The basic sequence problem

by

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Abstract. We construct a quasi-Banach space $X$ which contains no basic sequence.

1. Introduction. It is a classical result in Banach space theory, known to Banach himself [1], that every (infinite-dimensional) Banach space contains a closed linear subspace with a basis, or, in other words, a basic sequence. The corresponding question for quasi-Banach spaces (and more general E-spaces) has, however, remained open. A number of equivalent formulations are known ([11], [14], [16], [17]); the question is also raised in a slightly disguised form in [28], p. 114.

In [11] and [17] it is shown that a quasi-Banach space $X$ contains a basic sequence if and only if there is a strictly weaker Hausdorff vector topology on $X$. Thus the existence of a space with no basic sequence is equivalent to the existence of a (topologically) minimal space (i.e. one on which there is no strictly weaker Hausdorff vector topology). See [3] and [4] for a discussion of minimal spaces. It further follows that $X$ contains a basic sequence if and only if there is some infinite-dimensional closed subspace with separating dual ([11], Theorem 4.4). Several positive results are known. For example, the work of Bastero [2] implies that every subspace of $L_p[0,1]$ ($0 < p < 1$) contains a basic sequence, while the author's results in [12] imply that every quotient of $L_p[0,1]$ contains a basic sequence. Bastero's result can be lifted to the wider class of so-called natural spaces and has further been extended by Tan [30] who shows that every complex quasi-Banach space with an equivalent plurisubharmonic norm contains a basic sequence. These results suggest that almost all "reasonable" spaces contain a basic sequence.

In this paper, we will prove

Theorem 1.1. There is a quasi-Banach space $Y$ with a one-dimensional subspace $L$ so that

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(1) if $Y_0$ is a closed infinite-dimensional subspace of $Y$ then $L \subseteq Y_0$, and
(2) $Y/L$ is isomorphic to the Banach space $\ell_1$.
In particular, $Y$ contains no basic sequence and is minimal.

It is clear that (1) would make it impossible for $Y$ to contain a basic sequence.

There are other applications of this space. A topological vector space $X$ is said to have the Hahn-Banach Extension Property (HBE) if whenever $X_0$ is a closed subspace of $X$ and $f$ is a continuous linear functional on $X_0$ then $f$ can be extended to a continuous linear functional on $X$. The author showed in [11], answering a question raised by Duren, Romberg and Shields [3] (see also [25], [29]) that for an $F$-space (complete metric linear space) (HBE) is equivalent to local convexity. It was very well known that metrisability is necessary in this theorem, but some partial results of Ribe [25] suggested that completeness might not be required. Ribe showed that if $X$ is a metric linear space so that $X$ is isomorphic to $X \otimes X$ then if $X$ has (HBE) it must be locally convex. More recently, the author [14] extended Ribe’s result to show

**Theorem 1.2.** Let $X$ be a decomposable quasi-Banach space (i.e. there is a bounded projection $P$ on $X$ so that neither $P$ nor $I-P$ has finite rank). Suppose $X_0$ is a dense subspace of $X$. Then $X_0$ has (HBE) if and only if $X$ is locally convex.

A proof of Theorem 1.2 is included in Section 6. The Hahn-Banach extension property for metrisable spaces is also discussed in [10].

However, if $Y$ is the space constructed above, we will show that any algebraic complement $Y_0$ of $L$ has (HBE). Thus we have

**Theorem 1.3.** There is a non-locally convex metric linear space $Y_0$ with the Hahn-Banach Extension Property.

In 1962, Klee [18] asked whether for every topological vector space $(X, \tau)$, the topology $\tau$ can be expressed as the supremum of two not necessarily Hausdorff vector topologies $\tau_1$ and $\tau_2$ so that (the Hausdorff quotient of) $(X, \tau_1)$ has a separating dual (i.e. is nearly convex) and $(X, \tau_2)$ has trivial dual. Recently Peck [22] has shown this to be true for certain twisted sums of a Banach space and a one-dimensional space (see also [23]). The space constructed here, $Y$, turns out to be a counterexample to Klee’s problem.

**Theorem 1.4.** There is a quasi-Banach space $Y$ so that the topology on $Y$ is not the supremum of a trivial dual topology and a nearly convex topology.

The construction of our example depends heavily on the recent remarkable developments in infinite-dimensional Banach spaces due to Gowers, Maurey, Odell and Schlamnicht [7], [8], [9], [20], [21]. It is perhaps a little ironic that the basic sequence question for quasi-Banach spaces turns out to be so closely related to the unconditional basic sequence problem for Banach spaces. However, it should be stressed that we use an example of a Banach space with an unconditional basis, very similar to that used by Gowers in [7]; the fundamental estimates we need are in [9].

Let us conclude this introduction by explaining the shortcomings of the example. It is still an open question whether every quasi-Banach space (or $F$-space) must contain a proper closed infinite-dimensional subspace. A space with no proper closed infinite-dimensional subspace is called atomic. The existence of an atomic quasi-Banach space is known to be equivalent to the existence of a quotient minimal quasi-Banach space, i.e. a space $X$ so that every quotient is minimal (this concept is due to Drewnowski [3]). See [14] or [16] for a discussion. Our example is quite far from an atomic space, and it is not clear at present whether it can be used towards making such a monster. We remark that Rease [24] has constructed an example of an “almost” atomic $F$-space, i.e. a space $X$ with a sequence of finite-dimensional subspaces $V_n$ with $\dim V_n = n$ so that if $x_n \in V_n$ is any sequence which is non-zero infinitely often then $\{x_n\} = X$. It is still unknown whether even this phenomenon can be reproduced in a quasi-Banach space. We suspect, however, that an atomic quasi-Banach space will eventually be found.

We would like to thank several colleagues for helpful comments and remarks during the course of this work, in particular P. Casazza, D. Kutzarova, M. Lampers, M. Mastny and N. T. Peck. We also want to thank B. Maurey for a substantial simplification of the last part of the argument which we have incorporated into the proof. We also wish to thank the referee for many very helpful suggestions and comments on improving the presentation of the paper.

2. Idea of the construction. In this section, we introduce the basic idea and notation and prove that the space $Y$ which will be constructed in Sections 3–5 yields solutions to the problems mentioned in the introduction.

We denote by $c_0$ the space of all finitely non-zero (real sequences. If $x \in c_0$, we denote its co-ordinates by $(x(j))_{j \in \mathbb{N}}$. We let $a(x) = \min\{j : x(j) \neq 0\}$ and $b(x) = \max\{j : x(j) \neq 0\}$. If $A$ is a subset of $N$ then $\chi_A(x) = x(j)\chi_A(j)$ where $\chi_A$ is the characteristic function of $A$. If $E_1, E_2$ are subsets of $N$ we write $E_1 < E_2$ if $\max E_1 < \min E_2$. We shall also write for $x, y \in c_0$ that $x < y$ if $b(x) < a(y)$. On the other hand, the natural co-ordinatewise order
(1) if $Y_0$ is a closed infinite-dimensional subspace of $Y$ then $L \subseteq Y_0$, and

(2) $Y/L$ is isomorphic to the Banach space $\ell_1$.

In particular, $Y$ contains no basic sequence and is minimal.

It is clear that (1) would make it impossible for $Y$ to contain a basic sequence.

There are other applications of this space. A topological vector space $X$ is said to have the Hahn-Banach Extension Property (HBEPP) if whenever $X_0$ is a closed subspace of $X$ and $f$ is a continous linear functional on $X_0$ then $f$ can be extended to a continuous linear functional on $X$. The author showed in [11], answering a question raised by Duren, Romberg and Shields [3] (see also [25], [29]) that for an F-space (complete metric linear space) (HBEPP) is equivalent to local convexity. It was very well known that metrizability is necessary in this theorem, but some partial results of Ribe [25] suggested that completeness might not be required. Ribe showed that if $X$ is a metric linear space so that $X$ is isomorphic to $X \otimes X$ then if $X$ has (HBEPP) it must be locally convex. More recently, the author [14] extended Ribe's result to show

**Theorem 1.2.** Let $X$ be a decomposable quasi-Banach space (i.e. there is a bounded projection P on X so that neither P nor 1-P has finite rank). Suppose $X_0$ is a dense subspace of $X$. Then $X_0$ has (HBEPP) if and only if $X$ is locally convex.

A proof of Theorem 1.2 is included in Section 6. The Hahn-Banach extension property for metrizable spaces is also discussed in [10].

However, if $Y$ is the space constructed above, we will show that any algebraic complement $Y_0$ of $L$ has (HBEPP). Thus we have

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2. Idea of the construction. In this section, we introduce the basic ideas and notation and prove that the space $Y$ which will be constructed in Sections 3-5 yields solutions to the problems mentioned in the introduction.

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on \( c_{00} \) will be denoted by \( x \leq y \), i.e., \( x \leq y \) if and only if \( x(j) \leq y(j) \) for all \( j \in \mathbb{N} \). Let \( c_{00}^\infty = \{ x \in c_{00} : x \geq 0 \} \).

For \( x, y \in c_{00} \) we will write \( (x, y) = \sum_{j=1}^{\infty} x(j) y(j) \). We will also use the same terminology when \( x \in c_{00}^\infty \) and \( y = \log v \) for some sequence \( v \in c_{00}^\infty \); in this case it will be understood that the pairing can take the value \(-\infty\) and that \( \log 0 = 0 \).

A sequence space \( X \) we will mean a subspace \( X \) of the space \( \omega \) of all sequences equipped with a lattice norm \( ||x||_X \) so that

1. \( c_{00} \subseteq X \),
2. if \( |x| \leq |y| \) in \( X \) then \( x \leq y \) in \( X \) and \( ||x||_X \leq ||y||_X \), and
3. if \( 0 \leq x_n \leq x \) and \( x_n \in X \) with \( \sup ||x_n||_X = \infty \) then \( x \in X \) with \( ||x||_X = \sup ||x_n||_X \) (the Fatou property).

The canonical basis vectors \( \{ e_n \}_{n=1}^{\infty} \) then form a 1-unconditional basis for the closure \( X_0 \) of \( c_{00} \). For convenience we will write \( X^* \) for the Köthe dual of \( X \), which coincides with the Banach space dual of \( X_0 \). We will denote the closed unit ball of a Banach space \( X \) by \( B_X \). We denote the canonical norm on \( \ell_p \) by \( ||y||_p \) for the cases \( p = 1 \) and \( p = \infty \).

Consider a map \( \Phi : c_{00} \to \mathbb{R} \). For any \( u_1, \ldots, u_n \) we define \( \Phi(u_1, \ldots, u_n) = \sum_{j=1}^{n} \Phi(u_j) - \Phi(\sum_{j=1}^{n} u_j) \). \( \Phi \) is called quasilinear if

4. \( \Phi(u_n) = \alpha u \) for all \( \alpha \in \mathbb{R} \), \( u \in c_{00} \), and
5. for a constant \( \beta = \beta(\Phi) \) we have \( |\alpha u| \leq \beta ||u||_1 + ||u||_1 \) whenever \( u \in c_{00} \).

Given a quasilinear map \( \Phi \) we can form the twisted sum \( Y = \mathbb{R} \oplus_{\Phi} \ell_1 \), which is defined to be the completion of \( \mathbb{R} \oplus c_{00} \) under the quasinorm

\[
|| (a,u) ||_\Phi = |a - \Phi(u)| + ||u||_1.
\]

It is readily verified that if \( L \) is the span of the vector \( e_0 = (1, 0) \) then \( Y/L \) is isomorphic to \( \ell_1 \). This construction was first used in [13] and [26] with explicit non-trivial twisted sums of \( \mathbb{R} \) and \( \ell_1 \) to deduce that local convexity is not a three-space property; see also [27].

**Theorem 2.1**. Let \( \Phi : c_{00} \to \mathbb{R} \) be a quasilinear map and let \( Y = \mathbb{R} \oplus_{\Phi} \ell_1 \).

Then the following conditions are equivalent:

1. \( Y \) contains no basic sequence.
2. If \( Y_0 \) is an infinite-dimensional closed subspace of \( Y \) then \( Y_0 \) contains \( e_0 \).
3. The quotient map \( \pi : Y \to \ell_1 \) is strictly singular.
4. \( Y \) is topologically minimal.
5. There is no infinite-dimensional subspace \( F \) of \( c_{00} \) so that for some constant \( K \) we have \( ||\Phi(u)|| \leq K ||u||_1 \) for all \( u \in F \).
6. If \( T : \ell_1 \to Y \) is a bounded operator then \( T \) is compact.

7. If \( T : Y \to Y \) is a bounded operator then \( T = \lambda I + S \) where \( \lambda \in \mathbb{R} \) and \( S \) is compact.

**Proof**. The equivalence of (1) and (4) is well known (see Theorem 4.2 of [11] and Theorem 3.2 of [17], or see [16]). (2) is clearly equivalent to (3) and implies (1). Conversely, if (3) fails then there is an infinite-dimensional closed subspace isomorphic to a subspace of \( \ell_1 \). Thus (1)–(4) are all equivalent.

Next we prove (2) implies (5). Suppose \( F \) is an infinite-dimensional subspace of \( c_{00} \) so that \( ||\Phi(u)|| \leq K ||u||_1 \) for all \( u \in F \). Let \( Y_0 \) be the closure of the subspace of all \( (0, x) \) for \( x \in F \). Suppose \( (0, x_n) \) converges to \( \Phi \). Then \( ||1 - \Phi(x_n)|| \) converges to zero, which is a contradiction.

Next assume (5) and suppose \( Y \) contains a basic sequence. By a perturbation argument we can suppose it contains a normalized basic sequence of the form \( (\alpha_n, u_n) \) where \( u_n \in c_{00} \). By passing to a subsequence we can suppose that \( u_1 < u_2 < \ldots \) and that \( \pi \) is not in the closed linear span of \( (u_n, u_n) \). It follows that \( \pi \) is an isomorphism on the span of this basic sequence so that for some \( K \) we have

\[
\sum_{i=1}^{n} \alpha_i t_i - \Phi \left( \sum_{i=1}^{n} t_i u_i \right) = \sum_{i=1}^{n} ||u_i||_1 \leq K \sum_{i=1}^{n} ||u_i||_1
\]

for all \( t_1, \ldots, t_n \). Let \( Y_0 \) be the subspace of the linear span of \( (u_n)_{n=1}^{\infty} \)

consisting of all \( \sum_{i=1}^{\infty} t_i u_i \) with \( \sum_{i=1}^{\infty} \alpha_i t_i = 0 \). Then \( ||\Phi|| \leq K ||u||_1 \) for all \( u \in F_0 \). Thus (5) implies (1).

(3) implies (6). If \( T : \ell_1 \to Y \) is bounded then \( \pi T \) is strictly singular and hence compact. If \( (x_n) \) is a sequence in the unit ball of \( \ell_1 \) then by passing to a subsequence we can suppose that \( \pi Tx_n \) converges. Hence there exist \( y_n \in Y \) so that \( (y_n) \) converges and \( \pi Tx_n = \pi y_n \). But then \( Tx_n - y_n \in L \) and so has a convergent subsequence.

