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The basic sequence problem

by

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Abstract. We construct a quasi-Banach space X which contains no basic sequence.

1. Introduction. It is a classical result in Banach space theory, known to Banach himself [1], that every (infinite-dimensional) Banach space contains a closed linear subspace with a basis, or, in other words, a basic sequence. The corresponding question for quasi-Banach spaces (and more general F-spaces) has, however, remained open. A number of equivalent formulations are known ([11], [14], [16], [17]); the question is also raised in a slightly disguised form in [28], p. 114.

In [11] and [17] it is shown that a quasi-Banach space X contains a basic sequence if and only if there is a strictly weaker Hausdorff vector topology on X. Thus the existence of a space with no basic sequence is equivalent to the existence of a (topologically) *minimal* space (i.e. one on which there is no strictly weaker Hausdorff vector topology). See [3] and [4] for a discussion of minimal spaces. It further follows that X contains a basic sequence if and only if there is some infinite-dimensional closed subspace with separating dual ([11], Theorem 4.4). Several positive results are known. For example, the work of Bastero [2] implies that every subspace of $L_p[0,1]$ $(0<p<1)$ contains a basic sequence, while the author's results in [12] imply that every quotient of $L_p[0,1]$ contains a basic sequence. Bastero's result can be lifted to the wider class of so-called natural spaces and has further been extended by Tam [30] who shows that every complex quasi-Banach space with an equivalent plurisubharmonic norm contains a basic sequence. These results suggest that almost all "reasonable" spaces contain a basic sequence.

In this paper, we will prove

THEOREM 1.1. *There is a quasi-Banach space Y with a one-dimensional subspace L so that*

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(1) *if* Y_0 *is a closed infinite-dimensional subspace of* Y *then* $L \subset Y_0$, *and*

(2) Y/L *is isomorphic to the Banach space* ℓ_1.

In particular, Y *contains no basic sequence and is minimal.*

It is clear that (1) would make it impossible for Y to contain a basic sequence.

There are other applications of this space. A topological vector space X is said to have the *Hahn–Banach Extension Property* (HBEP) if whenever X_0 is a closed subspace of X and f is a continuous linear functional on X_0 then f can be extended to a continuous linear functional on X. The author showed in [11], answering a question raised by Duren, Romberg and Shields [5] (see also [25], [29]) that for an F-space (complete metric linear space) (HBEP) is equivalent to local convexity. It was very well known that metrizability is necessary in this theorem, but some partial results of Ribe [25] suggested that completeness might not be required. Ribe showed that if X is a metric linear space so that X is isomorphic to $X \oplus X$ then if X has (HBEP) it must be locally convex. More recently, the author [14] extended Ribe's result to show

THEOREM 1.2. *Let* X *be a decomposable quasi-Banach space* (*i.e. there is a bounded projection* P *on* X *so that neither* P *nor* $I-P$ *has finite rank*). *Suppose* X_0 *is a dense subspace of* X. *Then* X_0 *has* (HBEP) *if and only if* X *is locally convex.*

A proof of Theorem 1.2 is included in Section 6. The Hahn–Banach extension property for metrizable spaces is also discussed in [10].

However, if Y is the space constructed above, we will show that any algebraic complement Y_0 of L has (HBEP). Thus we have

THEOREM 1.3. *There is a non-locally convex metric linear space* Y_0 *with the Hahn–Banach Extension Property.*

In 1962, Klee [18] asked whether for every topological vector space (X,τ), the topology τ can be expressed as the supremum of two not necessarily Hausdorff vector topologies τ_1 and τ_2 so that (the Hausdorff quotient of) (X,τ_1) has a separating dual (i.e. is *nearly convex*) and (X,τ_2) has trivial dual. Recently Peck [22] has shown this to be true for certain twisted sums of a Banach space and a one-dimensional space (see also [23]). The space constructed here, Y, turns out to be a counterexample to Klee's problem.

THEOREM 1.4. *There is a quasi-Banach space* Y *so that the topology on* Y *is not the supremum of a trivial dual topology and a nearly convex topology.*

The construction of our example depends heavily on the recent remarkable developments in infinite-dimensional Banach spaces due to Gowers, Maurey, Odell and Schlumprecht [7], [8], [9], [20], [21]. It is perhaps a little ironic that the basic sequence question for quasi-Banach spaces turns out to be so closely related to the *unconditional* basic sequence problem for Banach spaces. However, it should be stressed that we use an example of a Banach space with an unconditional basis, very similar to that used by Gowers in [7]; the fundamental estimates we need are in [9].

Let us conclude this introduction by explaining the shortcomings of the example. It is still an open question whether every quasi-Banach space (or F-space) must contain a proper closed infinite-dimensional subspace. A space with no proper closed infinite-dimensional subspace is called *atomic*. The existence of an atomic quasi-Banach space is known to be equivalent to the existence of a *quotient minimal* quasi-Banach space, i.e. a space X so that every quotient is minimal (this concept is due to Drewnowski [3]). See [14] or [16] for a discussion. Our example is quite far from an atomic space, and it is not clear at present whether it can be used towards making such a monster. We remark that Reese [24] has constructed an example of an "almost" atomic F-space, i.e. a space X with a sequence of finite-dimensional subspaces V_n with $\dim V_n > n$ so that if $x_n \in V_n$ is any sequence which is non-zero infinitely often then $[x_n] = X$. It is still unknown whether even this phenomenon can be reproduced in a quasi-Banach space. We suspect, however, that an atomic quasi-Banach space will eventually be found.

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2. Idea of the construction. In this section, we introduce the basic ideas and notation and prove that the space Y which will be constructed in Sections 3–5 yields solutions to the problems mentioned in the introduction.

We denote by c_{00} the space of all finitely non-zero (real) sequences. If $x \in c_{00}$ we denote its co-ordinates by $\{x(j)\}_{j=1}^{\infty}$. We let $a(x) = \min\{j : x(j) \neq 0\}$ and $b(x) = \max\{j : x(j) \neq 0\}$. If A is a subset of $\mathbb{N}$ then $Ax(j) = x(j)\chi_A(j)$ where χ_A is the characteristic function of A. If E_1, E_2 are subsets of $\mathbb{N}$ we write $E_1 < E_2$ if $\max E_1 < \min E_2$. We shall also write for $x, y \in c_{00}$ that $x < y$ if $b(x) < a(y)$. On the other hand, the natural co-ordinatewise order on c_{00} will be denoted by $x \le y$, i.e. $x \le y$ if and only if $x(j) \le y(j)$ for all $j \in \mathbb{N}$. Let $c_{00}^+ = \{x \in c_{00} : x \ge 0\}$.

For $x, y \in c_{00}$ we will write $\langle x, y\rangle = \sum_{j=1}^{\infty} x(j)y(j)$. We will also use the same terminology when $x \in c_{00}^+$ and $y = \log v$ for some sequence $v \in c_{00}^+$; in this case it will be understood that the pairing can take the value $-\infty$ and that $0 \log 0 = 0$.

By a *sequence space* X we will mean a subspace X of the space ω of all sequences equipped with a lattice norm $\|\ \|_X$ so that

(1) $c_{00} \subset X$,

(2) if $|x| \le |y| \in X$ then $x \in X$ and $\|x\|_X \le \|y\|_X$, and

(3) if $0 \le x_n \uparrow x$ and $x_n \in X$ with $\sup \|x_n\|_X < \infty$ then $x \in X$ with $\|x\|_X = \sup \|x_n\|_X$ (the Fatou property).

The canonical basis vectors $\{e_n\}_{n=1}^{\infty}$ then form a 1-unconditional basis for the closure X_0 of c_{00}. For convenience we will write X^* for the Köthe dual of X, which coincides with the Banach space dual of X_0. We will denote the closed unit ball of a Banach space X by B_X. We denote the canonical norm on ℓ_p by $\|\ \|_p$ for the cases $p = 1$ and $p = \infty$.

Consider a map $\Phi : c_{00} \to \mathbb{R}$. For any $u_1, \dots, u_n$ we define $\Delta_\Phi(u_1, \dots, u_n) = \sum_{i=1}^{n} \Phi(u_i) - \Phi(\sum_{i=1}^{n} u_i)$. Φ is called *quasilinear* if

(4) $\Phi(\alpha u) = \alpha u$ for $\alpha \in \mathbb{R}$, $u \in c_{00}$, and

(5) for a constant $\delta = \delta(\Phi)$ we have $|\Delta(u, v)| \le \delta(\|u\|_1 + \|v\|_1)$ whenever $u, v \in c_{00}$.

