V^{**}$. We denote by B the closed unit ball in V^{**}.

Fix a strictly decreasing sequence $(\delta_n)_{n=1}^\infty$ such that $\delta < \delta_n < \varepsilon/2$ for each n. The sequence $(y_n)_{n=1}^\infty$ will be constructed by induction, along with a doubly indexed sequence of continuous functionals $f_i^n \in V^*$, $n \geq 1$, $1 \leq$

$i \le n$, such that for all n and $i = 1, \ldots, n$ we have

$$f_i^n(y_i + \delta_n B) > 0 \tag{1}$$

and

$$f_i^n[y_1, \ldots, \widehat{y}_i, \ldots, y_n] = 0. \tag{2}$$

These conditions already imply

$$\operatorname{dist}(y_i, [y_1, \ldots, \widehat{y}_i, \ldots, y_n]) > \delta_n > \delta$$

($\widehat{y}_i$ denotes the omission of y_i) for all n, i, and the δ-independence of the sequence (y_n) directly follows.

However, besides the vector y_n and the functionals $f_1^n, \ldots, f_n^n$ we will have to single out two more auxiliary objects at each step of the construction—namely, a subsequence $\mathbf{x}^n = (x_j^n)_{j=1}^\infty$ of the original sequence $(x_j)_{j=1}^\infty$ and a weak* cluster point $a_n \in V^{**}$ of the infinite set $\operatorname{rng} \mathbf{x}^n = \{x_1^n, x_2^n, \ldots\}$. Some additional conditions they should satisfy will be formulated in the course of the proof.

To start the induction procedure let us denote by $\mathbf{x}^0 = (x_j^0)_{j=1}^\infty$ the original sequence $(x_j)_{j=1}^\infty$. Now, take an arbitrary $n \ge 0$ and suppose that we already have selected the terms $y_1, \ldots, y_n$ from the sequence (x_j), as well as the functionals $f_i^n \in V^*$, $1 \le i \le n$, satisfying (1) and (2). Further, as for $n = 0$ this condition is vacuous, hence trivially satisfied, we will assume that we have a subsequence $\mathbf{x}^n$ of the sequence $\mathbf{x}^0$ such that

$$\lim_{j \to \infty} f_i^n(x_j^n) = 0 \tag{3}$$

holds for all $i = 1, \ldots, n$.

By the Banach–Alaoglu Theorem there is a weak* cluster point a_{n+1} in V^{**} of the bounded infinite set $\operatorname{rng} \mathbf{x}^n$. As the space $[y_1, \ldots, y_n, a_{n+1}]$ is finite-dimensional, $\delta_n < \varepsilon/2$ and $\mathbf{x}^n$ is bounded and ε-separated, there is obviously a $k \ge 1$ such that for all $j \ge k$ we have

$$\operatorname{dist}(x_j^n, [y_1, \ldots, y_n, a_{n+1}]) > \delta_n. \tag{4}$$

From (3) it follows that at the same time we can require

$$|f_i^n(x_j^n)| \le (\delta_n - \delta_{n+1}) \|f_i^n\| \tag{5}$$

whenever $1 \le i \le n$ and $j \ge k$. We put $y_{n+1} = x_k^n$.

Now, the Hahn–Banach Theorem for the locally convex space V^{**} with the weak* topology, applied to (4), gives us an $f_{n+1}^{n+1} \in V^*$ such that

$$f_{n+1}^{n+1}(y_{n+1} + \delta_{n+1} B) > 0 \tag{6}$$

and

$$f_{n+1}^{n+1}[y_1, \ldots, y_n, a_{n+1}] = 0. \tag{7}$$

Without loss of generality we can assume

$$f_{n+1}^{n+1}(y_{n+1}) = 1. \tag{8}$$

Consequently,

$$|f_{n+1}^{n+1}(\delta_{n+1} z)| < f_{n+1}^{n+1}(y_{n+1}) = 1$$

for all $z \in B$, hence

$$\|f_{n+1}^{n+1}\| \le \frac{1}{\delta_{n+1}}. \tag{9}$$

We define the remaining functionals $f_1^{n+1}, \ldots, f_n^{n+1} \in V^*$ by

$$f_i^{n+1} = f_i^n - f_i^n(y_{n+1}) f_{n+1}^{n+1} \tag{10}$$

for $1 \le i \le n$. Then (2), (7) and (8) yield

$$f_i^{n+1}[y_1, \ldots, \widehat{y}_i, \ldots, y_n, y_{n+1}] = 0, \tag{11}$$

and (7), (11) imply condition (2) with n replaced by $n + 1$.

Next we show the inequality

$$f_i^{n+1}(y_i + \delta_{n+1} B) > 0 \tag{12}$$

for all $i = 1, \ldots, n$. Take any $u, v \in B$. Then

$$\|\delta_{n+1} u - (\delta_n - \delta_{n+1}) v\| \le \delta_n,$$

hence

$$\delta_{n+1} u - (\delta_n - \delta_{n+1}) v \in \delta_n B.$$

Thus by (1) we have

$$f_i^n(y_i + \delta_{n+1} u - (\delta_n - \delta_{n+1}) v) > 0,$$

implying

$$f_i^n(y_i + \delta_{n+1} u) > (\delta_n - \delta_{n+1}) f_i^n(v).$$

Hence

$$\inf_{u \in B} f_i^n(y_i + \delta_{n+1} u) \ge (\delta_n - \delta_{n+1}) \|f_i^n\|.$$

Now, using the definition (10) and the facts (7), (9) we obtain

$$\begin{aligned}
\inf_{u \in B} f_i^{n+1}(y_i + \delta_{n+1} u) &\ge \inf_{u \in B} f_i^n(y_i + \delta_{n+1} u) - f_i^n(y_{n+1}) \sup_{v \in B} f_{n+1}^{n+1}(y_i + \delta_{n+1} v) \\
&\ge (\delta_n - \delta_{n+1}) \|f_i^n\| - f_i^n(y_{n+1}) \delta_{n+1} \|f_{n+1}^{n+1}\| \\
&\ge (\delta_n - \delta_{n+1}) \|f_i^n\| - f_i^n(y_{n+1}).
\end{aligned}$$

The last inequality and (5) already imply (12). Then (6) and (12) give precisely condition (1) with n replaced by $n + 1$.

