Plurisubharmonic functions on quasi-Banach spaces

by

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Abstract. We study conditions under which a quasi-Banach space can be equipped with an equivalent plurisubharmonic quasi-norm. We show this is equivalent to validity of a weak form of the Maximum Modulus Principle for analytic functions valued in the space. We examine the relationship between the existence of a plurisubharmonic quasi-norm and the existence of "good" tensor products. We also prove that in a quasi-Banach algebra the spectral radius is plurisubharmonic, extending a theorem of Vesentini, and give some applications to the study of holomorphic functions on non-locally convex spaces.

1. Introduction. In [8] Eiter observed that in the spaces $L_p$ the natural quasi-norm

$$
\|f\| = \left\{ \int |f(t)|^p \, dt \right\}^{1/p}
$$

is plurisubharmonic, i.e. for $f, g \in L_p$

$$
\|f\| \leq (2\pi)^{-\frac{1}{2}} \int_0^{2\pi} \|f + e^{i\theta} g\| \, d\theta.
$$

Recently several authors have considered this notion. Aleksandrov [1] calls a complex quasi-Banach space locally holomorphic if it has an equivalent plurisubharmonic quasi-norm. Peetre [17] calls such a space locally pseudo-convex and Davis, Garling and Tomczak-Jaegermann [5] call a space equipped with a plurisubharmonic quasi-norm PL-convex. See also Edgar [7].


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An $A$-convex space satisfies the Maximum Modulus Principle. Let $A$ denote the open unit disc in $C$ and let $T$ be the unit circle so that $A = A \cup T$. Suppose $X$ is $A$-convex and $f : \bar{A} \to X$ is a continuous map which is analytic on $A$ (cf. [14], [21] or Section 2 for the definition of analytic functions in a non-locally convex setting). Then we have

$$\|f(0)\| \leq C \max_{w \in T} \|f(w)\|$$

where $C$ is a constant independent of $f$. Peetre [17] asks essentially whether the converse is true, and one of our first results (Theorem 4.1) shows that this is the case, i.e. if $X$ satisfies the Maximum Modulus Principle then $X$ is $A$-convex. We refer to [14] for a general discussion.

We relate $A$-convex spaces to the so-called natural spaces introduced by the author in [12]. A quasi-Banach space is natural if and only if it embeds into an $l_p$-product of $L_p$-spaces for some fixed $p$, $0 < p < 1$ (cf. Theorem 4.2 below). Natural spaces are $A$-convex and the converse is true for lattices (Theorem 4.4; compare the earlier results of Peetre [17]). We give an example to show that the converse is not true in general.

In Section 5, we first prove an annular version of the Maximum Modulus Principle which holds for every quasi-Banach space. Using this we study first quasi-Banach algebras and show that a theorem of Vesentini [23] can be generalized to this setting, so that the spectral radius is always a pluriharmonic function. Thus quasi-Banach algebras have a special structure amongst general quasi-Banach spaces.

In Section 6, we study conditions on a $p$-normable space $X$ so that whenever $Y$ is a $p$-normable space there exists a $p$-normed space $Z$ and a bilinear form $B : X \times Y \to Z$ so that

$$k \|x\| \|y\| \leq \|B(x, y)\| \leq k \|x\| \|y\|, \quad x, y \in X.$$ 

This is related to the existence of a $p$-normed tensor product. We show that if $X$ satisfies these conditions for some fixed $p$, $0 < p < 1$, then $X$ is $A$-convex; conversely, if $X$ is natural then it will satisfy these conditions for small enough $p$.

In Section 7, we sketch the foundations of a theory of holomorphic functions on a complex quasi-Banach space. A sample result is that, essentially, every holomorphic function on an $A$-trivial space is entire. Here an $A$-trivial space is a space such as $L_p/H_p$ ($0 < p < 1$) which admits no continuous operators into an $A$-convex space.

I would like to thank Jean Bourgain for suggesting the example in Section 4 and Stephen Dilworth for many helpful comments.

2. Basic definitions. Throughout this paper all vector spaces are assumed to be complex. If $X$ is a vector space then a map $x \mapsto \|x\|$ ($X \to \mathbb{R}_+$) is called a quasi-seminorm if

(i) $\|ax\| = |a| \|x\|$, $a \in C$, $x \in X$,

(ii) $\|x_1 + x_2\| \leq C \|x_1\| + \|x_2\|$, $x_1, x_2 \in X$,

where $C$ is a constant independent of $x_1, x_2$.

We call $\|\|\|$ a $p$-seminorm where $0 < p \leq 1$ if in addition

(iii) $\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p$, $x_1, x_2 \in X$.

If further $\|x\| = 0$ implies $x = 0$ then $\|\|\|$ is called a quasi-norm (if (i) and (ii) are satisfied) or a $p$-norm (if (i), (ii) and (iii) are satisfied). Since every quasi-norm is equivalent to a $p$-norm for some $p$, $0 < p \leq 1$, we shall always assume that quasi-norms are $p$-norms for some $p$ and are, in particular, continuous.

If $\|\|\|$ is a quasi-norm on $X$ defining a complete metrizable topology then $X$ is called a quasi-Banach space.

An upper-semicontinuous function $\varphi : X \to [-\infty, \infty)$ is called plurisubharmonic if for every $x_1, x_2 \in X$

$$\varphi(x_1) \leq \int \varphi(x_1 + w x_2) d\lambda(w)$$

where $\lambda$ denotes the normalized Haar measure on $T$, i.e. $d\lambda = (2\pi)^{-1} d\theta$.

If the quasi-norm $\|\|$ on a quasi-Banach space $X$ is plurisubharmonic then $X$ is called PL-convex by Davis, Garling and Tomczak-Jaegermann [5]. If $X$ is equivalently normed with a plurisubharmonic quasi-norm then we shall say that $X$ is $A$-convex (the term locally pseudo-convex and locally holomorphic have been used by Peetre [17] and Aleksandrov [1]). We also say that $X$ is $A$-trivial if there are no nonzero continuous plurisubharmonic quasi-seminorms on $X$. Thus $L_p (0 < p < 1)$ is $A$-convex (Etter [8]) and $L_p/H_p$ is $A$-trivial (Aleksandrov [1]). Clearly if $X$ is $A$-trivial and $Y$ is $A$-convex then $L_p(Y, X) = \{0\}$ as noted by Aleksandrov [1].

If $\varOmega$ is an open subset of $C$ then a map $f : \varOmega \to X$ is called analytic if for every $z_0 \in \varOmega$ there exists $\delta > 0$ so that $f$ can be expanded in a power series for $|z - z_0| < \delta$.

$$f(z) = \sum_{n=0}^{\infty} x_n(z - z_0)^n, \quad |z - z_0| < \delta.$$ 

Clearly $x_n = (1/n!) f^{(n)}(z_0)$ (see [14] and [21]). $A_0(X)$ will denote the space of all functions $f : A \to X$ which are continuous on $A$ and analytic on $A$ where $A = \{z : |z| < 1\}$.

$X$ is said to satisfy the Maximum Modulus Principle (MMP) if there exists a constant $M > 0$ so that for all $f \in A_0(X)$ we have

$$\|f(0)\| \leq M \max_{|z|=1} \|f(z)\|.$$ 

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Clearly if this is the case we can show also
\[ \|f(z_0)\| \leq M \max_{|z|=1} \|f(z)\| \]
for any \( z_0, |z_0| < 1 \). We also noted that it suffices to establish (**) for polynomials, since these are dense in \( A_0(X) \) with the topology of uniform convergence on \( \Delta \). Then if \( f_r(z) = f(rz) \) for \( 0 < r < 1 \) then \( f_r \) can be expanded uniformly in a power series and \( \max_{|z|=1} \|f_r(z) - f(z)\| \to 0 \).

Spaces satisfying (MMP) are considered by Peetre [17]. The following lemma will be useful later on.