Given a quasilinear map Φ we can form the twisted sum $Y = \mathbb{R} \oplus_\Phi \ell_1$, which is defined to be the completion of $\mathbb{R} \oplus c_{00}$ under the quasinorm

$$\|(\alpha, u)\|_\Phi = |\alpha - \Phi(u)| + \|u\|_1.$$

It is readily verified that if L is the span of the vector $e_0 = (1, 0)$ then Y/L is isomorphic to ℓ_1. This construction was first used in [13] and [26] with explicit non-trivial twisted sums of $\mathbb{R}$ and ℓ_1 to deduce that local convexity is not a three-space property; see also [27].

THEOREM 2.1. *Let $\Phi : c_{00} \to \mathbb{R}$ be a quasilinear map and let $Y = \mathbb{R} \oplus_\Phi \ell_1$. Then the following conditions are equivalent*:

(1) *Y contains no basic sequence.*

(2) *If Y_0 is an infinite-dimensional closed subspace of Y then Y_0 contains e_0.*

(3) *The quotient map $\pi : Y \to \ell_1$ is strictly singular.*

(4) *Y is topologically minimal.*

(5) *There is no infinite-dimensional subspace F of c_{00} so that for some constant K we have $|\Phi(u)| \le K\|u\|_1$ for all $u \in F$.*

(6) *If $T : \ell_1 \to Y$ is a bounded operator then T is compact.*

(7) *If* $T : Y \to Y$ *is a bounded operator then* $T = \lambda I + S$ *where* $\lambda \in \mathbb{R}$ *and* S *is compact.*

Proof. The equivalence of (1) and (4) is well known (see Theorem 4.2 of [11] and Theorem 3.2 of [17], or see [16]). (2) is clearly equivalent to (3) and implies (1). Conversely, if (3) fails then there is an infinite-dimensional closed subspace isomorphic to a subspace of ℓ_1. Thus (1)–(4) are all equivalent.

Next we prove (2) implies (5). Suppose F is an infinite-dimensional subspace of c_{00} so that $|\Phi(u)| \le K\|u\|_1$ for $u \in E$. Let Y_0 be the closure of the subspace of all $(0, x)$ for $x \in E$. Suppose $(0, x_n)$ converges to e_0. Then $|1 - \Phi(x_n)|$ and $\|x_n\|_1$ converge to zero, which is a contradiction.

Next assume (5) and suppose Y contains a basic sequence. By a perturbation argument we can suppose it contains a normalized basic sequence of the form (α_n, u_n) where $u_n \in c_{00}$. By passing to a subsequence we can suppose that $u_1 < u_2 < \ldots$ and that e is not in the closed linear span of (α_n, u_n). It follows that π is an isomorphism on the span of this basic sequence so that for some K we have

$$\Big|\sum_{i=1}^{n} \alpha_i t_i - \Phi\Big(\sum_{i=1}^{n} t_i u_i\Big)\Big| \le K\Big\|\sum_{i=1}^{n} t_i u_i\Big\|_1$$

for all $t_1, \ldots, t_n$. Let F_0 be the subspace of the linear span of the $(u_n)_{n=1}^{\infty}$ consisting of all $\sum_{i=1}^{n} t_i u_i$ with $\sum_{i=1}^{n} \alpha_i t_i = 0$. Then $|\Phi(u)| \le K\|u\|_1$ for $u \in F_0$. Thus (5) implies (1).

(3) implies (6). If $T : \ell_1 \to Y$ is bounded then πT is strictly singular and hence compact. If (x_n) is a sequence in the unit ball of ℓ_1 then by passing to a subsequence we can suppose that $\pi T x_n$ converges. Hence there exist $y_n \in Y$ so that (y_n) converges and $\pi T x_n = \pi y_n$. But then $Tx_n - y_n \in L$ and so has a convergent subsequence.

(6) implies (7). If $T : Y \to Y$ is a bounded operator then since L is the intersection of the kernels of all continuous linear functionals on Y we must have $T(L) \subset L$. Thus $Te = \lambda e$ for some λ. Let $S = T - \lambda I$; then $S = S_0\pi$ where $S_0 : Y/L \to Y$ is compact by (6).

(7) implies (3). If π is not strictly singular, there is a subspace Y_0 of Y of infinite codimension and isomorphic to ℓ_1. Hence there is an isomorphic embedding $V : \ell_1 \to Y$. Then suppose $V\pi = \lambda I + S$ where S is compact. Let $\pi_0 : Y \to Y/Y_0$ be the quotient map. Then $\lambda\pi_0 = -S\pi_0$ is compact. Hence $\lambda = 0$, but this contradicts the fact that V is an isomorphism.

THEOREM 2.2. *If* Y *satisfies the equivalent conditions of Theorem* 2.1 *then any algebraic complement of* L *has the Hahn–Banach Extension Property.*

Proof. Let Z be an algebraic complement of L. The continuous linear functionals on Z separate points, so that any linear functional on a finite-dimensional subspace can be extended continuously to Z. Now let Z_0 be a closed infinite-dimensional subspace of Z and suppose f is a continuous linear functional on Z_0. Let W be the closure of Z_0 in Y and let f denote the extension of f to W. Then W and $f^{-1}(0)$ contain L by (2) and so f factors to a continuous linear functional on $W/L \subset Y/L$, which is a Banach space. Hence by the Hahn–Banach theorem f can be extended continuously to Y and hence also to Z.

THEOREM 2.3. *If Y satisfies the conditions of Theorem* 2.1 *then the topology τ on Y cannot be the supremum of two vector topologies τ_1, τ_2 so that (Y, τ_1) is nearly convex and (Y, τ_2) has trivial dual.*

Proof. Clearly e_0 must be in the closure of $\{0\}$ for τ_1. Let E be the closure of $\{0\}$ for τ_2. If $e_0 \notin E$ then Theorem 2.1 implies that E is finite-dimensional and that Y^* separates the points of E. Hence $Y = Y_0 \oplus E$ for some closed subspace Y_0 of Y. Now Y_0 contains no basic sequence and so its topology is minimal; however, τ_2 is Hausdorff on Y_0 so that it must agree with the original topology. This implies that $Y_0^* = \{0\}$, but in fact Y_0^* is infinite-dimensional. This contradiction establishes the theorem.

We now review the method of approach to the example. Theorem 2.1 reduces the problem to a type of distortion question expressed by (4). The recent results of the author [15] show that there is a close relationship between quasilinear maps on c_{00} and sequence spaces (see Theorem 6.8 of [15]). We will explain the connection in the next section and show how the recent spaces discovered by Gowers and Maurey ([7] and [9]) enable us to construct a pathological Φ.

3. Indicators of sequence spaces. We now introduce some ideas from [15]. Suppose X is a sequence space. We define the *indicator* Φ_X (called the *entropy map* in [21]) on c_{00} by $\Phi_X(u) = \langle u, \log x\rangle$ where $u = x^*x$ is the (unique) *Lozanovskiĭ factorization* of u, i.e. $x \in B_X^+$ and $x^* \in X^*$ satisfy $\langle x, x^*\rangle = \|x^*\|_{X^*} = \|u\|_1$ and $\operatorname{supp} x$, $\operatorname{supp} x^* \subset \operatorname{supp} u$. The Lozanovskiĭ factorization originates in [19].

Clearly $\Phi_X(\alpha u) = \alpha\Phi_X(u)$ for $u \in c_{00}$. Furthermore, if $u, v \in c_{00}$ we also have

$$(1) \qquad |\Delta(u, v)| \le \frac{4}{e}(\|u\|_1 + \|v\|_1)$$

where $\Delta = \Delta_{\Phi_X}$ (see Lemma 5.6 of [15]). If $u \in c_{00}^+$ then we can characterize the Lozanovskiĭ factorization as the solution of an optimization problem so that

$$(2) \qquad \Phi_X(u) = \max_{x \in B_X^+} \langle u, \log x\rangle.$$

This idea originates with Gillespie [6]. Furthermore, for $u_1, \ldots, u_n \in c_{00}^+$ we have the inequalities

$$(3) \qquad 0 \le \Delta(u_1, \ldots, u_n) \le \sum_{i=1}^{n} \|u_i\|_1 \log \frac{S}{\|u_i\|_1}$$

where $S = \sum_{i=1}^{n} \|u_i\|_1$; see [15], Lemma 5.5.

