

(i) If $A \in (\text{BP})$, then there exist two sequences $1 = N_0 < N_1 < \dots$ and $0 = K_0 < K_1 < \dots$ of integers such that

$$\|A; N_r, N_{r+1}\| \leq 2^{-r} \quad (r = 1, 2, \dots),$$

$$(4.5) \quad \sup_{\varphi} \left\{ \left(\max_{0 \leq n \leq N_r} \left| \sum_{k=K_r+1}^{K_{r+1}} a_{nk} \varphi_k(x) \right|^2 dx \right)^{1/2} \right\} \leq 2^{-r}.$$

Setting $\mu_{nk} = r+1$ for $N_r \leq n < N_{r+1}$ and $K_r < k \leq K_{r+1}$ ($r = 0, 1, \dots$), it is easy to check that $\|\mu A\| < \infty$.

(ii) Similarly, if $A \notin (\text{BP})$, then there exist two sequences $1 = N_0 < N_1 < \dots$ and $0 = K_0 < K_1 < \dots$ of integers such that

$$\|A; N_r, N_{r+1}\| \geq 2^r \quad (r = 1, 2, \dots)$$

and (4.5) is satisfied. Now we set $\mu_{nk} = (r+1)^{-1}$ for $N_r \leq n < N_{r+1}$ and $K_r < k \leq K_{r+1}$ ($r = 0, 1, \dots$) and conclude that $\|\mu A\| = \infty$.

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BOLYAI INSTITUTE, UNIVERSITY OF SZEGED
Aradi vértanúk tere 1, 6720 Szeged, Hungary

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Analytic functions in non-locally convex spaces and applications

by

N. J. KALTON * (Columbia, Mo.)

Abstract. The aim of this paper is to determine, for a general p -normable space X , what can in general be said about X -valued analytic functions on the disc. The results obtained are used to solve a problem raised by Turpin [17] on tensor products of quasi-Banach spaces.

1. Summary of main results. Suppose Ω is an open subset of the complex plane C and X is a quasi-Banach space. A map $f: \Omega \rightarrow X$ is said to be *analytic* if for every $z_0 \in \Omega$ there exists $r > 0$ such that f can be expanded in a power series for $|z - z_0| < r$, i.e.

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n$$

for $|z - z_0| < r$. This definition of analyticity is forced on us by simple examples which demonstrate that complex differentiability of f does not suffice to produce reasonable properties (cf. Aleksandrov [3], p. 39 or Turpin [16], Chapitre IX).

A key property of analytic functions is ([16], p. 195) that if $f: \Omega \rightarrow X$ is analytic and $\Omega_0 \subset \Omega$ is open and relatively compact in Ω then there is a Banach space B , an analytic function $g: \Omega_0 \rightarrow B$ and a bounded linear operator $T: B \rightarrow X$ so that $f(z) = T(g(z))$, $z \in \Omega_0$. From this many of the standard properties of analytic functions in a Banach space can be lifted to quasi-Banach spaces.

In this paper we will primarily be concerned with the case $\Omega = \Delta$, the open unit disc. In this case one has, for example,

$$f(z) = \sum_{n=0}^{\infty} x_n z^n, \quad |z| < 1,$$

where $\limsup \|x_n\|^{1/n} \leq 1$.

It seems that the main obstacle to developing the theory of analytic functions for non-locally convex spaces is the failure of the Maximum Modulus Principle. It has been observed by several authors (Etter [7], Aleksandrov [3], Peetre [14], Davis–Garling–Tomczak [5]) that some standard spaces, e.g. L_p for $0 < p < 1$, have a plurisubharmonic quasi-norm and hence if $f: \bar{\Delta} \rightarrow L_p$ is analytic on Δ and

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continuous on \bar{A} then

$$\|f(z)\| \leq \max_{|\zeta|=1} \|f(\zeta)\|,$$

for all $z \in \Delta$. By contrast, however, Aleksandrov notes that if we define $J_{p,0}$ to be the closed linear span in $L_p(\mathcal{T})$ of the Cauchy kernels $\varphi_z(w) = (1-wz)^{-1}$ and let $Q: L_p \rightarrow L_p/J_{p,0}$ be the quotient map then we can define

$$v(z) = Q(u(z)), \quad |z| \leq 1,$$

where $u(z) = (1-wz)^{-1}$. The map $v: \bar{A} \rightarrow L_p/J_{p,0}$ is analytic on Δ , continuous on \bar{A} and vanishes on T .

The precise conditions on X so that such a phenomenon can occur will be investigated in a separate paper. Our aim in this paper is to determine, for a general p -normable space X , with no additional assumptions such as plurisubharmonicity of the quasi-norm, what can in general be said about X -valued analytic functions on the disc. In particular, we define, for $\sigma > 0$, $V_\sigma(X)$ to be the space of all analytic $f: \Delta \rightarrow X$ so that, for some constant C ,

$$\|f(z)\| \leq C(1-|z|)^\sigma.$$

We show that (Theorem 4.6) v defined above belongs to $V_\sigma(L_p/J_{p,0})$ where $\sigma = 1/p-1$.

It turns out (Theorem 6.7) that if X is p -normable and $\sigma > 1/p-1$ then $V_\sigma(X) = \{0\}$, so that v represents extremal behaviour. In Theorem 8.3 it is shown further that if

$$\lim_{r \rightarrow 1} (1-r^2)^{1-1/p} \left\{ \int_0^{2\pi} \|f(re^{i\theta})\|^p d\theta \right\}^{1/p} = 0$$

then $f = 0$.

The key result, however, is that $V_\sigma(X)$ is nontrivial if and only if there is a nontrivial linear operator $T: L_p/H_p \rightarrow X$ (Theorem 7.3).

These results can be used to solve a problem raised by Turpin [17] on tensor products of quasi-Banach spaces. Turpin showed that if X is a p -Banach space and Y is a q -Banach space then there is an r -convex tensor quasi-norm on $X \otimes Y$ if $1/r = 1/p + 1/q - 1$, and asked whether this can be improved. The author showed that in the case $p = q$ we cannot always have $r = p$ [9], by showing that there is no nonzero bilinear $B: L_p/H_p \times L_p/H_p \rightarrow Z$ where Z is p -normable. Here we show Turpin's result is best possible by showing that if Z is s -normable, where $1/s < 1/p + 1/q - 1$, then there is no nonzero bilinear form $B: L_p/H_p \times L_q/H_q \rightarrow Z$ (Corollary 9.2).

The method employed is to establish a correspondence between certain classes of analytic functions and linear operators. The first such result is Theorem 5.1 which identifies the space of linear operators $\mathcal{L}(H_q, X)$ where X is p -normable and $0 < q < p$. This theorem is essentially a translation of a result of Coifman and Rochberg [4]. In Theorem 7.1 we similarly identify $\mathcal{L}(L_p, X)$ and hence $\mathcal{L}(L_p/H_p, X)$. We use these results to give an "atomic decomposition" of

L_p in the spirit of Coifman-Rochberg [4] and to extend Aleksandrov's theorem ([1], [2]) that $L_p = H_p + \bar{H}_p$ for $0 < p < 1$, by showing that if $f \in L_p(\mathcal{T})$ then we can find $g_1, g_2 \in H_p$ so that

$$g_1(e^{i\theta}) + g_2(e^{-i\theta}) = f(e^{i\theta}),$$

$$\int_1^{\infty} |g_j'(w)|(1-|w|^2)^{p-1} d\lambda(w) < \infty$$

for $j = 1, 2$ (where λ is the planar measure on Δ) (see Theorems 8.1, 8.2).

In Section 6 we also give some applications to the general theory of vector-valued analytic functions. For example, we show Liouville's theorem holds (Theorem 6.2), in the form that if $f: C \rightarrow X$ is entire and bounded then f is constant (see [16], [18] for a similar result when f is analytic on the Riemann sphere $C \cup \{\infty\}$). We also show (Theorem 6.3) that the uniform limit of analytic functions on Δ is again analytic.

2. Notation and plan of the paper. Throughout this paper all vector spaces are assumed complex. By definition, a *quasi-normed space* is a vector space X with a quasi-norm $x \rightarrow \|x\|$ satisfying:

- (i) $\|x\| > 0, \quad x \neq 0,$
- (ii) $\|\alpha x\| = |\alpha| \|x\|, \quad \alpha \in C, x \in X,$
- (iii) $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|), \quad x_1, x_2 \in X,$

for some C independent of x_1, x_2 . In fact we will always assume that the quasi-norm is *p-subadditive* for some $p > 0$, i.e.

$$(iv) \|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p, \quad x_1, x_2 \in X.$$

This assumption is justified by the Aoki-Rolewicz theorem [15] that every quasi-norm is equivalent to a p -subadditive quasi-norm where $C = 2^{1/p-1}$.

If X is complete, we say it is a *quasi-Banach space*; if it has a p -subadditive quasi-norm, we say it is a *p-Banach space*. For convenience of exposition, we shall always assume that X is a p -Banach space without specifying the fact (while spaces Y, Z , etc. need not be p -Banach spaces but must be q -Banach spaces for some q).

If X and Y are quasi-Banach spaces then $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators $T: X \rightarrow Y$ with the usual quasi-norm $\|T\| = \sup(\|Tx\|: \|x\| \leq 1)$.

Suppose $0 < p < 1$. We let $L_p = L_p(\mathcal{T})$ be the space of all complex-valued Borel functions $f: \mathcal{T} \rightarrow C$ satisfying

$$\|f\|_p^p = (2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta < \infty.$$

In general we shall use w as the independent variable when considering function spaces. Let H_p be the closed linear span in L_p of $\{w^n: n = 0, 1, 2, \dots\}$. \bar{H}_p denotes the closed linear span of $\{\bar{w}^n: n = 0, 1, 2, \dots\}$ and $\bar{H}_{p,0}$ denotes the closed linear span of $\{\bar{w}^n: n = 1, 2, 3, \dots\}$ or $\{w^n: n = -1, -2, -3, \dots\}$. The intersections

are denoted by $J_p = H_p \cap \bar{H}_p$ and $J_{p,0} = H_p \cap \bar{H}_{p,0}$. $J_{p,0}$ is the closed linear span of the Cauchy kernels

$$u(z) = (1 - wz)^{-1}$$

for $z \in T$ (see [1]). We shall also need the spaces $J_{p,0}^{(m)}$ spanned by the functions $u^{(m)}(z) = m! w^m (1 - wz)^{-(m+1)}$ provided $m < 1/p - 1$.

It is known that the quotient spaces $L_p/H_p, L_p/\bar{H}_p, L_p/\bar{H}_{p,0}, L_p/J_p, H_p/J_p, H_p/J_{p,0}$ are all isomorphic (cf. [3], [10]). The first three spaces in this list are isomorphic by constructing simple automorphisms of L_p which map H_p to \bar{H}_p or $\bar{H}_{p,0}$. For the other spaces one needs the theorem of Aleksandrov [1] that $H_p + \bar{H}_{p,0} = L_p$.

For $p < q \leq 1$, the q -Banach envelope of H_p has been identified in [3] and [4]. This is the Bergman space $B_{p,q}$ of all analytic functions f defined on the unit disc \mathcal{A} in the complex plane and satisfying

$$\int_{\mathcal{A}} |f(w)|^q (1 - |w|^2)^{q/p - 2} d\lambda(w) = \|f\|_{p,q}^q < \infty$$

where λ is the planar Lebesgue measure.