**Lemma 2.1.** Let \( X \) be a quasi-Banach space and let \( f, g : X \to \mathbb{R}_+ \), \( G : X \to \mathbb{R}_+ \) be two functions. Suppose that \( f \) is upper-semicontinuous, and satisfies \( F(\alpha x) = |\alpha| F(x) \) for \( x \in C, \alpha \in X \). Then the following conditions on \( G \) are equivalent:

1. For every polynomial \( f : C \to X \)
   \[ G(f(0)) \leq \max_{|z|=1} F(f(z)). \]
2. For every \( f \in A_0(X) \)
   \[ G(f(0)) \leq \max_{|z|=1} F(f(z)). \]
3. For every \( f \in A_0(X) \)
   \[ G(f(0)) \leq \exp \left\{ \int_{T} \log F(f(w))d\lambda(w) \right\}. \]

**Proof.** (i) \( \Rightarrow \) (ii). This is a simple density argument.

(ii) \( \Rightarrow \) (iii). Suppose that \( f \in A_0(X) \) and that \( K = f(T) \). Let \( H \) be any continuous function on \( K \) so that \( H(x) > F(x) \) for all \( x \in K \). Note in particular \( H(x) > 0 \) for \( x \in K \) and hence \( \log H \) is continuous on \( K \). Thus \( \log H(f(w)) \) is continuous on \( T \); if we define
\[ u(z) = -\int_{T} \left( \frac{w+z}{w-z} \right) \log H(f(w))d\lambda(w) \]
for \( |z| < 1 \) and set \( u(z) = -\log H(f(z)) \) for \( |z| = 1 \), then \( u \) is continuous on \( \Delta \).

Now define \( h \in H^\infty \) by
\[ h(z) = \exp \left\{ -\int_{T} \left( \frac{w+z}{w-z} \right) \log H(f(w))d\lambda(w) \right\}. \]
Then \( |h(z)| = \exp(u(z)) \) for \( z \in \Delta \).

For \( r < 1 \) set \( h_r(z) = h(rz) \). Then \( h(0)^{-1} h_r(z) f(z) \in A_0(X) \) and hence
\[ G(f(0)) \leq \max_{|z|=1} |h(0)|^{-1} |h_r(z)| F(f(z)). \]

Letting \( r \to 1 \) we obtain
\[ G(f(0)) \leq |h(0)|^{-1} \int_{T} H(f(w))d\lambda(w). \]

As \( H > F \) is arbitrary and \( F \) is u.s.c., the lemma follows.

**Lemma 2.2.** Let \( X \) be a quasi-Banach space and suppose \( F : X \to \mathbb{R}_+ \) is upper-semicontinuous and satisfies \( F(\alpha x) = |\alpha| F(x) \) for \( \alpha \in C, x \in X \). Then the following conditions on \( F \) are equivalent:

1. \( F \) is plurisubharmonic.
2. For every \( f \in A_0(X) \)
   \[ F(f(0)) \leq \max_{|z|=1} F(f(z)). \]
3. \( F \) is plurisubharmonic.

This is immediate from Lemma 2.1.

**3. Plurisubharmonic functions on a quasi-Banach space.**

**Lemma 3.1.** Let \( X \) be a quasi-Banach space and suppose \( f, g : C \to X \) are continuous functions. Suppose \( f, g : C \to X \) are continuous functions. Then
\[ \lim_{n \to \infty} F(f(w) + w^n g(w)) = \lim_{n \to \infty} F(f(w) + z(w))d\lambda(z) \]

**Proof.** We suppose
\[ f(z) = \sum_{k=0}^{N} x_k z^k, \quad g(z) = \sum_{k=0}^{N} y_k z^k, \]
and let \( X_0 \) be a finite-dimensional subspace of \( X \) containing \( f(T) + g(T) \). Let \( K = f(T) + g(T) \). Then we define linear functionals \( \mu_x : C(K) \to C \) and \( \mu : C(K) \to C \) by
\[ \mu_x(f) = \int_{T} F(f(w) + w^n g(w))d\lambda(w), \]
\[ \mu(f) = \int_{T} F(f(w) + z(w))d\lambda(z)d\lambda(w). \]

Note that \( \mu_x(1) = \mu(1) = 1 \) and \( \|\mu_x\| = \|\mu\| = 1 \). In order to show that \( \mu_x \to \mu \) weak* as required, we need only check on a dense subset of \( F \)'s. We therefore consider \( F \) of the form
\[ F(x) = \prod_{j=1}^{l} \phi_j(x) \prod_{j=0}^{m} \phi_j^{\infty}(x) \]
where \( \phi_1, \ldots, \phi_m \in X_0^\infty \). By the Stone-Weierstrass theorem the linear span of
such $F$, and 1, is dense in $C(K)$. Let

$$
\Phi(x) = \prod_{j=1}^{l} \phi_j(x) \prod_{j=l+1}^{m} \phi_j(x)
$$

for $x \in X_{0}$. $\Phi$ is then linear in the first $l$ coordinates and conjugate-linear in the remaining $m-l$. Note $F(x) = \Phi(x, \ldots, x)$.

For $A \subset \{1, 2, \ldots, l\}$ and $B \subset \{l+1, l+2, \ldots, m\}$ set $h_j(w) = f(w)$ if $j \notin A \cup B$ and $h_j(w) = g(w)$ if $j \in A \cup B$. Then

$$
F_f + w^*g = \sum_{A,B} \prod_{j \in A} \phi_j h_j(w) \prod_{j \notin A} \phi_j(w).
$$

By the Riemann–Lebesgue lemma,

$$
\lim_{n \to \infty} \int F_f d\lambda(w) = \int \Phi h \, d\lambda(w).
$$

Similarly

$$
F_f + zg = \sum_{A,B} \prod_{j \in A} \phi_j h_j(w) \prod_{j \notin A} \phi_j(w)
$$

and so

$$
\int F_f d\lambda(z) = \sum_{A,B} \prod_{j \in A} \phi_j h_j(w) \prod_{j \notin A} \phi_j(w)
$$

and the lemma follows.

**Lemma 3.2.** Let $X$ be a normed quasi-Banach space. Let $\varphi: T \to X$ be a bounded Borel function. Suppose $\epsilon > 0$ and $0 < \gamma < 1$. Then there exists a polynomial $g: C \to X$ so that

(a) $g(0) = 0$.

(b) $\|g(x)\| < \epsilon$ for $\|x\| \leq \gamma$.

(c) There exists a Borel map $v: T \to T$ so that

$$
\int T \|g(w) - v(w)\| \varphi(w) \, d\lambda(w) < \epsilon^p.
$$

**Proof.** It clearly suffices to consider the case when $\varphi$ is simple, i.e. $\varphi(w) = \sum_{j=1}^{N} x_j 1_{E_j}$

where $x_j \in X (1 \leq j \leq N)$ and $E_1, \ldots, E_N$ are disjoint Borel sets in $T$ so that $E_1 \cup \ldots \cup E_N = T$.

For each $j$, there exists a polynomial $u_j: C \to C$ so that $u_j(0) = 0$ and

$$
\int T \|u_j(w) - 1_{E_j}(w)\| \varphi(w) \, d\lambda(w) < N^{-1} \epsilon^p \|\varphi\|_{C}^p.
$$

We let

$$
g(z) = z^m \sum_{j=1}^{N} u_j(z) x_j
$$

where $m$ is chosen so large that $\|g(z)\| < \epsilon$ for $\|z\| \leq \gamma$.

Let $v: T \to T$ be a Borel map satisfying

$$
v(w) = u_j(w), \quad w \in E_j.
$$

If $w \in E_k$ then

$$
\|g(w) - v(w)\| \varphi(w) \, d\lambda(w) = \sum_{j \in E_k} \|u_j(w) - 1_{E_j}(w)\| \varphi(w) \, d\lambda(w).
$$

Clearly $g$ satisfies the lemma.

We now suppose $X$ is $p$-normed and set $F_0(x) = \|x\|^p$. For each $n \in N$ define

$$
F_n(x) = \inf_{y \in X} \int F_{n-1}(x + wy) \, d\lambda(w), \quad G_n(x) = \inf_{y \in X} \int F_{n-1}(\varphi(w)) \, d\lambda(w)
$$

where the infimum is taken over all polynomials $\varphi$ such that $\varphi(0) = x$.

Note that if $x_1, x_2 \in X$ then

$$
|G_n(x_1) - G_n(x_2)| \leq \|x_1 - x_2\|^p.
$$

**Lemma 3.3.** For all $n \in N$ and $x_1, x_2$

$$
|F_n(x_1) - F_n(x_2)| \leq \|x_1 - x_2\|^p, \quad |G_n(x_1) - G_n(x_2)| \leq \|x_1 - x_2\|^p.
$$

This is an easy induction proof which we omit. In particular each $F_n, G_n$ is continuous. Note also that $F_n$ is monotone decreasing and so $\lim_{n \to \infty} F_n = F_{\infty}$ exists and further

$$
|F_{\infty}(x_1) - F_{\infty}(x_2)| \leq \|x_1 - x_2\|^p
$$

for $x_1, x_2 \in X$.