Suppose $f : [1, \infty) \to [1, \infty)$ is any increasing map with $f(1) = 1$ and so that $f(t) \le t$ for all $t \ge 1$. We will say that a sequence space X has a *lower f-estimate on blocks* if, whenever $x_1 < \ldots < x_n \in c_{00}$, then

$$\|x_1 + \ldots + x_n\|_X \ge \frac{1}{f(n)} \sum_{i=1}^{n} \|x_i\|_X,$$

and an *upper f-estimate on blocks* if, whenever $x_1 < \ldots < x_n \in c_{00}$, then

$$\|x_1 + \ldots + x_n\|_X \le f(n) \max_{1 \le i \le n} \|x_i\|_X.$$

LEMMA 3.1. *Suppose X satisfies an upper f-estimate on blocks. Then for $u_1 < \ldots < u_n$ in c_{00}^+ we have*

$$\Delta(u_1, \ldots, u_n) \le \log f(n)(\|u_1\|_1 + \ldots + \|u_n\|_1).$$

Proof. Let $u_i = x_i x_i^*$ be the Lozanovskiĭ factorizations. Then since $f(n)^{-1}(x_1 + \ldots + x_n) \in B_X$ we have by (2),

$$\Phi_X(u_1 + \ldots + u_n) \ge \Big\langle \sum_{i=1}^{n} u_i, \log \Big(f(n)^{-1} \sum_{i=1}^{n} x_i \Big) \Big\rangle$$

so that the lemma follows.

The following is a special case of Lemma 5.8 of [15]. Unfortunately, as the referee has pointed out, Lemma 5.8 in [15] is misstated with the inequality reversed, and in the proof the maximum should be replaced by the minimum. This lemma is used in Theorem 5.7 of [15], which is correct although an inequality is again reversed. In view of this we will sketch a simple direct proof.

LEMMA 3.2. *Suppose $s_1, \ldots, s_n, t_1, \ldots, t_n \ge 0$ and let $\sum_{i=1}^{n} s_i = S$ and $\sum_{i=1}^{n} t_i = T$. Then*

$$\sum_{i=1}^{n} \Big(s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i} \Big) \le S \log \frac{S + T}{S} + T \log \frac{S + T}{T}.$$

Remark. The summand is zero if either s_i or t_i vanishes.

Proof. We will seek to maximize the function

$$u(s_1, \ldots, s_n, t_1, \ldots, t_n) = \sum_{i=1}^{n} \Big(s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i} \Big)$$

subject to the constraints $\sum_{i=1}^n s_i = S$ and $\sum_{i=1}^n t_i = T$ and $s_i \geq 0,\ t_i \geq 0$ for $1 \leq i \leq n$. By continuity, there is a point where the maximum is attained. We can suppose $s_i t_i > 0$ for $1 \leq i \leq m$ and $s_i t_i = 0$ if $m+1 \leq i \leq n$. By the method of Lagrange multipliers it is easy to show that s_i/t_i is constant for $1 \leq i \leq m$. But then

$$u(s_1, \ldots, s_n, t_1, \ldots, t_n) = S_0 \log \frac{S_0 + T_0}{S_0} + T_0 \log \frac{S_0 + T_0}{T_0}$$

where $S_0 = \sum_{i=1}^m s_i \leq S$ and $T_0 = \sum_{i=1}^m t_i \leq T$. This expression is monotone increasing in S_0 and T_0 and so the result follows.

Let $D = B_{\ell_1} \cap c_{00}^+$.

LEMMA 3.3. *Suppose X satisfies an upper f-estimate on blocks and suppose $u \in D$. Let $u = \sum_{i=1}^n u_i$ where $u_1 < \ldots < u_n$. Let A be any subset of $\mathbb{N}$ and let $t = \|Au\|_1$. Then*

$$\begin{aligned}\Delta(u_1, \ldots, u_n) - (1-t)\log f(n) - \varphi(t) &\leq \Delta(Au_1, \ldots, Au_n) \\ &\leq \Delta(u_1, \ldots, u_n) + \varphi(t),\end{aligned}$$

where $\varphi(t) = t\log(1/t) + (1-t)\log(1/(1-t))$ $(\leq \log 2)$.

Proof. Let $\mathbb{N} \setminus A = B$. Then

$$\begin{aligned}&\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) \\ &\qquad = \Delta(Au_1, \ldots, Au_n) + \Delta(Bu_1, \ldots, Bu_n) + \Delta(Au, Bu).\end{aligned}$$

Similarly

$$\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) = \Delta(u_1, \ldots, u_n) + \sum_{i=1}^n \Delta(Au_i, Bu_i).$$

Since $\Delta(Bu_1, \ldots, Bu_n), \Delta(Au, Bu) \geq 0$ we deduce

$$\Delta(Au_1, \ldots, Au_n) \leq \Delta(u_1, \ldots, u_n) + \sum_{i=1}^n \Delta(Au_i, Bu_i).$$

Now we use (3) and Lemma 3.2. We have

$$\begin{aligned}\sum_{i=1}^n \Delta(Au_i, Bu_i) &\leq \sum_{i=1}^n \left(\|Au_i\|_1 \log \frac{\|u_i\|_1}{\|Au_i\|_1} + \|Bu_i\|_1 \log \frac{\|u_i\|_1}{\|Bu_i\|_1} \right) \\ &\leq t \log \frac{1}{t} + (1-t) \log \frac{1}{1-t}.\end{aligned}$$

For the former inequality we observe that $\Delta(Bu_1, \ldots, Bu_n) \leq \log f(n)\|Bu\|_1$. Hence

$$\Delta(Au_1, \ldots, Au_n) \geq \Delta(u_1, \ldots, u_n) - (1-t)\log f(n) - \Delta(Au, Bu),$$

and the second inequality follows.

LEMMA 3.4. *Suppose $u \in c_{00}^+$ with $\|u\|_1 \le 1$. Suppose $u = xx^*$ where $x \in B_X^+$, $x^* \in B_{X^*}^+$. Then $\Phi_X(u) - \langle u, \log x\rangle \le \|u\|_1 \log(1/(\|u\|_1))$ $(\le 1/e)$.*

Proof. We can suppose that the supports of x, x^* coincide with the support of u. Define Z by $\|z\|_Z = \max(\|z\|_X, \|u\|_1^{-1}\langle |z|, x^*\rangle)$. Then $\|z\|_X \le \|z\|_Z \le \|u\|^{-1}\|z\|_X$ so that $\Phi_X(v) + \|v\|_1 \log \|u\|_1 \le \Phi_Z(v) \le \Phi_X(v)$ for $v \in c_{00}^+$. However, $\|x\|_Z \le 1$ and $\|x^*\|_{Z^*} \le \|u\|_1$ so that $u = xx^*$ is the Lozanovskiĭ factorization for u. Thus $\Phi_Z(u) = \langle u, \log x\rangle$ and the lemma follows.

The next lemma is essentially due to Odell and Schlumprecht [21].

LEMMA 3.5. *Given $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $\eta > 0$ so that if $u_1 < \ldots < u_n$ are in D, $u = (1/n)(u_1 + \ldots + u_n)$ and $\delta = (1/n)\Delta(u_1, \ldots, u_n) < \eta$ then for the Lozanovskiĭ factorizations $u = xx^*$ and $u_i = x_i x_i^*$ we have $\|Au\|_1 < \varepsilon$ where $A = \{j : y(j) > (1+\varepsilon)x(j)\}$ and $y = x_1 + \ldots + x_n$.*

Proof. By Proposition 2.3 of Odell and Schlumprecht [21], given $\varepsilon > 0$ there exists $\nu > 0$ so that if $v \in D$ and $z \in B_X^+$ are such that $\langle v, \log z\rangle > \Phi_X(v) - \nu$ then if $v = z_0 z_0^*$ is the Lozanovskiĭ factorization then $\|Bv\|_1 < \varepsilon$ where $B = \{j : z_0(j) > (1+\varepsilon)z(j)\}$. Let $\eta = \nu/n$. Then if $\delta < \eta$ we have

$$\sum_{i=1}^{n} (\Phi_X(u_i) - \langle u_i, \log x\rangle) < \nu$$

and since each term is positive we conclude that $\|A_i u_i\|_1 < \varepsilon$ where $A_i = \{j : x_i(j) > (1+\varepsilon)x(j)\}$. This quickly implies that $\|Au\|_1 < \varepsilon$.