(6) implies (7). If \( T : Y \to Y \) is a bounded operator then since \( T \) is the intersection of the kernels of all continuous linear functionals on \( Y \) we must have \( T(L) \subseteq L \). Thus \( T = \lambda I \) for some \( \lambda \). Let \( S = T - \lambda I \); then \( S = S_0 \pi \) where \( S_0 : Y/L \to Y \) is compact by (6).

(7) implies (3). If \( \pi \) is not strictly singular, there is a subspace \( Y_0 \) of \( Y \) of infinite codimension and isomorphic to \( \ell_1 \). Hence there is an isomorphic embedding \( V : \ell_1 \to Y \). Then suppose \( V \pi = \lambda I + S \) where \( S \) is compact. Let \( \pi_0 : Y \to Y_0 \) be the quotient map. Then \( \lambda \pi_0 = -S \pi_0 \) is compact. Hence \( \lambda = 0 \), but this contradicts the fact that \( V \) is an isomorphism.

**Theorem 2.2**. If \( Y \) satisfies the equivalent conditions of Theorem 2.1 then any algebraic complement of \( L \) has the Hahn-Banach Extension Property.

**Proof**. Let \( Z \) be an algebraic complement of \( L \). The continuous linear functionals on \( Z \) separate points, so that any linear functional on a finite-
on $c_0$ will be denoted by $x \leq y$, i.e. $x \leq y$ if and only if $x(j) \leq y(j)$ for all $j \in \mathbb{N}$. Let $c_0^\infty = \{ x \in c_0 : x \geq 0 \}$.

For $x, y \in c_0^\infty$ we will write $\langle x, y \rangle = \sum_{j=1}^\infty x(j)y(j)$. We will also use the same terminology when $x \in c_0^\infty$ and $y = \log v$ for some sequence $v \in c_0^\infty$; in this case it will be understood that the pairing can take the value $-\infty$ and that $0 \log 0 = 0$.

By a sequence space $X$ we will mean a subspace $X$ of the space $\omega$ of all sequences equipped with a lattice norm $\| \|_X$ so that

1. $c_0 \subset X$,
2. $|x| \leq |y|$ in $X$ then $x \leq y$ and $\|x\|_X \leq \|y\|_X$, and
3. if $0 \leq x_n \uparrow x$ and $x_n \in X$ with $\|x\|_X = \sup \|x_n\|_X$ then $x \in X$ with

The canonical basis vectors $\{ e_n \}_{n=1}^\infty$ then form a 1-unconditional basis for the closure $X_0$ of $c_0$. For convenience we will write $X^*$ for the Köthe dual of $X$, which coincides with the Banach space dual of $X_0$. We will denote the closed unit ball of a Banach space $X$ by $B_X$. We denote the canonical norm on $\ell_p$ by $\| \|_p$ for the cases $p = 1$ and $p = \infty$.

Consider a map $\Phi : c_0 \to \mathbb{R}$. For any $u_1, \ldots, u_n$ we define $\Delta(\Phi(u_1, \ldots, u_n)) = \sum_{i=1}^n \Phi(u_i) - \Phi(\sum_{i=1}^n u_i)$. $\Phi$ is called quasi-linear if

1. $\Phi(\alpha u) = \alpha \Phi(u)$ for all $\alpha \in \mathbb{R}$, $u \in c_0$, and
2. for a constant $\delta = \delta(\Phi)$ we have $|\Delta(\Phi, u, v)| \leq \delta(\|u\|_1 + \|v\|_1)$ whenever $u, v \in c_0$.

Given a quasi-linear map $\Phi$ we can form the twisted sum $Y = \mathbb{R} \oplus_{\Phi} \ell_1$, which is defined to be the completion of $\mathbb{R} \oplus c_0$ under the quasi-norm $\| (a, b) \|_{\Phi} = |a - \Phi(b)| + \|b\|_1$.

It is readily verified that if $L$ is the span of the vector $e_0 = (1, 0)$ then $Y/L$ is isomorphic to $\ell_1$. This construction was first used in [13] and [26] with explicit non-trivial twisted sums of $\mathbb{R}$ and $\ell_1$ to deduce that local convexity is not a three-space property; see also [27].

**Theorem 2.1.** Let $\Phi : c_0 \to \mathbb{R}$ be a quasi-linear map and let $Y = \mathbb{R} \oplus_{\Phi} \ell_1$. Then the following conditions are equivalent:

1. $Y$ contains no basic sequence.
2. If $Y_0$ is an infinite-dimensional closed subspace of $Y$ then $Y_0$ contains $e_0$.
3. The quotient map $\pi : Y \to \ell_1$ is strictly singular.
4. $Y$ is topologically minimal.
5. There is no infinite-dimensional subspace $F$ of $c_0$ so that for some constant $K$ we have $|\Phi(u)| \leq K\|u\|_1$ for all $u \in F$. 
6. If $T : \ell_1 \to Y$ is a bounded operator then $T$ is compact.
7. If $T : Y \to Y$ is a bounded operator then $T = \lambda I + S$ where $\lambda \in \mathbb{R}$ and $S$ is compact.

**Proof.** The equivalence of (1) and (4) is well known (see Theorem 4.2 of [11] and Theorem 3.2 of [17], or see [16]). (2) is clearly equivalent to (3) and implies (1). Conversely, if (3) fails then there is an infinite-dimensional closed subspace isomorphic to a subspace of $\ell_1$. Thus (1)–(4) are all equivalent.

Next we prove (2) implies (5). Suppose $F$ is an infinite-dimensional subspace of $c_0$ so that $|\Phi(u)| \leq K\|u\|_1$ for $u \in F$. Let $Y_0$ be the closure of the subspace of all $(0, x)$ for $x \in E$. Suppose $(0, x_n)$ converges to $0$. Then $|1 - \Phi(x_n)|$ and $\|x_n\|_1$ converge to zero, which is a contradiction.

Next assume (5) and suppose $Y$ contains a basic sequence. By a perturbation argument we can suppose it contains a normalised basic sequence of the form $(\alpha_n, u_n)$ where $u_n \in c_0$. By passing to a subsequence, we may suppose that $u_1 < u_2 < \ldots$ and that $\varepsilon$ is not in the closed linear span of $(\alpha_n, u_n)$. It follows that $\pi$ is an isomorphism on the span of this basic sequence so that for some $K$ we have

$$\left| \sum_{i=1}^n \alpha_i t_i - \Phi \left( \sum_{i=1}^n t_i u_i \right) \right| \leq K \left\| \sum_{i=1}^n t_i u_i \right\|_1$$

for all $t_1, \ldots, t_n$. Let $F_0$ be the subspace of the linear span of the $(u_n)_{n=1}^\infty$ consisting of all $\sum_{i=1}^n t_i u_i$ with $\sum_{i=1}^\infty \alpha_i t_i = 0$. Then $\Phi(u) \leq K\|u\|_1$ for $u \in F_0$. Thus (5) implies (1).

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(6) implies (7). If $T : Y \to Y$ is a bounded operator then $F$ is an infinite-dimensional closed subspace of $Y$ and $T$ is a compact operator. Hence $T$ is a compact operator. Let $\pi_0 : Y \to Y/Y_0$ be the quotient map. Then $\lambda_0 = -S\pi_0$ is compact. Hence $\lambda = 0$, but this contradicts the fact that $V$ is an isomorphism.

**Theorem 2.2.** If $Y$ satisfies the equivalent conditions of Theorem 2.1 then any algebraic complement of $L$ has the Hahn-Banach Extension Property.

**Proof.** Let $Z$ be an algebraic complement of $L$. The continuous linear functionals on $Z$ separate points, so that any linear functional on a finite-
dimensional subspace can be extended continuously to \( Z \). Now let \( Z_0 \) be a closed infinite-dimensional subspace of \( Z \) and suppose \( f \) is a continuous linear functional on \( Z_0 \). Let \( W \) be the closure of \( Z_0 \) in \( Y \) and let \( f \) denote the extension of \( f \) to \( W \). Then \( W \) and \( f^{-1}(0) \) contain \( L \) by (2) and so \( f \) factors to a continuous linear functional on \( W/L \subset Y/L \), which is a Banach space. Hence by the Hahn–Banach theorem \( f \) can be extended continuously to \( Y \) and hence also to \( Z \).

**Theorem 2.3.** If \( Y \) satisfies the conditions of Theorem 2.1 then the topology \( \tau \) on \( Y \) cannot be the supremum of two vector topologies \( \tau_1, \tau_2 \) so that \( (Y, \tau_1) \) is nearly convex and \( (Y, \tau_2) \) has trivial dual.

**Proof:** Clearly \( e_0 \) must be in the closure of \( \{0\} \) for \( \tau_1 \). Let \( E \) be the closure of \( \{0\} \) for \( \tau_2 \). If \( e_0 \notin E \) then Theorem 2.1 implies that \( E \) is finite-dimensional and that \( Y^* \) separates the points of \( E \). Hence \( Y = Y_0 \oplus E \) for some closed subspace \( Y_0 \) of \( Y \). Now \( Y_0 \) contains no basic sequence and so its topology is minimal; however, \( \tau_2 \) is Hausdorff on \( Y_0 \) so that it must agree with the original topology. This implies that \( Y_0^* = \{0\} \), but in fact \( Y_0^* \) is infinite-dimensional. This contradiction establishes the theorem.

We now review the method of approach to the example. Theorem 2.1 reduces the problem to a type of distortion question expressed by (4). The recent results of the author [18], show that there is a close relationship between quasi-linear maps on \( c_0 \) and sequence spaces (see Theorem 6.8 of [15]). We will explain the connection in the next section and show how the recent spaces discovered by Gowers and Maurey ([7] and [9]) enable us to construct a pathological \( \Phi \).

### 3. Indicators of sequence spaces.

We now introduce some ideas from [15]. Suppose \( X \) is a sequence space. We define the *indicator* \( \Phi_X \) (called the *entropy map* in [21]) on \( c_0 \) by \( \Phi_X(u) = (u, \log x) \) where \( u = x^* \) is the (unique) Lozanovskii factorization of \( u \), i.e. \( x \in B_X^* \) and \( x^* \in X^* \) satisfy \( \|x^*\|_{X^*} = \|u\|_1 \) and \( \text{supp } x, \text{ supp } x^* \subset \text{supp } u \). The Lozanovskii factorization originates in [19].

Clearly \( \Phi_X(u_x) = \alpha \Phi_X(u) \) for \( u \in c_0 \). Furthermore, if \( u, v \in c_0 \) we also have

\[
\Delta(u, v) \leq \frac{4}{e} (\|u\|_1 + \|v\|_1)
\]

where \( \Delta = \Delta_{\Phi_X} \) (see Lemma 5.6 of [15]). If \( u \in c_0 \) then we can characterize the Lozanovskii factorization as the solution of an optimization problem so that

\[
\Phi_X(u) = \max_{x \in B_X^*} (u, \log x).
\]

This idea originates with Gillespie [6]. Furthermore, for \( u_1, \ldots, u_n \in c_0 \), we have the inequalities

\[
0 \leq \Delta(u_1, \ldots, u_n) \leq \sum_{i=1}^n \|u_i\|_1 \log \frac{S}{\|u_i\|_1}
\]

where \( S = \sum_{i=1}^n \|u_i\|_1 \); see [15], Lemma 5.5.

Suppose \( f : [1, \infty) \rightarrow [0, \infty) \) is any increasing map with \( f(1) = 1 \) and so that \( f(t) \leq 1 \) for all \( t \geq 1 \). We will say that a sequence space \( X \) has a lower \( f \)-estimate on blocks if, whenever \( x_1 < \ldots < x_n \in c_0 \), then

\[
\|x_1 + \ldots + x_n\|_X \geq \frac{1}{f(n)} \sum_{i=1}^n \|x_i\|_X,
\]

and an upper \( f \)-estimate on blocks if, whenever \( x_1 < \ldots < x_n \in c_0 \), then

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\]

**Lemma 3.1.** Suppose \( X \) satisfies an upper \( f \)-estimate on blocks. Then for \( u_1 < \ldots < u_n \in c_0 \), we have

\[
\Delta(u_1, \ldots, u_n) \leq f(n) (\|u_1\|_1 + \ldots + \|u_n\|_1).
\]

**Proof:** Let \( u = x^* \) be the Lozanovskii factorization. Then since \( f(n)^{-1}(x_1 + \ldots + x_n) \in H_X \) we have by (2),

\[
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\]

so that the lemma follows.

The following is a special case of Lemma 5.8 of [15]. Unfortunately, as the referee has pointed out, in Lemma 5.8 in [15] is misstated with the inequality reversed, and in the proof the minimum should be replaced by the maximum. This lemma is used in Theorem 5.7 of [15], which is correct although an inequality is again reversed. In view of this we will sketch a simple direct proof.

**Lemma 3.2.** Suppose \( s_1, \ldots, s_n, t_1, \ldots, t_n \geq 0 \) and let \( \sum_{i=1}^n s_i = S \) and \( \sum_{i=1}^n t_i = T \). Then

\[
\sum_{i=1}^n \left( s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i} \right) \leq S \log \frac{S + T}{S} + T \log \frac{S + T}{T}.
\]

**Remark:** The summand is zero if either \( s_i \) or \( t_i \) vanishes.

**Proof:** We will seek to maximize the function

\[
u(s_1, \ldots, s_n, t_1, \ldots, t_n) = \sum_{i=1}^n \left( s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i} \right)
\]
dimensional subspace can be extended continuously to Z. Now let \( Z_0 \) be a closed infinite-dimensional subspace of \( Z \) and suppose \( f \) is a continuous linear functional on \( Z_0 \). Let \( W \) be the closure of \( Z_0 \) in \( Y \) and let \( f \) denote the extension of \( f \) to \( W \). Then \( W \) and \( f^{-1}(0) \) contain \( L \) by (2) and so \( f \) factors to a continuous linear functional on \( W/L \subset Y/L \), which is a Banach space. Hence by the Hahn–Banach theorem \( f \) can be extended continuously to \( Y \) and hence also to \( Z \).

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\Phi_X(u) = \max_{x \in B_Y^*} (u, \log x).
\]

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\[
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Suppose \( f : [1, \infty) \to [1, \infty) \) is any increasing map with \( f(1) = 1 \) and so that \( f(t) \leq t \) for \( t \geq 1 \). We will say that a sequence space \( X \) has a lower \( f \)-estimate on blocks if, whenever \( x_1 < \ldots < x_n \in c_0 \), then

\[
\|x_1 + \ldots + x_n\|_X \geq \frac{1}{f(n)} \sum_{i=1}^n \|x_i\|_X,
\]

and an upper \( f \)-estimate on blocks if, whenever \( x_1 < \ldots < x_n \in c_0 \), then

\[
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\]

**Lemma 3.1.** Suppose \( X \) satisfies an upper \( f \)-estimate on blocks. Then for \( u_1 < \ldots < u_n \in c_0 \), we have

\[
\Delta(u_1, \ldots, u_n) \leq \log f(n)(\|u_1\|_1 + \ldots + \|u_n\|_1).
\]

**Proof.** Let \( u = x_1^* \) be the Lozanovski factorizations. Then since \( f(n)^{-1}(x_1 + \ldots + x_n) \in H_X \) we have by (2),

\[
\Phi_X(u_1 + \ldots + u_n) \geq \left( \sum_{i=1}^n \log \left( f(n)^{-1} \sum_{i=1}^n \right) \right)
\]

so that the lemma follows.