**Lemma 3.4 (Edgar [7]).** $F_{\infty}$ is plurisubharmonic.

We remark that this follows from the fact that

$$
F_{\infty}(x) \leq F_{n+1}(x) \leq \int F_{n}(x + wy) \, d\lambda(w)
$$

for all $n \in N, \, x, \, y \in X$. 
Lemma 3.5. For every \( n \in \mathbb{N} \), \( G_n = G_{n+1} \).

Proof. Suppose \( x \in X \) and \( \varepsilon > 0 \). Then there is a polynomial \( f \colon C \to X \) so that
\[
\int \frac{F_n(f(w)) \, d\lambda(w)}{f} < G_{n+1}(x) + \varepsilon.
\]
Now we can partition \( T \) into \( N \) disjoint Borel sets \( E_1, \ldots, E_N \) and find \( w_j \in E_j \) so that
\[
\|f(w) - f(w_j)\| \leq \varepsilon, \quad w \in E_j.
\]
Pick \( y_j \in X \) (\( 1 \leq j \leq N \)) so that
\[
\int \frac{F_n(f(w_j) + y_j) \, d\lambda(w)}{f} < G_n(f(w_j)) + \varepsilon.
\]
Thus
\[
\int \frac{\int F_{n-1}(f(w) + zy_j) \, d\lambda(z) \, d\lambda(w)}{E_j} < \int \frac{F_n(f(w)) \, d\lambda(w)}{E_j} + 3\varepsilon \lambda(E_j).
\]
Thus if we let \( \varphi \colon T \to X \) be defined by \( \varphi(w) = y_j \), \( w \in E_j \), we have
\[
\int \frac{\int F_{n-1}(f(w) + z\varphi(w)) \, d\lambda(z) \, d\lambda(w)}{E_j} < \int \frac{F_n(f(w)) \, d\lambda(w)}{E_j} + 3\varepsilon.
\]
Now use Lemma 3.2 to find a polynomial \( g \colon C \to X \) so that \( g(0) = 0 \) and
\[
\int \frac{\|g(w) - \varphi(w)\| \, d\lambda(w)}{f} < \varepsilon
\]
where \( \varphi \colon T \to T \) is a Borel function.

Let \( f_m(z) = f(z) + z^n g(z) \). Then
\[
\lim_{m \to \infty} \int \frac{F_{n-1}(f_m(w)) \, d\lambda(w)}{f} = \int \frac{\int F_{n-1}(f(w) + zg(w)) \, d\lambda(z) \, d\lambda(w)}{f} \quad \text{(by Lemma 3.1).}
\]
Now
\[
\int \frac{\int F_{n-1}(f(w) + zg(w)) \, d\lambda(z) \, d\lambda(w)}{f} < \varepsilon + \int \frac{F_{n-1}(f(w) + z\varphi(w)) \, d\lambda(z) \, d\lambda(w)}{f} \quad \text{(by Lemma 3.1).}
\]
Hence for large enough \( m \)
\[
\int \frac{F_{n-1}(f_m(w)) \, d\lambda(w)}{f} < \int \frac{F_n(f(w)) \, d\lambda(w)}{f} + 4\varepsilon < G_{n+1}(x) + 5\varepsilon.
\]
As \( \varepsilon > 0 \) is arbitrary, \( G_n(x) \leq G_{n+1}(x) \). As \( G_n \geq G_{n+1} \) trivially the lemma is proved.

Lemma 3.6. \( G_1(x) = F_\varphi(x) \) for \( x \in X \).

Proof. Clearly \( G_n \leq F_\varphi \) so that by Lemma 3.5, \( G_1 \leq F_\varphi \). Conversely, if \( \varphi \) is a polynomial then \( F_\varphi \circ \varphi \) is subharmonic and so
\[
\int \frac{F_\varphi(\varphi(0)) \, d\lambda(w)}{f} \leq \int \frac{F_\varphi(\varphi(w)) \, d\lambda(w)}{f} \leq \int ||\varphi(w)|| \, d\lambda(w).
\]
Hence \( F_\varphi \leq G_1 \).

Theorem 3.7. Let \( X \) be a p-normed quasi-Banach space. For \( x \in X \) define
\[
||x||_A = \inf_{d=1} \max_{||x||=1} ||\varphi(x)||
\]
where the infimum is taken over all \( \varphi \in A_0(X) \) so that \( \varphi(0) = x \). Then
(i) \( ||\cdot||_A \) is a p-seminorm on \( X \).
(ii) \( ||\varphi(0)||_A \leq \exp \int \log ||\varphi(w)|| \, d\lambda(w) \) for all polynomials \( \varphi \colon C \to X \).
(iii) \( ||\cdot||_A \) is plurisubharmonic.
(iv) \( \log ||\cdot||_A \) is plurisubharmonic.

Proof. (i) is trivial. (ii) follows from Lemma 2.1. (iii) From (ii) it follows that
\[
||\varphi(0)||_A \leq \int \frac{||\varphi(w)|| \, d\lambda(w)}{f}
\]
and hence \( ||x||_A \leq G_1(x) \), but \( G_1(x) \leq ||x||_A \) by definition so that \( ||\cdot||_A \) is plurisubharmonic.

(iv) This follows from Lemma 2.2.

We remark that \( ||\cdot||_A \) is a PL-convex quasi-seminorm in the sense of [4]. We note also that \( ||\cdot||_A \) is the largest plurisubharmonic function dominated by \( ||\cdot|| \). \( X \) is A-trivial if and only if \( ||x||_A = 0 \) for every \( x \in X \). \( X \) is A-convex if and only if \( ||\cdot|| \) and \( ||\cdot||_A \) are equivalent.

For a general space \( X \), we may form a space \( X_A \) by quotienting by \( N = \{ x : ||x||_A = 0 \} \) and then completing \( X/N \) with respect to \( ||\cdot||_A \). \( X_A \) is A-convex and it is easy to see that every operator \( T : X \to Y \) where \( Y \) is A-convex factors through \( X_A \). In particular, \( X \) is A-trivial if and only if every operator from \( X \) into an A-convex space is zero.

4. Natural spaces and A-convex spaces. Our first result is a very simple application of Theorem 3.7, which partially answers a question of Peetre [17].
Theorem 4.1. In order that a quasi-Banach space $X$ is $A$-convex it is necessary and sufficient that $X$ satisfies the Maximum Modulus Principle (MMP).

Proof. $X$ satisfies (MMP) if and only if $\|\cdot\|_A$ is equivalent to $\|\cdot\|$. Next we turn to quasi-Banach lattices motivated by the results of Peetre [17]. Here a complex quasi-Banach lattice $X$ is simply the complexification of a real lattice which we denote $\text{Re}X$. If $x_1, \ldots, x_n \geq 0$ then it is possible to define unambiguously $(x_1 \cdots x_n)^{1/n}$ as an element of $X$ using the Krivine calculus (see [16]). We recall that a quasi-Banach lattice is $L$-convex [12] if there exists $\delta > 0$ so that if $u \in X_+$, $\|u\| = 1$ and $0 \leq x_i \leq u$ are such that $(1/n)(x_1 + \ldots + x_n) \geq 1 - \delta u$ then

$$\max_{1 \leq i \leq n} \|x_i\| \geq \delta.$$ 

A quasi-Banach space is called natural [13] if it is isomorphic to a subspace of an $L$-convex quasi-Banach lattice. In [12] it is shown, essentially, that a natural quasi-Banach lattice is automatically $L$-convex so that this definition is consistent. The following theorem [13] characterizes natural spaces and can be thought of as supplying an alternative definition.

Theorem 4.2. Let $X$ be a quasi-Banach space. Then $X$ is natural if and only if there is a constant $M > 0$ and $q > 0$ so that if $x \in X$ is nonzero, there exists a probability space $(\Omega, \Sigma, P)$ and an operator $T: \Omega \to L_q(\Omega, \Sigma, P)$ with $\|T\| \leq M\|x\|^{-1}$ and $Tx = 1$. If $X$ is $p$-normal, any $q < p$ suffices.

Before our main theorem we prove a simple lemma.