If we identify H_p in the usual way as a space of analytic functions on \mathcal{A} , then $H_p \subset B_{p,q}$ and the inclusion is continuous. Furthermore, if Y is any q -Banach space and $T: H_p \rightarrow Y$ is a bounded linear operator then T can be extended to a bounded linear operator $\bar{T}: B_{p,q} \rightarrow Y$. Thus $\mathcal{L}(H_p, Y)$ and $\mathcal{L}(B_{p,q}, Y)$ are naturally isomorphic.

We now discuss the plan of the paper. In Sections 3–4 we describe a theory of integration in non-locally convex spaces originally developed by Turpin and Waelbroeck ([16], [18], [19]); roughly speaking, a function can be integrated successfully if it is sufficiently smooth. We introduce in Section 4 the class $C_\sigma(T, X)$ of “ σ -differentiable” functions $f: T \rightarrow X$ where $\sigma > 0$. In particular, we study the function $u: T \rightarrow L_p$ given by

$$u(z) = (1 - wz)^{-1}.$$

In Section 5, we prove our main representation theorem for operators on H_q , and apply these results in Section 6 to give results on analytic functions taking their values in an arbitrary quasi-Banach space X . In Section 7, these results and the Turpin–Waelbroeck integral are used to give a representation theorem for operators on $L_p(T)$ and on $L_p/\bar{H}_{p,0}$. In Section 8, we give some applications to the space L_p , and in Section 9 we give applications to tensor products.

Convention. Throughout the paper we adopt the convention that C is a constant which may vary from line to line and may depend on the parameters p, q, σ, ν , etc., but is independent of f, x, K , etc.

3. The class C_σ . In this section we give a self-contained treatment of an integration theory developed by Turpin and Waelbroeck ([16], [18], [19]). Our

approach goes a little further than that of Turpin and Waelbroeck as we shall need to cover the case when σ , as specified below, is an integer.

Suppose X is a p -Banach space where $0 < p \leq 1$. Let K be any fixed closed bounded interval in \mathbb{R} . Let $f: K \rightarrow X$ be any continuous function. Then for any closed subinterval I of K we define

$$\|f\|_I = \max_{t \in I} \|f(t)\|.$$

Now suppose $\sigma > 0$ and N is an integer with $N \geq \sigma - 1$; suppose further that $l > 0$. We shall say that a continuous function $f: K \rightarrow X$ is in $C_\sigma^{l,N}(K, X)$ if there is a constant $\gamma > 0$ with the property that for any closed subinterval I of K with length $|I| \leq l$ there is a polynomial $\phi_I: I \rightarrow X$ of degree at most N so that

$$(3.1) \quad \|f - \phi_I\|_I \leq \gamma |I|^\sigma.$$

Note that if $\sigma < 1$ and $N = 0$ then this simply implies that f is Lipschitz of order σ .

Before proceeding we observe two crucial facts. The first is a lemma due to Peck [13]: if F is any m -dimensional complex p -Banach space then there is a norm $\| \cdot \|$ on F satisfying

$$\| \|x\| \| \leq \|x\| \leq (2m)^{1/p-1} \| \|x\| \|, \quad x \in F.$$

Note here that the real dimension of F is $2m$.

The second observation is that there is a constant $C = C(p, N)$ so that for any interval I and any polynomial ϕ of degree N we have

$$\|\phi^{(k)}\|_I \leq C |I|^{-k} \|\phi\|_I$$

for any $k \leq N$. This is proved by standardizing to an interval of length one and using Peck’s lemma.

In the next proposition we let $\nu = [\sigma]$ be the largest integer in σ .

PROPOSITION 3.1. (i) *The spaces $C_\sigma^{l,N}(K, X)$ are independent of $l > 0$ and $N \geq \sigma - 1$. Let $C_\sigma(K, X)$ denote this class.*

(ii) *If $\sigma > 1$ and $f \in C_\sigma(K, X)$ then f is continuously differentiable and $f' \in C_{\sigma-1}(K, X)$.*

(iii) *If $\sigma \notin \mathbb{N}$ then $f \in C_\sigma(K, X)$ if and only if f is ν times continuously differentiable on K and, for some $\beta > 0$,*

$$\left\| f(t) - \sum_{k=0}^{\nu} \frac{f^{(k)}(s)}{k!} (t-s)^k \right\| \leq \beta |t-s|^\sigma$$

for $s, t \in K$.

(iv) *If $\sigma \in \mathbb{N}$ (i.e. $\sigma = \nu$) and $f^{(\nu-1)}$ is Lipschitz then $f \in C_\sigma(K, X)$ if and only if, for some $\beta > 0$,*

$$\left\| f(t) - \sum_{k=0}^{\nu-1} \frac{f^{(k)}(s)}{k!} (t-s)^k \right\| \leq \beta |t-s|^\sigma.$$

Proof: Suppose $f \in C_a^{l,N}(K, X)$. As in (3.1) we suppose that if I is a closed subinterval of K we can find a polynomial φ_I of degree at most N so that $\|f - \varphi_I\|_I \leq \gamma|I|^\sigma$. If I and J are intersecting intervals then

$$\|\varphi_I - \varphi_J\|_{I \cap J} \leq C\gamma(|I|^\sigma + |J|^\sigma),$$

and hence if $k \leq N$ then

$$(3.2) \quad \|\varphi_I^{(k)} - \varphi_J^{(k)}\|_{I \cap J} \leq C\gamma|I \cap J|^{-k}(|I|^\sigma + |J|^\sigma).$$

Fix $s \in I \cap J$. Then for $t \in I \cup J$

$$(3.3) \quad \begin{aligned} \|\varphi_I(t) - \varphi_J(t)\| &= \left\| \sum_{k=0}^N \frac{\varphi_I^{(k)}(s) - \varphi_J^{(k)}(s)}{k!} (t-s)^k \right\| \\ &\leq C\gamma(|I \cup J|/|I \cap J|)^N(|I|^\sigma + |J|^\sigma). \end{aligned}$$

Now note that if L is an interval with $l \leq |L| \leq \frac{3}{2}l$ then we can write $L = I \cup J$ where $|I| \leq l$, $|J| \leq l$ and $|I \cap J| \geq \frac{1}{2}l$. Thus for $t \in L$

$$\|\varphi_I(t) - \varphi_J(t)\| \leq 3^N C\gamma|I|^\sigma$$

and so

$$\|f(t) - \varphi_I(t)\| \leq C\gamma|L|^\sigma$$

for $t \in L$. Thus $C_a^{l,N} = C_a^{3l/2,N}$, and it follows quickly that $C_a^{l,N}$ is independent of l . Henceforward we take $l = |K|$.

Now suppose $t \in K$ and $0 < h < |K| = \delta$ say. We can find a polynomial $\varphi_h = \varphi_{t,h}$ of degree N so that $\|f(s) - \varphi_h(s)\| \leq C\gamma h^\sigma$ if $|s-t| \leq h$ and $s \in K$.

Using (3.3) we see that if $1 \leq \alpha \leq 2$ and $\alpha h \leq \delta$ then $\|\varphi_h(s) - \varphi_{\alpha h}(s)\| \leq C\gamma h^\sigma$ if $|s-t| \leq h$ and $s \in K$. It follows that if $0 \leq k \leq N$ then

$$(3.4) \quad \|\varphi_h^{(k)}(t) - \varphi_{\alpha h}^{(k)}(t)\| \leq C\gamma h^{\sigma-k}.$$

Now if $2^{-n}\delta \leq h < 2 \cdot 2^{-n}\delta$ where $n \in \mathbb{N}$ then we can obtain $\varphi_h^{(k)}(t)$ as

$$(3.5) \quad \varphi_h^{(k)}(t) = \varphi_\delta^{(k)}(t) - \sum_{j=1}^n P_j$$

where

$$P_j = \varphi_{2^{-j}\delta}^{(k)}(t) - \varphi_{2^{-j+1}\delta}^{(k)}(t) \quad \text{for } j < n,$$

$$P_n = \varphi_{2^{-n}\delta}^{(k)}(t) - \varphi_h^{(k)}(t).$$

We note that $\|P_j\| \leq C\gamma(2^{-j}\delta)^{\sigma-k}$.

We consider first the case $k > \sigma$. In this case

$$\|\varphi_h^{(k)}(t) - \varphi_\delta^{(k)}(t)\| \leq C\gamma h^{\sigma-k}.$$

However, $\|\varphi_\delta^{(k)}(t)\| \leq C\delta^{-k}\|\varphi\|_K \leq C\delta^{-k}(\|f\|_K + \gamma\delta^\sigma)$. Hence

$$\|\varphi_h^{(k)}(t)\| \leq C(\delta^{-k}\|f\|_K + \gamma h^{\sigma-k}).$$

Thus

$$\left\| \sum_{k=v+1}^N \varphi_h^{(k)}(t)(s-t)^k/k! \right\| \leq C(\|f\|_K + \gamma)h^\sigma$$

if $|s-t| \leq h$, and so if $s \in K$ then

$$\|f(s) - \sum_{k=0}^v \varphi_h^{(k)}(t)(s-t)^k/k!\| \leq \gamma' h^\sigma$$

where γ' is independent of t and h . Thus (i) is established; we may take $N = v$ in the definition. Henceforward we assume $N = v$ so that $\deg \varphi_h \leq v$ for all t, h .

Returning to (3.5) we see that if $k < \sigma$ then $x_k = \lim_{h \rightarrow 0} \varphi_h^{(k)}(t)$ exists, and

furthermore

$$(3.6) \quad \|x_k - \varphi_h^{(k)}(t)\| \leq C\gamma h^{\sigma-k}.$$

If $\sigma \in \mathbb{N}$ and $k = \sigma$ then (3.4) and (3.5) yield

$$\|\varphi_h^{(k)}(t) - \varphi_\delta^{(k)}(t)\| \leq C\gamma(\log(\delta/h) + 1)^{1/p}$$

and hence

$$\|\varphi_h^{(k)}(t)\| \leq C(1 + \log(\delta/h))^{1/p}(\|f\|_K + \gamma),$$

where $C = C(p, N, \delta)$. Let us write

$$g_t(s) = \sum_{k < \sigma} x_k(s-t)^k/k!$$

and define $\varrho_{t,h}$ by

$$\begin{aligned} \varrho_{t,h}(s) &= 0 && \text{if } \sigma \notin \mathbb{N}, \\ \varrho_{t,h}(s) &= \varphi_h^{(\sigma)}(t)(s-t)^\sigma/\sigma! && \text{if } \sigma \in \mathbb{N}. \end{aligned}$$

Let $\psi_{t,h}(s) = g_t(s) + \varrho_{t,h}(s)$. Then $\|\varphi_h(s) - \psi_{t,h}(s)\| \leq C\gamma h^\sigma$ if $|s-t| \leq h$ by (3.6). Thus

$$\|f(s) - \psi_{t,h}(s)\| \leq C\gamma h^\sigma$$

if $|s-t| \leq h$ and $s \in K$.