The following is a special case of Lemma 5.8 of [15]. Unfortunately, as the referee has pointed out, in Lemma 5.8 in [15] is misstated with the inequality reversed, and in the proof the maximum should be replaced by the minimum. This lemma is used in Theorem 5.7 of [15], which is correct although an inequality is again reversed. In view of this we will sketch a simple direct proof.

**Lemma 3.2.** Suppose \( s_1, \ldots, s_n, t_1, \ldots, t_n \geq 0 \) and let \( \sum_{i=1}^n s_i = S \) and \( \sum_{i=1}^n t_i = T \). Then

\[
\sum_{i=1}^n \left( s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i} \right) \leq S \log \frac{S + T}{S} + T \log \frac{S + T}{T}.
\]

Remark: The summand is zero if either \( s_i \) or \( t_i \) vanishes.

**Proof.** We will seek to maximize the function

\[
u(s_1, \ldots, s_n, t_1, \ldots, t_n) = \sum_{i=1}^n \left( s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i} \right)
\]
subject to the constraints \( \sum_{i=1}^{n} s_i = S \) and \( \sum_{i=1}^{n} t_i = T \) and \( s_i \geq 0, t_i \geq 0 \) for \( 1 \leq i \leq n \). By continuity, there is a point where the maximum is attained. We can suppose \( s_i t_i > 0 \) for \( 1 \leq i \leq m \) and \( s_i t_i = 0 \) if \( m + 1 \leq i \leq n \). By the method of Lagrange multipliers it is easy to show that \( s_i/t_i \) is constant for \( 1 \leq i \leq m \). But then

\[
u(s_1, \ldots, s_n, t_1, \ldots, t_n) = S_0 \log \frac{S_0 + T_0}{S_0 + T_0} + T_0 \log \frac{S_0 + T_0}{T_0}
\]

where \( S_0 = \sum_{i=1}^{n} s_i \leq S \) and \( T_0 = \sum_{i=1}^{n} t_i \leq T \). This expression is monotone increasing in \( S_0 \) and \( T_0 \) and so the result follows.

Let \( D = B_\| \cdot \|_1 \cap c_0^\| \cdot \|_1 \).

**Lemma 3.3.** Suppose \( X \) satisfies an upper f-estimate on blocks and suppose \( u \in D \). Let \( u = \sum_{i=1}^{n} u_i \) where \( u_1 < \ldots < u_n \). Let \( A \) be any subset of \( \mathbb{N} \) and let \( t = \| Au \|_1 \). Then

\[
\Delta(u_1, \ldots, u_n) - (1 - t) \log f(n) - \varphi(t) \leq \Delta(Au_1, \ldots, Au_n) \\
\leq \Delta(u_1, \ldots, u_n) + \varphi(t),
\]

where \( \varphi(t) = t \log(1/t) + (1 - t) \log(1/(1 - t)) \leq \log 2 \).

**Proof.** Let \( N \setminus A = B \). Then

\[
\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) = \Delta(Au_1, \ldots, Au_n) + \Delta(Bu_1, \ldots, Bu_n) + \Delta(Au, Bu).
\]

Similarly

\[
\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) = \Delta(u_1, \ldots, u_n) + \sum_{i=1}^{n} \Delta(Au_i, Bu_i).
\]

Since \( \Delta(Bu_1, \ldots, Bu_n), \Delta(Au, Bu) \geq 0 \) we deduce

\[
\Delta(Au_1, \ldots, Au_n) \leq \Delta(u_1, \ldots, u_n) + \sum_{i=1}^{n} \Delta(Au_i, Bu_i).
\]

Now we use (3) and Lemma 3.2. We have

\[
\sum_{i=1}^{n} \Delta(Au_i, Bu_i) \leq \sum_{i=1}^{n} \left( \| Au_i \|_1 \log \frac{\| Au_i \|_1}{\| Au \|_1} + \| Bu_i \|_1 \log \frac{\| Bu \|_1}{\| Bu_i \|_1} \right) \\
\leq t \log \frac{1}{1 - t} + (1 - t) \log \frac{1}{1 - t}.
\]

For the former inequality we observe that \( \Delta(Bu_1, \ldots, Bu_n) \leq \log f(n) \| Bu \|_1 \). Hence

\[
\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) \geq \Delta(u_1, \ldots, u_n) - (1 - t) \log f(n) - \Delta(Au, Bu),
\]

and the second inequality follows.

**Lemma 3.4.** Suppose \( u \in c_0^\| \cdot \|_1 \) with \( \| u \|_1 \leq 1 \). Suppose \( u = xx^* \) where \( x \in B_\| \cdot \|_X \), \( x^* \in B_\| \cdot \|_{X^*} \). Then \( \Phi_X(u(t)) = \| u(t) \|_1 \log(1/(\| u(t) \|_1)) \leq 1/e \).

**Proof.** We can suppose that the supports of \( x, x^* \) coincide with the support of \( u \). Define \( Z = \max \{2\|z\|_X, \|u^*\|_1^{-1}\} \). Then \( \|z\|_X \leq \|z\|_Z \leq \|u^*\|_1^{-1}\|z\|_X \) so that \( \Phi_X(u) = \|u\|_1 \log \|u\|_1 \leq \Phi_X(u) \leq \Phi_X(u) \) for \( v \in c_0^\| \cdot \|_1 \). However, \( \|z\|_Z \leq 1 \) with \( \|z\|_Z \leq \|u^*\|_1^{-1} \) so that \( u = xx^* \) is the Lozanovskii factorization for \( u \). Thus \( \Phi_Z(u) = \langle u, \log u \rangle \) and the lemma follows.

The next lemma is essentially due to Odell and Schlumprecht [21].

**Lemma 3.5.** Given \( \epsilon > 0 \) and \( n \in \mathbb{N} \) there exists \( \eta > 0 \) so that if \( u_1 < \ldots < u_n \) are in \( D, u = (1/n)(u_1 + \ldots + u_n) \) and \( \delta = (1/n)\Delta(u_1, \ldots, u_n) < \eta \) then the Lozanovskii factorizations \( u = xx^* \) and \( u_i = x_i x_i^* \) we have \( \| Au \|_1 < \epsilon \) where \( A = \{j : y(j) > (1 + \epsilon)x(j)\} \) and \( y = x_1 + \ldots + x_n \).

**Proof.** By Proposition 2.3 of Odell and Schlumprecht [21], given \( \epsilon > 0 \) there exists \( \nu > 0 \) so that if \( v \in D \) and \( x \in B_\| \cdot \|_X \) are such that \( \langle v, \log x \rangle > \Phi_X(y) - \nu \) then if \( v = x x^* \) is the Lozanovskii factorization then \( \| Bu \|_1 < \epsilon \) where \( B = \{j : x(j) > (1 + \epsilon)x(j)\} \). Thus \( \eta = \nu/n \). Then if \( \delta < \eta \) we have

\[
\sum_{i=1}^{n} \langle \Phi_X(u_i) - \langle u_i, \log u_i \rangle \rangle < \nu
\]

and since each term is positive we conclude that \( \| Au \|_1 < \epsilon \) where \( A_i = \{j : x_i(j) > (1 + \epsilon)x_i(j)\} \). This quickly implies that \( \| Au \|_1 < \epsilon \).

4. The Gowers–Maurey space. At this point we let \( f(x) = \log(2(x + 1)) \) and introduce as in [9] the class \( F \) of functions \( g : [1, \infty) \to [1, \infty) \) having the following properties:

(1) \( g(1) = 1 \) and \( g(x) < x \) for \( x > 1 \).

(2) \( g \) is strictly increasing and unbounded.

(3) \( \lim_{x \to \infty} x^{-g(x)} = 0 \) for any \( g > 0 \).

(4) \( x/g(x) \) is concave and non-decreasing.

(5) \( g \) is submultiplicative, i.e. \( g(xy) \leq g(x)g(y) \) for \( x, y \geq 1 \).

Clearly \( f \in F \) and so is \( \sqrt{f} \).

Now suppose \( X \) is a sequence space. If \( n \in \mathbb{N} \) and \( \kappa > 0 \) we define \( \lambda_X(n, \kappa) \) to be the set of \( x \in c_0^\| \cdot \|_1 \) so that \( \| x \|_X = 1 \) and \( x = (1/n)(x_1 + \ldots + x_n) \) where \( x_1 < \ldots < x_n \) and \( \| x_i \|_X \leq \kappa \) for \( 1 \leq i \leq n \). (Thus \( x \) is an \( l_1 \) average with constant \( \kappa \), in the sense of [9]; note that we restrict ourselves to non-negative sequences and to spaces \( X \) for which the canonical basis is unconditional.)
subject to the constraints $\sum_{i=1}^n s_i = S$ and $\sum_{i=1}^m t_i = T$ and $s_i \geq 0$, $t_i \geq 0$ for $1 \leq i \leq n$. By continuity, there is a point where the maximum is attained. We can suppose $s_i t_i > 0$ for $1 \leq i \leq m$ and $s_i t_i = 0$ if $m + 1 \leq i \leq n$. By the method of Lagrange multipliers it is easy to show that $s_{i^*} t_{i^*}$ is constant for $1 \leq i \leq m$. But then

$$u(s_1, \ldots, s_n, t_1, \ldots, t_m) = S_0 \log \frac{S_0 + T_0}{S_0 + T_0} = T_0 \log \frac{S_0 + T_0}{T_0},$$

where $S_0 = \sum_{i=1}^n s_i \leq S$ and $T_0 = \sum_{i=1}^m t_i \leq T$. This expression is monotone increasing in $S_0$ and $T_0$ and so the result follows.

Let $D = B_t \cap c_0^0$.

**Lemma 3.3.** Suppose $X$ satisfies an upper $f$-estimate on blocks and suppose $u \in D$. Let $u = \sum_{i=1}^n u_i$ where $u_1 \leq \cdots \leq u_n$. Let $A$ be any subset of $N$ and let $t = \|Au\|_1$. Then

$$\Delta(u_1, \ldots, u_n) - (1-t) \log f(n) - \varphi(t) \leq \Delta(Au_1, \ldots, Au_n) \leq \Delta(u_1, \ldots, u_n) + \varphi(t),$$

where

$$\varphi(t) = t \log(1/t) + (1-t) \log(1/(1-t)) \leq \log 2.$$

**Proof.** Let $N \setminus A = B$. Then

$$\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) = \Delta(Au_1, \ldots, Au_n) + \Delta(Bu_1, \ldots, Bu_n) + \Delta(Au, Bu).$$

Similarly

$$\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) = \Delta(u_1, \ldots, u_n) + \sum_{i=1}^n \Delta(Au_i, Bu_i).$$

Since $\Delta(Bu_1, \ldots, Bu_n), \Delta(Au, Bu) \geq 0$ we deduce

$$\Delta(Au_1, \ldots, Au_n) \leq \Delta(u_1, \ldots, u_n) + \sum_{i=1}^n \Delta(Au_i, Bu_i).$$

Now we use (3) and Lemma 3.2. We have

$$\sum_{i=1}^n \Delta(Au_i, Bu_i) \leq \sum_{i=1}^n \left( \|Au_i\|_1 \log \frac{\|Au_i\|_1}{\|Au\|_1} + \|Bu_i\|_1 \log \frac{\|Bu_i\|_1}{\|Bu\|_1} \right) \leq t \log \frac{1}{1-t} + (1-t) \log \frac{1}{1-t}.$$
We then define $\text{RIS}_X(n; \kappa)$ to be the collection of sequences $x_1 < \ldots < x_n$ in $c_{00}$ satisfying $x_i \in \lambda_X(M_i; \kappa)$ where $M_i \geq 4^{k/2} \cdot 3^{3/2} a_k^{\frac{3}{2}}$ and $M_{k+1} \geq \frac{2^{k}(a_k^{\frac{1}{2}})^{k/2}}{k}$ for $k \geq 1$ where $a_k = \min(\kappa - 1, 1)$. We next define $\lambda_X(n; \kappa)$ to be the collection of $x \in c_{00}$ of the form $x = \langle x_1, \ldots, x_n \rangle$ where $(x_1, \ldots, x_n) \in \text{RIS}_X(n; \kappa)$. This definition differs slightly but insignificantly from that of [9]. In fact, we will only really require the case $\kappa \geq 2$ when $\gamma = 1$; this is in contrast to [9] where values of $\kappa$ close to one are important.

At the same time if $g \in F$ we define $\mathcal{H}_{X}(g; m)$ to be the collection of $(m, g)$-forms, i.e. $x^* \in \mathcal{H}_X(g; m)$ if and only if $x^* = (g(m))^{-1}(x_1^* \ldots + x_m^*)$ where $x_1^* < \ldots < x_m^*$ are in $c_{00}$ and $\|x^*\|_{X^*} \leq 1$ for $1 \leq i \leq m$.

We will require certain lemmas from [9].

**Lemma 4.1 (Lemma 4 of [9]).** Suppose $x \in \lambda_X(N; \kappa)$ and $x^* \in \mathcal{H}_X(g; M)$ where $\gamma \in F$. Then $\langle x, x^* \rangle \leq \kappa(1 + 2\|g\|N)M^{-\frac{1}{2}}$.

**Lemma 4.2 (Lemma 5 of [9]).** Suppose $X$ satisfies a lower $f$-estimate on blocks and $g \in F$ with $g \geq f^{1/2}$. Suppose $N \in N$, $N > 1$. Suppose $M \geq 2^{3N^{2/3}}$ and that $x \in \Lambda(N; \kappa)$, $x^* \in \mathcal{H}_X(g; M)$. Then $\langle x, x^* \rangle \leq \kappa(1 + 1)^{f(N)}N \leq (\kappa + 1)^{f(N)}N$.

**Remark.** For our statement of Lemma 4.2, observe that since $X$ has a lower $f$-estimate, for any $\{x_i\}_{i=1}^N$ in $\text{RIS}_X(N; \kappa)$ we have $\|\sum_{i=1}^N x_i\|_X \geq N^{f(N)}$.

**Lemma 4.3.** Suppose $X$ satisfies a lower $f$-estimate on blocks and $g \in F$ with $g \geq f^{1/2}$. Suppose $\kappa \geq 2$ and $(x_1, \ldots, x_N) \in \text{RIS}_X(N; \kappa)$. Let $x = \sum_{i=1}^N x_i$ and suppose that for every interval $E$ with $\|E\| \geq 1$ we have (1)

$$\|Ex\|_X \leq \sup\{\langle Ex, x^* \rangle : x^* \in \mathcal{H}_X(g; M), M \geq 2\}.$$

Then $\|x\|_X \leq \kappa(1 + N^{f(N)}$.

**Proof.** We introduce the length of an interval $E$ as in [9]. Let $x_i \in \lambda_X(n_i; \kappa)$ for $1 \leq i \leq N$. Suppose $x_i$ is written as $(1/n_i)^{\sum_{j=1}^{n_i} x_{ij}}$ where $x_{i1} < \ldots < x_{in_i}$ and $\|x_{i}\|_X \leq n_i^{-1}$. If $E$ is any interval which intersects the support of $\sum_{i=1}^N x_i$ we let $k \leq l$ be the least and greatest indices $i$ such that $E_{xy} \neq 0$. Then we let $p$ be the least index such that $\|x_{pi}\|_X \neq 0$ and $q$ the greatest index such that $\|x_{qj}\|_X \neq 0$. Define $\|E\| = 1 - k - qn_k^{-1} - p_n^k$. If $E$ does not meet the support of $\sum_{i=1}^N x_i$, then $\|E\| = 0$.