Lemma 4.3. Let $(\Omega, P)$ be a probability space and suppose $f: \Omega \to \mathbb{R}$ is a random variable, $0 \leq f \leq 1$ a.e. Then

$$\delta \leq f \leq 1 \text{ a.e.} \implies \delta \leq f \leq 1 \text{ a.e.} \implies \delta \leq f \leq 1 \text{ a.e.}$$

Proof. Let $W \subseteq L_1(\Omega)$ be the set of $g$ so that $0 \leq g \leq 1$ and $\delta(g) = \delta(f)$. Note that for any $x, a \in R$, we have

$$\log x \leq \log a + \frac{1}{a}(x - a).$$

For all $g \in W$,

$$\delta(g) \leq \delta(f) + \delta(f^{-1}(g - f))$$

and so there is an extreme point $g$ so that

$$\delta(g) \leq \delta(f).$$

At an extreme point, $g = 0$ on a set of measure $\frac{1 - \delta(f)}{1 - \delta}$ and $g = 1$ on a set of measure $\frac{\delta(f) - \delta}{1 - \delta}$. Hence

$$\delta(f) \geq \frac{1 - \delta(f)}{1 - \delta} \log \theta.$$ 

Theorem 4.4. Let $X$ be a complex quasi-Banach lattice. Then the following conditions are equivalent:

(i) $X$ is $A$-convex.

(ii) There exists a constant $C$ so that if $x_1, \ldots, x_n \geq 0$ in $X$ then

$$\|x_1 \cdots x_n\| \leq C \max_{1 \leq i \leq n} \|x_i\|$$

(iii) $X$ is $L$-convex.

Proof. We begin with a well-known remark. Suppose $u \in X$ and $u \geq 0$. Then the order interval $[-u, u]$ in $\text{Re}X$ generates in a natural way an abstract Hausdorff space $\Omega = \Omega_u$ and a lattice embedding $\varphi: C_0(\Omega) \to \text{Re}X$ so that $\varphi u = u$, $\varphi$ extends naturally to a map $\varphi: C(\Omega) \to X$. We refer to $\varphi$ as the Katunari map associated to $u$.

(i) $\Rightarrow$ (ii). We suppose $X$ satisfies (MMP) with constant $M$. Suppose $x_1, \ldots, x_n \geq 0$ and $u = x_1 \vee \ldots \vee x_n$. Let $\varphi: C(\Omega) \to X$ be the Katunari map associated to $u$. Suppose $\delta > 0$.

Let $\varphi: T \to C(\Omega)$ be a $C^\infty$-map so that $\delta 1 \leq \varphi(w) \leq 1$ for all $w \in T$. We write $\varphi(w, s)$ for $\varphi(w)(s)$ when $w \in T$, $s \in \Omega$. For $x \in A$, $s \in \Omega$ set

$$f(x, s) = \exp\left\{\int \frac{w + z}{w - z} \log \varphi(w, s) d\lambda(w)\right\}.$$ 

Then $f$ is analytic into $C(\Omega)$ and extends continuously to $\bar{A}$. On $T$,

$$f(0, s) = \exp\left\{\int \log \varphi(w, s) d\lambda(w)\right\} = \varphi_0(s)$$

say. By the Maximum Modulus Principle in $X$,

$$\|\varphi_0\| \leq M \max_{|w| = 1} \|\varphi(w, s)\|.$$ 

Now pick $\xi_1, \ldots, \xi_n \in C(\Omega)$ so that $\varphi(\xi) = x_1 \vee \delta u$. Clearly $\delta 1 \leq \xi_1 \leq 1$. We may pick $C^\infty$-functions $v_1, \ldots, v_n: T \to [0, 1]$ so that

(a) $\sum_{j=1}^n v_j(w) = 1, \quad w \in T$. 

Thus there exists $k \leq n$ so that $C^p (||x_k||^p + \delta^{2p}) \geq \delta^{4p}$, i.e.

$$||x_k||^p \geq C^{-p} \delta^{4p} - \delta^{2p} \geq \delta^p.$$ 

Thus $X$ is $L$-convex.

(iii) $\Rightarrow$ (i). If $X$ is $L$-convex then there exists $q > 0$ and a constant $M > 0$ so that if $x \in X$, with $||x|| = 1$, there exists a probability space $(\Omega, \Sigma', P)$ and an operator $T_x : X \to L_q(\Omega, \Sigma')$ so that $T_x x = 1_\Omega$ and $||T_x|| \leq M$.

Note that $y \to M^{-1} ||T_x y||$ is a $q$-seminorm on $X$ dominated by $||\cdot||$ which is plurisubharmonic. Hence

$$M^{-1} ||T_x y|| \leq ||y||_q, \quad y \in X.$$ 

In particular, $M^{-1} \leq ||x||_q$ if $||x|| = 1$. Thus $||\cdot||_q$ is equivalent to $||\cdot||$.

**Corollary 4.5.** Every natural quasi-Banach space is $A$-convex.

We conclude this section by giving an example of an $A$-convex quasi-Banach space which fails to be natural. The example is the Schatten class $S_p$, where $0 < p < 1$; that this might provide an example was suggested to the author by Jean Bourgain.

We shall need first a preparatory theorem which is probably of independent interest. Let us recall that a quasi-Banach space $X$ is of type $p$ ($0 < p \leq 2$) if there is a constant $C$ so that

$$\left[ \left( \sum_{i=1}^n \epsilon_i x_i \right)^p \right]^{1/p} \leq C \left( \sum_{i=1}^n ||x_i||^p \right)^{1/p}$$

where $\epsilon_1, \ldots, \epsilon_n$ is any sequence of independent Bernoulli random variables with $P(\epsilon_1 = 1) = P(\epsilon_1 = -1) = 1/2$. For $0 < p < 1$, $X$ is of type $p$ if (and only if) $X$ is $p$-normable; if $p > 1$ and $X$ is of type $p$ then $X$ is a Banach space (cf. [15], p. 99 and p. 107).

**Theorem 4.6.** Suppose $X$ and $Y$ are quasi-Banach spaces; suppose $X$ is of type $p$ and $Y$ is of type $q$ where $0 < p, q \leq 2$. Suppose $0 < r < 1$. Then there is a constant $C = C(p, q, r, X, Y)$ so that if $(\Omega, \Sigma', \mu)$ is a probability measure space and $B : X \times Y \to L_1(\Omega, \Sigma', \mu)$ is a bilinear form then for $x_1, \ldots, x_n \in X$, $y_1, \ldots, y_n \in Y$

$$\left\| \sum_{j=1}^n B(x_j, y_j) \right\|_r \leq C \|B\| \left( \sum_{j=1}^n ||x_j||^p ||y_j||^q \right)^{1/n}$$

where $1/s = 1/p + 1/q$.

**Proof.** (In the ensuing argument, $C$ will represent a constant depending on $X$, $Y$ and $r$, which may vary from line to line.) Let $\epsilon_1, \ldots, \epsilon_n$, $\eta_1, \ldots, \eta_n$ be any sequence of $2n$ independent Bernoulli random variables defined on some probability space $(\Omega', \Sigma', P)$. Then for $\omega' \in \Omega'$

$$\left\| B \left( \sum_{j=1}^n \epsilon_j(\omega') x_j, \sum_{j=1}^n \eta_j(\omega') y_j \right) \right\|_r \leq \|B\| \left( \sum_{j=1}^n \epsilon_j(\omega') \right) \left( \sum_{j=1}^n \eta_j(\omega') \right) \left\| x_j \right\| \left\| y_j \right\|_r.$$
Let \( f_{jk} = B(x_j, y_k) \). Then
\[
\sigma\left(\| \sum_{j=1}^{n} \sum_{k=1}^{n} e_j \eta_k f_{jk} \| \right) \leq \| B \|^{\alpha} \sigma\left(\| \sum_{j=1}^{n} e_j x_j \| \right)^{\alpha} \sigma\left(\| \sum_{j=1}^{n} \eta_j y_j \| \right)^{\alpha}
\]
\[
= \| B \|^{\alpha} \sigma\left(\| \sum_{j=1}^{n} e_j x_j \| \right) \sigma\left(\| \sum_{j=1}^{n} \eta_j y_j \| \right)
\]
\[
\leq C \| B \|^{\alpha} (\sum_{j=1}^{n} \| x_j \|^{\alpha} (\sum_{j=1}^{n} \| y_j \|^{\alpha})^{\alpha/\beta}).
\]

