Convexity, type and the three space problem

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Abstract. A twisted sum of two quasi-Banach spaces $X$ and $Y$ is a quasi-Banach space $Z$ with a closed subspace $X_0 \cong X$ such that $Z/X_0 \cong Y$.

We show that if $X$ is $p$-convex and $Y$ is $q$-convex where $p \neq q$, then $Z$ is $\min(p, q)$ convex. Similarly, if $X$ is a type $p$ Banach space and $Y$ is a type $q$ Banach space where $p \neq q$ then $Z$ is type $\min(p, q)$.

If $X$ and $Y$ are Banach spaces, we show that $Z$ is log convex, i.e., for some $C < \infty$

$$\|x_1 + \ldots + x_n\| \leq C \left( \sum_{i=1}^{n} \|x_i\| \left( 1 + \log \frac{1}{\|x_i\|} \right) \right)$$

where $|x_1| + \ldots + |x_n| = 1$. Conversely, every log convex space is the quotient of a subspace of a twisted sum of two Banach spaces.

If $X$ and $Y$ are type $p$ Banach spaces ($1 < p < 2$) and one is the quotient of a subspace of some $L_p$-space, then $Z$ is type $p$, i.e.,

$$\int \left( |x_1(t)| + \ldots + |x_n(t)| \right)^{1/p} dt \leq C \left( \sum \|x_i\|^p \left( 1 + \log \frac{1}{\|x_i\|^p} \right) \right)^{1/2p}$$

where $|x_1| + \ldots + |x_n| = 1$. This result is best possible in a certain sense.

We also show that if $p < 1$ type $p$ implies $p$-convexity, but if $p > 1$ a type 1 space need not be convex.

We investigate which Orlicz sequence spaces and Köthe sequence spaces are $\mathcal{K}$-spaces, i.e., such that every twisted sum with $K$ is a direct sum.

1. Introduction. A quasi-Banach space $Z$ is a twisted sum of $X$ and $Y$ if it has a subspace $X_0 \cong X$ such that $Z/X_0 \cong Y$. The so-called three space problem is to study the properties of $Z$ in terms of those of $X$ and $Y$.

In [1], Enflo, Lindenstrauss and Pisier showed that a Banach space which is a twisted sum of two Hilbert spaces need not be a Hilbert space. Independently, the author [6], Ribe [15] and Roberts [16] showed that a twisted sum of a line and a Banach space need not be locally convex. In [9] the author and Peck showed that these results are related by describing a general construction which shows that for every $p$, $0 < p < \infty$, there is a twisted sum of $l_p$ with itself which is not a direct sum. In particular, for $0 < p < 1$, there is a non-$p$-convex space which is a twisted sum of two $p$-convex spaces.

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In contrast to these negative results there are a number of theorems which say that a twisted sum cannot be too bad. In [1] the twisted sum of two type 2 Banach spaces is shown to be type \( p \) for all \( p < 2 \). In [6] it is shown that if \( X \) is a Banach space and \( Y \) is a type \( p \) Banach space for some \( p > 1 \), then a twisted sum of \( X \) and \( Y \) (in that order) is convex (i.e., a Banach space). Also if \( X \) is \( p \)-convex \((0 < p \leq 1)\) and \( Y \) is \( q \)-convex \((0 < q < p)\), then any twisted sum of \( X \) and \( Y \) (in that order) is again \( q \)-convex.

These suggest a general principle, if we regard \( p \)-convexity \((0 < p \leq 1)\) or type \( p \) \((0 < p \leq 2)\) as an index of "roundness". The twisted sum of two spaces of differing degrees of roundness will retain the properties of the less round space; the twisted sum of two spaces of equal roundness may however be less round than either. The main aim of this paper is to study the case of two Banach spaces of equal roundness, and to examine more precisely the case of equal roundness.

First in Section 3, we introduce a new class of quasi-Banach spaces which we name log convexa. A space \( X \) is log convex if either of the following two equivalent conditions holds for some \( C, \quad 0 < \infty \)

\[
\left\| x_1 + \ldots + x_n \right\| \leq C \sum_{j=1}^{n} \left( 1 + \log \left( \frac{1}{\|x_j\|} \right) \right)
\]

wherever \( \left\| x_1 \right\| + \ldots + \left\| x_n \right\| = 1 \), \( x_1, \ldots, x_n \in X \) or

\[
\left\| x_1 + \ldots + x_n \right\| \leq C^* \sum_{j=1}^{n} \left( 1 + \log j \right).
\]

Log convex spaces play an important role in this paper; they are, in a sense, the next best thing to being Banach spaces. An example is the space \( L(1, \infty) \) (i.e., weak \( L_1 \)).