Note first that $f'(t) = \lim_{h \rightarrow 0} \psi_{t,h}(t) = g_t(t)$. Thus if $\sigma > 1$ then $f''(t)$ exists and

$$f'(t) = g_t'(t) = x_1.$$

Suppose $s, t \in K$ and $\delta \geq h \geq |s-t|$. Then if $|\tau-t| \leq h$ and $\tau \in K$, we have

$$\|\psi_{t,h}(\tau) - \psi_{s,h}(\tau)\| \leq C\gamma h^\sigma$$

and hence

$$\|\psi'_{t,h}(\tau) - \psi'_{s,h}(\tau)\| \leq C\gamma h^{\sigma-1}.$$

In particular, if $\tau = s$ then

$$(3.7) \quad \|\psi'_{t,h}(s) - f'(s)\| \leq C\gamma h^{\sigma-1}.$$

Letting $h = |s-t|$ and $s \rightarrow t$ we see that f' is continuous. Furthermore, $f' \in C_{\sigma-1}$ since (3.7) holds if $s \in K$ and $|s-t| \leq h$. Thus (ii) is established.

Now for (iii) we can repeat this argument ν times to show f is ν times continuously differentiable and

$$\psi_{t,h}(s) = \sum_{k=0}^{\nu} f^{(k)}(t)(s-t)^k/k!$$

In case (iv), we repeat the argument $\nu-1$ times to deduce that $f^{(\nu-1)}$ is in class C_1 , and $\|\psi_{t,h}^{(\nu-1)}(s) - f^{(\nu-1)}(s)\| \leq C\gamma h$ if $|s-t| \leq h$. Now

$$\psi_{t,h}^{(\nu-1)}(s) = x_{\nu-1} + \phi_h^{(\nu)}(t)(s-t) = f^{(\nu-1)}(t) + \phi_h^{(\nu)}(t)(s-t).$$

Hence

$$\|f^{(\nu-1)}(s) - f^{(\nu-1)}(t) - \phi_h^{(\nu)}(t)(s-t)\| \leq C\gamma h.$$

If $f^{(\nu-1)}$ is Lipschitz we conclude, by taking $s = t+h$ or $t-h$, that $\|\phi_h^{(\nu)}(t)\| \leq C$ where C is independent of t and h . Hence

$$\|\varrho_{t,h}(s)\| \leq C|s-t|^\nu$$

and (iv) follows. The proposition is proved.

From now we shall choose $l = |K|$ and $N = \nu$ in our definition of $C_\sigma(K, X)$. We let $\gamma_\sigma(f)$ be the infimum of all possible constants γ in (3.1). Then we set

$$\|f\|_{K,\sigma} = \|f\|_K + \gamma_\sigma(f).$$

If $f \in C_\sigma(K, X)$ we say that f is of rank m if $f(K)$ is contained in an m -dimensional subspace.

LEMMA 3.2. *There is a constant $C = C(p, \sigma)$ so that if $f \in C_\sigma(K, X)$ and $m \geq 2(\nu+1)$ then there exists $g_m \in C_\sigma(K, X)$ with $\text{rank } g_m \leq m$ and*

$$\|f - g_m\|_K \leq Cm^{-\sigma} \|f\|_{K,\sigma} |K|^\sigma \quad \text{and} \quad \|g_m\|_{K,\sigma} \leq C \|f\|_{K,\sigma}.$$

PROOF. We prove the statement of the lemma if $m = (N+1)(\nu+1)$ where $N \in \mathbb{N}$. The general statement then follows easily.

First suppose $\psi: \mathbf{R} \rightarrow \mathbf{R}$ is a C^∞ -function so that $\text{supp } \psi \subset [-1, 1]$, $0 \leq \psi \leq 1$, $\psi(0) = 1$ and

$$\sum_{n \in \mathbf{Z}} \psi(t-n) = 1, \quad t \in \mathbf{R}.$$

Let $K = [a, b]$ and let $I_j = [a + (j-1)\delta/N, a + j\delta/N]$ for $0 \leq j \leq N+1$ where $\delta = b-a = |K|$. Define $\psi_0, \dots, \psi_N: \mathbf{R} \rightarrow \mathbf{R}$ by

$$\psi_j(t) = \psi(N\delta^{-1}(t-a)-j).$$

Then

$$\sum_{j=0}^N \psi_j(t) = 1, \quad t \in K,$$

and $\text{supp } \psi_j \subset I_j \cup I_{j+1}$. For $l \leq \nu+1$,

$$|\psi_j^{(l)}(t)| \leq CN^l \delta^{-l}.$$

Pick polynomials $\varphi_0, \dots, \varphi_N \in C_\sigma(K, X)$ of degree at most ν so that

$$\|f(s) - \varphi_j(s)\| \leq \|f\|_{K,\sigma} (2\delta/N)^\sigma$$

for $s \in (I_j \cup I_{j+1}) \cap K$. Let $g = g_m = \sum_{j=0}^N \psi_j \varphi_j$. Then $g \in C_\sigma(K, X)$ and if $s \in I_j$ then

$$\begin{aligned} \|f(s) - g(s)\| &= \|\psi_{j-1}(s)(f(s) - \varphi_{j-1}(s)) + \psi_j(s)(f(s) - \varphi_j(s))\| \\ &\leq C \|f\|_{K,\sigma} N^{-\sigma} \delta^\sigma. \end{aligned}$$

Furthermore,

$$g(s) = \psi_{j-1}(s)\varphi_{j-1}(s) + \psi_j(s)\varphi_j(s) = \varphi_{j-1}(s) + \psi_j(s)(\varphi_j(s) - \varphi_{j-1}(s)).$$

Thus

$$g^{(\nu+1)}(s) = \frac{\partial^{\nu+1}}{\partial s^{\nu+1}} (\psi_j(\varphi_j - \varphi_{j-1})).$$

By (3.2), $\|\varphi_j^{(l)}(s) - \varphi_{j-1}^{(l)}(s)\| \leq C \|f\|_{K,\sigma} (\delta/N)^{\sigma-l}$. Hence

$$\|g^{(\nu+1)}(s)\| \leq C \|f\|_{K,\sigma} (\delta N^{-1})^{\sigma-\nu-1}.$$

Let J be any closed subinterval of K . If $|J| \geq N^{-1}|K|$ then, since $\|f - g\|_K \leq C \|f\|_{K,\sigma} N^{-\sigma} \delta^\sigma$, there is a polynomial φ_J so that $\text{deg } \varphi_J \leq \nu$ and

$$\|g - \varphi_J\| \leq C \|f\|_{K,\sigma} (N^{-\sigma} \delta^\sigma + |J|^\sigma) \leq C \|f\|_{K,\sigma} |J|^\sigma.$$

Now suppose $|J| \leq N^{-1}\delta$. Let s be the midpoint of J and define

$$\varphi(t) = \sum_{j=0}^{\nu} g^{(j)}(s)(t-s)^j/j!$$

Then for $t \in J$

$$\|g(t) - \varphi(t)\| \leq C |t-s|^{\nu+1} \max_{t \in J} \|g^{(\nu+1)}(t)\|.$$

Here we use the fact that $g|_J$ takes its values in a fixed $3(\nu+1)$ -dimensional space, so that by Peck's lemma the quasi-norm is uniformly equivalent to a norm. Thus

$$\|g(t) - \varphi(t)\| \leq C \|f\|_{K,\sigma} (\delta N^{-1})^{\sigma-\nu-1} |J|^{\nu+1} \leq C \|f\|_{K,\sigma} |J|^\sigma.$$

Now note that $\text{rank } g_m \leq (N+1)(\nu+1)$.

Remark. Suppose, as we will later, that $K = [-2\pi, 2\pi]$ and that f is 2π -periodic. Then if N is even in the above argument, then g_m is also 2π -periodic.

Now suppose μ is a regular Borel measure on K . If $g \in C_\sigma(K, X)$ and $\text{rank } g < \infty$ then we may define the finite-dimensional integral $\int_K g d\mu$. By Peck's lemma

we clearly obtain

$$\left\| \int_K g d\mu \right\| \leq (2m)^{1/p-1} \|g\|_K \|\mu\|$$

if rank $g = m$.

If $f \in C_\sigma(K, X)$ we can define g_m for $m \geq 2(\nu + 1)$ with rank $g_m \leq m$ and so that

$$\|f - g_m\|_K \leq C m^{-\sigma} |K|^\sigma \|f\|_{K,\sigma}.$$

Then, for $m \leq n \leq 2m$, $\|g_m - g_n\|_K \leq C m^{-\sigma} |K|^\sigma \|f\|_{K,\sigma}$, and hence

$$\left\| \int_K g_m d\mu - \int_K g_n d\mu \right\| \leq C m^{1/p-1-\sigma} |K|^\sigma \|f\|_{K,\sigma}.$$

Now it is easy to show that $\lim_{n \rightarrow \infty} \int_K g_{2^n} d\mu$ exists. It follows that $\lim_{m \rightarrow \infty} \int_K g_m d\mu$ exists and is independent of the choice of the approximating sequence g_m as long as $\|f - g_m\| \leq C m^{-\sigma}$. Furthermore,

$$(3.7) \quad \left\| \int_K f d\mu - \int_K g_n d\mu \right\| \leq C n^{1/p-1-\sigma} \|f\|_{K,\sigma} |K|^\sigma \|\mu\|.$$

However, $\|g_n\|_K \leq C(\|f\|_K + |K|^\sigma n^{-\sigma} \|f\|_{K,\sigma})$, and hence for all $n \geq 2(\nu + 1)$ we have

$$(3.8) \quad \left\| \int_K f d\mu \right\| \leq C(n^{1/p-1} \|f\|_K + n^{1/p-1-\sigma} |K|^\sigma \|f\|_{K,\sigma}) \|\mu\|.$$

Taking n to be fixed, say $2(\nu + 1)$, we obtain:

LEMMA 3.3.

$$\left\| \int_K f d\mu \right\| \leq C(\|f\|_K + |K|^\sigma \|f\|_{K,\sigma}) \|\mu\|,$$

where $C = C(p, \sigma)$ is independent of f, μ and K .

We can now state the main properties of the Turpin-Waelbroeck integral.

THEOREM 3.4. (i) Suppose $f_n \in C_\sigma(K, X)$ where $\sigma > 1/p - 1$. Suppose $\|f_n - f\|_K \rightarrow 0$ and $\sup_n \|f_n\|_{K,\sigma} < \infty$. Then

$$\lim_{n \rightarrow \infty} \int_K f_n d\mu = \int_K f d\mu.$$

(ii) Suppose $\mu_n \in M(K)$ and $\mu_n \rightarrow \mu$ weak*. Then if $f \in C_\sigma(K, X)$ where $\sigma > 1/p - 1$ then

$$\lim_{n \rightarrow \infty} \int_K f d\mu_n = \int_K f d\mu.$$

Proof. (i) We omit the simple proof that $f \in C_\sigma(K, X)$. By 3.8 we note that if

$m, n \in \mathbb{N}$ then

$$\left\| \int_K (f - f_m) d\mu \right\| \leq C(n^{1/p-1} \|f - f_m\|_K + n^{1/p-1-\sigma} \|f - f_m\|_{K,\sigma}).$$

However, it is easily seen from the definition that $\sup_m \|f - f_m\|_{K,\sigma} < \infty$. Letting $m \rightarrow \infty$ we obtain

$$\limsup_{m \rightarrow \infty} \left\| \int_K (f - f_m) d\mu \right\| \leq C n^{1/p-1-\sigma}$$

for all $n \in \mathbb{N}$ and the result follows.