Finally, let us recall that $f_{n+1}^{n+1}(a_{n+1}) = 0$ and a_{n+1} is a weak* cluster point of the set $\operatorname{rng} \mathbf{x}^n$. Thus $\mathbf{x}^n$ has a subsequence $\mathbf{x}^{n+1}$ such that

$$\lim_{j\to\infty} f_{n+1}^{n+1}(x_j^{n+1}) = 0.$$

Now, (3) and (10) imply

$$\lim_{j\to\infty} f_i^{n+1}(x_j^{n+1}) = 0$$

for all $i = 1, \ldots, n$, too.

This completes the description of the induction procedure as well as the proof of Theorem 1.1.

2. The quasinormed case—finite version. There is also a finite version of Theorem 1.1 which surprisingly holds even in a large number of nonlocally convex spaces, namely in the quasinormed ones. Given any $p > 0$, a quasinorm $\|\cdot\|$ on a real or complex vector space V will be called a *p-norm* if it is homogeneous and its pth power satisfies the triangle inequality. By the fundamental theorem of Aoki–Rolewicz [1], [11], every quasinorm is equivalent to a p-norm for some $0 < p \le 1$. Thus we can restrict attention to the p-normed spaces, which in turn allows us to formulate our result in direct analogy with Theorem 1.1.

The basic tool in proving our next result will be the finite version of Ramsey's Theorem, guaranteeing to any $k, m, r \in \mathbb{N}$ the existence of an $n \in \mathbb{N}$ such that for any sets A and C with n and r elements, respectively, and every function (coloring) $F : \mathcal{P}_k(A) \to C$ there is an m-element subset $X \subseteq A$ such that X is F-homogeneous; i.e., $F(u) = F(v)$ for all u, v in $\mathcal{P}_k(X)$, where $\mathcal{P}_k(A)$ denotes the set of all k-element subsets of A.

The smallest n guaranteed to k, m, r by Ramsey's Theorem will be denoted by $R(k, m, r)$.

In what follows, every set X is regarded as the system of elements $(x)_{x\in X}$, so that the notions of ε-separateness and δ-independence apply in the obvious way.

THEOREM 2.1. *Let $(V, \|\cdot\|)$ be a p-normed space for some $0 < p \le 1$, and K, ε be any positive real numbers. Then for each positive $\delta < \varepsilon/2^{1/p}$ and each $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every K-bounded ε-separated set $X \subseteq V$ with at least n elements contains a δ-independent subset Y with at least m elements.*

Proof. First we establish the result for vector spaces over the field $\mathbb{R}$, then we will show how the proof can be modified to handle the complex case. Also, to avoid trivialities, we will always assume $m > 1$.

Let $<$ be a fixed linear order of V. Whenever we write $Z = \{z_1, \ldots, z_k\}$ for a k-element subset Z of V, we will assume that Z is listed in an increasing sequence with respect to the order $<$.

For $\alpha \in \mathbb{R}^m$ we denote by $F_\alpha : \mathcal{P}_m(V) \to \{0, 1\}$ the coloring given by

$$F_\alpha(Z) = \begin{cases} 0 & \text{if } \|\sum_{i=1}^m \alpha_i z_i\| < \varepsilon/2^{1/p}, \\ 1 & \text{otherwise}, \end{cases}$$

for any $Z = \{z_1, \ldots, z_m\} \in \mathcal{P}_m(V)$. We claim that whenever a set $\Gamma \subseteq \mathbb{R}^m$ is appropriately chosen and Z is a $(2m-1)$-element subset of the K-bounded ε-separated set $X \subseteq V$, which is F_α-homogeneous for each $\alpha \in \Gamma$, then Z already contains an m-element δ-independent subset Y. By Ramsey's Theorem, Z, and hence Y as well, can be arbitrarily large provided X is large enough.

To be more precise, let us denote by s a positive integer which will be specified later on, put

$$J_s = \left\{\frac{k}{s} : k \in \mathbb{Z} \;\&\; -s \le k \le s\right\}, \qquad S = \{\alpha \in J_s^m : \max_{1\le i\le m} |\alpha_i| = 1\},$$

and take for Γ any subset of S such that for any $\alpha \in S$ either $\alpha \in \Gamma$ or $-\alpha \in \Gamma$, but not both. Obviously, any such Γ has exactly

$$t = \frac{1}{2}((2s+1)^m - (2s-1)^m)$$

elements. Then the family $\{F_\alpha : \alpha \in \Gamma\}$ of t colorings can be regarded as a coloring $F : \mathcal{P}_m(V) \to \{0,1\}^\Gamma$ by 2^t colors, given by $F(Z)(\alpha) = F_\alpha(Z)$ for all $Z \in \mathcal{P}_m(V)$, $\alpha \in \Gamma$.

It follows that whenever $n \ge R(m, 2m-1, 2^t)$ and X is an n-element subset of V, then X contains a $(2m-1)$-element F-homogeneous subset $Z = \{z_1, \ldots, z_{2m-1}\}$. Then Z is F_α-homogeneous for each $\alpha \in \Gamma$. Putting $y_i = z_{2i-1}$ for $i = 1, \ldots, m$, we arrive at an m-element subset $Y = \{y_1, \ldots, y_m\}$ of X. We will conclude the proof by showing that Y is δ-independent provided X is K-bounded ε-separated and s has been chosen in an appropriate way.

Assume the contrary; then without loss of generality we can assume

$$\Big\|\sum_{i=1}^m \alpha_i y_i\Big\| < \delta$$

for some $\alpha \in \mathbb{R}^m$ such that $\alpha_k = 1$ for some $k \le m$ and $|\alpha_i| \le 1$ for each $i \le m$. Then there is a $\beta \in J_s^m$ such that $\beta_k = \alpha_k = 1$ and $|\beta_i - \alpha_i| \le 1/2s$ for all i. Then $\beta \in S$ and either $\beta \in \Gamma$ or $-\beta \in \Gamma$. Let us restrict to the first case; the second one can be handled in exactly the same way, passing from α to $-\alpha$. We compute

$$\left\|\sum_{i=1}^{m}\beta_i y_i-\sum_{i=1}^{m}\alpha_i y_i\right\|^p=\left\|\sum_{i\neq k}(\beta_i-\alpha_i)y_i\right\|^p$$
$$\le\sum_{i\neq k}|\beta_i-\alpha_i|^p\|y_i\|^p\le\frac{K^p(m-1)}{(2s)^p}.$$