In Section 4 we show that if \( p < 1 \), type \( p \) is equivalent to \( p \)-convexity, so that we reduce the study of type to the case \( 1 \leq p \leq 2 \).

Section 5 contains some initial technical results in twisted sums which contain very little that is new. Lemma 5.2 essentially reproduces a result of [1] in rather more generality.

In Section 6 we show that if \( X \) and \( Y \) are \( p \)-convex and \( q \)-convex, respectively, where \( p < q \), then any twisted sum of \( X \) and \( Y \) or of \( Y \) and \( X \) is \( p \)-convex. (One half of this result is in [6], see above.) In a similar vein, if \( X \) and \( Y \) are type \( p \) and type \( q \), respectively, where \( 1 \leq p < q < 2 \), then any twisted sum \( Z \) of \( X \) and \( Y \) or \( Y \) and \( X \) is type \( p \). Let us remark here that the methods of [1] (cf. also [13]) show that in either case if \( x_1, \ldots, x_n \in Z \)

\[
\left\{ \int_0^1 \left( \sum_{j=1}^{n} \left\| x_j \right\|^p \right)^{p/p'} dt \leq \left( \sum_{j=1}^{n} \left\| x_j \right\|^p \right)^{p/p'}
\]

the main step in the argument here is to pass from this inequality to establishing type \( p \) (but only for twisted sums where the other space is type \( q \geq p \)). As shown in [13] (1.0.3) implies type \( r \) for \( r < p \). We note that a type 1 space need not be convex.

In Section 7 we study the case \( p = q \). We show that any twisted sum of two Banach spaces is log convex, and this result is best possible. In fact, a space is log convex if and only if it is a quotient of some subspace of a twisted sum of two Banach spaces. The corresponding results for type \( p \) are right if we assume that one of the spaces \( X \) and \( Y \) is a quotient of a subspace of a space \( L_p(m) \). In that case any twisted sum \( Z \) is log type \( p \), i.e.,

\[
\left\{ \int_0^1 \left( \sum_{j=1}^{n} \left\| x_j \right\|^p \right)^{p/p'} dt \leq \left( 1 + \sum \left( \left\| x_j \right\|^p \right)^{p/p'} \right)^{p/p'}
\]

whenever \( \left\| x_1 \right\|^p + \ldots + \left\| x_n \right\|^p = 1 \), or equivalently

\[
\left\{ \int_0^1 \left( \sum_{j=1}^{n} \left\| x_j \right\|^p \right)^{p/p'} dt \leq \left( \sum \left( \left\| x_j \right\|^p (1 + \log k) \right)^{p/p'} \right)^{p/p'}
\]

for \( x_1, \ldots, x_n \in Z \). Furthermore this is best possible for the twisted sum of \( L_p \) and \( L_q \) \((1 \leq p \leq 2)\) constructed in [9] contains a copy of the Orlicz space \( \ell_\infty \), where

\[
u(t) = t^p \left[ 1 + \log \left( \frac{1}{t} \right) \right]^p
\]

near zero, and for this space (1.0.4) cannot be improved.

In Section 8 we examine twisted sums of \( R \) and \( l_1 \) more closely, showing in particular that the examples in [8] and [13] are non-equivalent but both are best possible in a certain sense.

In Section 9 we classify those non-locally convex Orlicz spaces \( l_p \subset l_1 \) which are \( X \)-spaces, i.e., for which every twisted sum of \( R \) and \( l_p \) is a direct sum. In particular, this applies to examining "gall" conditions of the type \( \sum \left( \left\| x_j \right\|^p \right) < \infty \Rightarrow \sum \left\| x_j \right\| \) converge) which are preserved under twisted sums with \( R \). It is shown that if \( f \) is submultiplicative, this condition will be preserved if and only if \( f(x) \geq c x^p \) for some \( p < 1 \).