Now our hypotheses differ from Lemma 7 of [9] in that we assume (1) whenever $\|E\| \geq 1$, while [9] assumes (1) whenever $\|E\| \geq 1$; we, however, assume $\kappa \geq 2$. Our hypotheses imply that (1) holds if $\|E\| \geq 2$ since then there exists at least one $x_i$ with support contained entirely in $E$. As in [9]

$$G(t) = t/\gamma(t)$$

for $t \geq 1$ and $G(t) = t$ for $t \leq 1$. Then if $\kappa n^{-1} \leq \gamma(E) \leq 1$ we have $\|Ex\|_X \leq (\kappa + 1)G(\gamma(E))$ as in [9]. We claim the same inequality if $1 \leq \gamma(E) \leq 2$; in fact, in this situation we can see that $E$ intersects the supports of at most three $x_i$ and so $\|Ex\|_X \leq 3 \leq (\kappa + 1)G(\gamma(E))$. The proof can now be completed by applying Lemma 7 of [9].

We will now define a Gowers-Maurey space $Z$, very similar to the construction in [9]; in fact, essentially the same space is considered by Gowers in [7] as a counterexample to the hyperplane problem, and also as a space in which all operators are strictly singular perturbations of a diagonal map. We will suppose that $\mathcal{P} = \{p_k\}_{k=1}^\infty$ is an increasing sequence of natural numbers satisfying $f(p_k) > 256$, $\log \log \log p_k > \log^2 p_k$, $p_k > \log^5 p_k$, for all $k$. We shall also require that $f(p_k^2) p_k^2 \leq \kappa^{-1}/2$, which doubtless follows from our other hypotheses. For convenience we suppose each $p_k$ is a square. We partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1 = \{p_k^2 : k \in \mathbb{N}\}$ and $\mathcal{P}_2 = \{p_k\}_{k=1}^\infty$.

Let $\mathbb{Q}_+$ denote the countable collection of $u \in c_{00}$ which have only rational coefficients and let $\sigma$ be an injection from the collection of all finite subsets of $\mathbb{Q}_+$, $(z_1, \ldots, z_N)$ where $z_1 < \ldots < z_N$, to $\mathcal{P}_2$ which satisfy the condition $(z_1, \ldots, z_N) \geq 2^{10|z_i|}$. We then define $Z$ implicitly by the formula

$$\|x\|_Z = \max\{\|x\|_{\sigma}, \|x\|_{\sigma} \langle x, x^* \rangle \}$$

where

$$\|x\|_{\sigma} = \sup\{\langle x, x^* \rangle : x^* \in \mathcal{H}_Z(f; M), M \geq 2\},$$

$$\|x\|_{\sigma} = \sup\left\{ f(k)^{-1/2} \left\{ \sum_{m=1}^{k} \langle x, x^* \rangle \right\} \right\},$$

with the supremum being over all $k \in \mathcal{P}_1$ and special sequences $(z_1^*, \ldots, z_N^*)$, i.e., such that $z_1^* < \ldots < z_N^*$, with $x_i \in \mathbb{Q}_+ \cap \mathcal{H}_Z(f; p_k)$ and then for $j > 0$, $z_{j+1}^* \in \mathbb{Q}_+ \cap \mathcal{H}_Z(f; o(z_1^*, \ldots, z_N^*))$.

This implicit definition can be justified by an inductive construction as in [9]. Precisely we set $\|x\|_{\sigma_n} = \|x\|_{\sigma}$ for $x \in c_{00}$ and then define for $N \geq 1$,

$$\|x\|_{\sigma_N} = \max\{\|x\|_{\sigma_{N-1}}, \|x\|_{\sigma_{N-1}} \langle x, x^* \rangle \},$$

where

$$\|x\|_{\sigma_N} = \sup\{\langle x, x^* \rangle : x^* \in \mathcal{H}_Z(f; M), M \geq 2\},$$

$$\|x\|_{\sigma_N} = \sup\left\{ f(k)^{-1/2} \left\{ \sum_{m=1}^{k} \langle x, x^* \rangle \right\} \right\},$$

with the supremum being over all $k \in \mathcal{P}_1$ and $(z_1^*, \ldots, z_N^*)$, i.e., such that $z_1^* < \ldots < z_N^*$, with $x_i \in \mathbb{Q}_+ \cap \mathcal{H}_Z(f; p_k)$ and then for $j \geq 1$, $z_{j+1}^* \in \mathbb{Q}_+ \cap \mathcal{H}_Z(f; o(z_1^*, \ldots, z_N^*))$. We claim that $\|x\|_{\sigma_n} = \|x\|_{\sigma}$ for $x \in c_{00}$ and $N \geq 1$.
We then define $\text{RIS}_X(n; \kappa)$ to be the collection of sequences $x_1 < \ldots < x_n$ in $c_0^+ \cap c_0^-$ satisfying $x_i \in \chi_X(M_n, \kappa)$ where $M_1 \geq 4\kappa^{-2} 2^{-3\kappa^{-2}}$ and $M_{k+1} \geq 4^{k-1} m_k^{-2}$ for $k \geq 1$ where $m_0 = \min(\kappa - 1, 1)$. We next define $\chi_X(n; \kappa)$ to be the collection of $x \in c_0^+$ of the form $x = \|x_1 + \ldots + x_n\|_X^{1/(x_1 + \ldots + x_n)}$ where $(x_1, \ldots, x_n) \in \text{RIS}_X(n; \kappa)$. This definition differs slightly but inessential from that of $[9]$, in fact, we will only really require the case $\kappa \geq 2$ when $\kappa = 1$; this is in contrast to $[9]$ where values of $\kappa$ close to one are important.

As the same time if $f \in F$ we define $\mathcal{H}_X(f; m)$ to be the collection of $(m, g)$-forms, i.e. $x^* \in H_X(f; g)$ if and only if $x^* = (g(m))^{-1}(x_1 + \ldots + x_m)$ where $x^*_1 < \ldots < x^*_m$ are in $c_0^+$ and $\|x_i\|_X \leq 1$ for $1 \leq i \leq m$.

We will require certain lemmas from $[9]$.

**Lemma 4.1** (Lemma 4 of $[9]$). Suppose $x \in \chi_X(N; \kappa)$ and $\sigma^* \in H_X(g; M)$ where $g \in F$. Then $(x, \sigma^*) \leq \kappa(4 + 2M/N)g(N)^{-1}$.

**Lemma 4.2** (Lemma 5 of $[9]$). Suppose $X$ satisfies a lower $f$-estimate on blocks and $x \in F$ with $g \geq f^{1/2}$. Suppose $N \in N$ and $k > 1$. Suppose $M \geq 236\kappa^{-2}$ and that $x \in \chi_X(N; \kappa)$, $\sigma^* \in H_X(g; M)$. Then $(x, \sigma^*) \leq (\kappa + 1)g(N)N \leq (\kappa + 1)g(N)$.

Remark. For our statement of Lemma 4.2, observe that since $X$ has a lower $f$-estimate, for any $(x_1, \ldots, x_N) \in \text{RIS}_X(N; \kappa)$ we have $\sum_{i=1}^N x_i \geq Nf(N)^{-1}$.

Our next lemma is a slight modification of Lemma 7 of $[9]$.

**Lemma 4.3**. Suppose $X$ satisfies a lower $f$-estimate on blocks and $g \in F$ with $g \geq f^{1/2}$. Suppose $\kappa \geq 2$ and $(x_1, \ldots, x_N) \in \text{RIS}_X(N; \kappa)$. Let $x = \sum_{i=1}^N x_i$ and suppose that for every interval $E$ in $X$ with $x \geq Nf(N)^{-1}$ we have

\[ \|Ex\|_X \leq \sup \{(Ex, \sigma^*) : \sigma^* \in H_X(g; M), M \geq 2\}. \]

Then $\|x\|_X \leq (\kappa + 1)g(N)$.

Proof. We introduce the length of an interval $E$ as in $[9]$. Let $x_i \in \chi_X(n_i; \kappa)$ for $1 \leq i \leq N$. Suppose $x_i$ is written as $1/n_i \sum_{j=1}^{n_i} x_{ij}$ where $x_{1i} < \ldots < x_{n_i}$ and $\|x_{ij}\|_X \leq \kappa n_i^{-1}$. If $E$ is any interval which intersects the support of $\sum_{i=1}^N x_i$ we let $k \leq l$ be the least and greatest indices such that $E \cap x_{i} \neq 0$. Then let $p_i$ be the least index such that $E \cap x_{i}$ is defined so that $E \cap x_{i} \neq 0$. Define $\ell(E) = 1 + k + \kappa n_i^{-1} - \kappa n_i^{-1}$.

If $E$ does not meet the support of $\sum_{i=1}^N x_i$ then $\ell(E) = 0$.

Now our hypotheses differ from Lemma 7 of $[9]$ in that we assume $(\ast)$ whenever $\|Ex\|_X \geq 1$, while $[9]$ assumes $(\ast)$ whenever $\ell(E) > 1$; we, however, assume $\kappa \geq 2$. Our hypotheses imply that $(\ast)$ holds if $\ell(E) \geq 2$ since then there exists at least one $x_i$ with support contained entirely in $E$. As in $[9]$ let $G(t) = t/g(t)$ for $t \geq 1$ and $G(t) = t$ for $t \leq 1$. Then if $\kappa n_i^{-1} < \ell(E) \leq 1$ we have $\|Ex\|_X \leq (\kappa + 1)G(\ell(E))$ as in $[9]$. We claim the same inequality if $1 \leq \ell(E) \leq 2$; in fact, in this situation we can see that $E$ intersects the supports of at most three $x_i$ and so $\|Ex\|_X \leq 3(\kappa + 1)G(\ell(E))$. The proof can now be completed by applying Lemma 7 of $[9]$.

We will now define a Gowers–Maurey space $Z$, very similar to the construction in $[9]$; in fact, essentially the same space is considered by Gowers in $[7]$ as a counterexample to the hyperplane problem, and also as a space in which all operators are strictly singular perturbations of a diagonal map. We will suppose that $P = (p_k)_{k=1}^\infty$ is an increasing sequence of natural numbers satisfying $f(p_k) > 256, \log \log p_k > 2p_{k-1}, p_k > 4p_{k-1}^2$, for all $k$. We shall also require that $f(p_k) p_k^2 \leq k^{-3}/2$, which doubtless follows from our other hypotheses. For convenience we suppose each $p_k$ is a square. We partition $P = P_1 \cup P_2$ where $P_1 = (p_{2k-1})_{k=1}^\infty$ and $P_2 = (p_{2k})_{k=1}^\infty$.

Let $\mathcal{Q}_k$ denote the countable collection of $u \in c_0^+$ which have only rational coefficients and let $\sigma$ be an injection from the collection of all finite subsets of $\mathcal{Q}_k, \{x_1, \ldots, x_n\}$ where $x_1 < \ldots < x_n$, to $P_2$ which satisfy the condition $\sigma(x_1, \ldots, x_n) \geq 10^{10(\kappa^2s^2)}$.

We then define $Z$ implicitly by the formula

\[ \|z\|_Z = \max(\|z\|_{\mathcal{Q}_k}, \|z\|_{P_1}, \|z\|_{P_2}) \]

where

\[ \|z\|_{\mathcal{Q}_k} = \sup \{\|x, x^*\|^* : \sigma^* \in H_{\mathcal{Q}_k}(f; M), M \geq 2\}, \]

\[ \|z\|_{P_1} = \sup \left\{ f(k)^{-1/2} \sum_{i=1}^{k} \|x_i\|_{\mathcal{Q}_k} : \sigma^* \in H_{\mathcal{Q}_k}(f; M), M \geq 2\} \],

with the supremum being over all $k \in P_1$ and special sequences $(x_1^*, \ldots, x_k^*)$, i.e. such that $x_1^* < \ldots < x_k^*$, with $x_1^* \in \mathcal{Q}_k \cap H_{\mathcal{Q}_k}(f; p_1)$ and then for $j > 1$, $x_j^* \in \mathcal{Q}_k \cap \mathcal{H}_{\mathcal{Q}_k}(f; p_{2j-1})$.

This implicit definition can be justified by an inductive construction as in $[9]$. Precisely we set $\|z\|_{P_1} = \|z\|_{\mathcal{Q}_k}$ for $x \in c_0$ and then define for $N \geq 1$,

\[ \|z\|_{\mathcal{Q}_k} = \max(\|z\|_{\mathcal{Q}_{k-1}}, \|z\|_{P_1}) \]

where

\[ \|z\|_{P_1} = \sup \{\|x, x^*\|^* : \sigma^* \in H_{\mathcal{Q}_k}(f; M), M \geq 2\}, \]

\[ \|z\|_{P_2} = \sup \left\{ f(k)^{-1/2} \sum_{i=1}^{k} \|x_i\|_{\mathcal{Q}_k} : \sigma^* \in H_{\mathcal{Q}_k}(f; M), M \geq 2\} \],

with the supremum being over all $k \in P_1$ and $(x_1^*, \ldots, x_k^*)$, i.e. such that $x_1^* < \ldots < x_k^*$, with $x_1^* \in \mathcal{Q}_k \cap H_{\mathcal{Q}_k}(f; p_1)$ and then for $j \geq 1$, $x_{j+1}^* \in \mathcal{Q}_k \cap \mathcal{H}_{\mathcal{Q}_k}(f; p_{2j})$. The basic sequence problem.
Q_+ \cap \mathcal{H}_\mathbb{N}(f; \sigma(x_1^*, \ldots, x_j^*))$. It is then easily verified that $\| \|_\mathbb{N}$ is an increasing sequence of norms, bounded above by the $\ell_1$ norm, and that the sets $H_\mathbb{N}(f; M)$ also increase in $N$. We set $\| x \|_\mathbb{N} = \lim_{N \to \infty} \| x \|_\mathbb{N}$.

We emphasize that this space is an unconditional version of the counterexample constructed in [9], but share some of the same features. We will need versions for $Z$ of certain lemmas proved in [9] for the Gowers-Maurey space. Fortunately the same basic techniques go through more or less unchanged.

Let us note first that $Z$ satisfies a lower $f$-estimate. This follows immediately from the definition of $\| x \|_\mathbb{N}$. We also note that, by induction, it follows that $\| e_n \|_Z = 1$ for all $n$.

\textbf{Lemma 4.4.} Suppose $(x_j)^N_{j=1} \in \text{RIS}_2(\mathbb{N}; \kappa)$ where $\kappa \geq 1$. Then $\| \sum_{j=1}^N x_j \|_Z < 1$.

\textbf{Proof.} We have $x_j \in \lambda_2(M_j, \kappa)$ where $M_j \geq 4\kappa$ by the definition of \text{RIS}_2(\mathbb{N}; \kappa)$. Hence $\| x_j \|_\infty \leq M_j^{-1} \kappa < 1$ and the lemma follows.