Then

$$\left\|\sum_{i=1}^{m}\beta_i y_i\right\|^p\le\left\|\sum_{i=1}^{m}\alpha_i y_i\right\|^p+\frac{K^p(m-1)}{(2s)^p}\le\delta^p+\frac{K^p(m-1)}{(2s)^p}.$$

Now, one can easily check what we need in order to derive the contradiction; namely

$$\delta^p+\frac{K^p(m-1)}{(2s)^p}<\frac{\varepsilon^p}{2}$$

should hold. This is equivalent to

$$s>\frac{K}{2}\left(\frac{m-1}{\varepsilon^p/2-\delta^p}\right)^{1/p}.$$

With such an s we have

$$\left\|y_k+\sum_{i\neq k}\beta_i y_i\right\|^p<\frac{\varepsilon^p}{2}.$$

Let j be any even index such that $1\le j\le 2m-1$ and $y_k\in\{z_{j-1},z_{j+1}\}$. As $\beta\in\Gamma$, by the F_β-homogeneity of Z we also have

$$\left\|z_j+\sum_{i\neq k}\beta_i y_i\right\|^p<\frac{\varepsilon^p}{2}.$$

Consequently, $\|y_k-z_j\|^p<\varepsilon^p$, contradicting the ε-separateness of X.

If the field of scalars is $\mathbb{C}$, the proof can follow exactly the same pattern. However, we have to define

$$J_s=\left\{\frac{k+\mathrm{i}l}{s}:k,l\in\mathbb{Z}\ \&\ k^2+l^2\le s^2\right\},$$

and relate the definitions of S and Γ to this new set J_s. Without falling into details, let us denote by $q(s)$ the number of elements of J_s. Then the set Γ has exactly

$$t=\frac{1}{2}(q(s)^m-q(s-1)^m)$$

elements. As one can easily check, for each $\alpha\in\mathbb{C}^m$ such that $\alpha_k=1$ for some k and $|\alpha_i|\le1$ for each i between 1 and m, there is a $\beta\in\Gamma$ satisfying $\beta_k=\alpha_k$ and $|\beta_i-\alpha_i|<\sqrt{5}/(2s)$ for all i, or the same conditions with α replaced by $-\alpha$. Again, it suffices to deal with the first option.

A similar computation to the real case gives

$$\left\|\sum_{i=1}^{m}\beta_i y_i-\sum_{i=1}^{m}\alpha_i y_i\right\|^p<\left(\frac{K\sqrt{5}}{2s}\right)^p(m-1).$$

Then for any

$$s\ge\frac{K\sqrt{5}}{2}\left(\frac{m-1}{\varepsilon^p/2-\delta^p}\right)^{1/p}$$

the needed contradiction can be derived.

For any positive real numbers K, ε, $\delta<\varepsilon/2^{1/p}$, $p\le1$, and any natural number m, let us denote by $\varrho^{\mathbb{R}}(m,K,\varepsilon,\delta,p)$, resp. $\varrho^{\mathbb{C}}(m,K,\varepsilon,\delta,p)$, the smallest natural number n such that in every quasinormed vector space $(V,\|\cdot\|)$ over $\mathbb{R}$, resp. over $\mathbb{C}$, where $\|\cdot\|$ is a p-norm on V, every K-bounded ε-separated set with m elements contains an n-element δ-independent subset.

In course of the above proof we have established the following huge estimate, most probably not the best possible. Recall that $R:\mathbb{N}^3\to\mathbb{N}$ denotes the Ramsey function, and for $a\in\mathbb{R}$, $\lfloor a\rfloor$ is the biggest integer $\le a$, $\lceil a\rceil$ is the smallest integer $\ge a$, and for $s\in\mathbb{N}$, $q(s)$ denotes the number of integer solutions (k,l) of the inequality $k^2+l^2\le s^2$.

COROLLARY 2.2. *For both $\varrho=\varrho^{\mathbb{R}}$ and $\varrho=\varrho^{\mathbb{C}}$, and any admissible parameters K, ε, δ, p, m, we have*

$$\varrho(m,K,\varepsilon,\delta,p)\le R(m,2m-1,2^t),$$

where

$$t=\frac{1}{2}((2s+1)^m-(2s-1)^m),\qquad s=\left\lfloor K\left(\frac{2^{1-p}(m-1)}{\varepsilon^p-2\delta^p}\right)^{1/p}\right\rfloor+1$$

in the real case, and

$$t=\frac{1}{2}(q(s)^m-q(s-1)^m),\qquad s=\left\lceil K\sqrt{5}\left(\frac{2^{1-p}(m-1)}{\varepsilon^p-2\delta^p}\right)^{1/p}\right\rceil$$

in the complex case.

3. The quasinormed case—infinite version. Our next results deal with the possibility of generalizing Theorem 1.1 or at least of its weaker version, mentioned in the Remark, to some nonlocally convex quasinormed spaces.

DEFINITION. A bounded sequence (x_n) in a quasi-Banach space V will be called *irreducible* if $[x_n]_{n\in M}=[x_n]_{n\in\mathbb{N}}$ for every infinite subset M of $\mathbb{N}$. We will say that (x_n) is *completely* (resp. *null*) *irreducible* if $[x_n+g_n]_{n\in M}=[x_n]_{n\in\mathbb{N}}$ for every infinite subset M of $\mathbb{N}$ and every sequence (g_n) in $[x_n]_{n\in\mathbb{N}}$

with relatively compact range (resp. with $\lim \|g_n\| = 0$). Finally, we will say that (x_n) is *fundamental* if $[x_n]_{n\in\mathbb{N}} = V$.

Note that the trivial sequence $x_n = 0$ is completely irreducible. A constant sequence is null irreducible. Sequences which are fundamental and irreducible are also called *overfilling* in the literature ([9]). It is relatively easy to find fundamental irreducible sequences, as we note in the following proposition, which is due to Klee [8]:

PROPOSITION 3.1. *Every separable quasi-Banach space contains a fundamental irreducible sequence.*

Proof. The following proof is due to Lyubich (cf. [9]). Let (y_n) be any sequence dense in the unit ball of V. Let $x_n = \sum_{k=1}^{\infty} 2^{-kn} y_k$. It is easy to verify that (x_n) is irreducible and fundamental.

Note that the sequence constructed in Proposition 3.1 is a null sequence and therefore has no uniformly separated subsequence.