Similarly
\[
\sigma\left(\| \sum_{j=1}^{n} \sum_{k=1}^{n} e_j \eta_k f_{jk} \| \right) \leq \| B \|^{\alpha} \sigma\left(\| \sum_{j=1}^{n} e_j x_j \| \right) \sigma\left(\| \sum_{j=1}^{n} \eta_j y_j \| \right)
\]
\[
\leq \| B \|^{\alpha} \sigma\left(\| \sum_{j=1}^{n} e_j x_j \|^{2\alpha} \right)^{1/2} \sigma\left(\| \sum_{j=1}^{n} \eta_j y_j \|^{2\alpha} \right)^{1/2}
\]
\[
\leq C \| B \|^{\alpha} \left( \sum_{j=1}^{n} \| x_j \|^{\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{\alpha} \right)^{1/\beta}.
\]

Now
\[
\left\| \sum_{j=1}^{n} \sum_{k=1}^{n} e_j \eta_k f_{jk} \right\|_r \leq \left( \sum_{j=1}^{n} \sum_{k=1}^{n} e_j (\omega) \eta_k (\omega) f_{jk}(\omega) \right)^{1/r} d\mu(\omega).
\]

By a generalization of Khintchine's inequality due to Bonami [3] (cf. also Pisier [19]) we have
\[
\sigma\left(\| \sum_{j=1}^{n} \sum_{k=1}^{n} e_j (\omega) \eta_k (\omega) f_{jk}(\omega) \| \right) \leq C \left( \sum_{j,k} \| f_{jk}(\omega) \|^{2\alpha} \right)^{1/2}
\]
and hence we conclude
\[
\left\| \left( \sum_{j=1}^{n} \sum_{k=1}^{n} f_{jk} \right)^{1/2} \right\|_r \leq C \| B \| \left( \sum_{j=1}^{n} \| x_j \|^{\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{\alpha} \right)^{1/\beta}.
\]

In particular
\[
\left\| \left( \sum_{j=1}^{n} |f_{jk}|^2 \right)^{1/2} \right\|_r \leq C \| B \| \left( \sum_{j=1}^{n} \| x_j \|^{2\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{2\alpha} \right)^{1/\beta}
\]
and hence
\[
\left\| \left( \sum_{j=1}^{n} f_{jk} + f_{kd} \right)^{1/2} \right\|_r \leq C \| B \| \left( \sum_{j=1}^{n} \| x_j \|^{2\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{2\alpha} \right)^{1/\beta}.
\]

Now
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} e_j \eta_k f_{jk} = \sum_{k=1}^{n} f_{kk} + \sum_{j,k}^* e_j \eta_j (f_{jk} + f_{kj}).
\]

Again by use of Bonami's theorem
\[
\left( \sum_{j,k} \| e_j \eta_k (f_{jk} + f_{kj}) \|^{1/\beta} \right)^{1/\alpha} \leq C \left( \sum_{j,k} \| f_{jk} + f_{kj} \|^{1/\beta} \right)^{1/\alpha}.
\]

Thus
\[
\left\| \sum_{k=1}^{n} f_{kk} \right\|_r \leq C \| B \| \left( \sum_{j=1}^{n} \| x_j \|^{1/\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{1/\alpha} \right)^{1/\beta}.
\]

Now for any \( \alpha_j > 0 \)
\[
\left\| \sum_{k=1}^{n} f_{kk} \right\|_r \leq C \| B \| \left( \sum_{j=1}^{n} \| x_j \|^{1/\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{1/\alpha} \right)^{1/\beta}.
\]

Let \( \alpha_j = \| y_j \|^{1/\alpha} / \| x_j \|^{1/\alpha} \). The result then follows.

**Corollary 4.7.** Suppose \( X \) is a quasi-Banach space of type \( p \), \( Y \) is a quasi-Banach space of type \( q \) and \( Z \) is a natural quasi-Banach space. Then there is a constant \( C = C(X, Y, Z) \) so that if \( B: X \times Y \rightarrow Z \) is a bilinear form then
\[
\left\| \sum_{j=1}^{n} B(x_j, y_j) \right\|_r \leq C \| B \| \left( \sum_{j=1}^{n} \| x_j \|^{2\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{2\alpha} \right)^{1/\beta}
\]
where \( 1/s = 1/p + 1/q \).

**Proof.** This is an immediate consequence of Theorem 4.2 and Theorem 4.6.

Now let \( H \) be a separable infinite-dimensional Hilbert space and let \( S_p \) denote the Schatten \( p \)-class where \( 0 < p < 1 \). Thus \( T \in S_p \) if and only if \( \| T \|_{L^p(H)} = \sum_{n} s_n(T)^{p} \) is a compact operator whose singular values \( s_n(T) \) satisfy
\[
\| T \|_{L^p(H)} = \sum_{n} s_n(T)^{p} \leq C \| B \| \left( \sum_{j=1}^{n} \| x_j \|^{\alpha} \right)^{1/\beta} \left( \sum_{j=1}^{n} \| y_j \|^{\alpha} \right)^{1/\beta}.
\]

\( S_p \) is a \( p \)-normable operator-ideal (cf. Pietsch [18], pp. 216, 255).

**Theorem 4.8.** Suppose that \( 0 < p < 1 \) and that \( Z \) is a natural quasi-Banach space. Then there is a constant \( C \) so that if \( T: S_p \rightarrow Z \) is a bounded linear operator then
\[
\| T(A) \| \leq C \| T \| \| A \|_{L^1}.
\]
and hence \( T \) factors in the form \( T = T_0 J \) where \( T_0: S_1 \rightarrow Z \) is bounded and \( J: S_p \rightarrow S_1 \) is the inclusion map.
Proof. Consider the bilinear form $B: H \times H^p \to Z$ given by

$$B(h, h^*) = T(h^* \otimes h).$$

Then $\|B\| \leq \|T\|$. Hence if $A \in S_p$ then we can write

$$A = \sum_{j=1}^m \sigma_j e_j^* \otimes f_j$$

where $\sum |\sigma_j|^p = \|A\|_p^p$, $\|e_j^*\| = \|f_j\| = 1$. Thus

$$T(A) = \sum_{j=1}^m \sigma_j B(f_j, e_j)$$

and hence

$$\|T(A)\| \leq C \|B\| \sum_{j=1}^m |\sigma_j| \leq C \|T\| \|A\|_{1,1}.$$  

**Corollary 49.** For $0 < p < 1$, $S_p$ is not natural.

**Theorem 4.10.** $S_p$ is $A$-convex for $0 < p < 1$; in fact $\|\cdot\|_p$ is plurisubharmonic.

**Proof.** It will suffice to show that if $A, B \in S_p$ are of finite rank and $F(z) = A + zB$, then $\|F(z)\|_p$ is subharmonic on $C$.

For any $z_0 \in C$ there is an isometry $U$ of $H$ so that $U F(z_0)$ is a positive hermitian operator. Let $H_0 = \Re \{ U F(z_0) \}$ and suppose $\dim H_0 = m = \text{rank } F(z_0)$. Let $P$ be the orthogonal projection of $H$ onto $H_0$ and define $G: C \to \mathcal{L}(H_0)$ by

$$G(z)(h) = P U F(z)(h), \quad h \in H_0.$$  

$G(z_0)$ is invertible on $H_0$ and has eigenvalues $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$. There exists $\delta > 0$ so that if $|z - z_0| < \delta$ then the spectrum of $G(z)$ is contained in some compact subset $C$ of $\{ z : Rez > 0 \}$. The function $\varphi(z) = z^p$ can be defined to be analytic on this half-plane with $\varphi(x) = x^p$ if $x > 0$.

Now consider $\varphi(G(z))$ for $|z - z_0| < \delta$. Precisely,

$$\varphi(G(z)) = (2\pi i)^{-1} \int \varphi(w) \left( w - G(z) \right)^{-1} dw$$

where $\gamma$ is any contour in the right half-plane around $C$. The map $z \to \varphi(G(z))$ is analytic and hence so is the map $z \to \text{tr} \varphi(G(z))$. If $0 < r < \delta$ then

$$\text{tr} \varphi(G(z_0)) = \int \text{tr} \varphi(G(z_0 + rw)) \, d\lambda(w).$$

Now $\text{tr} \varphi(G(z_0 + rw)) = \sum_{i=1}^m \sigma_i^r x_i$ where $x_1, \ldots, x_m$ are the eigenvalues of $G(z_0 + rw)$. Now

$$\sum |\sigma_i|^p \leq \|G(z_0 + rw)\|_{F^p} \leq \|F(z_0 + rw)\|_{F^p}$$

and hence

$$\text{tr} \varphi(G(z_0)) = \|G(z_0)\|_{F^p} = \|F(z_0)\|_{F^p} \leq \left( \int \|F(z_0 + rw)\|_{F^p} \, d\lambda(w) \right)^{\frac{1}{p}}$$

so that $\|F\|_{p^*}$ is plurisubharmonic.