It now follows as in Lemma 10 of [9]:

\textbf{Lemma 4.5.} Suppose $\kappa \geq 2$. Suppose $N \in \mathbb{N}$ and $\log N \leq n \leq \exp N$. Then if $(x_1, \ldots, x_n) \in \text{RIS}_2(\mathbb{N}; \kappa)$ we have $\| \sum_{j=1}^n x_j \|_Z \leq (\kappa + 1)f(n)^{-1}$.

\textbf{Proof.} The key point, proved in [16], Lemma 9, is that there exists $g \in \mathcal{F}$ with $f^{1/2} \leq g \leq f$ such that $g(x) = f(x)$ for $\log N \leq n \leq \exp N$ and $g(k) = f^{1/2}(k)$ when $k \in \mathbb{N}$. Thus if $x \in c_{00}$ and $\| x \|_Z > \| x \|_\mathbb{N}$ then $\| x \|_Z = \sup \{ \| Ex \cdot x^* \| : x^* \in H_\mathbb{Z}(g; M), M \geq 2 \}$.

Now, by the preceding lemma if $\sum_{j=1}^n x_j$ and $E$ is any interval then $\| Ex \|_\mathbb{N} \leq 1$. We can therefore apply Lemma 4.3 to obtain the result.

The next lemma is simply a rudimentary form of Lemma 11 from [9].

\textbf{Lemma 4.6.} Suppose $\kappa \geq 2$ and $N \in \mathbb{N}$. If $x \in A_2(\mathbb{N}, \kappa)$ then $x \in \lambda(\sqrt{N}, \kappa + 1)$.

\textbf{Proof.} Suppose $(x_j)^N_{j=1} \in \text{RIS}_2(\mathbb{N}; \kappa)$ and that $x = \sum_{j=1}^N x_j \in \lambda(\sqrt{N}, \kappa + 1)$. We break $\lambda(\sqrt{N}, \kappa + 1)$ into $\sqrt{N}$ intervals $E_j$ each of length $\sqrt{N}$, which is an integer by hypothesis. Note that $(x_j)_{j \in E_j} \subset \text{RIS}_2(\sqrt{N}, \kappa)$ if $y_j = \sum_{j \in E_j} x_j$, then, by Lemma 4.5, $\| y_j \|_\infty \leq (\kappa + 1)\sqrt{N}$ Also $\| \sum_{j=1}^n x_j \|_Z \geq N/f(N)$, by the lower $f$-estimate on $K$. Now $z = (1/\sqrt{N}) \sum_{j=1}^N x_j$ where $x_j = (\sum_{i=1}^N x_i) \sqrt{N}$. But $\| x \|_Z \leq (\kappa + 1)\sqrt{N}/(f(N)) \leq 2(\kappa + 1)$.

Our next result is a modification of Lemma 12 of [9]. In fact, this lemma appears to be incorrectly stated in [9] and so some modification is necessary. In the proof of the lemma in [9] it is claimed without justification that $(x_1^*, \ldots, x_k^*)$ is a RIS of length $k$ and constant $1 + \varepsilon$. For the applications some modification similar to that given below seems adequate, however.

\textbf{Lemma 4.7.} Let $\kappa \geq 2$. Suppose $k \in \mathbb{N}$ with $f(k) > 100\kappa^2$. Suppose $E_1, \ldots, E_k$ are intervals with $E_1 \subset \ldots \subset E_k$. Let $(x_1^*, \ldots, x_k^*)$ be special sequence with $\sum_{j=1}^k x_j^* \subset E_j$. Let $M = p_{k, \varepsilon}$ and $M_{j+1} = \sigma(x_1^*, \ldots, x_j^*)$ for $1 \leq j \leq k - 1$. Let $A$ be any subset of $\{1, 2, \ldots, k\}$ and suppose for each $j \in A$ we have $x_j \in c_{00}$ with $\sum_{j=1}^k x_j \in E_j$ so that $x_i, x_j$ are disjoint.

\[ \| \sum_{i \in A} x_i \|_Z \leq 16 \varepsilon \sqrt{f(k)}^{-1}. \]

\textbf{Proof.} We have $x_j \in \lambda(\sqrt{M_{j+1}}, 4\kappa)$, by Lemma 4.6. Note that $\sqrt{M_{j+1}} \geq 4\kappa$. We also have $\sqrt{M_{j+1}} > 4\kappa$. Now assume $A$ contains no two consecutive integers. Then if $j \in A$ we have $\sqrt{M_j} > 4\kappa(f(k))^{1/2}$ for $j \geq 2$ so $(x_j)_{j \in A} \in \text{RIS}_2(|A|, 4\kappa)$. As in [9] we use Lemma 4.3.

Note first that there exists $h \in \mathcal{F}$ with $\sqrt{h} \leq h \leq f$, so that $h(n) = \sqrt{h(n)}$ if $n \in \mathbb{N}$, while $h(n) = f(n)$ if $n \in \mathbb{N} \setminus \{h\}$. This fact follows from Lemma 9 of [9].

Let $x = \sum_{i \in A} x_i$ and suppose that for some interval $E$ we have $\| Ex \|_Z \geq 1$, and

\[ \| Ex \|_Z > \sup \{ (Bx, x^*) : x^* \in H_\mathbb{Z}(g; M), M \geq 2 \}. \]

Since $h \leq f$ this implies that $\| Ex \|_Z > \| Ex \|_\mathbb{N}$. On the other hand, since $(x_j)_{j \in A} \in \text{RIS}_2(|A|, 4\kappa)$ we can apply Lemma 4.4 to deduce that $\| Ex \|_Z > \| Ex \|_\mathbb{N}$. The conclusion is that $\| Ex \|_Z \geq \| Ex \|_\mathbb{N}$. Thus there is a special sequence $(x_1^*, \ldots, x_k^*)$ with $i \in \mathbb{N} \setminus \{h\}$ so that

\[ \| Ex \|_Z = f(l)^{-1/2} \left( \sum_{i=1}^k x_i^* \right). \]

However, $f(l)^{-1/2} = h(l)$ unless $l = k$. We conclude $l = k$ and

\[ 1 \leq \| Ex \|_Z \leq f(k)^{-1/2} \sum_{j=1}^k |x_j^*|. \]

Let $l$ be the greatest integer so that $x_l^* = x_l^*$ (with $t = 0$ if no such integer exists). If $i < t$ it is clear that $(x_i, x_i^*) = 0$ for all $j$. Similarly if $t < j$ it is also clear that $(x_i, x_j^*) = 0$ for all $i$. If $i = t$, then $(x_i, x_j^*) = 0$ unless $i = j$ or when of course $(x_i, x_i^*) = 0$. If $t + 1 \leq i \in A$ and $t + 1 \leq j \leq k$ then, unless $t + 1 = i = j$, we have $x_i \in A_2(M_i, \kappa)$ and $x_j \in H_\mathbb{Z}(g; M_j)$ where $M_i, M_j \in \mathbb{N}$. However, $M_i, M_j \in \mathbb{N}$ are not equal. It follows from the separation conditions
Q_∞ \cap H_{2N}(f; \sigma(x_1^*, \ldots, x_j^*)) it is then easily verified that \( \| x_N \|_Z \) is an increasing sequence of norms, bounded above by the \( \ell_1 \)-norm, and that the sets \( H_{2N}(f; M) \) also increase in \( N \). We set \( \| x \|_{Z} = \lim_{N \to \infty} \| x_N \|_{Z} \).

We emphasize that this space is an unconditional version of the counterexample constructed in [9], but shares some of the same features. We will need versions for \( Z \) of certain lemmas proved in [9] for the Gowers-MAurer space. Fortunately the same basic techniques go through more or less unchanged.

Let us note first that \( Z \) satisfies a lower \( f \)-estimate. This follows immediately from the definition of \( \| x \|_n \). We also note that, by induction, it follows that \( \| e_n \|_Z = 1 \) for all \( n \).

**Lemma 4.4.** Suppose \( \{ x_j \}_{j=1}^{2N} \in \text{RIS}_Z(\nu; \kappa) \) where \( \kappa \geq 1 \). Then \( \sum_{j=1}^{2N} x_j \|_Z \leq 1 \).

**Proof.** We have \( x_j \in \lambda_2(M_j, \kappa) \) where \( M_j \geq 4\kappa \) by the definition of \( \text{RIS}_Z(\nu; \kappa) \). Hence \( \| x_j \|_Z \leq M_j^{-1} \kappa \leq 1 \) and the lemma follows.

It now follows as in Lemma 10 of [9]:

**Lemma 4.5.** Suppose \( \kappa \geq 2 \). Suppose \( N \in \mathbb{P}_2 \) and log \( N \) \( \leq n \leq \exp N \). Then if \( \{ x_1, \ldots, x_n \} \in \text{RIS}_Z(\nu; \kappa) \) we have \( \sum_{j=1}^{n} x_j \|_Z \leq (\kappa + 1) n f(n)^{-1} \).

**Proof.** The key point, proved in [9], Lemma 9, is that there exists \( g \in \mathcal{F} \) with \( f^{1/2} \leq g \leq f \) such that \( g(x) = f(x) \) for log \( N \) \( \leq n \leq \exp N \) and \( g(k) = f^{1/2}(k) \) when \( k \in \mathbb{P}_2 \). Thus if \( x \in \mathcal{E}_0 \) and \( \| x \|_Z \leq \| x \|_\infty \) then

\[ \| x \|_Z \leq \sup \{ \langle E x, x^* \rangle : x^* \in \mathcal{H}_Z(g; M), M \geq 2 \}. \]

Now, by the preceding lemma if \( x = \sum_{j=1}^{n} x_j \) and \( E \) is any interval then \( \| E x \|_{\kappa} < 1 \). We can therefore apply Lemma 4.3 to obtain the result.

The next lemma is simply a cruder form of Lemma 11 from [9].

**Lemma 4.6.** Suppose \( \kappa \geq 2 \) and \( N \in \mathbb{P}_2 \). If \( x \in A_2(\mathbb{N}, \kappa) \) then \( x \in \lambda(\sqrt{N}, (\kappa + 1)/2) \).

**Proof.** Suppose \( \{ x_j \}_{j=1}^{2N} \in \text{RIS}_Z(\mathbb{N}, \kappa) \) and that \( x = \sum_{j=1}^{N} x_j \|_Z^{-1} \times \sum_{j=1}^{N} x_j \). We break \([1, N]\) into \( \sqrt{N} \) intervals \( E_j \) each of length \( \sqrt{N} \), which is an index by hypothesis. Note that \( x_j \in \text{RIS}_Z(\sqrt{N}, \kappa) \).\( y_j = \sum_{i \in E_j} x_i \) then, by Lemma 4.5, \( \| y_j \|_Z \leq (\kappa + 1) \sqrt{N} \). Also \( \sum_{j=1}^{N} x_j \|_Z \geq N/f(N) \). By the lower \( f \)-estimate on \( X \). Now \( z = (1/\sqrt{N}) \sum_{j=1}^{N} x_j \) where \( x_j \in (\sum_{j=1}^{N} x_0 \|_Z^{-1}) \sqrt{N} y_j \). But \( \| x_j \|_Z \leq (\kappa + 1)(N f(N)/(N f(\sqrt{N})) \leq 2(\kappa + 1) \).

Our next result is a modification of Lemma 12 of [9]. In fact, this lemma appears to be incorrectly stated in [9] and so some modification is necessary. In the proof of the lemma in [9] it is claimed without justification that \( \{ x_1, \ldots, x_k \} \) is an RIS of length \( k \) and constant \( 1 + \varepsilon \). For the applications some modification similar to that given below seems adequate, however.

**Lemma 4.7.** Let \( \kappa \geq 2 \). Suppose \( k \in \mathbb{P}_1 \) with \( f(k) > 100\kappa^2 \). Suppose \( E_1, \ldots, E_k \) are intervals with \( E_1 \leq \cdots \leq E_k \). Let \( \{ x_1, \ldots, x_k \} \) be a special sequence with \( \text{supp} x_i \subseteq E_i \). Let \( M_i = p_{x_i} \) and \( M_{i+1} = \sigma(x_1^*, \ldots, x_j^*) \) for \( 1 \leq j \leq k - 1 \). Let \( A \) be any subset of \( \{ 1, 2, \ldots, k \} \) and suppose for each \( f \in A \) we have \( x_f \in \mathcal{E}_0 \) with \( \text{supp} x_f \subseteq \mathcal{E}_f \) so that \( x_i, x_j \) are disjoint and \( x_f \in A(M_i, \kappa) \). Then

\[ \sum_{i \in A} x_i \|_Z \leq 16 \kappa f(k)^{-1} \]

**Proof.** We have \( x_j \in \lambda_2(\mathbb{R}^M, 4\kappa^2) \), by Lemma 4.6. Note that \( \sqrt{M_i} = \kappa f(k)^{-1} \). We also have \( \sqrt{M_{j+1}} > g^{-1}(\kappa^2) \).

Now assume \( A \) contains no two consecutive integers. Then if \( f \in A \) we have \( \sqrt{M_f} \geq 2g(x_i \|_Z)^2 \) for \( j \geq 2 \) and so \( \{ x_f \}_{f \in A} \in \text{RIS}_Z(A, 4\kappa^2) \). As in [9] we use Lemma 4.3.

Note first that there exists \( h \in \mathcal{F} \) with \( f^2 \leq h \leq f \) so that \( h(n) = \sqrt{f(n)} \) if \( n \in \mathbb{P}_2 \setminus \{ k \} \) while \( h(n) = f(n) \) if \( n \in \mathbb{P}_2 \). This fact follows from Lemma 9 of [9].

Let \( x = \sum_{i \in A} x_i \) and suppose that for some interval \( E \) we have \( \| E x \|_{1/2} \geq 1 \), and

\[ \| E x \|_{1/2} > \sup \{ \langle E x, x^* \rangle : x^* \in \mathcal{H}_Z(g; M), M \geq 2 \}. \]

Since \( h \leq f \) this implies that \( \| E x \|_{1/2} > \| E x \|_{1/2} \). On the other hand, since \( \{ x_f \}_{f \in A} \in \text{RIS}_Z(A, 4\kappa^2) \) we can apply Lemma 4.4 to deduce that \( \| E x \|_{1/2} > \| E x \|_{1/2} \). The conclusion is that \( \| E x \|_{1/2} = \| E x \|_{1/2} \). Thus there is a special sequence \( \{ x_1^*, \ldots, x_k^* \} \), with \( i \in \mathbb{P}_1 \), such that

\[ \| E x \|_{1/2} = f(l)^{-1/2} \langle E x, x_i \|_Z \rangle. \]

However, \( f(l)^{-1/2} = h(l) \) unless \( i = k \). We conclude \( l = k \) and

\[ 1 \leq \| E x \|_{1/2} \leq f(k)^{-1/2} \sum_{i \in A} \langle x_i \|_Z \rangle. \]

Let \( l \) be the greatest integer so that \( x_i^* = x_i^* \) (with \( t = 0 \) if no such integer exists). If \( i < l \) it is clear that \( \langle x_i, x_i^* \rangle = 0 \) for all \( j \). Similarly if \( i \geq l \) it is also clear that \( \langle x_i, x_i^* \rangle = 0 \) for all \( i \). If \( t = r \), then \( x_i, x_i^* = 0 \) unless \( f = t + 1 \) when of course \( \langle x_i, x_i^* \rangle \leq 1 \). If \( t + 1 \leq i \in A \) and \( t + 1 \leq j \leq k \) then, unless \( t + 1 = i = j \), we have \( x_i \in A_2(M_i, \kappa) \) and \( x_i^* \in H_2(g; M_i^*) \) where \( M_i, M_i^* \in \mathbb{P}_2 \) are not equal. It follows from the separation conditions
on $\mathcal{P}_2$ that we can apply either Lemma 4.1 or Lemma 4.2; if $M'_j < M_i$, then by Lemma 4.1,
\[
\langle x_i, z^*_j \rangle \leq 24\kappa f(M'_j)^{-1} \leq 24\kappa f(p_{2k})^{-1},
\]
or if $M'_j > M_i$, then $M'_j \geq 2^{\log_2 M_i}$ and by Lemma 4.2,
\[
\langle x_i, z^*_j \rangle \leq 2\kappa f(M_i)/M_i \leq 2\kappa f(p_{2k})^{-1}.
\]
In either case we have $\langle x_i, z^*_j \rangle \leq 2\kappa^{-2}$. If $i = j = i + 1$ then $\langle x_i, z^*_j \rangle \leq 1$.