LEMMA 3.2. *Let (x_n) be a null irreducible sequence; then (x_n) is completely irreducible if and only if $[x_n]_{n\in\mathbb{N}}$ has trivial dual.*

Proof. Suppose first that $E = [x_n]_{n\in\mathbb{N}}$ has trivial dual. Let (g_n) be any relatively compact sequence and M any infinite subset of $\mathbb{N}$. Then there is an infinite subset P of M so that $\lim_{n\in P} g_n = g$ exists. Now

$$[x_n + g_n - g]_{n\in P} = E$$

and hence $[x_n + g_n]_{n\in P}$ has codimension at most one in E, so by assumption coincides with E.

Conversely, suppose φ is a nonzero bounded linear functional on E. Fix $u \in E$ with $\varphi(u) = 1$. Then $[x_n - \varphi(x_n)u]_{n\in\mathbb{N}}$ is a proper closed subspace of E; hence (x_n) is not completely irreducible.

Remark. In particular, the only completely irreducible sequence in a Banach space is the trivial sequence!

In the next section, we will show that the space $L_p[0,1]$ has a fundamental null irreducible sequence. The following proposition is then immediate.

PROPOSITION 3.3. *For each $0 < p < 1$, L_p has a fundamental completely irreducible sequence.*

LEMMA 3.4. *Let (x_n) be a null irreducible sequence in a quasi-Banach space V. Then there exists a $u \in [x_n]_{n\in\mathbb{N}}$ and a bounded sequence of scalars (a_n) so that $(x_n + a_n u)$ is completely irreducible.*

Proof. Let $E = [x_n]_{n\in\mathbb{N}}$. If $E^* = \{0\}$, we apply Lemma 3.2 and set $u = 0$. If $E^* \neq \{0\}$, pick any nonzero vector $\varphi \in E^*$ and let $E_0 = \varphi^{-1}(0)$. We will show that $E_0^* = \{0\}$. Indeed, if not then there exists an $\psi \in E^*$ which is linearly independent of φ. By elementary compactness considerations there exist constants c_1, c_2 not both zero and an infinite subset M of $\mathbb{N}$ so that $\lim_{n\in M} \theta(x_n) = 0$, where $\theta = c_1\varphi + c_2\psi$. But then we can choose $(g_n)_{n\in M}$ so that $\lim_{n\in M} \|g_n\| = 0$ and $\theta(x_n + g_n) = 0$ for all $n \in M$. This contradicts null irreducibility of (x_n). Hence $E_0^* = \{0\}$.

Pick $u \in E$ with $\varphi(u) = 1$ and set $a_n = -\varphi(x_n)u$. Then $x_n + a_n u \in E_0$; we show this sequence is null irreducible in E_0 and hence is completely irreducible. Suppose $g_n \in E_0$ and $\lim \|g_n\| = 0$; suppose also M is an infinite subset of $\mathbb{N}$. Then the set $\{u, (x_n + a_n u + g_n)_{n\in M}\}$ is fundamental for E and so $[x_n + a_n u + g_n]_{n\in M} = E_0$, completing the argument.

PROPOSITION 3.5. *Let V, W be quasi-Banach spaces and $T : V \to W$ be a bounded linear operator. Suppose (x_n) is a bounded sequence in V and M is an infinite subset of $\mathbb{N}$ such that $\lim_{n\in M} \|Tx_n\| = 0$. Then either*

(1) $\lim_{n\in M} \operatorname{dist}(x_n, T^{-1}(0)) = 0$, *or*

(2) *there is an infinite subset P of M so that $(x_n)_{n\in P}$ is uniformly independent.*

Proof. Let $E = T^{-1}(0)$ and let $\pi : V \to V/E$ be the quotient map. We may factor $T = T_0\pi$, where $T_0 : V/E \to W$. If (1) fails there is an infinite subset Q of M so that $\inf_{n\in Q} \|\pi x_n\| > 0$. However, T_0 is one-one on V/E and so there is a weaker Hausdorff vector topology on V/E for which $\lim_{n\in Q} \pi x_n = 0$. (Simply take the inverse image under T_0 of the given topology on W.) By [7, Theorem 4.7] there is an infinite subset P of Q so that $(\pi x_n)_{n\in Q}$ is strongly regular and M-basic. In particular, $(\pi x_n)_{n\in Q}$ is uniformly independent and the same is also true for $(x_n)_{n\in Q}$.

PROPOSITION 3.6. *Let V, W be quasi-Banach spaces and $T : V \to W$ be a bounded linear operator. Suppose (x_n) is a completely irreducible sequence in V and the set $\{Tx_n : n \in M\}$ is relatively compact in W for any infinite subset M of $\mathbb{N}$. Then $Tx_n = 0$ for all $n \in \mathbb{N}$.*

Proof. It suffices to consider the case when $V = [x_n]_{n\in\mathbb{N}}$. We may pass to an infinite subset P of M so that $\lim_{n\in P} Tx_n = y$ exists. Let F be the subspace of W generated by y and let $\pi : W \to W/F$ be the quotient map. Then $\lim_{n\in P} \pi Tx_n = 0$ and hence $\lim_{n\in P} \operatorname{dist}(x_n, T^{-1}(F)) = 0$. (Note that since (x_n) is completely irreducible, the second alternative in Proposition 3.5 cannot hold.) Thus we can find $(g_n)_{n\in P}$ with $\lim_{n\in P} \|g_n\| = 0$ so that $x_n + g_n \in T^{-1}(F)$ for all n. By the complete irreducibility of (x_n) this implies that $V = T^{-1}(F)$ and hence T has at most rank one. By Lemma 3.1 we have $V^* = \{0\}$ and thus $T = 0$.

PROPOSITION 3.7. *Let (x_n) be a completely irreducible sequence in a quasi-Banach space V. Suppose (g_n) is any sequence in V with relatively* *compact range. Then* $[x_n]_{n\in\mathbb{N}} \subseteq [x_n + g_n]_{n\in M}$ *for any infinite subset* M *of* $\mathbb{N}$.

Remark. The definition of complete irreducibility allows compact perturbations of the sequence $[x_n]_{n\in\mathbb{N}}$.

Proof of Proposition 3.7. Let $G = [x_n + g_n]_{n\in M}$ and let $\pi : V \to V/G$ be the quotient map. Then $\{\pi(x_n) : n \in M\}$ is relatively compact and hence $\pi(x_n) = 0$ for all n, so that $x_n \in G$ for all n.