4. The Gowers–Maurey space. At this point we let $f(x) = \log_2(x+1)$ and introduce as in [9] the class $\mathcal{F}$ of functions $g : [1,\infty) \to [1,\infty)$ having the following properties:

(1) $g(1) = 1$ and $g(x) < x$ for $x > 1$.
(2) g is strictly increasing and unbounded.
(3) $\lim_{x\to\infty} x^{-q} g(x) = 0$ for any $q > 0$.
(4) $x/g(x)$ is concave and non-decreasing.
(5) g is submultiplicative, i.e. $g(xy) \le g(x)g(y)$ for $x, y \ge 1$.

Clearly $f \in \mathcal{F}$ and so is $\sqrt{f}$.

Now suppose X is a sequence space. If $n \in \mathbb{N}$ and $\kappa > 1$ we define $\lambda_X(n,\kappa)$ to be the set of $x \in c_{00}^+$ so that $\|x\|_X = 1$ and $x = (1/n)(x_1 + \ldots + x_n)$ where $x_1 < \ldots < x_n$ and $\|x_i\|_X \le \kappa$ for $1 \le i \le n$. (Thus x is an ℓ_{1+}^n average with constant κ, in the sense of [9]: note that we restrict ourselves to non-negative sequences and to spaces X for which the canonical basis is unconditional.)

We then define $\mathrm{RIS}_X(n;\kappa)$ to be the collection of sequences $x_1 < \ldots < x_n$ in c_{00}^+ satisfying $x_i \in \lambda_X(M_i,\kappa)$ where $M_1 \geq 4\kappa\varrho^{-1}2^{36n^2\varrho^{-2}}$ and $M_{k+1} \geq 2^{4b(x_k)^2\varrho^{-2}}$ for $k \geq 1$ where $\varrho = \min(\kappa-1,1)$. We next define $\Lambda_X(n;\kappa)$ to be the collection of $x \in c_{00}^+$ of the form $x = \|x_1+\ldots+x_n\|_X^{-1}(x_1+\ldots$ $\ldots + x_n)$ where $(x_1,\ldots,x_n) \in \mathrm{RIS}_X(n,\kappa)$. This definition differs slightly but inessentially from that of [9]. In fact, we will only really require the case $\kappa \geq 2$ when $\varrho = 1$; this is in contrast to [9] where values of κ close to one are important.

At the same time if $g \in \mathcal{F}$ we define $\mathcal{H}_X(g;m)$ to be the collection of (m,g)-forms, i.e. $x^* \in \mathcal{H}_X(g;m)$ if and only if $x^* = g(m)^{-1}(x_1^*+\ldots+x_m^*)$ where $x_1^* < \ldots < x_m^*$ are in c_{00}^- and $\|x_i^*\|_{X^*} \leq 1$ for $1 \leq i \leq m$.

We will require certain lemmas from [9].

Lemma 4.1 (Lemma 4 of [9]). *Suppose $x \in \lambda_X(N,\kappa)$ and $x^* \in \mathcal{H}_X(g;M)$ where $g \in \mathcal{F}$. Then $\langle x,x^*\rangle \leq \kappa(1+2M/N)g(M)^{-1}$.*

Lemma 4.2 (Lemma 5 of [9]). *Suppose X satisfies a lower f-estimate on blocks and $g \in \mathcal{F}$ with $g \geq f^{1/2}$. Suppose $N \in \mathbb{N}$ and $\kappa > 1$. Suppose $M \geq 2^{36N^2\varrho^{-2}}$ and that $x \in \Lambda(N,\kappa)$, $x^* \in \mathcal{H}_X(g,M)$. Then $\langle x,x^*\rangle \leq (\kappa+\varrho)f(N)/N \leq (\kappa+1)f(N)/N$.*

Remark. For our statement of Lemma 4.2, observe that since X has a lower f-estimate, for any $\{x_i\}_{i=1}^N \in \mathrm{RIS}_X(N,\kappa)$ we have $\|\sum_{i=1}^N x_i\|_X \geq Nf(N)^{-1}$.

Our next lemma is a slight modification of Lemma 7 of [9].

Lemma 4.3. *Suppose X satisfies a lower f-estimate on blocks and $g \in \mathcal{F}$ with $g \geq f^{1/2}$. Suppose $\kappa \geq 2$ and $(x_1,\ldots,x_N) \in \mathrm{RIS}_X(N,\kappa)$. Let $x = \sum_{i=1}^N x_i$ and suppose that for every interval E with $\|Ex\|_X \geq 1$ we have*

$$(*) \qquad \|Ex\|_X \leq \sup\{\langle Ex,x^*\rangle : x^* \in \mathcal{H}_X(g;M),\ M \geq 2\}.$$

Then $\|x\|_X \leq (\kappa+1)N/g(N)$.

Proof. We introduce the length of an interval E as in [9]. Let $x_i \in \lambda_X(n_i,\kappa)$ for $1 \leq i \leq N$. Suppose x_i is written as $(1/n_i)\sum_{j=1}^{n_i} x_{ij}$ where $x_{i1} < \ldots < x_{in_i}$ and $\|x_{ij}\|_X \leq \kappa n_i^{-1}$. If E is any interval which intersects the support of $\sum_{i=1}^N x_i$ we let $k \leq l$ be the least and greatest indices i such that $Ex_i \neq 0$. Then we let p be the least index such that $Ex_{kp} \neq 0$ and q the greatest index such that $Ex_{lq} \neq 0$. Define $\ell(E) = l-k+qn_l^{-1}-pn_k^{-1}$. If E does not meet the support of $\sum_{i=1}^N x_i$ then $\ell(E) = 0$.

Now our hypotheses differ from Lemma 7 of [9] in that we assume $(*)$ whenever $\|Ex\|_X \geq 1$, while [9] assumes $(*)$ whenever $\ell(E) \geq 1$; we, however, assume $\kappa \geq 2$. Our hypotheses imply that $(*)$ holds if $\ell(E) \geq 2$ since then there exists at least one x_i with support contained entirely in E. As in [9] let $G(t)=t/g(t)$ for $t\geq 1$ and $G(t)=t$ for $t\leq 1$. Then if $\kappa n_1^{-1}\leq \ell(E)\leq 1$ we have $\|Ex\|_X\leq(\kappa+1)G(\ell(E))$ as in [9]. We claim the same inequality if $1\leq \ell(E)\leq 2$; in fact, in this situation we can see that E intersects the supports of at most three x_i and so $\|Ex\|_X\leq 3\leq(\kappa+1)G(\ell(E))$. The proof can now be completed by applying Lemma 7 of [9].

We will now define a Gowers–Maurey space Z, very similar to the construction in [9]; in fact, essentially the same space is considered by Gowers in [7] as a counterexample to the hyperplane problem, and also as a space in which all operators are strictly singular perturbations of a diagonal map. We will suppose that $\mathcal{P}=\{p_k\}_{k=1}^\infty$ is an increasing sequence of natural numbers satisfying $f(p_1)>256$, $\log\log\log p_k>4p_{k-1}^2$, $p_k>k^6 2^{100k^2}$, for all k. We shall also require that $f(p_{2k})p_{2k}^{-1}\leq k^{-3}/2$, which doubtless follows from our other hypotheses. For convenience we suppose each p_k is a square. We partition $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$ where $\mathcal{P}_1=\{p_{2k-1}\}_{k=1}^\infty$ and $\mathcal{P}_2=\{p_{2k}\}_{k=1}^\infty$.

Let $\mathbb{Q}_+$ denote the countable collection of $u\in c_{00}^+$ which have only rational coefficients and let σ be an injection from the collection of all finite subsets of $\mathbb{Q}_+$, $\{z_1,\ldots,z_s\}$ where $z_1<\ldots<z_s$, to $\mathcal{P}_2$ which satisfy the condition $\sigma(z_1,\ldots,z_s)\geq 2^{10b(z_s)^2}$.

We then define Z implicitly by the formula

$$\|x\|_Z=\max(\|x\|_\infty,\|x\|_\alpha,\|x\|_\beta)$$

where

$$\|x\|_\alpha=\sup\{\langle|x|,x^*\rangle : x^*\in\mathcal{H}_Z(f;M),\ M\geq 2\},$$

$$\|x\|_\beta=\sup\Big\{f(k)^{-1/2}\sum_{i=1}^k\langle|x|,x_i^*\rangle\Big\},$$

with the supremum being over all $k\in\mathcal{P}_1$ and *special sequences* $(x_1^*,\ldots,x_k^*)$, i.e. such that $x_1^*<\ldots<x_k^*$, with $x_1^*\in\mathbb{Q}_+\cap\mathcal{H}_Z(f;p_{2k})$ and then for $j\geq 1$, $x_{j+1}^*\in\mathbb{Q}_+\cap\mathcal{H}_Z(f;\sigma(x_1^*,\ldots,x_j^*))$.