In Section 10, we examine those locally convex \( X \)-spaces \( X \) which are not locally bounded, but are \( X \)-spaces, so that they have the property that if \( Y \) is locally convex any twisted sum of \( Y \) and \( X \) is locally convex. It is shown that every nuclear space is an \( X \)-space, and Köthe spaces which are \( X \)-spaces are characterized exactly.

2. Quasi-Banach spaces. Throughout this paper all vector spaces will be real, although most arguments may be modified without difficulty to the complex case.
A quasi-norm on a real vector space $X$ is a map $x \mapsto \|x\|$ ($X \to \mathbb{R}$) such that for some $K < \infty$,
\begin{align}
(2.0.1) & \quad \|x\| > 0, \quad x \neq 0, \quad x \in X, \\
(2.0.2) & \quad \|tx\| = |t|\|x\|, \quad t \in \mathbb{R}, \quad x \in X, \\
(2.0.3) & \quad \|x + y\| \leq K(\|x\| + \|y\|), \quad x, y \in X.
\end{align}

A quasi-norm induces a locally bounded topology on $X$ and conversely any locally bounded topology is given by a quasi-norm. A complete quasi-normed space is called a quasi-Banach space. If in addition we have for some $0 < p \leq 1$
\begin{equation}
(2.0.4) \quad \|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X,
\end{equation}
then $X$ is called a $p$-Banach space (or if $p = 1$ a Banach space).

A quasi-Banach space $X$ is said to be $p$-convex for some $0 < p \leq 1$ if there is a constant $A$ such that
\begin{equation}
(2.0.5) \quad \|x_1 + x_2 + \ldots + x_n\| \leq A(\|x_1\|^p + \ldots + \|x_n\|^p)^{1/p}
\end{equation}
for $x_1, \ldots, x_n \in X$. If $X$ is $p$-convex, it may be equivalently quasi-normed to be a $p$-Banach space. A theorem of Aoki and Rolewicz (see [17]) states that every quasi-Banach space is $p$-convex for some $p > 0$. We shall repeatedly exploit this by assuming the quasi-norm on a given space satisfies (2.0.4) for some $p > 0$.

We denote by $(\epsilon_n; n \in \mathbb{N})$ a sequence of independent random variables (or measurable functions) on $[0, 1]$ such that $\lambda(\epsilon_n = +1) = \lambda(\epsilon_n = -1) = \frac{1}{2}$ where $\lambda$ is Lebesgue measure. We then say that a quasi-Banach space $X$ is type $p$ ($0 < p < 2$) ([12], [13]) if for some constant $\lambda < \infty$ we have
\begin{align}
(2.0.6) & \quad \left\{ \int_0^1 \left[ \sum_{i=1}^{n} \epsilon_i(t)x_i^p \right] dt \right\}^{1/p} \leq K \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}.
\end{align}

If $X$ is $p$-convex, then $X$ is certainly type $p$.

We remark here that Kahane [4] shows that for a Banach space $X$ and $0 < p < q < \infty$ there is a constant $K = K(p, q)$ such that
\begin{align}
(2.0.7) & \quad \left\{ \int_0^1 \sum_{i=1}^{n} \epsilon_i x_i^p \right\}^{1/p} \leq \left\{ \int_0^1 \sum_{i=1}^{n} \epsilon_i x_i^q \right\}^{1/2^q} \\
& \quad \leq K \left\{ \int_0^1 \sum_{i=1}^{n} \epsilon_i x_i^p \right\}^{1/p}.
\end{align}

This means that we can change the exponent on the left of (2.0.6) without altering the definition.

In fact, (2.0.7) holds for quasi-Banach spaces; the modifications in Kahane's argument are minor but we include a proof for completeness.

**Theorem 2.1.** Let $X$ be a quasi-Banach space. Then (2.0.7) holds.

**Proof.** Let $L_\infty(X)$ be the space of $X$-valued simple functions on $[0, 1]$ equipped with the topology of convergence in measure. Let $\text{Rad}(X)$ be the subspace of functions of the form $\epsilon_1x_1 + \ldots + \epsilon_nx_n$ for $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$. We show that $\text{Rad}(X)$, the $L_\infty$-topology coincides with the stronger topology induced by any quasi-norm
\[ f \mapsto \left\{ \frac{1}{t} \int_0^t |f(t)|^p dt \right\}^{1/p}. \]

We see this, we need only show that the set of $f \in \text{Rad}(X)$ with $\lambda(\|f\| \geq 1) < \frac{1}{2}$ is bounded in each $L_\infty$-norm.