Hence
\[
\left\langle \sum_{c \in \mathcal{A}} a_i \sum_{f=1}^k z^*_f \right\rangle \leq 2 + \kappa \leq 3\kappa.
\]

This implies that
\[
\|Ex\|_2 \leq 3\kappa f(k)^{-1/2} < 3/10
\]
contrary to assumption. The conclusion from Lemma 4.3 is then that
\[
\|\pi\|_2 \leq 8\kappa|A|\hbar(A)|^{-1} \leq 8\kappa f(k)^{-1}.
\]

The general result follows by splitting $A$ into two subsets obeying the condition that no two consecutive integers are contained in either.

5. **The main result.** We now let $X = Z^*$ and consider the indicator $\Phi_X$. We will need the elementary fact, which follows from duality, that $X$ satisfies an upper $f$-estimate, i.e. if $x_1 < \ldots < x_n \in \mathbb{C}_0$ then $\|x_1 + \ldots + x_n\|_X \leq f(n)\max_{1 \leq i \leq n} |x_i|_X$. It also follows from the definition of $Z$ that if $x_1, \ldots, x_n$ is a special sequence (with $n \in \mathcal{P}_1$) then $\|x_1 + \ldots + x_n\|_X \leq f(n)^{1/2}$.

Our main result, which combined with the results of Section 2 establishes Theorems 1.1, 1.3 and 1.4, is the following:

**Theorem 5.1.** For every infinite-dimensional subspace $G$ of $c_0$ we have
\[
\sup\{|\Phi_X(u)| : \|u\|_1 = 1, u \in G\} = \infty.
\]

**Remark.** The following proof has been substantially simplified according to a suggestion of B. Maurey.

**Proof of Theorem 5.1.** We will start from the assumption that there is a subspace $G$ of infinite dimension so that $|\Phi_X(u)| \leq |X|u_1$ for $u \in G$. We may suppose that if $u \in G$ then $\langle u, x \rangle = 0$ where $x$ is the constantly zero sequence. Then by induction we can pick $\xi_1 < \xi_2 < \xi_3 < \ldots$ in $G$ with $\|\xi_j\|_1 = 1$. We split $\xi_j$ into positive and negative parts: $\xi_j = c^n_j - c^n_j^*$, where $c^n_j, c^n_j^*$ are disjoint and non-negative. Then $c^n_j, c^n_j^* \in D$. We let $R$ be the union of the supports of the $c^n_j$ and $S$ be the union of the supports of the $c^n_j^*$. Let $W$ be the linear span of $\{c^n_j\}_{j=1}^\infty$.

Notice first that $X$ satisfies an upper $f$-estimate on blocks where $f(x) = \log_2(x + 1)$. If $\gamma > 0$ and $n \in \mathbb{N}$ we define $\Gamma(n, \gamma)$ to be the set of $w \in D$ such that there exist $w_1 < \ldots < w_n \in D$ with $w = (1/n)(w_1 + \ldots + w_n)$ and $(1/n)\Delta_i(w_1, \ldots, w_n) \leq \gamma$.

**Lemma 5.2.** Given any $m, n \in \mathbb{N}$ and $\delta > 0$ there exists $w \in W \cap \Gamma(n, \delta)$ with $\|w\| < a(w)$.

**Proof.** For $n \in \mathbb{N}$ let $c_n$ be the infimum of all constants $\gamma$ so that if $m \in \mathbb{N}$ there exists $w \in W \cap \Gamma(n, \gamma)$ with $\|w\| < a(w)$. It is easy to see that $c_{n+1} \geq c_n + \gamma$ for any $n$, and that from Lemma 3.1, $c_n \leq f(n)$. Hence $pc_n \leq \gamma c_n \leq f(n)$ and so letting $p \to \infty$ we obtain $c_n = 0$ for all $n$ and the lemma follows.

We now turn to estimates on the Lozanovskij factorization of $w \in \Gamma(n, \delta)$.

**Lemma 5.3.** For fixed $n$ and $0 < \epsilon < 1/2$ there exists $\eta > 0$ so that if $w \in \Gamma(n, \eta)$ and $w = \lambda\pi$ is the Lozanovskij factorization of $w$, then there exists $A \subset [0(w), b(w)]$ with $\|Aw\|_1 > 1 - \epsilon$ and such that $\|A\pi^*\|_2, \|A\pi^*\|_2 < \lambda_\eta(n, 2)$.

**Proof.** If $w \in \Gamma(n, \delta)$ then $w = (1/n)\sum_{i=1}^n w_i$ where $w_1 < \ldots < w_n \in D$ are such that $(1/n)\Delta_1(w_1, \ldots, w_n) \leq \delta$. Let $w_i = x_i \pi^*$ be the Lozanovskij factorizations of each. Let $\pi_0 = \pi_1 + \ldots + \pi_n$. If $\epsilon > 1$ then $A = \{j : \pi_0(j) \leq \pi_n(j) \leq \pi_0(j) + \epsilon > 0\}$. Then $A\pi^* \leq c/n^2 A[x_1 + \ldots + x_n]^*$ and hence if $A_1 = nA[0(w), b(w)]$ then $\|A_1\pi^*\|_2 \leq c/n$. Now $\|A\pi^*\|_2 \geq \|Aw\|_1$ and so $\|A\pi^*\|_2, \|A\pi^*\|_2 \leq \lambda_\eta(n, c')$ where $c' < \epsilon\|Aw\|_1^{-1}$. Now, according to Lemma 3.5, if $\delta > 0$ is sufficiently small we can choose $c'$ close enough to $1$ so that the conclusions follow.

Using the preceding lemma we describe a construction. Suppose $N \in \mathcal{P}_2$ and $\epsilon > 0$. Then given any $m \in \mathbb{N}$ in any $M_1 \geq 2^{\log_2 N^2 + 4}$ we can construct two sequences $\{w_j\}_{j=1}^N$ and $\{c_j\}_{j=1}^N$ and a sequence of integers $(M_j)_{j=1}^N$ so that

1. $m < a(w_j)$,
2. $w_1 < c_1 < c_2 < \ldots < c_N$,
3. $w_i \in \Gamma(M_j, w_j) \cap W$ where $0 < \eta_i < \epsilon$ is sufficiently small so that there exists $A_j \subset [a(w_j), b(w_j)]$ with $\|A_jw_j\|_1 > 1 - \epsilon$ and $x_j = \|A_j c_j^*\|_2, \|A_j c_j^*\|_2 \leq \lambda_\eta(M_j, 2)$ where $x_j = x_j c_j^*$ is the Lozanovskij factorization of $w_j$,
4. $c_j \in \lambda_\eta(M_j, 2)$,
5. $M_j+1 > 2^d|c_j|^a$.

We will call the resulting sequence $\{w_j\}_{j=1}^N$ an $(\mathbb{N}, \epsilon)$-sequence, and $w = (1/N)(w_1 + \ldots + w_N)$ the associated $(\mathbb{N}, \epsilon)$-average. The sequence $\{c_j\}_{j=1}^N$ is called the ballast sequence; it is present simply for technical reasons to
on $P_2$ that we can apply either Lemma 4.1 or Lemma 4.2; if $M_j' < M_i$, then by Lemma 4.1,
\[
\langle x_i, z_i^* \rangle \leq 24 \kappa f(M_i) M_i^{-1} \leq 24 \kappa f(p_{2k})^{-1},
\]
or if $M_j' > M_i$, then $M_j' \geq 2^{8k} M_i^2$ and by Lemma 4.2,
\[
\langle x_i, z_i^* \rangle \leq 2 \kappa f(M_i) M_i \leq 2 \kappa f(p_{2k}) p_{2k}.
\]
In either case we have $\langle x_i, z_i^* \rangle \leq k^{-2}$. If $i = j = i + 1$ then $\langle x_i, z_i^* \rangle \leq 1$.
Hence
\[
\left( \sum_{i \in A} \sum_{j=1}^k z_i^* \right) \leq 2 + \kappa \leq 3 \kappa.
\]
This implies that
\[
\|Ex\|_2 \leq 3 \kappa f(k)^{-1/2} < 3/10
\]
contrary to assumption. The conclusion from Lemma 4.3 is then that
\[
\|\pi\|_2 \leq 8 \kappa f(A) h(A)^{-1} \leq 8 k \kappa f(k)^{-1}
\]
The general result follows by splitting $A$ into two subsets obeying the condition that no two consecutive integers are contained in either.

5. The main result. We now let $X = X'$ and consider the indicator $\Phi_X$. We will need the elementary fact that follows from duality, that $X$ satisfies an upper $f$-estimate, i.e. if $x_1 < \cdots < x_n \in G_0$ then $\|x_1 + \cdots + x_n\|_X \leq f(n) \max_{1 \leq n} |x_i|_X$. It also follows from the definition of $Z$ that if $x_1, \ldots, x_n$ is a special sequence (with $n \in P_1$) then $\|x_1 + \cdots + x_n\|_X \leq f(n)^{1/2}$.

Our main result, which combined with the results of Section 2 establishes Theorems 1.1, 1.3 and 1.4, is the following:

**Theorem 5.1.** For every infinite-dimensional subspace $G$ of $c_0$ we have
\[
\sup \{ |\Phi_X(u)| : \|u\|_1 = 1, u \in G \} = \infty.
\]

**Remark.** The following proof has been substantially simplified according to a suggestion of B. Maurey.

**Proof of Theorem 5.1.** We will start from the assumption that there is a subspace $G$ of infinite dimension so that $|\Phi_X(u)| \leq X_1/\|u\|_1$ for $u \in G$. We may suppose that if $u \in G$ then $\langle u, \chi \rangle = 0$ where $\chi$ is the constantly one sequence. Then by induction we can pick $\xi_1 < \xi_2 < \xi_3 < \cdots \in G$ with $|\xi_i|_1 = 1$. We split $\xi_i$ into positive and negative parts: $\xi_i = \xi_i^+ - \xi_i^-$, where $\xi_i^+ \xi_i^-$ are disjoint and non-negative. Then $\xi_i \xi_i^+ \in D$. We set $R$ to be the union of the supports of the $\xi_i$ and $S$ be the union of the supports of the $\xi_i^-$. Let $W$ be the linear span of $\{\xi_i \xi_i^+ \}_i$.

Notice first that $X$ satisfies an upper $f$-estimate on blocks where $f(x) = \log_2(x + 1)$. $\gamma > 0$ and $n \in N$ we define $\Gamma(n, \gamma)$ to be the set of $w \in D$ such that there exist $w_1 < \cdots < w_n \in D$ with $w = (1/n)(w_1 + \cdots + w_n)$ and $(1/n)\Delta(w_1, \ldots, w_n) < \gamma$.

**Lemma 5.2.** Given any $m, n \in N$ and $\delta > 0$ there exists $w \in W \cap \Gamma(n, \delta)$ with $m < a(w)$.

**Proof.** For $n \in N$ let $c_n$ be the infimum of all constants $\gamma$ so that if $m \leq n$ there exists $w \in W \cap \Gamma(n, \gamma)$ with $m < a(w)$. It is easy to see that $c_{np} \leq c_m + c_p$ for any $n, p$ and that from Lemma 3.1, $c_n \leq f(n)$. Hence $pc_n \leq c_m \leq f(m)$ and so letting $p \to \infty$ we obtain $c_n = 0$ for all $n$ and the lemma follows.

We now turn to estimates on the Lozanovskii factorization of $w \in \Gamma(n, \delta)$.

**Lemma 5.3.** For fixed $n$ and $0 < \varepsilon < 1/2$ there exists $\eta > 0$ so that if $w \in \Gamma(n, \eta)$ and $w = \sum_{i=1}^{\infty} x_i^* w_i$ is the Lozanovskii factorization of $w$, then there exists $A \subset [a(w), b(w)]$ with $\|Ax\|_1 > 1 - \varepsilon$ and such that $\|Ax\|_1 > 1 - \varepsilon$ and that the conclusions follow.

**Proof.** If $w \in \Gamma(n, \delta)$ then $w = (1/n)\sum_{i=1}^{\infty} x_i^* w_i$ where $w_1 < \cdots < w_n \in D$ such that $(1/n)\Delta(w_1, \ldots, w_n) \leq \delta$. Let $w_i = x_i^* w_i$ be the Lozanovskii factorizations of each. Let $y = x_1 + \cdots + x_n$. If $y > 1$ let $A = \{j : \|y_j\|_2 \leq \varepsilon \xi_j, -\xi_j \subset 0\}$. Then $\|Ax\|_1 \leq \varepsilon$ and hence $A_1 = A \cap [a(w), b(w)]$ then $\|A_1 x\|_2 \leq c/n$. Now $\|A_1 x\|_2 \geq \|Ax\|_1$ and so $\|A_1 x\|_2 / \|A_1 x\|_1 \leq \lambda_2(n, c')$ and $c' \leq c \|\|A\|\|_1^{-1}$. Now, according to Lemma 3.5, if $\delta > 0$ is sufficiently small we can choose $c$ close enough to 1 so that the conclusions follow.

Using the preceding lemma we describe a construction. Suppose $N \in P_2$ and $\delta > 0$. Then given any $m \in N$ and any $M_i > 2^{2M_2^2 + 4}$ we can construct two sequences $\{w_j\}_{j=1}^{N_1}$ and $\{\xi_j\}_{j=1}^{N_1}$ and a sequence of integers $(M_i)_{i=1}^{\infty}$ so that
\begin{enumerate}
\item $m < a(w_j)$,
\item $w_1 < \xi_1 < w_2 < \xi_2 < \ldots < w_N < \xi_N$,
\item $w_j \in \Gamma(M_j, \eta_j) \cap W$ where $\eta_j < \varepsilon$ is sufficiently small so that there exists $A_j \subset [a(w), b(w)]$ with $\|A_j w_j\|_1 > 1 - \varepsilon$ and $x_j = |A_j x_j|_2 / \|A_j x_j\|_1 \leq \lambda_2(M_j, 2)$ where $w_j = x_j x_j^*$ is the Lozanovskii factorization of $w_j$,
\item $w_j \in \lambda_2(M_j, 2)$
\item $M_j+1 > 2^{4M_2^2}$
\end{enumerate}

We will call the resulting sequence $\{w_j\}_{j=1}^{N_1}$ an $(N, \varepsilon)$-sequence, and $w = (1/N)(w_1 + \cdots + w_N)$ the associated $(N, \varepsilon)$-average. The sequence $\{\xi_j\}_{j=1}^{N_1}$ is called the ballast sequence; it is present simply for technical reasons to
provide ballast in the argument. Let $H$ be the union of the supports of the ballast sequence.