This implicit definition can be justified by an inductive construction as in [9]. Precisely we set $\|x\|_{Z_0}=\|x\|_\infty$ for $x\in c_{00}$ and then define for $N\geq 1$,

$$\|x\|_{Z_N}=\max(\|x\|_{Z_{N-1}},\|x\|_{\alpha_{N-1}},\|x\|_{\beta_{N-1}})$$

where

$$\|x\|_{\alpha_N}=\sup\{\langle|x|,x^*\rangle : x^*\in\mathcal{H}_{Z_N}(f;M),\ M\geq 2\},$$

$$\|x\|_{\beta_N}=\sup\Big\{f(k)^{-1/2}\sum_{i=1}^k\langle|x|,x_i^*\rangle\Big\}$$

with the supremum being over all $k\in\mathcal{P}_1$ and $(x_1^*,\ldots,x_k^*)$, i.e. such that $x_1^*<\ldots<x_k^*$, with $x_1^*\in\mathbb{Q}_+\cap\mathcal{H}_{Z_N}(f;p_{2k})$ and then for $j\geq 1$, $x_{j+1}^*\in$

$\mathbb{Q}_+ \cap \mathcal{H}_{Z_N}(f;\sigma(x_1^*,\ldots,x_j^*))$. It is then easily verified that $\|\ \|_{Z_N}$ is an increasing sequence of norms, bounded above by the ℓ_1-norm, and that the sets $H_{Z_N}(f;M)$ also increase in N. We set $\|x\|_Z = \lim_{N\to\infty}\|x\|_{Z_N}$.

We emphasize that this space is an unconditional version of the counterexample constructed in [9], but shares some of the same features. We will need versions for Z of certain lemmas proved in [9] for the Gowers–Maurey space. Fortunately the same basic techniques go through more or less unchanged.

Let us note first that Z satisfies a lower f-estimate. This follows immediately from the definition of $\|x\|_\alpha$. We also note that, by induction, it follows that $\|e_n\|_Z = 1$ for all n.

LEMMA 4.4. *Suppose* $(x_j)_{j=1}^n \in \mathrm{RIS}_Z(n;\kappa)$ *where* $\kappa\ge 1$. *Then* $\|\sum_{j=1}^n x_j\|_\infty < 1$.

Proof. We have $x_j \in \lambda_Z(M_j,\kappa)$ where $M_j \ge 4\kappa$ by the definition of $\mathrm{RIS}_Z(n;\kappa)$. Hence $\|x_j\|_\infty \le M_j^{-1}\kappa < 1$ and the lemma follows.

It now follows as in Lemma 10 of [9]:

LEMMA 4.5. *Suppose* $\kappa \ge 2$. *Suppose* $N \in \mathcal{P}_2$ *and* $\log N \le n \le \exp N$. *Then if* $\{x_1,\ldots,x_n\} \in \mathrm{RIS}_Z(n,\kappa)$ *we have* $\|\sum_{i=1}^n x_i\|_Z \le (\kappa+1)nf(n)^{-1}$.

Proof. The key point, proved in [9], Lemma 9, is that there exists $g\in\mathcal{F}$ with $f^{1/2}\le g\le f$ such that $g(x)=f(x)$ for $\log N\le x\le \exp N$ and $g(k)=f^{1/2}(k)$ when $k\in\mathcal{P}_1$. Thus if $x\in c_{00}$ and $\|x\|_Z > \|x\|_\infty$ then

$$\|x\|_Z = \sup\{\langle Ex, x^*\rangle : \ x^*\in\mathcal{H}_Z(g;M),\ M\ge 2\}.$$

Now, by the preceding lemma if $x=\sum_{j=1}^n x_j$ and E is any interval then $\|Ex\|_\infty < 1$. We can therefore apply Lemma 4.3 to obtain the result.

The next lemma is simply a cruder form of Lemma 11 from [9].

LEMMA 4.6. *Suppose* $\kappa \ge 2$ *and* $N\in\mathcal{P}_2$. *If* $x\in \Lambda_Z(N,\kappa)$ *then* $x\in \lambda(\sqrt{N}, 2(\kappa+1))$.

Proof. Suppose $\{x_i\}_{i=1}^N \in \mathrm{RIS}_Z(N,\kappa)$ and that $x = \|\sum_{i=1}^N x_i\|_Z^{-1}\times \sum_{i=1}^N x_i$. We break $[1,N]$ into $\sqrt{N}$ intervals E_j each of length $\sqrt{N}$, which is an integer by hypothesis. Note that $\{x_i\}_{i\in E_j}\in \mathrm{RIS}_Z(\sqrt{N},\kappa)$. If $y_j = \sum_{i\in E_j} x_i$ then, by Lemma 4.5, $\|y_j\|_Z \le (\kappa+1)\sqrt{N}$. Also $\|\sum_{j=1}^N x_j\|_Z \ge N/f(N)$, by the lower f-estimate on X. Now $x = (1/\sqrt{N})\sum_{j=1}^{\sqrt{N}} z_j$ where $z_j = (\|\sum_{i=1}^N x_i\|_Z)^{-1}\sqrt{N}y_j$. But $\|z_j\|_Z \le (\kappa+1)(Nf(N))/(Nf(\sqrt{N})) \le 2(\kappa+1)$.

Our next result is a modification of Lemma 12 of [9]. In fact, this lemma appears to be incorrectly stated in [9] and so some modification is necessary. In the proof of the lemma in [9] it is claimed without justification that

$\{x_1,\ldots,x_k\}$ is a RIS of length k and constant $1+\varepsilon$. For the applications some modification similar to that given below seems adequate, however.

LEMMA 4.7. *Let $\kappa \ge 2$. Suppose $k \in \mathcal{P}_1$ with $f(k) > 100\kappa^2$. Suppose $E_1,\ldots,E_k$ are intervals with $E_1 < \ldots < E_k$. Let $\{x_1^*,\ldots,x_k^*\}$ be a special sequence with $\operatorname{supp} x_j^* \subset E_j$. Let $M_1 = p_{2k}$ and $M_{j+1} = \sigma(x_1^*,\ldots,x_j^*)$ for $1 \le j \le k-1$. Let A be any subset of $\{1,2,\ldots,k\}$ and suppose for each $j \in A$ we have $x_j \in c_{00}^+$ with $\operatorname{supp} x_j \subset E_j$ so that x_j, x_j^* are disjoint and $x_j \in \Lambda(M_j,\kappa)$. Then*

$$\Big\|\sum_{i\in A} x_i\Big\|_Z \le 16\kappa k f(k)^{-1}.$$

Proof. We have $x_j \in \lambda_Z(\sqrt{M_j},4\kappa)$, by Lemma 4.6. Note that $\sqrt{M_1} = \sqrt{p_{2k}} \ge 4\kappa 2^{36k^2}$. We also have $\sqrt{M_{j+1}} > 2^{4b(x_j^*)^2}$.

Now assume A contains no two consecutive integers. Then if $j \in A$ we have $\sqrt{M_j} \ge 2^{4b(x_{j-2})^2}$ for $j \ge 2$ and so $\{x_j\}_{j\in A} \in \mathrm{RIS}_Z(|A|,4\kappa)$. As in [9] we use Lemma 4.3.

Note first that there exists $h \in \mathcal{F}$ with $\sqrt{f} \le h \le f$, so that $h(n) = \sqrt{f(n)}$ if $n \in \mathcal{P}_1 \setminus \{k\}$ while $h(n) = f(n)$ if $n \in \mathcal{P}_2 \cup \{k\}$. This fact follows from Lemma 9 of [9].