Suppose $f = \epsilon_1x_1 + \ldots + \epsilon_nx_n$ and
\[ \lambda(\|f\| > r) = \alpha. \]

Let
\[ M(t) = \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \epsilon_i(t)x_i \right\|, \quad 0 \leq t \leq 1, \]
\[ N(t) = \max_{1 \leq k \leq n} \left\| \sum_{i=k+1}^{n} \epsilon_i(t)x_i \right\|, \quad 0 \leq t \leq 1. \]

Let $A_k$ ($1 \leq k \leq n$) be the set of $t$ such that
\[ \left\| \sum_{i=1}^{k} \epsilon_i(t)x_i \right\| < Kr, \quad 1 \leq k \leq n-1, \]
\[ \left\| \sum_{i=k+1}^{n} \epsilon_i(t)x_i \right\| \geq Kr \]
(where $K$ is the modulus of concavity of the quasi-norm given by (2.0.3)). Since $f$ has the same distribution as
\[ f^* = \epsilon_1x_1 + \ldots + \epsilon_kx_k - \epsilon_{k+1}x_{k+1} - \ldots - \epsilon_nx_n \]
and
\[ \lambda(A_k \cap \{||f|| \geq 2Kr\}) \leq \lambda(A_k \cap \{||f|| > r\}) + \lambda(A_k \cap \{||f^*|| \geq r\}) = 2\lambda(A_k \cap \{||f|| > r\}) \]
and hence
\[ \lambda(A_k) \leq 2\lambda(A_k \cap \{||f|| > r\}), \]
so that, summing over $k$,
\[ \lambda(M > Kr) \leq 2\alpha. \]
Similarly, $\lambda(N > Kr) \leq 2\alpha$. Therefore, $\lambda(A_k \cap \{||f|| > r\}) = 0$, and hence $\lambda(\|f\| > r) = 0$. Therefore $f \in \text{Rad}(X)$.
Now if \( t \in A_k \) and \( \|f\| > 2K^2r \),
\[
\left\| \sum_{k=1}^n a_k e_k \right\| \geq K^2r
\]
and
\[
\left\| \sum_{k=1}^n a_k e_k \right\| \geq K^2r \quad \text{(since } \left\| \sum_{k=1}^n a_k e_k \right\| < K^2r)\]
Hence
\[
\lambda\left( A_k \cap (\|f\| > 2K^2r) \right) \leq 2 \lambda\left( A_k \cap (\left\| \sum_{k=1}^n a_k e_k \right\| > K^2r) \right) \leq 2a \lambda\left( A_k \right)
\]
since these sets are independent. Summing over \( k \),
\[
\lambda(\|f\| > 2K^2r) < 4a^2.
\]
Thus if
\[
\lambda(\|f\| > 1) < \frac{1}{4},
\]
\[
\lambda(\|f\| > (2K^2)^n) < 4^{n-1}(\frac{1}{2})^n < (\frac{1}{4})^n
\]
and
\[
\int_0^1 \|f\|^p dt < 1 + \sum_{n=0}^{\infty} (2K^2)^p(\frac{1}{2})^n = S < \infty.
\]
The 
\[
\text{gall} \ G(\mathcal{X})\text{ of a quasi-Banach space is the space of all sequences} \ \{a_n\} \text{ such that if } \|a_n\| \leq 1, \text{ then } \sum_{n=1} a_n e_n \text{ is bounded. } G(\mathcal{X}) \text{ is a quasi-Banach space when quasi-normed by}
\]
\[
\|\{a_1, a_2, \ldots\}\| = \sup_{\|y\| \leq 1} \sup_n \left\| \sum_{k=1}^n a_k e_k \right\|.
\]
\( \mathcal{X} \) is said to be 
\[
\text{galled by a space of sequences } \mathcal{E} \text{ if, given } \|a_n\| \leq 1, \text{ and } a_n \in \mathcal{E}, \text{ then } \sum_{k=1}^n a_k e_k \text{ is bounded (see Turán [17] for a more detailed study of these notions).}
\]
An 
\[
\mathcal{E}\text{-space} \text{ is a complete metric topological vector space. A twisted sum of two } \mathcal{E}\text{-spaces } \mathcal{X} \text{ and } \mathcal{Y} \text{ is a space } Z \text{ which has a closed subspace } \mathcal{X} \cong Z = \mathcal{X} \text{ such that } Z \cong \mathcal{Y} = \mathcal{Y}. \text{ Thus there is a short exact sequence}
\]
\[
0 \rightarrow \mathcal{X} \rightarrow Z \rightarrow \mathcal{Y} \rightarrow 0. \text{ If every twisted sum of } \mathcal{X} \text{ and } \mathcal{Y} \text{ is a direct sum (i.e. } \mathcal{Z} \cong \mathcal{X} \oplus \mathcal{Y} \text{ in the natural way) then we say that } (\mathcal{X}, \mathcal{Y}) \text{ splits (the order is important here). If } (\mathcal{R}, \mathcal{X}) \text{ splits, then } \mathcal{X} \text{ is a } \mathcal{R}\text{-space ([8]).}
\]
If \( \mathcal{X} \) is a locally convex \( \mathcal{X}\text{-space}, \text{ then every twisted sum of } \mathcal{Y} \text{ with } \mathcal{X} \text{ locally convex is also locally convex ([8], Theorem 4.10) and this property characterizes locally convex } \mathcal{X}\text{-spaces.}
\]