PROPOSITION 3.8. *Let* V, W *be quasi-Banach spaces and* $T : V \to W$ *be a bounded linear operator. Then for every completely irreducible sequence* (x_n) *in* V *the sequence* $(Tx_n)_{n\in\mathbb{N}}$ *is completely irreducible in* W.

Proof. Let $Z = [Tx_n]_{n\in\mathbb{N}}$. Let (g_n) be any relatively compact sequence in Z and let M be any infinite subset of $\mathbb{N}$. Let $E = [Tx_n + g_n]_{n\in M}$ and let $\pi : W \to W/E$ be the quotient map. Then $\{\pi Tx_n : n \in M\}$ is relatively compact and so $Tx_n \in E$ for all n as required.

Remark. Thus any quasi-Banach space V admitting a bounded linear operator $T : L_p \to V$ with dense range for some $0 < p < 1$ has a fundamental completely irreducible sequence. This class includes all quotients of the L_p spaces and any separable space on $[0,1]$ containing some L_p for $0 < p < 1$.

THEOREM 3.9. *Let* (x_n) *be a bounded sequence in a quasi-Banach space* V. *Then either*

(1) *there is an infinite subset* M *of* $\mathbb{N}$ *so that* $(x_n)_{n\in M}$ *is uniformly independent, or*

(2) *there is a sequence* (g_n) *with relatively compact range and an infinite subset* M *of* $\mathbb{N}$ *so that* $(x_n + g_n)_{n\in M}$ *is completely irreducible.*

Proof. We will assume the negation of (1) and prove (2), by transfinite induction. Let ω_1 be the first uncountable ordinal. For every $\alpha < \omega_1$ we will define an infinite subset $M_\alpha \subseteq \mathbb{N}$ and a sequence (g_n^α) in V so that $\lim_{n\to\infty} \|g_n^\alpha\| = 0$. For $\alpha < \omega_1$ and $n \in \mathbb{N}$ let $M_{\alpha,n} = \{m \in M_\alpha : m \ge n\}$ and $E_{\alpha,n} = [x_m + g_m^\alpha]_{m\in M_{\alpha,n}}$. We also let $E_\alpha = \bigcap_{n\in\mathbb{N}} E_{\alpha,n}$. The construction will be carried so that if $\alpha < \beta$, then for some n we have both $M_{\beta,n} \subseteq M_{\alpha,n}$ and $E_{\beta,k} \subseteq E_{\alpha,k}$, whenever $k \ge n$.

To start the induction let $M_1 = \mathbb{N}$ and $g_n^1 = 0$ for all n. Suppose $\alpha + 1$ is a nonlimit ordinal and that M_α, (g_n^α) have already been determined. In this case, if possible, we pick an infinite subset $M_{\alpha+1}$ of M_α and a sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \in E_{\alpha,n}$ and $\lim_{n\to\infty} \|h_n\| = 0$, so that if we set $g_n^{\alpha+1} = g_n^\alpha + h_n$, then $E_{\alpha+1}$ is strictly contained in E_α. If this is not possible, we simply set $M_{\alpha+1} = M_\alpha$ and $g_n^{\alpha+1} = g_n^\alpha$ for all n. It is easy to verify that if $(M_\beta, (g_n^\beta))_{\beta\le\alpha}$ satisfies the inductive hypotheses then so does $(M_\beta, (g_n^\beta))_{\beta\le\alpha+1}$.

Now suppose α is a limit ordinal, so that $\alpha = \sup_{n\in\mathbb{N}} \beta_n$, where $(\beta_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence. We will define a sequence $(m_n)_{n=1}^\infty$ of positive integers inductively. Pick $m_1 \in M_{\beta_1}$ so that $\|g_{m_1}^{\beta_1}\| < 1$. For each n, we may choose $m_n \in M_{\beta_n}$ so large that $m_n > m_{n-1}$, $\|g_{m_n}\| < 2^{-n}$, and $M_{\beta_n,j} \subseteq M_{\beta_k,j}$ and $E_{\beta_n,j} \subseteq E_{\beta_k,j}$ whenever $1 \le k \le n-1$ and $j \ge m_n$. Now set $M_\alpha = \{m_n : n \in \mathbb{N}\}$, $g_{m_n}^\alpha = g_{m_n}^{\beta_n}$ and $g_n = 0$ if $n \notin M_\alpha$.

It is easy to see that for each n and $k \ge n$ we have $m_k \in M_{\beta_n}$ and hence that if $\gamma < \alpha$ then there exists n so that $M_{\alpha,n} \subseteq M_{\gamma,n}$. Furthermore, if $k \ge n$ then $x_{m_k} + g_k^\alpha \in E_{\alpha,m_k} \subseteq E_{\beta_k,m_k} \subseteq E_{\beta_n,m_k}$. Hence if $k \ge m_n$, we have $E_{\alpha,k} \subseteq E_{\beta_n,k}$. It follows that if $\gamma < \alpha$ then for some n, we have $E_{\alpha,k} \subseteq E_{\gamma,k}$ whenever $k \ge n$. This completes the inductive construction.

To end the proof, we observe that $(E_\alpha)_{\alpha<\omega_1}$ is a decreasing family of closed subspaces of a separable quasi-Banach space. Hence, by Lindelöf's Theorem, there is a countable ordinal θ so that $E_\theta = E_\alpha$ for all $\alpha > \theta$. Let $M = M_\theta$ and $g_n = g_n^\theta$. Let $F_n = E_{n,\theta}$ and $F = E_\theta$.

We consider the quotient maps $\pi_n : V \to V/F_n$ and construct a linear map $T : V \to \ell_2(V/F_n)$ by setting $Tx = (2^{-n}\pi_n x)_{n=1}^\infty$. Then we have $\lim_{n\in M} T(x_n + g_n) = 0$, so that $\lim Tx_n = 0$. By Proposition 3.5 we conclude that $\lim_{n\in M} \operatorname{dist}(x_n, F) = 0$. It follows that we can pick a null sequence $(h_n)_{n=1}^\infty$ so that $x_n + h_n \in F$ for all $n \in M$. We argue that $(x_n + h_n)_{n\in M}$ is null irreducible. In fact, if $f_n \in F$ and $\lim \|f_n\| = 0$, then $h_n + f_n - g_n \in F_n$ for all n and so the inductive construction and properties of θ guarantee that for every infinite subset P of M we have $[x_n + h_n + f_n]_{n\in P} \supseteq F$.