**5. Applications to algebras.** We begin with a slight strengthening of a theorem of Coifman and Rochberg on Bergman spaces. Let us suppose $0 < q < p$, $\sigma = 1/4 - 1$ and $\nu = [\sigma]$. $B_{q, p}$ is the Bergman space of all analytic functions $\varphi$ on $A$ so that

$$\|\varphi\|_{q, p} = \int \varphi(z)^p (1 - |z|^2)^{\nu + 2} \, dm(z) < \infty,$$

where $m$ denotes the planar measure on $A$.

**Lemma 5.1.** If $0 < r < 1$ there is a constant $C = C(p, q, r)$ so that if $\varphi \in B_{q, p}$, then $\|\varphi\|_{q, p}$ is plurisubharmonic on $C$.

**Proof.** According to Theorem 2 of [4], there exist $\xi_k$, $k \geq 1$, with $|\xi_k| < 1$ and $|\xi_k| \to 1$, and $C = C(p, q)$ so that if $\varphi \in B_{q, p}$, then we can write

$$\varphi(w) = \sum_{k=1}^\infty \alpha_k \left( 1 - |\xi_k|^2 \right)^{1 + \sigma - \nu} (1 - w \xi_k)^{-\nu}$$

and

$$\sum_{k=1}^\infty |\alpha_k|^p \leq C \|\varphi\|_{q, p}.$$  

The $\xi_k$ are independent of $\varphi$.

Let $\Gamma$ denote the set of $z \in A$ with $|z| < 1$. We define a map $S: \Gamma \to B_{q, p}$ by

$$S(z) = \sum_{k=1}^\infty \alpha_k \left( 1 - |z|^2 \right)^{1 + \sigma - \nu} (1 - w \xi_k)^{-\nu}.$$  

Let $E$ be the range of $S$. Let $F$ be the finite-dimensional spanned by $(1 - w \xi_k)^{-\nu}$ for $|\xi_k| < r$. Then $E + F = B_{q, p}$.

Consider the function $\tilde{S} : I_p(\Gamma) \to F$. This map is an open mapping. Let $\varphi$ be any linear functional on $B_{q, p}$ so that $\varphi(E) = 0$. Then $\varphi(S \oplus f) = \varphi(f)$, where $S \oplus f$ is continuous. Hence $\varphi$ is continuous on $B_{q, p}$. Now let

$$h(x) = \varphi(1 - w z)^{-\nu}.$$  

$h$ is analytic on $A$ and vanishes for $r < |z| < 1$. Then $h = 0$ and so $\varphi = 0.$
Thus $E = B_{a,p}$ and $S$ is onto. The lemma follows by the Open Mapping Theorem.

Our next theorem is an annular maximum modulus principle valid for all quasi-Banach spaces.

We recall first that if $\sigma > 0$ and $v = [\sigma]$ then $A_{\sigma}(X)$ is defined to be the space of analytic functions $f : \Delta \to X$ satisfying

$$
\sup_{|z| \leq 1} (1 - |z|^2)^{v+1-\sigma} \|f^{(v+1)}(z)\| < \infty.
$$

It is shown in [14], Theorem 5.1, that if $f \in A_{\sigma}(X)$ and $X$ is $p$-normed then there exists $T \in L^p(B_{a,p}, X)$ such that $T((1-wz)^{-1}) = f(z)$.

**Theorem 5.2.** Let $X$ be a $p$-normed quasi-Banach space, and suppose $0 < r < 1$. Then there exists a constant $C = C(r, X)$ so that if $f \in A_{\sigma}(X)$ then

$$
\|f(0)\| \leq C \max_{|z| \leq r} \|f(z)\|.
$$

**Proof.** As above suppose $0 < q < p$, $\sigma = 1/q - 1$ and $v = [\sigma]$. For any $f \in A_{\sigma}(X)$, let

$$
f(z) = \sum_{n=0}^{\infty} x_n z^n, \quad z \in \Delta
$$

(this is permissible, see [21]), and

$$
F(z) = \sum_{n=0}^{\infty} x_n \frac{n!}{(n+v+1)!} z^{n+v+1}.
$$

Then $F \in A_{\sigma}(X)$ and so there is a bounded linear operator $T : B_{a,p} \to X$ so that

$$
T((1-wz)^{-1}) = F(z), \quad z \in \Delta
$$

(Theorem 5.1 of [14]). If $\varphi \in B_{a,p}$ suppose

$$
\varphi(w) = \sum_{n=0}^{\infty} a_n w^n.
$$

Note that $T(w^n) = 0$, $0 \leq k \leq v$, and hence $T\varphi = T(w^{v+1}\psi)$ where

$$
\psi(w) = \sum_{n=v+1}^{\infty} a_n w^{n-v-1}.
$$

Note that $\|\psi\|_{a,p} \leq C \|\varphi\|_{a,p}$ where $C = C(p, q)$. Now by Lemma 5.1 we can write

$$
\psi(w) = \sum_{k=1}^{\infty} a_k (1 - |z_k|^2)^{v+1-\sigma}(1 - w z_k)^{-v-2}.
$$

where $r \leq |z_k| < 1$ and

$$
\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \leq C \|\psi\|_{a,p} \leq C \|\varphi\|_{a,p}.
$$

Hence

$$
\psi(w) = \sum_{k=1}^{\infty} a_k (1 - |z_k|^2)^{v+1-\sigma}(1 - w z_k)^{-v-2}
$$

and so

$$
T\varphi = \left((v+1)\right)^{-1} \sum_{k=1}^{\infty} a_k (1 - |z_k|^2)^{v+1-\sigma} f^{v+1}(z_k)
$$

$$
= \left((v+1)\right)^{-1} \sum_{k=1}^{\infty} a_k (1 - |z_k|^2)^{v+1-\sigma} f(z_k).
$$

Hence since $v+1-\sigma \geq 0$,

$$
\|T\varphi\| \leq C(\max_{r \leq |z| \leq 1} ||f(z)||) \|\varphi\|_{a,p}
$$

where $C = C(p, q, r)$. In particular

$$
\|f(0)\| = ||T((v+1)! w^{v+1})|| \leq C \max_{r \leq |z| \leq 1} ||f(z)||.
$$

Now let us suppose that $B$ is a quasi-Banach algebra with identity. We recall that the spectral radius formula

$$
\rho(x) = \lim_{n \to \infty} \|x^n\|^n
$$

is valid in $B$ (cf. [24]).

We now generalize a result of Vesentini [23] (see also [2]).

**Theorem 5.3.** If $B$ is a unital quasi-Banach algebra then $\log q$ and $q$ are plurisubharmonic.

**Proof.** For any $0 < r < 1$, there exists a constant $C = C(r)$ so that if $f \in A_0(B)$ then

$$
\|f(0)\| \leq C(r) \max_{r \leq |z| \leq 1} ||f(z)||.
$$

Hence

$$
\|f(0)^n\| \leq C(r) \max_{r \leq |z| \leq 1} ||f(z)||^n
$$

since $z \to f(z)^n$ is also in $A_0(B)$. Thus

$$
\|f(0)^n\|^{1/n} \leq C(r)^{1/n} \max_{r \leq |z| \leq 1} ||f(z)||^{1/n}.
$$
Hence \( g(x,y) \geq 1 \) and \( \|x\|_A \|y\|_A \geq 1 \) so that \( \|x\|_A > 0 \). This contradiction shows \( x \in I \).

(iii) is immediate.

Remark. Note in particular that a quasi-Banach algebra with identity cannot be \( A \)-trivial. In fact, if an \( A \)-trivial space is embedded in a quasi-Banach algebra, it embeds into the radical. It is asked in \([11]\) whether a quasi-Banach algebra with identity can have trivial dual; this question is related to the question of whether \( L_p \) is prime for \( 0 < p < 1 \).