3. Logconvex spaces and related classes. Let \( \varphi \) denote the Orlicz function
\[
\varphi(t) = \begin{cases} t(1 + \log \frac{1}{t}), & 0 \leq t \leq 1, \\ t, & t \geq 1 \end{cases}
\]
(where 0 \( \log \infty = 0 \) \( \log 0 = 0 \) by convention). Then \( I_\varphi \) is a locally bounded but non-locally convex Orlicz sequence space. The quasi-norm inducing the topology on \( I_\varphi \) may be given by
\[
\|x_\varphi\| = \sup \left\{ \varepsilon : \sum_{n=1}^{\infty} \varphi(\varepsilon^{-1}\|x_n\|) \leq 1 \right\}.
\]
Our first result gives an equivalent quasi-norm.

**Theorem 3.1.** An equivalent quasi-norm on \( I_\varphi \) is given by
\[
\|x_\varphi\|^* = \|x_\varphi\| + \sum_{n=1}^{\infty} |x_n| \log \frac{1}{|x_n|}
\]
where \( \|x_\varphi\| = \sum_{n=1}^{\infty} |x_n| \).

**Proof.** Since \( \|x_\varphi\|^* \) is easily seen to be homogeneous, it suffices to show that
\[
0 < \inf \{ \|x_\varphi\|^* : \|x_\varphi\| = 1 \} \leq \sup \{ \|x_\varphi\|^* : \|x_\varphi\| = 1 \} < \infty.
\]
If \( \|x_\varphi\| = 1 \), then \( \sum \varphi(|x_i|) = 1 \) and hence
\[
\sum |x_i| \left( 1 + \log \frac{1}{|x_i|} \right) = 1.
\]
Hence \( \|x_\varphi\| < 1 \) and
\[
\|x_\varphi\|^* \leq \|x_\varphi\| + \sum_{n=1}^{\infty} |x_n| \log \frac{1}{|x_n|} = \|x_\varphi\| = 1.
\]
Conversely,
\[
\|x_\varphi\|^* = \|x_\varphi\| - \|x_\varphi\| \log \|x_\varphi\| = 1 - |x_\varphi| \log |x_\varphi| > 1 - \frac{1}{e}
\]
since \( 0 \leq |x_\varphi| \leq 1 \).

**Definition 3.2.** A quasi-Banach space \( \mathcal{X} \) is logconvex if it is galled by \( I_\varphi \), i.e., whenever \( x_n \in \mathcal{X} \) and
\[
\sum \varphi(\|x_n\|) < \infty
\]
then \( \sum x_n \) converges.
EXAMPLE. The space $L^\infty$ itself is logconvex; this follows easily from the fact that $\varphi$ is submultiplicative at 0 (cf. Turpin [19], p. 79).