Finally, by Lemma 3.4 we can find a $u \in F$ and a bounded sequence of scalars (a_n) so that $(x_n - h_n + a_n u)_{n\in M}$ is completely irreducible.

THEOREM 3.10. *Let* V *be a separable infinite-dimensional quasi-Banach space. Then the following conditions on* V *are equivalent*:

(1) V *contains a bounded uniformly separated sequence with no uniformly independent subsequence.*

(2) V *contains a uniformly separated irreducible sequence.*

(3) V *contains a nontrivial completely irreducible sequence.*

(4) V *contains a fundamental uniformly separated irreducible sequence.*

Proof. First we prove (1)⇒(3). If (x_n) is uniformly separated and has no uniformly independent subsequence, by 3.9 we may find a relatively compact sequence (g_n) and an infinite subset M so that $(x_n + g_n)_{n\in M}$ is completely irreducible. We claim that $[x_n + g_n]_{n\in M}$ has noncompact range; indeed, if not, the compactness of the range of (g_n) would imply that (x_n) has compact range, contradicting uniform separation.

We note that (4)⇒(2) and (2)⇒(1) are trivial. It remains to prove (3)⇒(4). Suppose (x_n) is a nontrivial completely irreducible sequence. Let

$E = [x_n]_{n\in\mathbb{N}}$. Consider the quotient space V/E. It contains a fundamental irreducible sequence and so we may pick $(y_n) \in V$ so that (πy_n) is fundamental and irreducible in V/E (if $E = V$, then $y_n = 0$ for all n). Now we show that $(x_n + 2^{-n}y_n)$ is irreducible and fundamental for V. In fact, $[x_n + 2^{-n}y_n]_{n\in M}$ contains E for any infinite subset M of $\mathbb{N}$ by Proposition 3.7 and $[\pi(x_n + 2^{-n}y_n]_{n\in M} = V/E$ by the choice of (y_n). Hence $[x_n + 2^{-n}y_n]_{n\in M} = V$.

It remains to argue that $(x_n + 2^{-n}y_n)$ has a uniformly separated subsequence. If this is not the case then its range is compact, and so is the range of (x_n); but this makes (x_n) the trivial completely irreducible sequence, i.e. $x_n = 0$ for all n.

4. The quasinormed case—the counterexample. In this section we construct a uniformly separated sequence with no uniformly independent subsequence. For this sequence, the second alternative in Theorem 3.9 must hold; in fact, more is true (see the statement of Theorem 4.2).

Let $(N_n)_{n=0}^{\infty}$ be the sequence of natural numbers defined by

$$\log_2 \log_2 \log_2 N_n = n.$$

Let χ be the constantly one function on the interval $[0,1]$.

LEMMA 4.1. *Suppose $0 < p < 1$. Let A be an infinite subset of the natural numbers. Then there is a sequence $(\xi_n)_{n=1}^{\infty}$ of independent identically distributed (i.i.d.) positive random variables in $L_p = L_p[0,1]$ and a constant $K = K(A)$ such that the following conditions hold:*

(1) *for any $m \ge 2$ we have*

$$\Big\|\sum_{k=1}^{m} \xi_k\Big\|_p \le Km\ln m;$$

(2) *if $n \notin A$ and $m \le N_n$, then*

$$\Big\|\sum_{k=1}^{m} \xi_k\Big\|_p \le KN_n \ln N_{n-1};$$

(3) *if $n \in A$, then*

$$\Big\|\sum_{k=1}^{N_n} \xi_k - (N_n \ln N_n)\chi\Big\|_p \le KN_n \ln N_{n-1}.$$

Proof. Let (ζ_n) be a sequence of independent random variables on $[0,1]$ with common distribution given by the density function $x^{-2}\chi_{\{x\ge1\}}$. For any $k \in \mathbb{N}$, let

$$\xi_k(t) = \begin{cases} \zeta_k(t) & \text{if } N_{n-1} \le \zeta_k(t) \le N_n \text{ for some } n \in A,\\ 1 & \text{otherwise.}\end{cases}$$

Obviously, (ξ_k) is i.i.d. Let $\xi = \xi(t)$ be any function on $[0,1]$ which has the distribution of ξ_1.

The estimate (1) is immediate. The sequence (ξ_k) is uniformly bounded in weak L_1, which is logconvex ([6], [13]); since for a suitable constant D we have $\|f\|_p \le D\|f\|_{1,\infty}$ the conclusion follows.

We now use Theorem 1 of [5]. For $x \ge 0$, let

$$\varphi(x) = \int \min(x^2\xi^2, x^p\xi^p)\,dt$$

and, for all real x,

$$\psi(x) = \int_{|x\xi|\le 1} x\xi\,dt.$$

Thus ψ is an odd function (note: in [5] the definition of ψ is inadvertently restricted to positive x). Then there is a constant C_0 so that for any finitely nonzero sequence $(a_k)_{k=1}^{\infty}$ and any real b we have

$$\Big\|b\chi + \sum_{k=1}^{\infty} a_k\xi_k\Big\|_p \le C_0(|b + F(a)| + \|a\|_\varphi),$$

where $\|a\|_\varphi$ is the (quasi)norm in the Orlicz space ℓ_φ and

$$F(a) = \|a\|_\varphi \sum_{k=1}^{\infty} \psi\Big(\frac{a_k}{\|a\|_\varphi}\Big).$$

It is easy to estimate that if $0 \le x \le 1$ then $x^2 \le \varphi(x) \le cx$ for a suitable constant c. Therefore $\|a\|_2 \le \|a\|_\varphi \le C_1\|a\|_1$ for all $a \in c_{00}$.

Suppose $n \notin A$. Then if $C_1^{-1}N_n^{-1} \le t \le N_n^{-1/2}$, we have

$$\psi(t) \le t(\ln N_{n-1} + \ln C_1).$$

Hence if $m \le N_n$, then

$$\Big\|\sum_{k=1}^{m} \xi_k\Big\|_p \le C_0(\ln N_{n-1} + \ln C_1 + C_1)N_n,$$

yielding our second estimate.