A quasi-Banach space \( X \) is called boundedly transitive (\([15]\), p. 151) if there exists a constant \( M \) so that if \( x, y \in X \) with \( \|x\| = \|y\| = 1 \) then there exists \( T \in \mathcal{L}(X) \) with \( Tx = y \) and \( \|T\| \leq M \). \( L_p \) is boundedly transitive if \( 0 < p < 1 \) (\([15]\), p. 126 or \([20]\), p. 253) and there is a space universal for separable quasi-Banach spaces which is boundedly transitive (\([10]\), Theorem 4.3).

Theorem 5.5. If \( X \) is a boundedly transitive quasi-Banach space then \( \mathcal{L}(X) \) is \( A \)-convex.

Proof. If \( T \in \mathcal{L}(X) \) there exists \( x \in X \) with \( \|x\| = 1 \) and \( \|Tx\| \geq \frac{1}{2} \|T\| \). If \( T \neq 0 \) we can find \( S \in \mathcal{L}(X) \) with \( \|S\| \leq M \) and \( STx = \|Tx\| x \). Thus \( g(ST) \geq \|STx\| \) and hence

\[
\|S\|_A \|T\|_A \geq \|ST\|_A \geq \frac{1}{2} \|T\|
\]

so that \( \|T\|_A \geq \frac{1}{2M} \|T\| \).

6. Applications to tensor products. We shall say that two quasi-Banach spaces \( X \), \( Y \) admit a \( p \)-normable weak tensor product if there exists a \( p \)-normable space \( Z \) and a bilinear form \( B : X \times Y \to Z \) with \( \|B\| \leq 1 \) so that for some \( k > 0 \)

\[
k \|x\| \|y\| \leq \|B(x, y)\| \leq \|x\| \|y\|
\]

for \( x \in X, y \in Y, B \) induces a linear map \( B : X \otimes Y \to Z, B(x) = x \). If \( B \) can be chosen to be one-one then \( X \), \( Y \) admit a \( p \)-normable tensor product.

In \([22]\) it is shown that if \( 0 < p \leq 1 \) and \( 0 < q \leq 1 \), \( X \) is \( p \)-normable and \( Y \) is \( q \)-normable then \( X \) and \( Y \) always admit an \( r \)-normable tensor product where \( 1/r = 1/p + 1/q - 1 \). This is best possible (cf. \([14]\)).

Lemma 6.1. In order that \( X \) and \( Y \) admit a weak \( p \)-normable tensor product it is necessary and sufficient that for some \( k > 0 \) and every \( x \in X, y \in Y \) there exists a bilinear form \( B : X \times Y \to Z \) where \( Z \) is \( p \)-normed with \( \|B\| = 1 \) and

\[
|B(x, y)| \geq k \|x\| \|y\|
\]
Proof. For each \( x \in X, \ y \in Y \) there exists \( T_{x,y} : X \otimes Y \to Z_{s,y} \) with
\[
\| T_{x,y}(x \otimes y) \| \geq k \| x \| \| y \|, \\
\| T_{x,y}(\xi \otimes \eta) \| \leq \| \xi \| \| \eta \|, \quad \xi \in X, \ \eta \in Y.
\]
Quasi-norm \( X \otimes Y \) by
\[
\| u \| = \sup_{x \in X, y \in Y} \| T_{x,y} u \|.
\]
Then the natural bilinear form \( (x, y) \to x \otimes y \) into \( X \otimes Y \) satisfies our definition.

We shall say that a quasi-Banach space \( X \) is \( p \)-tensorial if for every \( p \)-normable space \( Y \), \( X \) and \( Y \) admit a weak \( p \)-normable tensor product. We remark that \( L_p \) for \( 0 < p < 1 \), is \( p \)-tensorial (\( L_p(Y) \) is a \( p \)-normable tensor product if \( Y \) is \( p \)-normable) and hence every subspace of \( L_p \) is \( p \)-tensorial.

**Theorem 6.2.** Let \( X \) be a \( p \)-normable quasi-Banach space. Then

(i) If \( X \) is natural then \( X \) is \( q \)-tensorial for any \( q, \ 0 < q < p \).

(ii) If \( X \) is \( q \)-tensorial for some \( q, \ 0 < q < p \), then \( X \) is \( A \)-convex.

**Proof.** (i) There exists \( M = M(q) \) so that if \( x \in X \) with \( x \neq 0 \) there exists a probability measure space \( (\Omega, \Sigma, \mu) \) and a linear operator \( T_x : X \to L_1(\Omega, \Sigma, \mu) \) with \( \| T_x \| \leq M \| x \|^{-1} \) and \( T_x x = 1 \) (Theorem 4.2).

If \( Y \) is \( q \)-normable define \( B_x : X \otimes Y \to L_1(Y) \) by \( B_x(\xi, \eta) = M^{-1} \| x \|^{-1} \| \xi \| \| \xi \| \| \eta \| \), then \( \| B_x \| \leq 1 \) and \( \| B_x (x, y) \| = M^{-1} \| x \| \| y \| \) so that \( X \) and \( Y \) are \( q \)-tensorial by Lemma 6.1.

(ii) It clearly suffices to show that every separable subspace of \( X \) is \( A \)-convex. To do this suppose that \( X \) is separable. Let \( U \) be a boundedly transitive \( q \)-normed separable universal space ([10]). Then there exists a bilinear \( B : X \times U \to U \) so that
\[
k \| x \| \| u \| \leq \| B(x, u) \| \leq (k \| x \| \| u \|, \quad x \in X, u \in U,
\]
where \( k > 0 \). Now \( B \) induces a linear map \( T : X \to L(U) \) defined by
\[
(T_x)(u) = B(x, u)
\]

and \( k \| x \| \leq \| T_x \| \leq k \| x \| \). Thus \( X \) is isomorphic to a subspace of \( L(U) \) which is \( A \)-convex by Theorem 5.5.

7. Towards a theory of holomorphic functions. Throughout this section \( X \) and \( Y \) will denote fixed quasi-Banach spaces both of which are \( p \)-normed. We define a bounded homogeneous polynomial of degree \( n \geq 1 \), \( P : X \to Y \), to be a map of the form
\[
P(x) = \hat{P}(x, \ldots, x)
\]
where \( \hat{P} : X^n \to Y \) is a bounded symmetric \( n \)-linear form. The identification \( P \)

\[\rightarrow \hat{P} \]

between bounded homogeneous polynomials and symmetric \( n \)-linear forms is bijective as shown by the polarization formula ([6], p. 4). We set
\[
\| P \| = \sup_{\| x \| \leq 1} \| P(x) \|.
\]

For convenience a polynomial of degree zero is a constant map.

**Lemma 7.1.** Let \( P_n \) be a sequence of degree zero polynomials and suppose for some open set \( U \) we have
\[
\sup_{\| x \| \leq 1} \| P_n(x) \| < \infty
\]
whenever \( x \in U \). Then there exists \( \eta > 0 \) so that
\[
\sup_{\| x \| \leq 1} \| P_n(x) \| < \infty.
\]

**Proof.** (cf. [6], p. 9). By the Baire Category Theorem there exists \( M < \infty \) and \( x_0 \in U \), and \( \nu > 0 \) so that if \( \| x - x_0 \| < \nu \) then
\[
\| P_n(x) \| \leq M, \quad n \in \mathbb{N}.
\]

Let \( \eta = 2^{-1/n} \nu \) and choose \( r_0, 0 < r_0 < 1 \), so that \( (1 - r_0) \| x_0 \| < \eta \).

By Theorem 5.2 there exists a constant \( C = C(r_0, Y) \) so that if \( f \in A_q(Y) \) then
\[
\| f \| \leq C \max_{r_0 \leq |z| \leq 1} | f(z) |.
\]

If \( \xi \in X \) with \( ||\xi|| < \eta \) then for \( n \in \mathbb{N} \)
\[
\| P_n(\xi) \| \leq C \max_{r_0 \leq |z| \leq 1} | P_n(\xi + z x_0) |.
\]

since \( f(z) = P_n(\xi + z x_0) \) is analytic. Now if \( z = r e^{i \theta} \) with \( r_0 \leq r \leq 1 \) and
\[
0 \leq \theta \leq 2 \pi, \quad \text{then}
\]
\[
\| P_n(e^{i \theta} \xi + r x_0) \| = \| P_n(e^{i \theta} \xi + r x_0) \| = \| P_n(e^{i \theta} \xi - (1 - r) x_0 + x_0) \| \leq M.
\]

Hence \( \| P_n(\xi) \| \leq CM \) for \( \| \xi \| \leq \eta \).