Let $x = \sum_{i\in A} x_i$ and suppose that for some interval E we have $\|Ex\|_Z \ge 1$, and

$$\|Ex\|_Z > \sup\{\langle Ex, x^*\rangle : x^* \in \mathcal{H}_Z(h,m),\ m \ge 2\}.$$

Since $h \le f$ this implies that $\|Ex\|_Z > \|Ex\|_\alpha$. On the other hand, since $\{x_j\}_{j\in A} \in \mathrm{RIS}_Z(|A|,4\kappa)$ we can apply Lemma 4.4 to deduce that $\|Ex\|_Z > \|Ex\|_\infty$. The conclusion is that $\|Ex\|_Z = \|Ex\|_\beta$. Thus there is a special sequence $\{z_1^*,\ldots,z_l^*\}$, with $l \in \mathcal{P}_1$, so that

$$\|Ex\|_Z = f(l)^{-1/2}\Big\langle Ex, \sum_{i=1}^{l} z_i^*\Big\rangle.$$

However, $f(l)^{1/2} = h(l)$ unless $l = k$. We conclude $l = k$ and

$$1 \le \|Ex\|_Z \le f(k)^{-1/2} \sum_{i\in A}\sum_{j=1}^{k} \langle x_i, z_j^*\rangle.$$

Let t be the greatest integer so that $z_t^* = x_t^*$ (with $t = 0$ if no such integer exists). If $i < t$ it is clear that $\langle x_i, z_j^*\rangle = 0$ for all j. Similarly if $j \le t$ it is also clear that $\langle x_i, z_j^*\rangle = 0$ for all i. If $i = t$, then $\langle x_i, z_j^*\rangle = 0$ unless $j = t+1$ when of course $\langle x_t, z_{t+1}^*\rangle \le 1$. If $t+1 \le i \in A$ and $t+1 \le j \le k$ then, unless $t+1 = i = j$, we have $x_i \in \Lambda_Z(M_i,\kappa)$ and $z_j^* \in \mathcal{H}_Z(g; M_j')$ where $M_i, M_j' \in \mathcal{P}_2$ are not equal. It follows from the separation conditions on $\mathcal{P}_2$ that we can apply either Lemma 4.1 or Lemma 4.2; if $M_j' < M_i$, then by Lemma 4.1,

$$\langle x_i, z_j^* \rangle \le 24\kappa f(M_j')^{-1} \le 24\kappa f(p_{2k})^{-1},$$

or if $M_j' > M_i$, then $M_j' \ge 2^{36M_i^2}$ and by Lemma 4.2,

$$\langle x_i, z_j^* \rangle \le 2\kappa f(M_i)/M_i \le 2\kappa f(p_{2k})p_{2k}^{-1}.$$

In either case we have $\langle x_i, z_j^* \rangle \le \kappa k^{-2}$. If $i = j = t+1$ then $\langle x_i, z_j^* \rangle \le 1$.

Hence

$$\Big\langle \sum_{i\in A} x_i, \sum_{j=1}^{k} z_j^* \Big\rangle \le 2 + \kappa \le 3\kappa.$$

This implies that

$$\|Ex\|_Z \le 3\kappa f(k)^{-1/2} < 3/10$$

contrary to assumption. The conclusion from Lemma 4.3 is then that

$$\|x\|_Z \le 8\kappa |A| h(|A|)^{-1} \le 8\kappa k f(k)^{-1}.$$

The general result follows by splitting A into two subsets obeying the condition that no two consecutive integers are contained in either.

5. The main result. We now let $X = Z^*$ and consider the indicator Φ_X. We will need the elementary fact, which follows from duality, that X satisfies an upper f-estimate, i.e. if $x_1 < \ldots < x_n \in c_{00}$ then $\|x_1 + \ldots + x_n\|_X \le f(n) \max_{1\le i\le n} \|x_i\|_X$. It also follows from the definition of Z that if $x_1, \ldots, x_n$ is a special sequence (with $n \in \mathcal{P}_1$) then $\|x_1 + \ldots + x_n\|_X \le f(n)^{1/2}$.

Our main result, which combined with the results of Section 2 establishes Theorems 1.1, 1.3 and 1.4, is the following:

THEOREM 5.1. *For every infinite-dimensional subspace G of c_{00} we have* $\sup\{|\Phi_X(u)| : \|u\|_1 = 1,\ u \in G\} = \infty$.

Remark. The following proof has been substantially simplified according to a suggestion of B. Maurey.

Proof of Theorem 5.1. We will start from the assumption that there is a subspace G of infinite dimension so that $|\Phi_X(u)| \le K\|u\|_1$ for $u \in G$. We may suppose that if $u \in G$ then $\langle u, \chi \rangle = 0$ where χ is the constantly one sequence. Then by induction we can pick $\xi_1 < \xi_2 < \xi_3 < \ldots$ in G with $\|\xi_j\|_1 = 2$. We split ξ_i into positive and negative parts: $\xi_i = \xi_i' - \xi_i''$, where ξ_i', ξ_i'' are disjoint and non-negative. Then $\xi_i', \xi_i'' \in D$. We let R be the union of the supports of the ξ_i' and S be the union of the supports of the ξ_i''. Let W be the linear span of $\{|\xi_i|\}_{i=1}^{\infty}$.

Notice first that X satisfies an upper f-estimate on blocks where $f(x) = \log_2(x+1)$. If $\gamma > 0$ and $n \in \mathbb{N}$ we define $\Gamma(n,\gamma)$ to be the set of $w \in D$ such that there exist $w_1 < \ldots < w_n \in D$ with $w = (1/n)(w_1 + \ldots + w_n)$ and $(1/n)\Delta(w_1,\ldots,w_n) < \gamma$.

LEMMA 5.2. *Given any $m, n \in \mathbb{N}$ and $\delta > 0$ there exists $w \in W \cap \Gamma(n,\delta)$ with $m < a(w)$.*

Proof. For $n \in \mathbb{N}$ let c_n be the infimum of all constants γ so that if $m \in \mathbb{N}$ there exists $w \in W \cap \Gamma(n,\gamma)$ with $m < a(w)$. It is easy to see that $c_{np} \geq c_n + c_p$ for any n, p and that from Lemma 3.1, $c_n \leq \log f(n)$. Hence $pc_n \leq c_{n^p} \leq \log f(n^p)$ and so letting $p \to \infty$ we obtain $c_n = 0$ for all n and the lemma follows.

We now turn to estimates on the Lozanovskiĭ factorization of $w \in \Gamma(n,\delta)$.

LEMMA 5.3. *For fixed n and $0 < \varepsilon < 1/2$ there exists $\eta > 0$ so that if $w \in \Gamma(n,\eta)$ and $w = xx^*$ is the Lozanovskiĭ factorization of w, then there exists $A \subset [a(w), b(w)]$ with $\|Aw\|_1 > 1 - \varepsilon$ and such that $Ax^*/\|Ax^*\|_Z \in \lambda_Z(n,2)$.*

Proof. If $w \in \Gamma(n,\delta)$ then $w = (1/n)\sum_{i=1}^{n} w_i$ where $w_1 < \ldots < w_n \in D$ are such that $(1/n)\Delta(w_1,\ldots,w_n) \leq \delta$. Let $w_i = x_i x_i^*$ be the Lozanovskiĭ factorizations of each. Let $y = x_1 + \ldots + x_n$. If $c > 1$ let $A = \{j : y(j) \leq cx(j),\ x(j) > 0\}$. Then $Ax^* \leq cn^{-1}A(x_1^* + \ldots + x_n^*)$ and hence if $A_i = A \cap [a(w_i), b(w_i)]$ then $\|A_i x^*\|_Z \leq c/n$. Now $\|Ax^*\|_Z \geq \|Aw\|_1$ and so $\|Ax^*\|_Z^{-1} Ax^* \in \lambda_Z(n,c')$ where $c' \leq c\|Aw\|_1^{-1}$. Now, according to Lemma 3.5, if $\delta > 0$ is sufficiently small we can choose c close enough to 1 so that the conclusions follow.

Using the preceding lemma we describe a construction. Suppose $N \in \mathcal{P}_2$ and $\varepsilon > 0$. Then given any $m \in \mathbb{N}$ and any $M_1 > 2^{36N^2+4}$ we can construct two sequences $\{w_j\}_{j=1}^N$ and $\{\zeta_j\}_{j=1}^N$ and a sequence of integers $(M_j)_{j=1}^N$ so that

(1) $m < a(w_1)$,

(2) $w_1 < \zeta_1 < w_2 < \zeta_2 < \ldots < w_N < \zeta_N$,

(3) $w_j \in \Gamma(M_j,\eta_j) \cap W$ where $0 < \eta_j < \varepsilon$ is sufficiently small so that there exists $A_j \subset [a(w_j), b(w_j)]$ with $\|A_j w_j\|_1 > 1 - \varepsilon$ and $z_j = \|A_j x_j^*\|_Z^{-1} A_j x_j^* \in \lambda_Z(M_j,2)$ where $w_j = x_j x_j^*$ is the Lozanovskiĭ factorization of w_j,

(4) $\zeta_j \in \lambda_Z(M_j,2)$,

(5) $M_{j+1} > 2^{4b(\zeta_j)^2}$.