(ii) Here we use (3.7). Note that $\sup_n \|\mu_n\| < \infty$. Then for any m, n

$$\left\| \int f d\mu_m - \int g_n d\mu_m \right\| \leq C n^{1/p-1-\sigma} \|f\|_{K,\sigma}.$$

Now $\lim_{m \rightarrow \infty} \int g_n d\mu_m = 0$ for each n . Hence

$$\limsup_{m \rightarrow \infty} \left\| \int f d\mu_m \right\| \leq C n^{1/p-1-\sigma} \|f\|_{K,\sigma}$$

for all $n \in \mathbb{N}$, and the result follows.

4. The class $C_\sigma(T, X)$. Suppose $f \in C(T, X)$. We say $f \in C_\sigma(T, X)$ if $\tilde{f} \in C_\sigma(K, X)$ for any closed bounded subinterval K of \mathbb{R} where $\tilde{f}(\theta) = f(e^{i\theta})$. We set

$$\|f\|_{T,\sigma} = \|\tilde{f}\|_{K,\sigma}$$

where $K = [-2\pi, 2\pi]$. This interval has length greater than 2π to ensure smoothness at the end points.

Our first lemma translates the definition of C_σ into a statement about trigonometric polynomial approximation.

LEMMA 4.1. Suppose $f \in C(T, X)$ and suppose there exist $N \in \mathbb{N}, \sigma > 0, l > 0$ and $\gamma > 0$ so that for any subinterval I of $[-2\pi, 2\pi]$ with $|I| < l$ there is a trigonometric polynomial

$$\varphi_I(\theta) = \sum_{k=-N}^N x_k e^{ik\theta} \quad \text{with} \quad \|\tilde{f} - \varphi_I\|_I \leq \gamma |I|^\sigma.$$

Then $f \in C_\sigma(T, X)$.

Proof. First we note the existence of a constant $C = C(p, N)$ so that for α satisfying $0 < \alpha \leq 1$ and all $x_0, \dots, x_{2N} \in X$ we have

$$\max_{0 \leq k \leq 2N} \|x_k\| \leq C \max_{|l| \leq 1} \left\| \sum_{k=0}^{2N} x_k \left(\frac{e^{i\alpha k} - 1}{\alpha} \right)^k \right\|.$$

This is a simple consequence of the fact that $(2N + 1)$ -dimensional subspaces of X

are uniformly normable and $\lim_{\alpha \rightarrow 0} (e^{i\alpha t} - 1)/\alpha = it$. Thus

$$\max_{0 \leq k \leq 2N} \alpha^k \|x_k\| \leq C \max_{|t| \leq \alpha} \left\| \sum_{k=0}^{2N} x_k (e^{it} - 1)^k \right\|.$$

In the remainder of the argument we take $\varrho = \gamma + \|f\|$ for convenience. Let I be any interval of length at most l . Then let s be the midpoint of I . For $0 < \alpha \leq l/2$ there is a trigonometric polynomial g_α of degree at most N so that

$$\|\tilde{f}(s+t) - g_\alpha(t)\| \leq C\varrho\alpha^\sigma$$

for $|t| \leq \alpha$. Let

$$g_\alpha(t) = \sum_{k=0}^{2N} x_{k,\alpha} e^{-iNt} (e^{it} - 1)^k.$$

For $k \leq 2N$, $\|x_{k,l/2}\| \leq C\varrho$. Now arguing as in Proposition 3.1 we obtain

$$\|g_\alpha(t) - g_\beta(t)\| \leq C\varrho\alpha^\sigma, \quad |t| \leq \beta,$$

provided $\frac{1}{2}\alpha \leq \beta \leq \alpha$. Hence

$$\|x_{k,\alpha} - x_{k,\beta}\| \leq C\varrho\alpha^{\sigma-k}.$$

We conclude that if $k > \sigma$ then $\|x_{k,\alpha}\| \leq C\varrho\alpha^{\sigma-k}$, while if $k < \sigma$ then $\|x_{k,\alpha}\| \leq C\varrho$.

If $\sigma \in \mathbb{N}$ and $k = \sigma$ then arguing as in Proposition 3.1 we get

$$\|x_{k,\sigma}\| \leq C(\log(l/\alpha) + 1)^{1/p} \varrho.$$

Thus

$$\left\| \sum_{k=v+1}^{2N} x_{k,\alpha} e^{-iNt} (e^{it} - 1)^k \right\| \leq C\varrho\alpha^\sigma.$$

Select polynomials λ_k of degree v so that

$$|e^{-iNt} (e^{it} - 1)^k - \lambda_k(t)| \leq C|t|^{v+1}$$

for $|t| \leq 2\pi$. Then

$$\left\| \sum_{k=0}^v x_{k,\alpha} e^{-iNt} (e^{it} - 1)^k - \sum_{k=0}^v x_{k,\alpha} \lambda_k(t) \right\| \leq C\varrho\alpha^\sigma$$

if $|t| \leq \alpha$. Here if σ is an integer we need to observe that $\alpha^{v+1}(\log(l/\alpha) + 1)^{1/p} \leq C\alpha^v = C\alpha^\sigma$. Thus

$$\|\tilde{f}(s+t) - \sum_{k=0}^v x_{k,\alpha} \lambda_k(t)\| \leq C\varrho\alpha^\sigma$$

provided $|t| \leq \alpha$. Taking $\alpha = \frac{1}{2}|I|$ we obtain the lemma.

We will now turn to consideration of a specific example. We shall need the following general lemma.

LEMMA 4.2. *There is a constant C depending only on n, a_1, \dots, a_n so that if $z_j \in \bar{D}$, $1 \leq j \leq n$, and $0 < a_j < 1$ but $a_1 + \dots + a_n > 1$ then*

$$\int_{-\pi}^{\pi} \prod_{j=1}^n |1 - z_j e^{i\theta}|^{-a_j} d\theta \leq C\varrho^{1-(a_1+\dots+a_n)}$$

where $\varrho = \min_{j \neq k} |z_j - z_k|$.

Proof. Let A_j be the arc in T described by $|1 - z_j e^{i\theta}| < \frac{1}{2}\varrho$. Then

$$\int_{A_j} \prod_{j=1}^n |1 - z_j e^{i\theta}|^{-a_j} d\theta \leq \left(\frac{1}{2}\varrho\right)^{-\beta_j} \int_{A_j} |1 - z_j e^{i\theta}|^{-a_j} d\theta$$

where $\beta_j = \sum_{k \neq j} a_k$. Now using the estimate

$$|1 - re^{i\theta}|^{-1} \leq C((1-r)^2 + \theta^2)^{-1/2} \leq C|\theta|^{-1}$$

we see that

$$\int_{A_j} |1 - z_j e^{i\theta}|^{-a_j} d\theta \leq C^{a_j} \int_{C|\theta|^{-1} > \varrho/2} |\theta|^{-a_j} d\theta \leq C\varrho^{1-a_j}.$$

Thus

$$\sum_j \int_{A_j} \prod_{j=1}^n |1 - z_j e^{i\theta}|^{-a_j} d\theta \leq C\varrho^{1-(a_1+\dots+a_n)}.$$

Let B be the complement of $A_1 \cup \dots \cup A_n$ in T . Then on B

$$\begin{aligned} \prod_{j=1}^n |1 - z_j e^{i\theta}|^{-a_j} &\leq (\min_j |1 - z_j e^{i\theta}|)^{-(a_1+\dots+a_n)} \\ &\leq \left(\sum_j |1 - z_j e^{i\theta}|^{-1}\right)^{a_1+\dots+a_n}. \end{aligned}$$

Let $\alpha = a_1 + \dots + a_n > 1$. Then

$$\begin{aligned} \left\{ \int_B \prod_{j=1}^n |1 - z_j e^{i\theta}|^{-a_j} d\theta \right\}^{1/\alpha} &\leq \sum_j \left\{ \int_B |1 - z_j e^{i\theta}|^{-\alpha} d\theta \right\}^{1/\alpha} \\ &\leq \sum_j \left\{ \int_{|1 - z_j e^{i\theta}| > \varrho/2} |1 - z_j e^{i\theta}|^{-\alpha} d\theta \right\}^{1/\alpha} \\ &\leq \sum_j \left\{ \int \min(|1 - z_j e^{i\theta}|^{-\alpha}, (\frac{1}{2}\varrho)^{-\alpha}) d\theta \right\}^{1/\alpha} \\ &\leq C \left\{ \int \min(\theta^{-\alpha}, (\frac{1}{2}\varrho)^{-\alpha}) d\theta \right\}^{1/\alpha} \leq C\varrho^{1/\alpha-1} \end{aligned}$$

and the lemma follows.

Now we let

$$u(z) = (1 - wz)^{-1}$$

so that $u: \Delta \rightarrow L_p(T)$ is an analytic function. We note that

$$u^{(m)}(z) = m! w^m (1 - wz)^{-(m+1)}.$$

If $0 \leq m < 1/p - 1$ then $u^{(m)}$ extends continuously to $\bar{\Delta}$.

We now compare $u^{(m)}$ with its Taylor series. For $z, z + \zeta \in \bar{\Delta}$ let

$$\varrho^{(m)}(z, \zeta) = u^{(m)}(z + \zeta) - \sum_{j=0}^l u^{(m+j)}(z) \zeta^j / j!.$$

LEMMA 4.3. Suppose $1/p \notin \mathbb{N}$. Let $\sigma = 1/p - m - 1$ and $v = [\sigma]$. Then

$$\|\varrho_v^{(m)}(z, \zeta)\|_p \leq C |\zeta|^\sigma$$

where C is independent of z, ζ .

Proof. By direct calculation

$$\varrho_l^{(0)}(z, \zeta) = \zeta^{l+1} w^{l+1} (1 - wz)^{-(l+1)} (1 - w(z + \zeta))^{-1}$$

and $\varrho_v^{(m)} = (\partial^m / \partial z^m) \varrho_v^{(0)}$. Every term in $\varrho_v^{(m)}$ is thus of the type

$$\zeta^{v+1} w^{m+v+1} (1 - wz)^{-(v+j+1)} (1 - w(z + \zeta))^{-(m+1-j)}$$

for $0 \leq j \leq m$ and is thus $O(|\zeta|^\sigma)$ by Lemma 4.4.