**Lemma 5.4.** Suppose $\{w_1, \ldots, w_N\}$ is an $(N, \varepsilon)$-sequence as above with associated $(N, \varepsilon)$-average $w$ and ballast $\{\zeta_j\}_{j=1}^N$. Then there is a subset $A$ of $\{a(w), b(w)\}$ and $w \in \mathcal{H}_g(f; N) \cap \mathcal{Q}_+$ with $\supp x \subseteq \supp w$ so that

1. $\|Aw\|_{1} > 1 - \varepsilon$,
2. if $B \subset A$ then there exists $z \in \mathcal{A}_g(\mathcal{N}, 4)$ supported in $B \cup H$ so that $Bw \leq 10\delta z$,
3. if $B \subset A$ then $(Bw, \log z) > \Phi_{X}(Bw) - 4$.

**Proof.** Notice that $y = (1/\Phi(f(N))) (x_1 + \ldots + x_N) \in \mathcal{H}_g(f; N)$ and $\|y\|_1 \leq 1$, since $X$ has an upper $f$-estimate. Choose $x$ with rational coefficients so that $y/2 \leq x \leq y$. Let $A = A_1 \cup \ldots \cup A_N$ so that (6) immediately holds.

We recall that $x_j \in \mathcal{A}_g(M_j, 2)$ (condition (3)) for $1 \leq j \leq N$. It follows easily that if $B$ is a subset of $A$ then we can find $0 \leq \alpha_j \leq 1$ so that $\|Bw\|_{1} + \alpha_j \|x\|_{1} = 1$ and then $Bx_j^* + \alpha_j \zeta_j \in \mathcal{A}_g(M_j, 4)$. The sequence $\{Bx_j^* + \alpha_j \zeta_j\}_{j=1}^N$ thus belongs to $\mathcal{R}_g(N, 4)$ since $M_j > 2^{S(N, 4)}$ and so $\|\sum_{j=1}^N (Bx_j^* + \alpha_j \zeta_j)\|_{1} \leq 5N/\Phi(f(N))$, from Lemma 4.5.

Let $z$ be the normalized vector $(\sum_{j=1}^N Bx_j^* + \alpha_j \zeta_j)$ where, by the above, $z \geq f(N)/(5N)$. Then $z \in \mathcal{A}_g(N, 4)$ and $xz \geq y/2 \geq Bw_0/10$. This proves (7).

For (8) we notice that Lemma 3.4 now implies that $\Phi_{X}(Bw), (Bw, \log z) \leq 10/\varepsilon < 4$.

Let us suppose that $n \in \mathcal{P}_1$ is fixed and large, say $f(n) > \exp(8K + 4000)$, and let $\varepsilon = (\log f(n))^{-1}$. Let $M_1 = p_2 n$; we can construct an $(M_1, \varepsilon)$-sequence $\{w_i\}_{i=1}^M$ with $(M_1, \varepsilon)$-average $w_1 = M_1^{-1} \sum_{j=1}^M w_j$ and ballast $\{\zeta_j\}_{j=1}^M$. Let $x_1 \in \mathcal{Q}_+ \cap \mathcal{H}_g(f; M_1)$ and $A_1 \subset [a(w_1), b(w_1)]$ be such that the conclusions of Lemma 5.4 hold.

Next let $M_2 = \sigma(Rx_1)$ and construct an $(M_2, \varepsilon)$-sequence $\{w_i\}_{i=1}^M$ with $(M_2, \varepsilon)$-average $w_2$ and ballast $\{\zeta_j\}_{j=1}^M$ so that $\zeta_1, \ldots, \zeta_n$. Repeating this construction for $n$ steps we obtain sequences $(w_{i,j})_{i=1}^{M_j}$, $(\zeta_{i,j})_{i=1}^{M_j}$ for $i = 1, \ldots, n$, $(w_{i,j})_{i=1}^{M_j}$, $(\zeta_{i,j})_{i=1}^{M_j}$ for $i = 1, \ldots, n$, and $(x_{i,j})_{i=1}^{M_j}$ so that

1. $(w_{i,j})_{i=1}^{M_j}$ is an $(M_j, \varepsilon)$-sequence with associated $(M_j, \varepsilon)$-average $w_i$ and ballast $\{\zeta_{i,j}\}_{j=1}^{M_j}$ for $i \leq j \leq n$,
2. $w_1 < \zeta_{1,M_1}$, $w_2 < \zeta_{2,M_2}$, \ldots, $w_n < \zeta_{n,M_n}$,
3. $A_1 \subset [a(w_1), b(w_1)]$ for $1 \leq i \leq n$ and $|A_i w_i| > 1 - \varepsilon_i$,
4. $\supp x_i \subseteq \supp w_i$, $x_i \in \mathcal{H}_g(f_i; M_i) \cap \mathcal{Q}_+$, and so $\|x_i\|_1 \leq 1$,
5. $(Bw_i, \log x_i) > \Phi_{X}(Bw_i) - 4$ whenever $B \subset A_i$,
6. for any $B \subset A_i$ there exists $z \in A_g(M_i, 4)$ with $Bw_i \leq 10\delta z$,
7. $\Phi_{X}(Bw_i) \geq \log f(n)/(4) + 1$.
8. Thus $(1/n) \Delta(Pu_1, \ldots, Pu_n) \leq (1/2) \log f(n) + 8n$. Now $(1/n) \sum_{i=1}^n \|Pu_i\|_1 > 1 - 2\varepsilon$ so that by Lemma 3.3, and the choice of $\varepsilon$,
9. $\log f(n) + 11$.
10. On the other hand, by Lemma 5.4 we can find $z_i \in A_g(M_i, 4)$ supported on $(\supp w_i) \cap Q) \cup H_i$ so that $Qw_i \leq 10\delta z_i$. At this point we can invoke Lemma 4.7. Let $E_i = [a(w_i), b(w_i)]$ and notice that $Rw_i(x_i, z_i)$ both supported in $E_i$, but are disjoint. Since $f(n) > 1000$, $Rw_i, z_i$ is a special sequence and $z_i \in A_g(M_i, 4)$ where $M_i = p_2 n$ and $M_{i+1} = \sigma(x_i, z_i)$ for $1 \leq j \leq n - 1$ we can conclude that $\|\sum_{i=1}^n x_i\|_1 \leq 64n f(n)^{-1}$. At the same time, by the upper $f$-estimate on $X$, $\|\sum_{i=1}^n x_i\|_1 \leq f(n)$. Now we have

$$
\frac{1}{640} \log f(n) \leq \left( \frac{1}{f(n)} \right)^{n} \frac{n}{64n} \sum_{i=1}^{n} \zeta_i
$$

and we can apply Lemma 3.4 again to deduce that

$$
\Phi_{X}(Qw_i) \leq \sum_{i=1}^{n} (Qw_i, \log x_i) - \log f(n) \|Qw_i\|_1 + 640,
$$

and so, since $\|Qw_i\|_1 > 1/2 - \varepsilon$,

$$
\frac{1}{n} \Delta(Qw_i, \ldots, Qw_n) \geq \log f(n) \|Qw_i\|_1 - 640 \geq \frac{1}{2} \log f(n) - 640.
$$

Now recall that $Qw_i = 2w_i$. Hence $(1/n) \Delta(Qw_i, \ldots, Qw_n) \geq \log f(n) - 1282$ and we can apply Lemma 3.3 to deduce that

$$
\frac{1}{n} \Delta(u_1, \ldots, u_n) \geq \log f(n) - 1283.
$$

Notice that $u_i - u_j \in G$ for $1 \leq i \leq n$. Now we have $\|\Phi_{X}(u_i - u_j) \leq 2K$, for $1 \leq i \leq n$, and $\|\Phi_{X}(x_i - u_i) \leq 2K$. Hence $\|\Phi_{X}(u_i) - \Phi_{X}(u_j) \leq 2K + 8e^{-1} \leq \log f(n) - 1283$.
Lemma 5.4. Suppose \( \{w_1, \ldots, w_N\} \) is an \((N, \epsilon)\)-sequence as above with associated \((N, \epsilon)\)-average \( w \) and ballast \( \{\chi_j\}_1^M \). Then there is a subset \( A \) of \( [\lambda(w), \beta(w)] \), with \( \lambda(x) = \max(\lambda(\{w_i\}_1^n)\), \( \beta(x) = \max(\beta(\{w_i\}_1^n)\), and \( x \in H \), such that \( \forall \beta \geq f(N)/2N \), \( \beta \) is supported in \( B \) and \( \|w\|_1 \leq 12\beta \).

Proof. Notice that \( f = \frac{1}{2}(\frac{1}{f(N)} - \frac{1}{f(N)^2}) \times (w_1 + \ldots + w_N) \in H \), and \( \|w\|_1 \leq 1 \), since \( H \) has an upper \( f \)-estimate. Choose \( x \) with rational coefficients so that \( x \leq \frac{1}{f(N)} \). Let \( A = A_1 \cup \ldots \cup A_\ell \) so that \( \ell \) is immediately.

We recall that \( \chi_j \) is an \((M_j, \epsilon)\)-sequence as above with associated \((M_j, \epsilon)\)-average \( w_j \) and ballast \( \{\chi_j\}_1^M \). Then there is a subset \( A \) of \( \{w_1, \ldots, w_N\} \), such that \( \forall \beta \geq f(N)/2N \), \( \beta \) is supported in \( B \) and \( \|w\|_1 \leq 12\beta \).

Let us assume that \( n \in P \) is fixed and large, say \( f(n) > \exp(8K + 4000) \), and let \( \epsilon = \frac{1}{4}(f(n)^{-1}) \). Let \( M_1 = p_2 \), and let us construct an \((M_1, \epsilon)\)-sequence \( \{w_1\}_1^M \) with \( \lambda(w_1) = M_1^{-1} \sum_{j=1}^M \chi_j \), \( \beta(w_1) = \beta(w_1) \), and \( \{\chi_j\}_1^M \).

Next let \( M_2 = \sigma(R_{\epsilon}) \) and construct an \((M_2, \epsilon)\)-sequence \( \{w_2\}_1^M \) with \( \lambda(w_2) = M_2^{-1} \sum_{j=1}^M \chi_j \), \( \beta(w_2) = \beta(w_2) \), and \( \{\chi_j\}_1^M \).

Then we have the following:\n
(15) \( M_{i+1} = \sigma(R_{\epsilon}) \) for \( 1 \leq i \leq n - 1 \).

We also have\n
(16) \( \{R_{\epsilon}, \ldots, R_{\epsilon} \} \) is a special sequence of \( n \)-length \( u \) in \( X = Z \).

Let \( H \) be the union of the supports of the ballast at the \( i \)-th step. Let \( A = \bigcup_{j=1}^\ell A_j \) and then set \( P = A \cap R \) and \( Q = A \cap S \). We also define \( u_i = 2R_{u_i} \) (so that \( u_i, u_i' \in D \)) and then set \( w = \frac{1}{n} \langle u_1 + \ldots + u_n \rangle \) and \( w = \langle u_1 + \ldots + u_n \rangle \). If we set \( x = \frac{1}{2}(n)^{-1/2} \sum_{i=1}^n \chi_i \), then \( \|x\|_1 \leq 1 \), since \( \{R_{\epsilon}, \ldots, R_{\epsilon} \} \) is a special sequence. Hence, using (13) above,

\( \Phi_x(Pw) \geq \frac{1}{n} \sum_{i=1}^n \Phi_x(P_{u_i}) - 1/2 \geq f(n) \|Pw\|_1 - 4 \).

Thus \( (1/2)n \Delta(u_1, \ldots, u_n) \leq (1/2) \geq f(n) \). Now \( (1/2)n \sum_{i=1}^n \|P_{u_i}\|_1 \geq 1 - 2 \epsilon \) so that by Lemma 3.3, and the choice of \( \epsilon \),

(17) \( \frac{1}{n} \sum_{i=1}^n \Delta(u_1, \ldots, u_n) \leq \frac{1}{2} f(n) + 11 \).

On the other hand, we can find \( z \in \{R_{\epsilon}, \ldots, R_{\epsilon} \} \) supported on \( (\supp u_i) \cap \supp \) so that \( \supp u_i \leq 10z \). At this point we can invoke Lemma 4.7. Let \( E_j = [a(u_j), b(u_j)] \) and notice that \( R_{u_j}, z_j \) are both supported in \( E_j \), but are disjoint. Since \( f(n) \geq 1000, R_{u_1}, \ldots, R_{u_n} \) is a special sequence and \( x \in \{R_{\epsilon}, \ldots, R_{\epsilon} \} \), where \( \lambda(x) = \frac{1}{n} \sum_{i=1}^n z_i \), \( \lambda(x) \leq 64K/n \). At the same time, by the upper \( f \)-estimate on \( X \), \( \|z\| \leq f(n) \). Now we have

\( \frac{1}{n} \sum_{i=1}^n z_i \leq 64K/n \).

and we can apply Lemma 3.4 again to deduce that

\( \Phi_x(Qw) \leq \sum_{i=1}^n \Phi_x(u_i) - \log f(n) \|Qw\|_1 + 64K \).

and so, since \( \|Qw\|_1 \geq 1/2 - \epsilon \),

(18) \( \frac{1}{n} \sum_{i=1}^n \Delta(u_1, \ldots, u_n) \geq \log f(n) \|Qw\|_1 - 640 + \frac{1}{2} f(n) - 641 \).

Now recall that \( Qw = 2Qw \). Hence \( (1/2)n \Delta(u_1, \ldots, u_n) \geq \log f(n) - 1282 \) and we can apply Lemma 3.3 to deduce that

(19) \( \frac{1}{n} \sum_{i=1}^n \Delta(u_1, \ldots, u_n) \geq \log f(n) - 1283. \)

Notice that \( u_1 - u_\epsilon \in F \) for \( 1 \leq i \leq n \). Now we have \( |\Phi_x(u_1 - u_\epsilon) \leq 2K, \) for \( 1 \leq i \leq n, \) and \( |\Phi_x(u_\epsilon - u_\epsilon) \leq 2K. \) Hence \( |\Phi_x(u_1) - \Phi_x(u_\epsilon) \leq 2K + 8e^{-1} \leq 2K + 8e^{-1} \leq 2K + 8e^{-1} \).
$2K + 3$ for $1 \leq i \leq n$ and similarly $|\Phi_X(x) - \Phi_X(v)| \leq 2K + 3$. This implies that

$$\frac{1}{n} \Delta(v_1, \ldots, v_n) - \frac{1}{n} \Delta(u_1, \ldots, u_n) \leq 4K + 6.$$

Combining with (17) and (18) gives that $\log f(n) \leq 8K + 2600$, which contradicts our initial choice of $n$ and completes the proof of Theorem 5.1.