We will call the resulting sequence $\{w_j\}_{j=1}^N$ an *(N,ε)-sequence*, and $w = (1/N)(w_1 + \ldots + w_N)$ the associated *(N,ε)-average*. The sequence $\{\zeta_j\}_{j=1}^N$ is called the *ballast sequence*; it is present simply for technical reasons to provide ballast in the argument. Let H be the union of the supports of the ballast sequence.

LEMMA 5.4. *Suppose* $\{w_1,\ldots,w_N\}$ *is an* (N,ε)*-sequence as above with associated* (N,ε)*-average* w *and ballast* $\{\zeta_j\}_{j=1}^N$. *Then there is a subset* A *of* $[a(w),b(w)]$ *and* $x\in\mathcal{H}_Z(f;N)\cap\mathbb{Q}_+$ *with* $\operatorname{supp} x\subset\operatorname{supp} w$ *so that*

(6) $\|Aw\|_1>1-\varepsilon$,

(7) *if* $B\subset A$ *there exists* $z\in\Lambda_Z(N,4)$ *supported in* $B\cup H$ *so that* $Bw\le 10xz$,

(8) *if* $B\subset A$ *then* $\langle Bw,\log x\rangle>\Phi_X(Bw)-4$.

Proof. Notice that $y=(1/f(N))(x_1+\ldots+x_N)\in\mathcal{H}_Z(f;N)$ and $\|y\|_X\le 1$, since X has an upper f-estimate. Choose x with rational coefficients so that $y/2\le x\le y$. Let $A=A_1\cup\ldots\cup A_N$ so that (6) immediately holds.

We recall that $z_j\in\lambda_Z(M_j,2)$ (condition (3)) for $1\le j\le N$. It follows easily that if B is a subset of A then we can find $0\le\alpha_j\le 1$ so that $\|Bx_j^*+\alpha_j\zeta_j\|_Z=1$ and then $Bx_j^*+\alpha_j\zeta_j\in\lambda_Z(M_j,4)$. The sequence $\{Bx_j^*+\alpha_j\zeta_j\}_{j=1}^N$ thus belongs to $\mathrm{RIS}_Z(N,4)$ (since $M_1>2^{36N^2+4}$) and so $\|\sum_{j=1}^N(Bx_j^*+\alpha_j\zeta_j)\|_Z\le 5N/f(N)$, from Lemma 4.5.

Let z be the normalized vector $\beta(\sum_{j=1}^N Bx_j^*+\alpha_j\zeta_j)$ where, by the above, $\beta\ge f(N)/(5N)$. Then $z\in\Lambda_Z(N,4)$ and $xz\ge yz/2\ge Bw/10$. This proves (7).

For (8) we notice that Lemma 3.4 now implies that $\Phi_X(Bw)-\langle Bw,\log x\rangle\le 10/e<4$.

Let us suppose that $n\in\mathcal{P}_1$ is fixed and large, say $f(n)>\exp(8K+4000)$, and let $\varepsilon=(\log f(n))^{-1}$. Let $M_1=p_{2n}$; we can construct an (M_1,ε)-sequence $\{w_{1j}\}_{j=1}^{M_1}$ with (M_1,ε)-average $w_1=M_1^{-1}\sum_{j=1}^{M_1}w_{1j}$ and ballast $\{\zeta_{1j}\}_{j=1}^{M_1}$. Let $x_1\in\mathbb{Q}_+\cap\mathcal{H}_Z(f;M_1)$ and $A_1\subset[a(w_1),b(w_1)]$ be such that the conclusions of Lemma 5.4 hold.

Next let $M_2=\sigma(Rx_1)$ and construct an (M_2,ε)-sequence $\{w_{2j}\}_{j=1}^{M_2}$ with associated (M_2,ε)-average w_2 and ballast $\{\zeta_{2j}\}_{j=1}^{M_2}$ so that $\zeta_{1M_1}<u_2$. Repeating this construction for n steps we obtain sequences $(w_{ij})_{j=1}^{M_i}$, $(\zeta_{ij})_{j=1}^{M_i}$ for $i=1,\ldots,n$, $(w_i)_{i=1}^n$, $(M_i)_{i=1}^n$, $(A_i)_{i=1}^n$ and $(x_i)_{i=1}^n$ so that

(9) $(w_{ij})_{j=1}^{M_i}$ is an (M_i,ε)-sequence with associated (M_i,ε)-average w_i and ballast $\{\zeta_{ij}\}_{j=1}^{M_i}$ for $1\le i\le n$,

(10) $w_1<\zeta_{1M_1}<w_2<\zeta_{2M_2}<\ldots<w_n<\zeta_{nM_n}$,

(11) $A_i\subset[a(w_i),b(w_i)]$ for $1\le i\le n$ and $\|A_iw_i\|_1>1-\varepsilon_i$,

(12) $\operatorname{supp} x_i\subset\operatorname{supp} w_i$, $x_i\in\mathcal{H}_Z(f;M_i)\cap\mathbb{Q}_+$, and so $\|x_i\|_X\le 1$,

(13) $\langle Bw_i,\log x_i\rangle>\Phi_X(Bw_i)-4$ whenever $B\subset A_i$,

(14) for any $B\subset A_i$ there exists $z\in\Lambda_Z(M_i,4)$ with $Bw_i\le 10x_iz$,

(15) $\quad M_{i+1} = \sigma(Rx_1, \ldots, Rx_i)$ for $1 \le i \le n-1$.

We also have

(16) $\quad (Rx_1, \ldots, Rx_n)$ is a special sequence of length n in $X = Z^*$.

Let H_i be the union of the supports of the ballast at the ith step. Let $A = \bigcup_{i=1}^n A_i$ and then set $P = A \cap R$ and $Q = A \cap S$. We also define $u_i = 2Rw_i, v_i = 2Sw_i$ (so that $u_i, v_i \in D$) and then set $u = (1/n)(u_1 + \ldots + u_n)$, $v = (1/n)(v_1 + \ldots + v_n)$ and $w = (1/n)(w_1 + \ldots + w_n)$.

If we set $x = (f(n))^{-1/2} \sum_{i=1}^n Rx_i$ then $\|x\|_X \le 1$, since by (16), $\{Rx_1, \ldots, Rx_n\}$ is a special sequence. Hence, using (13) above,

$$\Phi_X(Pw) \ge \langle Pw, \log x \rangle \ge \frac{1}{n} \sum_{i=1}^n \Phi_X(Pw_i) - \frac{1}{2} \log f(n) \|Pw\|_1 - 4.$$

Thus $(1/n)\Delta(Pu_1, \ldots, Pu_n) \le (1/2)\log f(n) + 8$. Now $(1/n)\sum_{i=1}^n \|Pu_i\|_1 > 1 - 2\varepsilon$ so that by Lemma 3.3, and the choice of ε,

$$\frac{1}{n}\Delta(u_1, \ldots, u_n) \le \frac{1}{2}\log f(n) + 11. \tag{17}$$

On the other hand, by Lemma 5.4 we can find $z_i \in A_Z(M_i, 4)$ supported on $((\operatorname{supp} w_i) \cap Q) \cup H_i$ so that $Qw_i \le 10x_i z_i$. At this point we can invoke Lemma 4.7. Let $E_i = [a(w_i), b(\zeta_{i,M_i})]$ and notice that Rx_i, z_i are both supported in E_i, but are disjoint. Since $f(n) \ge 1600$, $Rx_1, \ldots, Rx_n$ is a special sequence and $z_i \in A_Z(M_j, 4)$ where $M_1 = p_{2n}$ and $M_{j+1} = \sigma(x_1, \ldots, x_j)$ for $1 \le j \le n-1$ we can conclude that $\|\sum_{j=1}^N z_j\|_Z \le 64nf(n)^{-1}$. At the same time, by the upper f-estimate on X, $\|\sum_{i=1}^n x_i\|_X \le f(n)$. Now we have

$$\frac{1}{640} Qw \le \left(\frac{1}{f(n)} \sum_{i=1}^n x_i\right)\left(\frac{f(n)}{64n} \sum_{i=1}^n z_i\right)$$

and we can apply Lemma 3.4 again to deduce that

$$\Phi_X(Qw) \le \sum_{i=1}^n \langle Qw, \log x_i \rangle - \log f(n) \|Qw\|_1 + 640,$$

and so, since $\|Qw\|_1 > 1/2 - \varepsilon$,

$$\frac{1}{n}\Delta(Qw_1, \ldots, Qw_n) \ge \log f(n) \|Qw\|_1 - 640 \ge \frac{1}{2}\log f(n) - 641.$$