If $1/p \in \mathbb{N}$ the situation is more complicated. For $0 < \varphi < \pi$ we write $h(\zeta, z)$ for the polynomial

$$\sum_{j=0}^{v-1} (1 - wz)^{-j+1} w^j \zeta^j + w^v \zeta^v (1 - wz)^{1-v} (1 - wze^{-i\varphi})^{-1} (1 - wze^{i\varphi})^{-1}.$$

LEMMA 4.4. If $1/p \in \mathbb{N}$ then there is a constant C so that if $\frac{1}{2} \sin \frac{1}{2} \varphi \leq |\zeta| \leq \sin \frac{1}{2} \varphi$ we have

$$\|u^{(m)}(z + \zeta) - (\partial^m h / \partial z^m)(\zeta, z)\| \leq C \varphi^\sigma.$$

Proof. For convenience we write $f_{a,b,c,d}$ for the function

$$(1 - wz)^{-a} (1 - w(z + \zeta))^{-b} (1 - wze^{i\varphi})^{-c} (1 - wze^{-i\varphi})^{-d}.$$

Now

$$u(z + \zeta) - h(\zeta, z) = \zeta^v w^v f_{v,1,1,1}((1 - \cos \varphi)wz + w\zeta - w^2 \zeta z).$$

Thus

$$u(z + \zeta) - h(\zeta, z) = ((1 - \cos \varphi)wz f_{v,1,1,1} + w \zeta f_{v-1,1,1,1}) \zeta^v w^v.$$

Since $1 - \cos \varphi = O(\varphi^2)$ and $|\zeta| = O(\varphi)$ we can use Lemma 4.2 to deduce

$$\|u^{(m)}(z + \zeta) - \partial^m h / \partial z^m\| \leq C \varphi^\sigma$$

provided $\frac{1}{2} \sin \frac{1}{2} \varphi \leq |\zeta| \leq \sin \frac{1}{2} \varphi$. This follows on checking each term in the derivative and noting that since $v + m < 1/p$ each term of the form $f_{a,b,c,d}$ satisfies $\|f_{a,b,c,d}\| \leq C \varphi^{(1/p) - a - b - c - d}$.

THEOREM 4.5. Let $\sigma = 1/p - m - 1$. Then $u^{(m)} \in C_\sigma(T, X)$.

Proof. For $1/p \notin \mathbb{N}$, this is immediate from Lemma 4.1 and Lemma 4.3.

If $1/p \in \mathbb{N}$, we use instead Lemma 4.4, which shows that by appropriate choice of z we can approximate $u^{(m)}(e^{i\theta})$ on any interval I with $|I| \leq \frac{1}{2}$ by a trigonometric polynomial $\varphi(\theta)$ of degree at most v so that

$$\|u^{(m)} - \varphi\|_I \leq C |I|^\sigma.$$

Now let $J_{p,0}^{(m)}$ be the closed linear span of the functions $u^{(m)}(z)$ for $|z| = 1$. We note that if $m \leq \beta \leq n < 1/p - 1$ then $w^\beta (1 - wz)^{-(n+1)}$ is in $J_{p,0}^{(m)}$. In fact this can be proved simply by induction. If $v(z) = w^\beta (1 - wz)^{-(n+1)}$ is in $J_{p,0}^{(m)}$ and $n + 1 < 1/p - 1$ then $v'(z) \in J_{p,0}^{(m)}$, i.e. $w^{\beta+1} (1 - wz)^{-(n+2)} \in J_{p,0}^{(m)}$. But then this also implies $w^\beta (1 - wz)^{-(n+1)} + w^{\beta+1} z (1 - wz)^{-(n+2)} = w^\beta (1 - wz)^{-(n+2)} \in J_{p,0}^{(m)}$.

Now if

$$h(z, w) = \sum_{j=1}^N a_j (1 - \alpha_j wz)^{-(k_j+1)}$$

where $|\alpha_j| = 1, 0 \leq k_j < 1/p - m - 1$ and $a_j \in C$, then $\partial^m h / \partial z^m \in J_{p,0}^{(m)}$. In particular, if h is the function defined before Lemma 4.4 then $\partial^m h / \partial z^m$ is in $J_{p,0}^{(m)}$ for all z . To see this simply write the last term in partial fractions.

Now by Lemmas 4.3 and 4.4 we immediately obtain:

THEOREM 4.6. Suppose $0 < p < 1$ and $0 \leq m < 1/p - 1$. Let $\sigma = 1/p - 1 - m$. Then for $0 < r < 1$

$$d(u^{(m)}(re^{i\theta}), J_{p,0}^{(m)}) \leq C(1-r)^\sigma$$

uniformly in θ .

5. Analytic functions and linear operators. As usual X will denote a p -Banach space where $0 < p < 1$. We denote by $A_0(X)$ the space of continuous functions $f: \bar{\Delta} \rightarrow X$ which are analytic in the open unit disc Δ . $A_0(X)$ is quasi-normed by

$$\|f\|_0 = \max_{|z| \leq 1} \|f(z)\|.$$

For $\sigma > 0$ and $\sigma \notin \mathbb{N}$, let $v = [\sigma]$. We let $A_\sigma(X)$ denote the space of analytic functions on Δ such that

$$\sup_{|z| < 1} \|f^{(v+1)}(z)\| (1 - |z|^2)^{v+1-\sigma} < \infty.$$

On $A_\sigma(X)$ we impose the quasi-norm

$$(5.1) \quad \|f\|_\sigma = \sup_{|z| < 1} \|f^{(v+1)}(z)\| (1 - |z|^2)^{v+1-\sigma} + \sum_{k=0}^v \|f^{(k)}(0)\|.$$

If σ is an integer we let $v = \sigma$ and define $A_\sigma(X)$ to be the space of analytic functions

f defined on Δ so that

$$\sup_{|z| < 1} (1 - |z|^2) \|f^{(v+1)}(z)\| < \infty.$$

The quasi-norm on $A_\sigma(X)$ is again defined by (5.1), with $v = \sigma$ of course.

Note that we have not asserted that the spaces $A_\sigma(X)$, $\sigma \geq 0$, are complete. This fact will, however, be established later.

Now suppose E is any quasi-Banach space of scalar-valued analytic functions on Δ containing the disc algebra $A(\Delta)$ and so that the inclusion $A(\Delta) \rightarrow E$ is bounded and has dense range. If $T \in \mathcal{L}(E, X)$ we define the analytic transform f_T to be the function $f_T: \Delta \rightarrow X$ given by

$$f_T(z) = T(u(z))$$

where $u(z) = (1 - wz)^{-1}$. It is clear that f_T is analytic on X and has the power series expansion

$$f_T(z) = \sum_{n=0}^{\infty} x_n z^n$$

where $T(w^n) = x_n$.

The analytic transform induces a one-one correspondence between $\mathcal{L}(E, X)$ and a certain space of X -valued analytic functions on Δ .

THEOREM 5.1. *Suppose $0 < q < p \leq 1$. Then the map $T \rightarrow f_T$ induces a linear isomorphism between the spaces $\mathcal{L}(H_q, X)$ and $A_\sigma(X)$ where $\sigma = 1/q - 1$.*

Proof. If $T \in \mathcal{L}(H_q, X)$ then $f_T(z) = T(u(z))$ where $u(z) = (1 - wz)^{-1}$. It follows that $f_T^{(v+1)}(z) = T(u^{(v+1)}(z))$ and the fact that $f_T \in A_\sigma(X)$ with $\|f_T\|_\sigma \leq C \|T\|$ follows easily from the fact that $u \in A_\sigma(H_q)$.

Conversely, let us suppose $f \in A_\sigma(X)$. Then f has a Maclaurin expansion

$$f(z) = \sum_{n=0}^{\infty} x_n z^n, \quad |z| < 1.$$

For $r < 1$ let $T_r \in \mathcal{L}(H_q, X)$ be defined by

$$T_r \varphi = \sum_{n=0}^{\infty} a_n r^n x_n$$

where $\varphi(w) = \sum_{n=0}^{\infty} a_n w^n \in H_q$. Since $|a_n| \leq C(n+1)^{1/q-1}$ it is clear that $T_r \varphi$ is well defined and T_r is bounded.

If $\varphi(w) = \sum_{n=0}^{\infty} a_n w^n$ in H_q define $\psi(w) \in H_q$ by

$$\psi(w) = \sum_{n=v+1}^{\infty} a_n w^{n-v-1}.$$

Then $\|\psi\|_q \leq C \|\varphi\|_q$ where $C = C(v, q)$. Now $\psi \in B_{q,p}$ and $\|\psi\|_{q,p} \leq C \|\varphi\|_q$. Hence by a theorem of Coifman and Rochberg [4] we can write in $B_{q,p}$

$$\psi(w) = \sum_{k=1}^{\alpha_1} \alpha_k (1 - |z_k|^2)^{v+1-\sigma} (1 - wz_k)^{-(v+2)}$$

where $z_k \in \Delta$ and (α_k) are so that

$$\left(\sum |\alpha_k|^p\right)^{1/p} \leq C \|\varphi\|_q.$$

Now T_r is bounded also on $B_{q,p}$ since $B_{q,p}$ is the containing p -Banach space of H_q . Thus if we write

$$w^{v+1} \psi(w) = ((v+1)!)^{-1} \sum_{k=1}^{\alpha_1} \alpha_k (1 - |z_k|^2)^{v+1-\sigma} u^{(v+1)}(z_k)$$

then

$$T_r(w^{v+1} \psi(w)) = ((v+1)!)^{-1} \sum_{k=1}^{\alpha_1} \alpha_k (1 - |z_k|^2)^{v+1-\sigma} f^{(v+1)}(rz_k).$$

Thus $\|T_r(w^{v+1} \psi(w))\| \leq C \|f\|_\sigma \|\varphi\|_q$. Now

$$\|T_r\left(\sum_{k=0}^v a_k w^k\right)\| \leq C \|f\|_\sigma \|\varphi\|_q$$

and hence

$$\|T_r \varphi\| \leq C \|f\|_\sigma \|\varphi\|_q.$$

Thus $\|T_r\| \leq C \|f\|_\sigma$. As $\lim_{r \rightarrow 1} T_r(w^n)$ exists for all $n \geq 0$, we can define a bounded linear operator T so that $T(w^n) = x_n$ and $\|T\| \leq C \|f\|_\sigma$. Clearly $f_T = f$.

COROLLARY 5.2. *If $\sigma > 1/p - 1$, then $A_\sigma(X)$ is complete.*

We now use the identification of Theorem 5.1 to derive some important facts about the class $A_\sigma(X)$ with no restriction on σ .

THEOREM 5.3. *Suppose $\sigma > 0$ and $f \in A_\sigma(X)$. Then*

- (i) *If m is an integer and $0 \leq m < \sigma$ then $f^{(m)} \in A_0(X)$, i.e., $f^{(m)}$ extends continuously to A . The map $f \rightarrow f^{(m)}$ is continuous from $A_\sigma(X)$ into $A_0(X)$.*
- (ii) *If m is an integer and $m = \sigma$, then for some constant $B = B(f)$ and exponent $\alpha > 0$,*

$$\|f^{(m)}(z)\| \leq B(1 + (\log(1 - |z|))^{-1})^\alpha.$$

- (iii) *For $0 \leq r \leq 1$, the functions f_r given by*

$$f_r(e^{i\theta}) = f(re^{i\theta})$$

are in $C_\sigma(T, X)$ and $\sup_{1/2 \leq r \leq 1} \|f_r\|_{T,\sigma} < \infty$.

(iv) If $\sigma > 1$ then

$$(d/d\theta) f(e^{i\theta}) = ie^{i\theta} f'(e^{i\theta}).$$

(v) Each of the spaces $A_\sigma(X)$ is complete.