If $n \in A$ note that $\varphi(N_n^{-1}) \ge N_n^{-2}(N_n - N_{n-1}) \ge (2N_n)^{-1}$. Since $\varphi(tx) \ge t^p x$ for $t \ge 1$, we obtain, for the basis vectors (e_k) in ℓ_φ,

$$\Big\|\sum_{k=1}^{N_n} e_k\Big\|_\varphi \ge 2^{-1/p}N_n.$$

Now if $C_1^{-1}N_n^{-1} \le t \le 2^{1/p}N_n^{-1}$, we get the estimate

$$\ln(N_n/2^{1/p}) - \ln N_{n-1} \le \psi(t)/t \le \ln C_1 N_n,$$

and hence

$$\left|F\Big(\sum_{k=1}^{N_n} e_k\Big) - N_n \ln N_n\right| \le C_2 N_n \ln N_{n-1},$$

and this yields our third estimate.

THEOREM 4.2. *For each $0 < p < 1$ there is a uniformly separated sequence of positive functions $(f_n)_{n=1}^\infty$ in L_p such that for any infinite subset M of $\mathbb{N}$ and any sequence $(g_n)_{n=1}^\infty$ with $\lim \|g_n\|_p = 0$ we have $[f_n + g_n]_{n\in M} = L_p$.*

Proof. Let $(A_j)_{j=1}^\infty$ be a partition of the natural numbers into countably many infinite sets. For each j there exists a sequence of i.i.d. positive random variables $(\xi_{jk})_{k=1}^\infty$ with the properties of Lemma 4.1, i.e. such that for some constant C_j we have:

(1) $\|\xi_{jk}\|_p \le C_j$;

(2) if $n \ge 2$ and $m \le n$, then

$$\Big\|\sum_{k=1}^{m} \xi_{jk}\Big\|_p \le C_j n \ln n;$$

(3) if $n \notin A_j$ and $m \le N_n$, then

$$\Big\|\sum_{k=1}^{m} \xi_{jk}\Big\|_p \le C_j N_n \ln N_{n-1};$$

(4) if $n \in A_j$, then

$$\Big\|\sum_{k=1}^{N_n} \xi_{jk} - N_n \ln N_n \chi\Big\|_p \le C_j N_n \ln N_{n-1}.$$

Pick any $\varepsilon_1 > 0$ so that $C_1^p \varepsilon_1^p < \frac{1}{2}$. Let $\delta = \varepsilon_1 \|\xi_{11} - \xi_{12}\|_p > 0$. We now pick $\varepsilon_j > 0$ for $j \ge 2$ so that we have both $\sum_{j=1}^\infty C_j^p \varepsilon_j^p < \frac{1}{2}$ and $\sum_{j=2}^\infty C_j^p \varepsilon_j^p < \frac{1}{8}\delta^p$.

Now pick any $p < r < 1$. We define functions $(h_k)_{k=1}^\infty$ as follows. Let $h_1 = 2^{-3/p}\delta\chi$; then if $2^n \le k \le 2^{n+1} - 1$ for $n \ge 1$, let

$$h_k = 2^{n/r - 3/p}\delta \chi_{[k/2^n - 1, (k+1)/2^n - 1]}.$$

Thus $\|h_1\|_p = 2^{-3/p}\delta$, and in general $\|h_k\|_p = 2^{-3/p}\delta 2^{n(1/r - 1/p)}$ when $2^n \le k \le 2^{n+1} - 1$.

We now define positive isometric embeddings $T_j : L_p \to L_p$ and positive functions $(f_j)_{j=1}^\infty$ by induction. We set $T_1 = I$ and $f_1 = \varepsilon_1 \xi_{11} + h_1$. If $(T_i)_{i=1}^{j-1}$ and $(f_i)_{i=1}^{j-1}$ have been defined, we let T_j be a positive isometric embedding with $T_j(\|f_{j-1}\|_p \chi) = f_{j-1}$. Let $f_j = \sum_{i=1}^{j} \varepsilon_i T_i \xi_{ij} + h_j$.

Notice first that $\|f_j\|_p \le (\|h_j\|_p^p + \sum_{i=1}^j C_i^p \varepsilon_i^p)^{1/p} < 1$ for all j. Secondly,

$$\|f_j - \varepsilon_1 \xi_{1j}\|_p \le \Big(\|h_j\|_p^p + \sum_{i=2}^{j} C_i^p \varepsilon_i^p\Big)^{1/p} \le 4^{-1/p}\delta.$$

Hence if $j \ne k$,

$$\|f_j - f_k\|_p \ge \frac{1}{2}\delta.$$

Thus the sequence (f_j) is uniformly separated.

Now let E be any finite subset of $\mathbb{N}$. Let $u = \min E$, $v = \max E$ and set $w = |E|$; suppose $w > 1$. We first note that:

$$(5)\qquad \sum_{k\in E} f_k = \sum_{k\in E}\sum_{i=1}^{k} \varepsilon_i T_i \xi_{ik} + \sum_{k\in E} h_k = \sum_{i=1}^{v} \varepsilon_i T_i\Big(\sum_{i\le k\in E} \xi_{ik}\Big) + \sum_{k\in E} h_k.$$

We first use (5) to prove that $[f_n]_{n\in\mathbb{N}} = L_p$. Using estimate (2) we have

$$\Big\|\sum_{i\le k\in E} \xi_{ik}\Big\|_p \le C_i w \ln w.$$

Hence

$$\Big\|\sum_{k\in E} f_k - \sum_{k\in E} h_k\Big\|_p \le w \ln w.$$

Let D be any dyadic interval of length 2^{-n}. If $m > n$ there is a subset E_m of $\{2^m, 2^m + 1, \ldots, 2^{m+1} - 1\}$ such that $|E_m| = 2^{m-n}$ and also such that

$$\chi_D = 2^{3/p}\delta^{-1} 2^{-m/r} \sum_{k\in E_m} h_k.$$

Hence

$$\Big\|\chi_D - 2^{3/p}\delta^{-1}2^{-m/r}\sum_{k\in E_m} f_k\Big\|_p \le 2^{3/p}\delta^{-1}(\ln 2) 2^{m-n-m/r}(m-n).$$

If we let $m \to \infty$, since $r < 1$, we obtain $\chi_D \in [f_n]_{n\in\mathbb{N}}$, and so $[f_n]_{n\in\mathbb{N}} = L_p$.