It is perhaps worth noting at this point that it is very simple to modify our example so that Theorem 1.1 holds with $L$ of any specified dimension.

**Theorem 5.5.** For any $n \in \mathbb{N}$ there is a quasi-Banach space $Y^{(n)}$ with a subspace $L$ of dimension $n$ so that $Y/L$ is isomorphic to $\ell_1$, and if $Y_n$ is a closed infinite-dimensional subspace of $Y^{(n)}$ then $L \subseteq Y_n$.

**Proof.** Let $A_k = \{u_j + k\}^n_{j=0} \subseteq \mathbb{N}$, for $k = 1, \ldots, n$. Define $S_k : c_{00} \to c_{00}$ by $S_k u = \sum_{j=0}^n u(j) e_{u_j + k}$. Define $\Phi : c_{00} \to c_{00}$ by $\Phi(u) = \{\Phi_X(S_k u)\}_{k=1}^n$. Then let $Y^{(n)}$ be the completion of $c_{00} \oplus c_{00}$ under the quasi-norm

$$\|\xi, u\| = \|\xi - \Phi(u)\|_{c_{00}} + \|u\|.$$

Let $L$ be the space of all $(\xi, 0)$ for $\xi \in c_{00}$. Clearly $Y^{(n)}/L$ is isomorphic to $\ell_1$. Now suppose $Y_n$ is a proper infinite-dimensional subspace of $Y$ so that $L \cap Y_n = \{0\}$. Then the non-trivial linear functional $f$ on $c_{00}$ so that $Y_n \cap L \subseteq Z = f^{-1}(0)$. Suppose $f(\xi) = \sum_{k=1}^n \beta_k e_k$. It is easy to verify that $Y/Z$ is isomorphic to the completion of $c_{00} \oplus c_{00}$ under the quasi-norm $\|\xi, u\|_{c_{00}} = |\xi| + \|u\|$ where $\Phi(u) = \sum_{k=1}^n \beta_k \Phi_X(S_k u)$. However, there is a constant $K$ depending only on $\beta_1, \ldots, \beta_n$ so that $|\Phi(u) - \Phi_X(S_k u)| \leq K \|u\|$. It follows easily that $\Phi$ is unbounded on every infinite-dimensional subspace of $c_{00}$ and hence that $Y_n + Z \subseteq L$ must contain $L/Z$, which is a contradiction to the fact that $Y_n \cap L = \{0\}$.

### 6. Some final remarks.

In this short final section we will present a proof of Theorem 1.2, which first appeared in [14], a reference which may now be readily available. Our proof here is slightly shorter. We begin with a lemma.

**Lemma 6.1.** Suppose $X$ is a quasi-Banach space with a dense subspace $V$ with (HBE). Suppose $L = \{x \in X : x^*(z) = 0 \forall z \in X^*\}$. Then:

1. If $L = \{0\}$, so that $X$ has a separating dual, then $X$ is locally convex.
2. If $X$ contains a basic sequence then $X$ is locally convex.
3. If $M$ is a closed subspace of $L$ then $X/M$ has a dense subspace with (HBE).

**Proof.** (1) [cf. [11]] Let $\|\cdot\|_c$ be the Banach envelope norm on $X$, i.e. $\|z\|_c = \sup\{x^*(z) : \|x^*\| \leq 1\}$. If $X$ is not locally convex we may choose $v_n \in V$ with $\|v_n\|_c \leq 4^{-n}$ and $\|v_n\| = 1$. Pick any $x \in V$ and consider the sequence $w_n = x + 2^{-n}x$. Then (see Theorem 4.7 of [16]) there is a subsequence $(w_{n_k})$ which is a Markushevich basis for its closed linear span in $X$. Pick $n_0$ large enough so that $x \notin [w_{n_0} : k \geq n_0]$. Then by (HBE) for $V$ there is a linear functional $x^* \in X^*$ with $x^*(w_{n_0}) = 0$ for $k \geq n_0$ but $x^*(x) = 1$. However, $\lim_{n \to \infty} \|x - 2^{-n}v_n\| = 0$ so that $x^*(x) = 0$, contrary to hypothesis.

(2) Pick any $u \in L$; we will show $u = 0$. Assume then that $u \neq 0$. Suppose $w \in V$ is non-zero, and $u, w$ are linear independent. Since $X$ contains a basic sequence and $V$ is dense in $X$ we can apply standard perturbation arguments to suppose that we have a bounded basic sequence $(x_n)$, with $x_n \in (u + w) + V$, any $x_n = n(u + w) - v_n$ where $v_n \in V$. Then there exists $n_0$ so that $[u, w] \cap [x_n]_{n \geq n_0} = (0)$. Thus there is a bounded linear functional $f$ on the span $Y$ of $u, w$ and $[x_n]_{n \geq n_0}$ with $f(u) = 1$, $f(w) = 0$ and $f(x_n) = 0$ for $n \geq n_0$. Since $V$ has (HBE) there is a bounded linear functional $x^* \in X^*$ on $x^*(v) = f(v)$ for $v \in V \cap Y$. Thus $x^*(w) = 0$ and $x^*(v_n) = -n$ since $u \in L$. Hence $x^*(x_n) = -n$, contradicting the boundedness of $x^*$. Now since $L = \{0\}$ we can apply (1) to deduce that $X$ is locally convex.

(3) Let $\pi : X \to X/M$ be the quotient map; we show $\pi(V)$ has (HBE). Indeed, if $E \subseteq \pi(V)$ is a subspace and $f$ is a continuous linear functional on $E$ then we can find $x^* \in X^*$ so that $x^*(v) = f(\pi(v))$ for $v \in \pi^{-1}E \cap V$. But then $x^*(x) = 0$ if $x \in M \subseteq L$ so that $x^*$ factors to a linear functional on $X/M$.

**Theorem 6.2.** Suppose $X$ is a decomposable quasi-Banach space. If $X$ has a dense subspace $V$ with (HBE) then $X$ is locally convex.

**Proof.** Let $P$ be a bounded projection on $X$ so that both $P$ and $Q = I - P$ have infinite rank. If $L$ is defined as in the previous lemma then $L$ is clearly invariant for $P$. From the hypotheses, $X^*$ has infinite dimension and hence so has $X/L$. Therefore either $P(X)/P(L)$ or $Q(X)/Q(L)$ has infinite dimension. Suppose the former; then consider $X/P(L)$, which has a dense subspace with (HBE) by Lemma 6.1(3). Then $P(X)/P(L)$ is isomorphic to a subspace of $X/L$ which has separating dual; since it has infinite dimension, it contains a basic sequence. By Lemma 6.1(2) this implies that $X/P(L)$ is locally convex and hence that $Q(X)$ is locally convex. But now $X$ itself must contain a basic sequence and Lemma 6.1(2) shows that $X$ is locally convex.

Let us conclude by mentioning that in [14] we raised the question of whether every quasi-Banach space $X$ with separating dual has a weakly closed subspace $W$ and a bounded linear functional $f$ on $W$ which cannot be extended to $X$. We proved that this is equivalent to the following:
$2K + 3$ for $1 \leq i \leq n$ and similarly $|\Phi_X(z) - \Phi_X(v)| \leq 2K + 3$. This implies that

$$\frac{1}{n} \Delta(v_1, \ldots, v_n) - \frac{1}{n} \Delta(u_1, \ldots, u_n) \leq 4K + 6.$$ 

Combining with (17) and (18) gives that $\log f(n) \leq 8K + 2600$, which contradicts our initial choice of $n$ and completes the proof of Theorem 5.1.

It is perhaps worth noting at this point that it is very simple to modify our example so that Theorem 1.1 holds with $L$ of any specified dimension.

**Theorem 5.5.** For any $n \in \mathbb{N}$ there is a quasi-Banach space $Y^{(n)}$ with a subspace $L$ of dimension $n$ so that $Y/L$ is isomorphic to $\ell_1$ and if $Y_0$ is a closed infinite-dimensional subspace of $Y^{(n)}$ then $L \not\subset Y_0$.

**Proof.** Let $A_k = \{\xi_k \mid k \leq 0\} \subseteq \mathbb{N}$ for $k = 1, \ldots, n$. Define $S_k : c_{00} \to c_{00}$ by $S_ku = \sum_{j=0}^\infty u(j)\epsilon_{j+k}$. Define $\Phi : c_{00} \to c_{00}$ by $\Phi(u) = \{\Phi_X(S_ku)\}_k$. Then let $Y^{(n)}$ be the completion of $c_{00} \oplus c_{00}$ under the quasinorm

$$\|\xi, u\|_0 = \|\xi\|_\infty + \|u\|_1.$$ 

Let $I$ be the space of all $(\xi, 0)$ for $\xi \in c_{00}$. Clearly $Y^{(n)} \oplus I$ is isomorphic to $\ell_1$. Now suppose $Y_0$ is a closed infinite-dimensional subspace of $Y$ so that $Y_0 \cap I$ is a proper subspace of $I$. Then there is a non-trivial linear functional $f$ on $c_{00}$ so that $Y_0 \cap I \subset Z = f^{-1}(0)$. Suppose $f(\xi) = \sum_{k=1}^n \beta_k \xi_k$. It is easy to verify that $Y/\mathbb{K}$ is isomorphic to the completion of $\mathbb{K} \oplus c_{00}$ under the quasinorm $\|(\sigma, u)\|_0 = |\sigma|_\infty + \|u\|_1$ where $\|\sigma\|_0 = \sum_{k=0}^{\infty} |\sigma_k|_1 \Phi_X(S_ku)$. However, there is a constant $K$ depending only on $\beta_1, \ldots, \beta_n$ so that $\|\xi, u\|_0 = \Phi_X(S_ku) \geq \|u\|_1$. It follows easily that $\mathbb{K}$ is unbounded on every infinite-dimensional subspace of $c_{00}$ and hence that $Y_0 + Z/\mathbb{K}$ must contain $I$, which is a contradiction to the fact that $Y_0 \cap I$ is contained in $Z$.

6. Some final remarks. In this short final section we will present a proof of Theorem 1.3, which first appeared in [14], a reference which may not be readily available. Our proof here is slightly shorter. We begin with a lemma:

**Lemma 6.1.** Suppose $X$ is a quasi-Banach space with a dense subspace $V$ with (HBEP). Suppose $L = \{x \in X : x^*(x) = 0 \forall x^* \in X^*\}$. Then:

1. If $L = \{0\}$, so that $X$ has a separating dual, then $X$ is locally convex.
2. If $X$ contains a basic sequence then $X$ is locally convex.
3. If $M$ is a closed subspace of $L$ then $X/M$ has a dense subspace with (HBEP).

**Proof.** (1) (cf. [11]) Let $\|\cdot\|_e$ be the Banach envelope norm on $X$, i.e. $\|x\|_e = \sup\{|x^*(x)| : \|x^*\|_1 \leq 1\}$. If $X$ is not locally convex we may choose

$$u_n \in V \text{ with } \|u_n\|_e \leq 4^{-n} \text{ and } \|v_n\|_e = 1.$$ 

Pick any $x \in V$ and consider the sequence $w_n = u_n + 2^{-n}x$. Then (see Theorem 4.7 of [16]) there is a subsequence $(w_{n_k})$ which is a Markushevich basis for its closed linear span in $X$. Pick $n_0$ large enough so that $x \notin \{w_{n_k} : k \geq n_0\}$. Then by (HBEP) for $V$ there is a linear functional $x^* \in X^*$ with $x^*(w_{n_k}) = 0$ for $k \geq n_0$ but $x^*(x) = 1$. However, $\lim_{n \to \infty} |x^* - 2^{-n}u_n|_e = 0$ so that $x^*(x) = 0$, contrary to hypothesis.

(2) Pick any $u \in L$; we will show $w = 0$. Assume then that $u \neq 0$. Suppose $w$ is non-zero, and $u, w$ are linearly independent. Since $X$ contains a basic sequence and $V$ is dense in $X$ we can apply standard perturbation arguments to suppose that we have a bounded basic sequence $(x_n)_n$ with $x_n \in u(u + w) + V$, say $x_n = n(u + w) - v_n$ where $v_n \in V$. Then there exists $n_0$ so that $[u, w] \cap [x_n]_{n \geq n_0} \neq (0)$. Thus there is a bounded linear functional $f$ on the span $\mathbb{K}$ of $u, w$ and $[x_n]_{n \geq n_0}$ with $f(u) = 1$, $f(w) = 0$ and $f(x_n) = 0$ for $n \geq n_0$. Since $V$ has (HBEP) there is a bounded linear functional $x^* \in X$ on $x^*(v) = f(v)$ for $v \in V \cap \mathbb{K}$. Thus $x^*(w) = 0$ and $x^*(v_n) = -n$ also $x^*(w) = 0$ since $u \in L$. Hence $x^*(x_n) = -n$, contradicting the boundedness of $x^*$. Now since $L = \{0\}$ we can apply (1) to deduce that $X$ is locally convex.

(3) Let $\pi : X \to X/M$ be the quotient map; we show $\pi(V)$ has (HBEP). Indeed, if $F \subset \pi(V)$ is a subspace and $f$ is a continuous linear functional on $E$ then we can find $x^* \in X^*$ so that $x^*(v) = f(\pi(v))$ for $v \in \pi^{-1}F \cap V$. But then $x^*(x) = 0$ if $x \in M \subset L$ so that $x^*$ factors to a linear functional on $X/M$.

**Theorem 6.2.** Suppose $X$ is a decomposable quasi-Banach space. If $X$ has a dense subspace $V$ with (HBEP) then $X$ is locally convex.

**Proof.** Let $P$ be a bounded projection on $X$ so that both $P$ and $Q = I - P$ have infinite rank. If $L$ is defined as in the previous lemma then $L$ is clearly invariant for $P$. From the hypotheses, $X^*$ has infinite dimension and hence so has $X/L$. Therefore either $P(X)/P(L)$ or $Q(X)/Q(L)$ has infinite dimension. Suppose the former; then consider $X/P(L)$, which has a dense subspace with (HBEP) by Lemma 6.1(3). Then $P(X)/P(L)$ is isomorphic to a subspace of $X/L$ which has separating dual; since it has infinite dimension, it contains a basic sequence. By Lemma 6.1(2) this implies that $X/P(L)$ is locally convex and hence that $Q(X)$ is locally convex. But now $X$ itself must contain a basic sequence and Lemma 6.1(2) shows that $X$ is locally convex.

Let us conclude by mentioning that in [14] we raised the question of whether every quasi-Banach space $X$ with separating dual has a weakly closed subspace $W$ and a bounded linear functional $f$ on $W$ which cannot be extended to $X$. We proved that this is equivalent to the following:
The basic sequence problem.

References


The basic sequence problem.

Problem. Suppose $X$ is a quasi-Banach space with separating dual and suppose that every quotient $X/E$ by an infinite-dimensional subspace $E$ is locally convex. Is $X$ locally convex?

Of course our main example $Y$ has every quotient $Y/E$ by an infinite-dimensional subspace locally convex, but fails to have a separating dual.

References


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