Proof. First select $n \in \mathbb{N}$ so that $\sigma + n > 1/p - 1$. Let $1/q = \sigma + n + 1$. Integrate f n times to produce $F \in A_{\sigma+n}$ with $F(0) = \dots = F^{(n-1)}(0) = 0$. Then $\|F\|_{\sigma+n} = \|f\|_\sigma$ and so there is a bounded linear operator $T: H_q \rightarrow X$ with $T(u(z)) = F(z)$ for $|z| < 1$, and $\|T\| \leq C \|f\|_\sigma$. Let $Q_r: H_q \rightarrow H_q$ be the map

$$Q_r \varphi(w) = \varphi(rw)$$

for $0 \leq r \leq 1$. Then $TQ_r(u(z)) = F(rz)$ and hence

$$TQ_r(u^{(n)}(z)) = r^n f(rz).$$

For (i) we observe, taking $r = 1$, that $u^{(m+n)}(z) \in A_0(H_q)$ and hence $T(u^{(m+n)}(z)) = f^{(m)}(z) \in A_0(X)$, and of course

$$\max \|f^{(m)}(z)\| \leq C \|T\| \leq C \|f\|_\sigma.$$

For (ii), note that $m+n = 1/q - 1$. Then in H_q

$$\|u^{(m+n)}(z)\| \leq C \left(1 + \left(\log \frac{1}{1-|z|} \right)^{1/q} \right)$$

and the result follows.

To prove (iii) note that $u^{(n)}(e^{i\theta}) \in C_\sigma(T, H_q)$ and hence, since $\|TQ_r\| \leq \|T\|$, $r^n f(re^{i\theta}) \in C_\sigma(T, X)$ and

$$\sup_{0 \leq r \leq 1} \|r^n f(re^{i\theta})\|_{T, \sigma} \leq C \|f\|_\sigma.$$

(iii) now follows.

To prove (iv) note that

$$\frac{d}{d\theta} f(e^{i\theta}) = T \left(\frac{d}{d\theta} u^{(n)}(e^{i\theta}) \right) = ie^{i\theta} T(u^{(n+1)}(e^{i\theta})) = ie^{i\theta} f'(e^{i\theta}).$$

For (v) suppose f_i is a Cauchy sequence in $A_\sigma(X)$. Let F_i be the corresponding n -fold integrals in $A_{\sigma+n}(X)$. Then F_i is a Cauchy sequence and converges in $A_{\sigma+n}(X)$ to a limit F by Corollary 5.2. Now $F_i^{(n)} \rightarrow F^{(n)}$ in $A_\sigma(X)$ and (v) is proved.

LEMMA 5.4. Suppose $\sigma > 0$. Then $f \in A_\sigma(X)$ if and only if $g \in A_\sigma(X)$ where $g(z) = zf(z)$.

Proof. For $n \in \mathbb{N}$

$$g^{(n)}(z) = zf^{(n)}(z) + nf^{(n-1)}(z).$$

Suppose $f \in A_\sigma(X)$. Pick $n = \nu + 1$. Then

$$\|g^{(\nu+1)}(z)\| \leq C(1-|z|^2)^{\sigma-\nu-1}$$

by applying Theorem 5.3 (i) or (ii) depending on whether $\sigma \notin \mathbb{N}$ or $\sigma \in \mathbb{N}$. Conversely, suppose $g \in A_\sigma(X)$. Then, by induction, $f^{(m)}$ is bounded if $0 \leq m < \sigma$. If $\sigma \notin \mathbb{N}$, $f^{(\nu)}$ is bounded, and hence, since

$$zf^{(\nu+1)}(z) = g^{(\nu+1)}(z) - (\nu+1)f^{(\nu)}(z),$$

we see that $f \in A_\sigma(X)$.

If $\sigma \in \mathbb{N}$, then $f^{(\nu-1)}$ is bounded and in this case

$$\|f^{(\nu)}(z)\| \leq C \left(1 + \left(\log \frac{1}{1-|z|} \right)^\alpha \right)$$

and hence again

$$\|f^{(\nu+1)}(z)\| \leq C(1-|z|)^{-1}$$

so that $f \in A_\sigma(X)$.

We are now ready to prove a limiting case of Theorem 5.3 when $\sigma = 1/p - 1$. For this we define C_p to be the space of functions f analytic on the open unit disc so that

$$\|f\| = |f(0)| + \left\{ \int_1^1 |f'(w)|^p (1-|w|^2)^{p-1} d\lambda(w) \right\}^{1/p} < \infty$$

(where λ denotes the planar measure $dx dy$). Then $f \in C_p$ if and only if $f' \in B_{r,p}$ where $1/r = 1/p + 1$. The natural integration operator J maps $B_{r,p}$ isomorphically onto the subspace $C_{p,0}$ of functions vanishing at 0. It is well known (Duren [6], p. 88) that J maps $B_{r,p}$ into H_p . Thus $C_p \subset H_p$. However, $C_p \cong B_{r,p} \oplus C \cong l_p$ (cf. [12], [20]) and hence $C_p \neq H_p$.

THEOREM 5.5. Suppose $0 < p < 1$. Then the mapping $T \rightarrow f_T$ defines a linear isomorphism between the spaces $\mathcal{L}(C_p, X)$ and $A_\sigma(X)$ where $\sigma = 1/p - 1$.

Proof. If $T \in \mathcal{L}(C_p, X)$ then $TJ \in \mathcal{L}(H_p, X)$. Let g be the analytic transform of TJ so that $g \in A_{\sigma+1}(X)$ and $\|g\|_{\sigma+1} \leq C \|T\|$. Now if $f = f_T$ is the analytic transform of T then

$$f(z) = z \frac{d}{dz} (zg(z))$$

and hence $f \in A_\sigma(X)$ by Lemma 5.4. By an application of the Uniform Boundedness Principle it may be seen that

$$\sup_{\|T\| \leq 1} \sup_{z \in D} (1-|z|^2)^{\nu+1-\sigma} \|f_T^{(\nu+1)}(z)\| < \infty$$

and for $k \leq \nu$

$$\sup_{\|T\| \leq 1} \|f_T^{(k)}(0)\| < \infty$$

so that $\|f_T\|_\sigma \leq C \|T\|$.

Conversely, suppose $f \in A_\sigma(X)$. Then $z^{-1}(f(z) - f(0)) \in A_\sigma(X)$ and hence if

$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(0)}{(n+1)(n+1)!} z^n$$

then $g \in A_{\sigma+1}(X)$. Thus there is an operator $S: B_{r,p} \rightarrow X$ so that $S(u(z)) = g(z)$. Now define $T: C_p \rightarrow X$ by $T\varphi = S\varphi' + f(0)\varphi(0)$. (Note $\varphi' = d\varphi/dw$.) Then

$$\begin{aligned} T(u(z)) &= S(z/(1-wz)^2) + f(0) \\ &= zS(1/(1-wz)) + z^2S(w/(1-wz)^2) + f(0) \\ &= zg(z) + z^2g'(z) + f(0) \\ &= \sum_{n=0}^{\infty} \frac{f^{(n+1)}(0)}{(n+1)!} \left(\frac{1}{n+1} + \frac{n}{n+1} \right) z^{n+1} + f(0) \\ &= \sum_{n=0}^{\infty} \frac{f^{(n+1)}(0)}{(n+1)!} z^{n+1} + f(0) = f(z). \end{aligned}$$

This shows that the map $T \rightarrow f_T$ is a surjection and by the Open Mapping Theorem it is also an isomorphism.

6. Remarks on analytic functions. We now give some applications of the results of Section 5 to the general theory of analytic functions in a quasi-Banach space.

THEOREM 6.1. *Suppose $n \in \mathbb{N}$ and $n > 1/p$. Then for any $f \in A_0(X)$ we have*

$$\|f^{(m)}(0)\| \leq C(m+n)! \|f\|$$

where C is independent of m and f .

Proof. Pick q so that $n+1 > 1/q > n$. Let

$$F(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{(n+k)!} z^{n+k}.$$

Then $F^{(n)} \in A_0(X)$ and so $F \in A_\sigma(X)$ where $\sigma = 1/q - 1$. Define $T \in \mathcal{L}(H_q, X)$ to be the operator with analytic transform F . Note that $\|F\|_\sigma \leq \|f\|$. Then

$$\left\| \frac{f^{(m)}(0)}{(m+n)!} \right\| = \|T(w^{(k+m)})\| \leq \|T\| \leq C\|f\|.$$

THEOREM 6.2 (Liouville). *Let $f: C \rightarrow X$ be a bounded entire function. Then f is constant (compare [16], [21]).*

Proof. Simply apply Theorem 6.1 to f_r where $f_r(z) = f(rz)$ for $0 < r < \infty$. One concludes that $f'(0) = f''(0) = \dots = 0$.

THEOREM 6.3. $A_0(X)$ is complete (i.e. it is a closed subspace of $C(\bar{D}, X)$).

Proof. Observe that the linear maps $f \rightarrow f^{(m)}(0)$ ($m \in \mathbb{N}$) extend to the closure of $A_0(X)$ and that, for fixed $z \in \Delta$, the series $\sum f^{(m)}(0) z^m/m!$ converges to $f(z)$ uniformly on the set $\{f \in A_0(X) : \|f\| \leq 1\}$. Hence if $g \in \overline{A_0(X)}$ then

$$g(z) = \sum_{m=0}^{\infty} x_m z^m, \quad |z| < 1,$$

for some $x_m \in X$.

THEOREM 6.4. *Suppose $\sigma > 1/p - 1$ and $f \in A_\sigma(X)$. Then for $m \geq 0$*

$$f^{(m)}(0) = \frac{m!}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-im\theta} d\theta,$$

while for $m < 0$

$$0 = \int_0^{2\pi} f(e^{i\theta}) e^{-im\theta} d\theta.$$

Here the integral is the Turpin-Waelbroeck integral as described in Section 3, off with respect to $d\mu = e^{-im\theta} d\theta$.

Proof. For $r < 1$, let

$$g_n(e^{i\theta}) = \sum_{k=0}^n f^{(k)}(0) r^k e^{ik\theta}/k!.$$

Then $\text{rank } g_n \leq n+1$ and if $f_r(e^{i\theta}) = f(re^{i\theta})$ then

$$\begin{aligned} \|g_n - f_r\|_T &\leq \left\{ \sum_{k=n+1}^{\infty} \|f^{(k)}(0)\|^p r^{kp}/(k!)^p \right\}^{1/p} \\ &\leq C \left\{ \sum_{k=n+1}^{\infty} ((s+k)!/k!)^p r^{kp} \right\}^{1/p} \end{aligned}$$

where $s \in \mathbb{N}$ and $s > 1/p$, by Theorem 6.1. Hence

$$\|g_n - f_r\|_T = O\left(\left(\frac{1+r}{2}\right)^n\right) = O(n^{-\sigma})$$

so that

$$\begin{aligned} \frac{m!}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-im\theta} d\theta &= r^m f^{(m)}(0), \quad m \geq 0, \\ \frac{m!}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-im\theta} d\theta &= 0, \quad m < 0. \end{aligned}$$

Now we use Theorem 3.4 (i) and Theorem 5.3 (i) and (iii) to let $r \rightarrow 1$, and obtain the result.