Now if \( (P_n)_{n \in \mathbb{N}} \) is a sequence of polynomials where \( \deg P_n = n \), we say that \( \sum P_n \) is a power series if for every \( x \in X \)
\[
\sup_{n \in \mathbb{N}} \| \sum P_n(x) \|^{1/n} < \infty.
\]

**Lemma 7.2.** If \( \sum P_n \) is a power series then there exists \( \eta > 0 \) so that the series \( \sum P_n(x) \) converges uniformly for \( \| x \| \leq \eta \).

**Proof.** It follows from the Baire Category Theorem that there exists an open set \( V \) and \( M > 0 \) so that
\[
\| P_n(x) \|^{1/n} \leq M, \quad x \in V, \ n \in \mathbb{N}.
\]
Let \( U = (1/2M) V \). For \( x \in U \),
\[
\| P_n(x) \| \leq (\delta)^n \leq 1.
\]
By Lemma 7.1 there exists \( \eta_1 > 0 \) and \( C < \infty \) so that
\[
\| P_n(x) \| \leq C, \quad \| x \| \leq \eta_1, \text{ } n \in \mathbb{N}.
\]
Hence \( \| P_n \| \leq C \eta_1^{-n} \). Let \( \eta = \frac{1}{2} \eta_1 \). Then for \( \| x \| \leq \eta \)
\[
\| P_n(x) \| \leq C (\delta)^n
\]
and so \( \sum P_n(x) \) converges uniformly.

Now if \( U \subset X \) is an open set we say that a map \( F : U \to X \) is
holomorphic if for every \( x_0 \in U \), there exists a power series \( \sum P_n \) and \( \delta > 0 \)
so that if \( \| \xi \| < \delta \) then \( x_0 + \xi \in U \) and
\[
F(x_0 + \xi) = \sum_{n=0}^{\infty} P_n(\xi).
\]
In the notation of [6], \( P_0(\xi) = F(x_0) \) and
\[
P_n(\xi) = \frac{d^n F(x_0)}{n!}(\xi).
\]

We now return to the consideration of power series. If \( \sum P_n \) is a power series we set
\[
q_n(x) = \sup_{k \geq n} \| P_k(x) \|^{1/k}
\]
and then let \( q_n^* \) be the upper-semicontinuous regularization of \( q_n \), i.e.
\[
q_n^*(x) = \limsup_{y \to x} q_n(y).
\]
Let \( q(x) = \lim_{n \to \infty} q_n^*(x) \). Then \( q \) is upper-semicontinuous and \( \sum P_n \) converges on
the open set \( \{ x : q(x) < 1 \} \). We call this set the domain of convergence of the series \( \sum P_n \).

**Theorem 7.3.** Let \( F(x) = \sum_{n=0}^{\infty} P_n(x) \) for \( q(x) < 1 \). Then \( F \) is holomorphic
on \( \{ x : q(x) < 1 \} \).

**Proof.** Let \( K \) be a compact subset of \( V = \{ x : q(x) < 1 \} \). Then since
\( q(x) < 1 \) on \( K \), there exists, by Dini’s theorem, \( N \) so that if \( x \in K \) then
\[
q_n^*(x) \leq \alpha < 1.
\]
Thus
\[
\| P_k(x) \| \leq \alpha^k, \quad k \geq N,
\]
and \( \sum P_k \) converges uniformly on \( K \).

Now suppose \( x_0 \in V \). Pick \( \delta > 0 \) so that if \( \| \xi \| < \delta \) then \( x_0 + \xi \in V \). Now
suppose \( \| \xi \| < \delta \). Then \( \xi = \alpha \xi_1 \) where \( \| \xi_1 \| = \delta \) and \( \alpha = \| \xi \| / \delta \). We define
\[
f_\alpha(z) = \sum_{k=0}^{\infty} P_k(x_0 + z \xi_1)
\]
for \( z \in C \). Then \( f_\alpha \) converges uniformly on \( \| z \| \leq 1 \) to \( f \) where \( f(z) = F(x_0 + z \xi_1) \). Each \( f_\alpha \) is a polynomial and hence \( f \in A_{\alpha}(Y) \) ([14], Theorem 6.3).

Furthermore for each \( m \)
\[
\lim_{n \to \infty} f^{(m)}(0) = f^{(m)}(0)
\]
([12], Theorem 6.1).

Now for fixed \( m \leq k \), let \( Q_{k,m} \) be the homogeneous polynomial of
degree \( m \),
\[
Q_{k,m}(x) = \frac{k!}{(k-m)!} P_k(x_0, \ldots, x, \ldots, x)
\]
where \( x_0 \) is repeated \( k-m \) times and \( x \) is repeated \( m \) times. Then
\[
f^{(m)}(0) = \sum_{m \leq k \leq n} Q_{k,m}(\xi_1).
\]
We conclude that
\[
\sum_{k=m}^{\infty} Q_{k,m}(\xi_1) = f^{(m)}(0).
\]
In particular the sequence of polynomials (each of degree \( m \)) \( \sum_{m \leq k \leq n} Q_{k,m} \) is
pointwise convergent. By Lemma 7.1
\[
\sup_{m \leq k \leq n} \| \sum_{m \leq k \leq n} Q_{k,m} \| < \infty
\]
and hence if we set \( Q_m(\xi) = \sum_{k=m}^{\infty} Q_{k,m}(\xi) \) then \( Q_m \) is a (continuous) polynomial
of degree \( m \). Now
\[
F(x_0 + \xi) = f(\alpha) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} \alpha^m = \sum_{m=0}^{\infty} \frac{Q_m(\xi_1)}{m!} \alpha^m
\]
\[
= \sum_{m=0}^{\infty} \frac{Q_m(\xi)}{m!}.
\]
Thus \( q \) is plurisubharmonic. Note that \( q_+(x) \leq C_0 R^n \|x\| \) and hence \( \varphi \leq C_1 \|x\| \). Thus

\[
\varphi \leq C_1 \|x\|.
\]

**Corollary 7.5.** Let \( X \) be a strictly convex subset of \( \mathbb{R}^n \). Let \( F: X \to Y \) be a holomorphic function. Then there is an entire holomorphic function \( G: \mathbb{C}^n \to Y \) such that \( G(x) = F(x) \) for \( x \in U \).

**Proof.** Fix \( x_0 \in U \). Let \( P_n = (1/n!) \sum_{|\alpha| \leq n} D^n \varphi \). Then the power series \( \sum P_n(z) \) converges for all \( z \in X \) by Theorem 7.4.

Set \( G(x) = \sum P_n(x - x_0) \). Then by Theorem 7.3, \( G \) is an entire holomorphic function. Let \( H(x) = G(x) - F(x) \) for \( x \in U \). Let \( W \) be the set of \( x \in U \) so that \( H(x) = 0 \) on a neighbourhood of \( x \). Then \( W \) is open in \( U \). Let \( x_1 \) be a boundary point of \( W \). Pick \( \delta > 0 \) so that \( 0 < \delta < \|x_1 - x\| \). Then \( x \in U \). Pick \( y \in W \) with \( \|y - x\| < \delta \) and define

\[
f(z) = H(x_1 + z(y - x)) \quad (z \in |z_1| < 1).
\]

Then \( f \) is analytic on a disc \( z \in |z_1| < 1 \). Since \( f \) vanishes on an open set, \( f \equiv 0 \) and hence \( H(x_1) = 0 \). Thus \( W \) is open and closed in \( U \). Since \( x_0 \in W \), \( W = U \).

**Remark.** If \( X = L_q/H_q \) where \( 1 \leq q < \infty \), we can go further. Every holomorphic function is a polynomial. In fact, if \( 1/p = 1/q - n \) then every homogeneous polynomial of degree \( n \) is identically zero (see [14]).

In a similar vein, if \( T: L_q/H_q \to A \) is a bounded linear operator and \( A \) is a quasi-Banach algebra, then there exists \( n \in \mathbb{N} \) so that \( \|Tx_k\| \leq 1 \) then

\[
(\|T_n(x)\| \leq 1), \quad n \in \mathbb{N}.
\]

**References**


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