To complete the proof we show that for any j, any null sequence (g_n) and any infinite subset M of $\mathbb{N}$ we have $f_j \in [f_n + g_n]_{n\in M} = Y$, say. First let $\sigma_u = \max\{\|g_n\|_p : n \ge u\}$, for $u \in \mathbb{N}$, so that $\lim_{u\to\infty} \sigma_u = 0$. We return to (5) and now suppose that E is any subset of $M \cap \{j+1, j+2, \ldots\}$ with $w = |E| = N_n$, where $n \in A_{j+1}$. We suppose that $l = \lfloor \log_2 u \rfloor$. Then if $k \in E$ we can estimate $\|h_k\|_p \le 2^{-3/p}\delta 2^{l/r - l/p}$ and hence

$$\Big\|\sum_{k\in E} h_k\Big\|_p \le 2^{-3/p}\delta^{-1} 2^{l/r - l/p} N_n^{1/p}.$$

Now for each $1 \le i \le v$, with $i \ne j+1$ we can estimate

$$\Big\|\sum_{i\le k\in E} \xi_{ik}\Big\|_p \le C_i N_n \ln N_{n-1}.$$

However,

$$\Big\|\sum_{k\in E}\xi_{j+1,k}-(N_n\ln N_n)\chi\Big\|_p\le C_{j+1}N_n\ln N_{n-1}.$$

Combining these estimates gives

$$\Big\|\sum_{k\in E}f_k-\varepsilon_{j+1}\|f_j\|_p^{-1}(N_n\ln N_n)f_j\Big\|_p^p\le(N_n\ln N_{n-1})^p+2^{-3}\delta^{-p}2^{lp/r-l}N_n.$$

We also have

$$\Big\|\sum_{k\in E}g_k\Big\|_p\le N_n^{1/p}\sigma_u.$$

We can choose E to be a subset of M with u and hence l arbitrarily large. Then

$$\operatorname{dist}(f_j,Y)\le\varepsilon_{j+1}^{-1}\|f_j\|_p\frac{\ln N_{n-1}}{\ln N_n}.$$

However, $\ln N_n=2^{2^n}\ln 2$ and this estimate holds for any $n\in A_{j+1}$, so that $f_j\in Y$, as required.

5. Nonstandard consequences. Theorem 2.1 also has some consequences for the theory of nonstandard topological vector spaces, solving a problem posed in an earlier paper by J. Náter, P. Pulmann and the last author [10], formulated, however, in the language of the alternative set theory (AST). A more direct proof, using the model-theoretical method of indiscernibles in AST, was given in [4].

In the sequel it is assumed that we are working inside an $\aleph_1$-saturated nonstandard universe and V is an internal vector space over an internal field $\mathbb{F}\in\{{}^*\mathbb{Q},{}^*\mathbb{R},{}^*\mathbb{C}\}$, with $\mathbb{IF}$ denoting the monad of infinitesimals and $\mathbb{BF}$ the galaxy of bounded (finitely large) elements in $\mathbb{F}$, determining its topology. Similarly, the topology on V is also given by a pair of (in general external) sets—M, called the monad of 0, and G, called the galaxy of 0, subject to the following conditions:

$$0\in M\subseteq G\subseteq V,$$
$$M+M\subseteq M,\qquad G+G\subseteq G,$$
$$\mathbb{BF}\cdot M\subseteq M,\qquad \mathbb{IF}\cdot G\subseteq M,\qquad \mathbb{BF}\cdot G\subseteq G$$

—cf. [3], [12]. Following the latter reference, the triple (V,M,G) will be called a *biequivalence vector space* if in addition M is Π_1^0-set (i.e. the intersection of countably many internal sets) and G is a Σ_1^0-set (i.e. the union of countably many internal sets).

A set $A\subseteq V$ in (V,M,G) will be called *separated* if $x-y\notin M$ for any distinct $x,y\in A$. A will be called *pseudocompact* if every hyperfinite separated set $H\subseteq A$ is finite.

A set $A\subseteq V$ in (V,M,G) will be called *independent* if $A\cap M=\emptyset$ and for every internal linear combination $\sum_{x\in H}\gamma(x)x$, where H is any hyperfinite (possibly infinite) subset of A and $\gamma:H\to\mathbb{F}$ is any internal function, from $\sum_{x\in H}\gamma(x)x\in M$ it follows that $\gamma(x)x\in M$ for each $x\in H$. The set A will be called *dimensionally compact* if every hyperfinite independent set $H\subseteq A$ is finite.

As every independent set $A\subseteq V$ is obviously separated, every pseudocompact set in (V,M,G) is dimensionally compact. The trivial example $\{ix:1\le i\le k\}$, where $k\in{}^*\mathbb{N}\setminus\mathbb{N}$ and $x\in V\setminus M$, shows that the reversed implication does not hold (except in the degenerate case $V=M$). The question is whether such a counterexample can entirely lie within the galaxy G. For biequivalence vector spaces this is excluded by our last result.

THEOREM 5.1. *Let (V,M,G) be a biequivalence vector space. Then any infinite hyperfinite separated set $H\subseteq G$ contains an infinite hyperfinite independent subset. Consequently, a set $A\subseteq G$ is dimensionally compact if and only if it is pseudocompact.*

Proof. In [12] an argument similar to the proof of the already quoted Aoki–Rolewicz Theorem ([1], [7], [11]) is given to prove that there are a noninfinitesimal $p\in{}^*\mathbb{R}$, $0<p\le 1$, and an internal p-norm $\|\cdot\|:W\to{}^*\mathbb{R}$, where W is an internal linear subspace of V, such that $G\subseteq W$ and

$$M=\{x\in W:\|x\|\in\mathbb{I}^*\mathbb{R}\},\qquad G=\{x\in W:\|x\|\in\mathbb{B}^*\mathbb{R}\}.$$

(Strictly speaking, the internal p-norm will only be a *pseudo* p-norm, in the sense that the internal subspace $W_0=\{x\in W:\|x\|=0\}$ of V need not be $\{0\}$. However, since $W_0\subseteq M$, this causes no problems.)

Due to the $\aleph_1$-saturation of the nonstandard model, there are standard positive numbers $K\in\mathbb{N}$ and $\varepsilon\in\mathbb{Q}$ such that H is both K-bounded and ε-separated. Choose any standard $\delta\in\mathbb{Q}$, $0<\delta<\varepsilon/2^{1/p}$. As the number of elements (internal cardinality) of H is bigger than each $n\in\mathbb{N}$, by Theorem 2.1 there is an m-element δ-independent subset $Y_m\subseteq H$, for each $m\in\mathbb{N}$. Using the $\aleph_1$-saturation again, we obtain an infinite hyperfinite δ-independent subset $Y\subseteq H$. The rest is trivial.

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