Now recall that for $\sigma \in \mathbb{R}$, $V_\sigma(X)$ consists of analytic $f: \Delta \rightarrow X$ so that for

some C

$$\|f(z)\| \leq C(1-|z|)^\sigma, \quad z \in \Delta.$$

LEMMA 6.5. *If $f \in V_\sigma(X)$ then $f' \in V_{\sigma-1}(X)$.*

PROOF. If $z_0 \in \Delta$, let $\alpha = \frac{1}{2}(1-|z_0|)$. Define $g(z) = f'(z_0 + \alpha z)$ for $|z| \leq 1$. Then $g \in A_0(X)$ and so

$$\|g'(0)\| \leq C \sup_{|z| \leq 1} \|g(z)\|, \quad \text{i.e.} \quad \|\alpha f'(z_0)\| \leq C\alpha^\sigma.$$

Hence $\|f'(z_0)\| \leq C(1-|z_0|)^{\sigma-1}$.

THEOREM 6.6. *For $\sigma > 0$, $f \in V_\sigma(X)$ if and only if $f' \in A_\sigma(X)$ and $f(z) = 0$ for $z \in T$.*

PROOF. Let $v = [\sigma]$. Then repeated application of Lemma 6.5 shows that if $f \in V_\sigma(X)$, then $f \in A_\sigma(X)$.

For the other direction, select n so that $\sigma + n > 1/p - 1$. Pick q so that $1/q = \sigma + n + 1$. Integrate f n times to produce $F: \Delta \rightarrow X$ with $F(0) = \dots = F^{(n-1)}(0) = 0$ and $\|F\|_{\sigma+n} = \|f\|_\sigma$. Then there is a bounded linear operator $T: H_q \rightarrow X$ with $T(u(z)) = F(z)$.

Note $T(u^{(m)}(z)) = f(z)$. If $f = 0$ on T then $T(J_{q,0}^{(m)}) = 0$ and so we can apply Theorem 4.6 to deduce $\|f(z)\| \leq C(1-|z|)^\sigma$.

THEOREM 6.7. *If $\sigma > 1/p - 1$ then $V_\sigma(X) = \{0\}$.*

PROOF. By Theorem 6.4 if $f \in V_\sigma(X)$ then $f^{(m)}(0) = 0$ for all m .

REMARK. We have seen that if $\sigma = 1/p - 1$ then $V_\sigma(X)$ can be nontrivial.

7. Operators on L_p . We now introduce the space $E_\sigma(X)$. This will consist of all functions $f: C \rightarrow X$ with the properties:

- (a) f continuously extends to the Riemann sphere $C^* = C \cup \{\infty\}$ if we set $f(\infty) = 0$.
- (b) f is analytic on $C^* \setminus T$.
- (c) $f \in A_\sigma(X)$ on the disc Δ .
- (d) $f(1/z) \in A_\sigma(X)$ on the disc Δ .

$E_\sigma(X)$ is quasi-normed by

$$\|f\|_{\sigma,E} = \max(\|f\|_\sigma, \|f(1/z)\|_\sigma).$$

We shall consider only the case $\sigma = 1/p - 1$. In this instance we have seen that $A_\sigma(X)$ is complete and the injection $A_\sigma(X) \hookrightarrow A_0(X)$ is bounded. From these observations we see that $E_\sigma(X)$ is complete.

If $T \in \mathcal{L}(L_p, X)$ where $0 < p < 1$ then its analytic transform f_T is defined by

$$f_T(z) = T((1-wz)^{-1}), \quad z \in C.$$

THEOREM 7.1. *Suppose $0 < p < 1$. Then the analytic transform $T \rightarrow f_T$ induces a linear isomorphism between $\mathcal{L}(L_p, X)$ and $E_\sigma(X)$ where $\sigma = 1/p - 1$.*

REMARK. As usual X is a p -normable space.

PROOF. One direction is very easy. If $T \in \mathcal{L}(L_p, X)$ then it is immediate that $f_T \in E_\sigma(X)$ and $\|f_T\|_{\sigma,E} \leq C\|T\|$. We now show that the map $T \rightarrow f_T$ is a surjection. Since it is trivially an injection the conclusion follows from the Open Mapping Theorem.

Let us suppose $f \in E_\sigma(X)$. Then f has a Taylor series expansion around the origin,

$$f(z) = \sum_{n=0}^{\infty} x_n z^n, \quad |z| < 1,$$

and a Laurent series expansion around ∞ ,

$$f(z) = \sum_{n=1}^{\infty} y_n z^{-n}, \quad |z| > 1.$$

It is readily seen that $f = f_T$ if and only if $T(w^n) = x_n$ for $n \geq 0$ and $T(w^{-n}) = -y_n$ for $n > 0$. We therefore need to show the existence of such an operator T .

Let us first suppose $x_0 = f(0) = 0$. Then we define two analytic functions on Δ by

$$F_1(z) = \sum_{n=1}^{\infty} \frac{x_n}{n} z^n, \quad F_2(z) = \sum_{n=1}^{\infty} \frac{y_n}{n} z^n.$$

Then $F_1'(z) = z^{-1}f(z)$ and hence, by Lemma 5.4, $F_1' \in A_\sigma$ and so $F_1 \in A_{\sigma+1}$. Similarly $F_2'(z) = z^{-1}f(z^{-1})$ and so $F_2 \in A_{\sigma+1}$. In particular, F_1 and F_2 extend continuously to $\bar{\Delta}$ (Theorem 5.3). Since $\sigma + 1 = 1/p > 1/p - 1$ we can utilize Theorem 6.4:

$$x_n/n = (2\pi)^{-1} \int_0^{2\pi} F_1(e^{i\theta}) e^{-in\theta} d\theta, \quad n \geq 1,$$

$$y_n/n = (2\pi)^{-1} \int_0^{2\pi} F_2(e^{i\theta}) e^{-in\theta} d\theta, \quad n \geq 1.$$

Also

$$\int_0^{2\pi} F_1(e^{i\theta}) e^{in\theta} d\theta = \int_0^{2\pi} F_2(e^{i\theta}) e^{in\theta} d\theta = 0.$$

Next we note, by Theorem 5.3 again, that

$$(d/d\theta) F_1(e^{-i\theta}) = -ie^{-i\theta} F_1'(e^{-i\theta}) = -if(e^{-i\theta}),$$

$$(d/d\theta) F_2(e^{i\theta}) = ie^{i\theta} F_2'(e^{i\theta}) = if(e^{i\theta}).$$

Now define $G \in C_{\sigma+1}(T, X)$ by $G(e^{i\theta}) = -F_1(e^{i\theta}) - F_2(e^{i\theta})$. Then $(d/d\theta)G(e^{i\theta}) = 0$. Now we use Proposition 3.1 (iii) or (iv), depending on

whether $\sigma \notin N$ or $\sigma \in N$, to deduce that

$$\|G(e^{i\varphi}) - G(e^{i\theta})\| \leq C|\varphi - \theta|^{1/n}.$$

Now it follows that there is an operator $T: L_p \rightarrow X$ with

$$T\chi_{(\theta, \varphi)} = (2\pi i)^{-1} (G(e^{i\varphi}) - G(e^{i\theta}))$$

where $\chi_{(\theta, \varphi)}(e^{it}) = 1$ if $\theta \leq t \leq \varphi$ and zero elsewhere (when $0 \leq \theta < \varphi \leq 2\pi$).

We now compute $T(w^n)$. To do this we introduce simple functions h_N ($N = 1, 2, \dots$). Let

$$h_N(e^{it}) = e^{in\theta_k}, \quad \theta_{k-1} < t \leq \theta_k,$$

where $\theta_k = 2k\pi/N$, for $k = 0, 1, \dots, N$. Then $Th_N \rightarrow Tw^n$. However,

$$\begin{aligned} Th_N &= (2\pi i)^{-1} \sum_{k=1}^N e^{in\theta_k} (G(e^{i\theta_k}) - G(e^{i\theta_{k-1}})) \\ &= (2\pi i)^{-1} \sum_{k=1}^N G(e^{i\theta_k}) (e^{in\theta_k} - e^{in\theta_{k+1}}) \end{aligned}$$

where $\theta_{N+1} = \theta_1$. Hence

$$Th_N = (2\pi i)^{-1} \int G(e^{i\theta}) d\mu_N(\theta)$$

where μ_N is the measure on $[0, 2\pi]$ given by

$$\mu_N = \sum_{k=1}^N (e^{in\theta_k} - e^{in\theta_{k+1}}) \delta(\theta_k).$$

In the weak*-topology $\mu_N \rightarrow \mu$ where $d\mu = -ine^{in\theta} d\theta$. At this stage we appeal to Theorem 3.4 (ii) to deduce that

$$\begin{aligned} Tw^n &= \frac{-n}{2\pi} \int_0^{2\pi} G(e^{i\theta}) e^{in\theta} d\theta \\ &= \frac{n}{2\pi} \int_0^{2\pi} (F_1(e^{-i\theta}) + F_2(e^{i\theta})) e^{in\theta} d\theta. \end{aligned}$$

If $n \geq 0$ then $Tw^n = x_n$, while for $n > 0$, $Tw^{-n} = -y_n$ as required. This settles the special case when $f(0) = 0$.

If $f(0) = x_0 \neq 0$, let X_0 be the one-dimensional space spanned by x_0 and let $Q: X \rightarrow X/X_0$ be the quotient map. Then $Qf(0) = 0$ and so there is a bounded linear operator $S: L_p(\mathcal{T}) \rightarrow X/X_0$ with $S(w^n) = Qx_n$ for $n \geq 0$ and $S(w^{-n}) = -Qy_n$ for $n > 0$. By results in [11], S has a unique lift $T: L_p \rightarrow X$ with $QT = S$. Let

$$T((1 - wz)^{-1}) = g(z).$$

Then $f - g$ has range in X_0 , i.e. $f(z) - g(z) = h(z)x_0$ where $h \in E_\sigma(C) = \{0\}$ by Liouville's Theorem. Hence T has analytic transform f .

The isomorphism between $\mathcal{L}(L_p, X)$ and $E_\sigma(X)$ follows from the Open Mapping Theorem.

An operator T on L_p vanishes on $\bar{H}_{p,0}$ if its analytic transform f_T vanishes for $|z| \geq 1$. Then we must have $f_T \in V_\sigma(X)$ on the disc. Conversely, if $f \in V_\sigma(X)$ then f can be continued over C to be zero outside the disc and hence there is an operator T on L_p so that $T(\bar{H}_{p,0}) = 0$ and $f_T = f$ in the open unit disc. Summarizing:

THEOREM 7.2. *There is a natural linear isomorphism between $\mathcal{L}(L_p/\bar{H}_{p,0}, X)$ (or $\mathcal{L}(H_p/J_{p,0}, X)$) and $V_\sigma(X)$ implemented by*

$$T(qu(z)) = f_T(z), \quad |z| < 1,$$

where $q: L_p \rightarrow L_p/\bar{H}_{p,0}$ is the quotient map.

THEOREM 7.3. *In order that $V_\sigma(X) \neq \{0\}$ it is necessary and sufficient that there exists a nonzero linear operator $T: L_p/H_p \rightarrow X$.*

8. Applications to L_p . We first apply our main theorem to extend a theorem due to Aleksandrov [1] that $L_p = H_p + \bar{H}_p$. Our extension uses the space C_p introduced in Section 5. As noted there, C_p is strictly contained in H_p .