Now recall that $Qv_i = 2Qw_i$. Hence $(1/n)\Delta(Qv_1, \ldots, Qv_n) \ge \log f(n) - 1282$ and we can apply Lemma 3.3 to deduce that

$$\frac{1}{n}\Delta(v_1, \ldots, v_n) \ge \log f(n) - 1283. \tag{18}$$

Notice that $u_i - v_i \in G$ for $1 \le i \le n$. Now we have $|\Phi_X(u_i - v_i)| \le 2K$, for $1 \le i \le n$, and $|\Phi_X(u - v)| \le 2K$. Hence $|\Phi_X(u_i) - \Phi_X(v_i)| \le 2K + 8e^{-1} \le$

$2K+3$ for $1 \le i \le n$ and similarly $|\Phi_X(u) - \Phi_X(v)| \le 2K+3$. This implies that

$$\frac{1}{n}\Delta(v_1,\ldots,v_n) - \frac{1}{n}\Delta(u_1,\ldots,u_n) \le 4K+6.$$

Combining with (17) and (18) gives that $\log f(n) \le 8K + 2600$, which contradicts our initial choice of n and completes the proof of Theorem 5.1.

It is perhaps worth noting at this point that it is very simple to modify our example so that Theorem 1.1 holds with L of any specified dimension.

THEOREM 5.5. *For any $n \in \mathbb{N}$ there is a quasi-Banach space $Y^{(n)}$ with a subspace L of dimension n so that Y/L is isomorphic to ℓ_1 and if Y_0 is a closed infinite-dimensional subspace of $Y^{(n)}$ then $L \subset Y$.*

Proof. Let $A_k = \{nj+k\}_{j=0}^{\infty} \subset \mathbb{N}$, for $k = 1,\ldots,n$. Define $S_k : c_{00} \to c_{00}$ by $S_k u = \sum_{j=0}^{\infty} u(j) e_{nj+k}$. Define $\Phi : c_{00} \to \ell_\infty^n$ by $\Phi(u) = \{\Phi_X(S_k u)\}_{k=1}^n$. Then let $Y^{(n)}$ be the completion of $\ell_\infty^n \oplus c_{00}$ under the quasinorm

$$\|(\xi,u)\|_\Phi = \|\xi - \Phi(u)\|_\infty + \|u\|_1.$$

Let L be the space of all $(\xi,0)$ for $\xi \in \ell_\infty^n$. Clearly $Y^{(n)}/L$ is isomorphic to ℓ_1. Now suppose Y_0 is an infinite-dimensional subspace so that $Y_0 \cap L$ is a proper subspace of L. Then there is a non-trivial linear functional f on ℓ_∞^n so that $Y_0 \cap L \subset Z = f^{-1}(0)$. Suppose $f(\xi) = \sum_{k=1}^n \beta_k \xi_k$. It is easy to verify that Y/Z is isomorphic to the completion of $\mathbb{R} \oplus c_{00}$ under the quasinorm $\|(\alpha,u)\|_\Psi = |\alpha - \Psi(u)| + \|u\|_1$ where $\Psi(u) = \sum_{k=1}^n \beta_k \Phi_X(S_k u)$. However, there is a constant K depending only on $\beta_1,\ldots,\beta_n$ so that $|\Psi(u) - \Phi_X(\sum_{k=1}^n \beta_k S_k u)| \le K\|u\|_1$. It follows easily that Ψ is unbounded on every infinite-dimensional subspace of c_{00} and hence that $(Y_0+Z)/Z$ must contain L/Z, which is a contradiction to the fact that $Y_0 \cap L$ is contained in Z.

6. Some final remarks. In this short final section we will present a proof of Theorem 1.2, which first appeared in [14], a reference which may not be readily available. Our proof here is slightly shorter. We begin with a lemma:

LEMMA 6.1. *Suppose X is a quasi-Banach space with a dense subspace V with* (HBEP). *Suppose $L = \{x \in X : x^*(x) = 0\ \forall x^* \in X^*\}$. Then:*

(1) *If $L = \{0\}$, so that X has a separating dual, then X is locally convex.*

(2) *If X contains a basic sequence then X is locally convex.*

(3) *If M is a closed subspace of L then X/M has a dense subspace with* (HBEP).

Proof. (1) (cf. [11]) Let $\|\ \|_c$ be the Banach envelope norm on X, i.e. $\|x\|_c = \sup\{|x^*(x)| : \|x^*\| \le 1\}$. If X is not locally convex we may choose

$v_n \in V$ with $\|v_n\|_c \le 4^{-n}$ and $\|v_n\| = 1$. Pick any $x \in V$ and consider the sequence $w_n = v_n + 2^{-n}x$. Then (see Theorem 4.7 of [16]) there is a subsequence (w_{p_n}) which is a Markushevich basis for its closed linear span in X. Pick n_0 large enough so that $x \notin [w_{p_k} : k \ge n_0]$. Then by (HBEP) for V there is a linear functional $x^* \in X^*$ with $x^*(w_{p_k}) = 0$ for $k \ge n_0$ but $x^*(x) = 1$. However, $\lim_{n\to\infty} \|x - 2^n w_n\|_c = 0$ so that $x^*(x) = 0$, contrary to hypothesis.

(2) Pick any $u \in L$; we will show $u = 0$. Assume then that $u \ne 0$. Suppose $w \in V$ is non-zero, and u, w are linearly independent. Since X contains a basic sequence and V is dense in X we can apply standard perturbation arguments to suppose that we have a bounded basic sequence (x_n) with $x_n \in n(u + w) + V$, say $x_n = n(u + w) - v_n$ where $v_n \in V$. Then there exists n_0 so that $[u, w] \cap [x_n]_{n \ge n_0} = \{0\}$. Thus there is a bounded linear functional f on the span Y of u, w and $[x_n]_{n\ge n_0}$ with $f(u) = 1$, $f(w) = 0$ and $f(x_n) = 0$ for $n \ge n_0$. Since V has (HBEP) there is a bounded linear functional x^* on X with $x^*(v) = f(v)$ for $v \in V \cap Y$. Thus $x^*(w) = 0$ and $x^*(v_n) = -n$; also $x^*(u) = 0$ since $u \in L$. Hence $x^*(x_n) = -n$, contradicting the boundedness of x^*. Now since $L = \{0\}$ we can apply (1) to deduce that X is locally convex.

(3) Let $\pi : X \to X/M$ be the quotient map; we show $\pi(V)$ has (HBEP). Indeed, if $E \subset \pi(V)$ is a subspace and f is a continuous linear functional on E then we can find $x^* \in X^*$ so that $x^*(v) = f(\pi v)$ for $v \in \pi^{-1}E \cap V$. But then $x^*(x) = 0$ if $x \in M \subset L$ so that x^* factors to a linear functional on X/M.

THEOREM 6.2. *Suppose X is a decomposable quasi-Banach space. If X has a dense subspace V with* (HBEP) *then X is locally convex.*

Proof. Let P be a bounded projection on X so that both P and $Q = I - P$ have infinite rank. If L is defined as in the previous lemma then L is clearly invariant for P. From the hypotheses, X^* has infinite dimension and hence so has X/L. Therefore either $P(X)/P(L)$ or $Q(X)/Q(L)$ has infinite dimension. Suppose the former; then consider $X/P(L)$, which has a dense subspace with (HBEP) by Lemma 6.1(3). Then $P(X)/P(L)$ is isomorphic to a subspace of X/L which has separating dual; since it has infinite dimension, it contains a basic sequence. By Lemma 6.1(2) this implies that $X/P(L)$ is locally convex and hence that $Q(X)$ is locally convex. But now X itself must contain a basic sequence and Lemma 6.1(2) shows that X is locally convex.

Let us conclude by mentioning that in [14] we raised the question of whether every quasi-Banach space X with separating dual has a weakly closed subspace W and a bounded linear functional f on W which cannot be extended to X. We proved that this is equivalent to the following:

PROBLEM. *Suppose X is a quasi-Banach space with separating dual and suppose that every quotient X/E by an infinite-dimensional subspace E is locally convex. Is X locally convex?*

Of course our main example Y has every quotient Y/E by an infinite-dimensional subspace locally convex, but fails to have a separating dual.

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