THEOREM 8.1 *There is a constant C so that if $f \in L_p(\mathcal{T})$ then there exist $g_1, g_2 \in C_p$ with $\|g_1\|_{C_p} \leq C\|f\|_p$, $\|g_2\|_{C_p} \leq C\|f\|_p$ and*

$$f(e^{i\theta}) = g_1(e^{i\theta}) + g_2(e^{-i\theta})$$

a.e. on \mathcal{T} .

Proof. We define a linear operator $W: C_p \oplus C_p \rightarrow L_p(\mathcal{T})$ by

$$W(h_1, h_2) = h_1(e^{i\theta}) + h_2(e^{-i\theta}).$$

(Note that each $h_i \in H_p$ and so has boundary values a.e. on \mathcal{T} .)

Let $N = W^{-1}(0)$, and let $Q: C_p \oplus C_p \rightarrow Y = C_p \oplus C_p/N$ be the quotient map. Define $f: C \rightarrow Y$ by

$$f(z) = \begin{cases} Q(u(z), 0), & |z| \leq 1, \\ Q(0, 1 - u(1/z)), & |z| \geq 1. \end{cases}$$

Then f is continuous on $C \cup \{\infty\}$ and it is readily verified to be in $E_\sigma(Y)$. Hence there is an operator $S: L_p(\mathcal{T}) \rightarrow Y$ with

$$S((1 - wz)^{-1}) = f(z), \quad z \in C.$$

It is easily seen that $SW = Q$ and hence Y is isomorphic to $L_p(\mathcal{T})$ and W is a surjection as required.

An immediate corollary of Theorem 8.1 is an atomic decomposition for $L_p(\mathcal{T})$ in the spirit of [4] which may also be regarded as a strengthening of Aleksandrov's theorem.

THEOREM 8.2. *Suppose $0 < p < 1$ and $\beta > 1/p$. Then there exists $\eta_0 = \eta_0(p, \beta)$ so that if $\eta < \eta_0$ and $(\zeta_n)_{n=1}^\infty$ is an η -lattice in Δ for the Bergman metric and $\zeta_n \neq 0$ then there is a constant C so that if $f \in L_p(T)$ then*

$$f(w) = \sum_{n=1}^\infty a_n(1 - \zeta_n w)^{-\beta} (1 - |\zeta_n|^2)^{\beta-1/p} + \sum_{n=1}^\infty b_n(1 - \bar{\zeta}_n \bar{w})^{-\beta} (1 - |\zeta_n|^2)^{\beta-1/p}$$

where $\sum |a_n|^p + \sum |b_n|^p \leq C \|f\|_p^p$.

Proof. First we note that if $C_{p,0}$ is the set of $g \in C_p$ so that $g(0) = 0$ then the decomposition in Theorem 8.1 can be achieved with $g_1, g_2 \in C_{p,0}$. Indeed, if not, there is a linear functional $\tau \neq 0$ on L_p so that $\tau(W(h_1, h_2)) = 0$ for $h_1, h_2 \in C_{p,0}$. Hence

$$\tau \circ W(h_1, h_2) = ah_1(0) + bh_2(0)$$

where $a, b \in C$. By the openness of W , τ is continuous on L_p and we have a contradiction.

Now we can write

$$f(w) = g_1(w) + g_2(\bar{w})$$

where $g_1, g_2 \in C_{p,0}$ and $\|g_i\| \leq C \|f\|_p$ ($i = 1, 2$). Then $g'_1, g'_2 \in B_{p,r}$ where $1/r = 1/p + 1$. Hence by Theorem 2 of [4],

$$g'_1(w) = \sum_{n=1}^\infty c_n(1 - \zeta_n w)^{-(\beta+1)} (1 - |\zeta_n|^2)^{\beta-1/p},$$

$$g'_2(w) = \sum_{n=1}^\infty d_n(1 - \bar{\zeta}_n w)^{-(\beta+1)} (1 - |\zeta_n|^2)^{\beta-1/p},$$

where $\sum |c_n|^p + \sum |d_n|^p \leq C \|f\|_p^p$. Thus

$$g_1(w) = \sum_{n=1}^\infty c_n \beta^{-1} \zeta_n^{-1} (1 - \zeta_n w)^{-\beta} (1 - |\zeta_n|^2)^{\beta-1/p},$$

$$g_2(w) = \sum_{n=1}^\infty d_n \beta^{-1} \bar{\zeta}_n^{-1} (1 - \bar{\zeta}_n w)^{-\beta} (1 - |\zeta_n|^2)^{\beta-1/p}.$$

Now let $a_n = c_n \zeta_n^{-1} \beta^{-1}$, $b_n = d_n \bar{\zeta}_n^{-1} \beta^{-1}$. Then

$$\sum |a_n|^p + \sum |b_n|^p \leq (\min |\zeta_n|)^{-p} \beta^{-p} \sum (|c_n|^p + |d_n|^p) \leq C \|f\|_p^p.$$

The result now follows easily.

THEOREM 8.3. *Suppose $f: \Delta \rightarrow X$ is analytic and*

$$\lim_{r \rightarrow 1} (1 - r^2)^{1-1/p} \left\{ \int_0^{2\pi} \|f(re^{i\theta})\|^p d\theta \right\}^{1/p} = 0$$

Then $f = 0$.

Proof. Define $g: \Delta \rightarrow L_p(T, X)$ by

$$g(z)(w) = f(wz).$$

Then g is analytic and $g \in V_\sigma(L_p(T, X))$ where $\sigma = 1/p - 1$. In fact, there is a monotone decreasing function $\varrho: [0, 1] \rightarrow \mathbf{R}$ so that $\lim_{r \rightarrow 1} \varrho(r) = 0$ and

$$\|g(z)\| \leq \varrho(|z|)(1 - |z|^2)^{1/p-1}.$$

By Theorem 6.1 we conclude that if $|z| \geq \frac{1}{2}$ then

$$\|g'(z)\| \leq C\varrho(2|z-1|)(1 - |z|^2)^{1/p-2},$$

and differentiating $v+1$ times where $v = [\sigma]$,

$$\|g^{(v+1)}(z)\| \leq C\varrho(2^{v+1}|z-2^{v+1}+1|)(1 - |z|^2)^{1/p-v-2}$$

for $|z| \geq 1 - 2^{-v+1}$.

Now by Theorem 8.2 pick a suitable η -lattice (ζ_n) and define a bounded linear operator $W: l_p \rightarrow L_p$ so that

$$We_{2n-1} = (1 - \zeta_n w)^{-(v+2)} (1 - |\zeta_n|^2)^{v+2-1/p},$$

$$We_{2n} = (1 - \bar{\zeta}_n \bar{w})^{-(v+2)} (1 - |\zeta_n|^2)^{v+2-1/p},$$

where (e_n) are the basic vectors of l_p . Then W maps l_p onto L_p . Define $W': l_p \rightarrow L_p$ by $W'(a) = (v+1)! w^{v+1} W(a)$; then W' is also onto.

By Theorem 7.1 there is an operator $T: L_p \rightarrow L_p(T, X)$ so that

$$T(u(z)) = g(z), \quad |z| < 1,$$

$$T(u(z)) = 0, \quad |z| \geq 1.$$

Now

$$TW'(e_{2n-1}) = g^{(v+1)}(\zeta_n)(1 - |\zeta_n|^2)^{v+2-1/p}$$

and so $\lim_{n \rightarrow \infty} \|TW'(e_{2n-1})\| = 0$. Also

$$TW'(e_{2n}) = (v+1)! T(w^{v+1}(1 - \bar{\zeta}_n \bar{w})^{-(v+2)}(1 - |\zeta_n|^2)^{v+2-1/p}).$$

However,

$$(1 - \bar{\zeta}_n \bar{w})^{-(v+2)} = \sum_{k=0}^\infty \frac{(v+k+1)!}{k!(v+1)!} \bar{\zeta}_n^k w^{-k}$$

in L_p and hence

$$TW'(e_{2n}) = (1 - |\zeta_n|^2)^{v+2-1/p} \sum_{k=0}^{v+1} \frac{(v+k+1)!}{k!} \bar{\zeta}_n^k T(w^{v+1-k}).$$

Thus $\lim_{n \rightarrow \infty} \|TW'(e_{2n})\| = 0$ since $v+2 > 1/p$ and $|\zeta_n| \rightarrow 1$.

It follows that TW' is compact on l_p and hence that T is compact on L_p . Now this means $T = 0$ (cf. [8]). Thus $g = f = 0$.

9. Applications to tensor products. Turpin [17] has shown that if X is p -normable and Y is q -normable then there is an r -convex quasi-norm on $X \otimes Y$ so that

$$\|x \otimes y\| = \|x\| \cdot \|y\|, \quad x \in X, y \in Y,$$

where $1/r = 1/p + 1/q - 1$. We now show this is best possible.

THEOREM 9.1. *Suppose $0 < p, q \leq 1$ and suppose $1/r = 1/p + 1/q - 1$. Let Z be an r -Banach space and suppose $B: L_p/H_p \times L_q/H_q \rightarrow Z$ is a nonzero bounded bilinear form. Then there is a nonzero linear operator $T: L_r/H_r \rightarrow Z$.*

Proof. We identify L_p/H_p with $L_p/\bar{H}_{p,0}$. Let $v_p: \Delta \rightarrow L_p/\bar{H}_{p,0}$ be defined by

$$v_p(z) = Q(u(z))$$

where Q is the quotient map. Then

$$\|v_p(z)\| \leq C(1 - |z|^2)^{1/p-1}$$

in $L_p/\bar{H}_{p,0}$ and similarly

$$\|v_q(z)\| \leq C(1 - |z|^2)^{1/q-1}.$$

Thus if $|\zeta| = 1$ then

$$\|B(v_p(z), v_q(\zeta z))\| \leq C(1 - |z|^2)^{1/r-1}.$$

Now by Theorem 7.3 if there is no nontrivial operator in $\mathcal{L}(L_r/H_r, Z)$ then

$$B(v_p(z), v_q(\zeta z)) = 0$$

for $z \in \Delta$ and $|\zeta| = 1$. If $|z| < 1$ the function $\zeta \rightarrow B(v_p(z), v_q(\zeta z))$ is analytic for $|\zeta| < |z|^{-1}$ and is zero for $|\zeta| = 1$. Since zeros of nontrivial analytic functions are isolated we have $B(v_p(z), v_q(\zeta z)) = 0$ for $|\zeta| < |z|^{-1}$ and hence $B(v_p(z_1), v_q(z_2)) = 0$ for $z_1, z_2 \in \Delta$. This implies $B = 0$, contrary to assumption.

COROLLARY 9.2. *If Z is s -normable where $1/s < 1/p + 1/q - 1$ then there is no nonzero bilinear form $B: L_p/H_p \times L_q/H_q \rightarrow Z$.*

COROLLARY 9.3. *Let B be a nontrivial n -linear form on $\prod_{j=1}^n L_{p_j}/H_{p_j}$, where $0 < p_j \leq 1$, whose range is contained in an r -normable space Z . Then*

$$\frac{1}{r} \geq \sum_{j=1}^n \frac{1}{p_j} - n.$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI COLUMBIA
Columbia, Mo. 65211, U.S.A.

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