Theorem 3.3. A quasi-Banach space $X$ is logconvex if and only if for some constant $C$ and any $x_1, \ldots, x_n \in X$

$$
\|x_1 + x_2 + \ldots + x_n\| \leq C \left( \sum_{i=1}^{n} |x_i| \log \frac{S}{|x_i|} \right)
$$

where $S = \sum_{i=1}^{n} |x_i|.$

Remark. (3.3.1) is equivalent to

$$
\|x_1 + \ldots + x_n\| \leq C \left( 1 + \sum_{i=1}^{n} |x_i| \log \frac{1}{|x_i|} \right)
$$

whenever $|x_1| + \ldots + |x_n| \leq 1.$

Proof. Let $I$ be an infinite set with $|I| = |X|$ and let $(x_i : i \in I)$ be the unit ball of $X.$ If $X$ is logconvex the map $T : l_1(I) \rightarrow X$ (where $l_1(I)$ is the generalized sequence space of all $(z_i : i \in I)$ such that $\sum |z_i| < \infty$ defined by

$$
T(z_i) = \sum_{i=1}^{\infty} z_i x_i
$$

is well-defined and continuous. Hence for some $C < \infty$

$$
\|T(z)\| \leq C \left( \sum_{i=1}^{\infty} |z_i| \log \frac{1}{|z_i|} \right)
$$

and (3.3.1) follows easily.

Conversely, if (3.3.1) holds and

$$
\sum_{n=1}^{\infty} \varphi(|x_n|) \leq 1,
$$

then

$$
\sum_{n=1}^{\infty} |x_n| \left( 1 + \log \frac{1}{|x_n|} \right) \leq 1,
$$

and hence

$$
\left\| \sum_{n=k+1}^{k+l} x_n \right\| \leq C \sum_{n=k+1}^{k+l} |x_n| \left( 1 + \frac{1}{\log |x_n|} \right) \rightarrow 0 \quad \text{as} \quad k, l \rightarrow \infty
$$

so that $\sum x_n$ converges.

$L(1, \infty)$ denotes the space of measurable functions on $[0, 1]$ such that

$$
\|f\| = \text{sup} \lambda(|f| > x) < \infty.
$$

Theorem 3.4. The space $L(1, \infty)$ is logconvex. [Added in proof: see [20].]

Proof. Suppose $x_1, \ldots, x_n \in L(1, \infty)$ and let

$$
f = x_1 + \ldots + x_n.
$$

Suppose also $|x_1| + \ldots + |x_n| = 1.$

Fix $r > 1$ and let $A = (0, 1)$ be a set of measure $\tau.$ For each $i = 1, 2, \ldots, n$ let

$$
E_i = \{|x_i| > 2^{-r} \}.
$$

The $\lambda(E_i) \leq \frac{1}{r} |x_i|,$ hence if $E = E_1 \cup \ldots \cup E_n,$ then $\lambda(E) \leq \frac{1}{r} \tau.$ Now

$$
\inf_{t \leq r} |f(t)| \leq \frac{1}{r} \int_{E \setminus S} \left| \int_{A \setminus S} f(\hat{t}) dt \right| dt
$$

$$
\leq \frac{2}{r} \sum_{i=1}^{n} \int_{A \setminus S} |x_i(\hat{t})| d\hat{t} \leq \frac{2}{r} \sum_{i=1}^{n} \int_{E_i} |x_i(\hat{t})| d\hat{t}
$$

$$
\leq \frac{2}{r} \sum_{i=1}^{n} \int_{A \setminus S} \min \{|x_i(\hat{t})|, 2^{-r} \} d\hat{t} \leq \frac{2}{r} \sum_{i=1}^{n} \int_{A \setminus S} \min \left( \frac{|x_i|}{|x_i|}, 2^{-r} \right) \hat{t} d\hat{t}
$$

$$
\leq \frac{2}{r} \sum_{i=1}^{n} \left( |x_i| + \int_{A \setminus S} \frac{|x_i|}{|x_i|} \hat{t} d\hat{t} \right) = \frac{2}{r} \sum_{i=1}^{n} \left( |x_i| + |x_i| \log \frac{2}{|x_i|} \right)
$$

$$
= \frac{1}{r} \left( 2 \log 2 + 2 \sum_{i=1}^{n} |x_i| \log \frac{1}{|x_i|} \right).
$$

Hence

$$
\|f\| \leq (2 \log 2 + 2 \sum_{i=1}^{n} \frac{|x_i|}{\log \frac{1}{|x_i|}})
$$

and so $L(1, \infty)$ is logconvex.

Example. Let $(y_i : i = 1, 2, \ldots)$ be a sequence of independent random variables each with the Cauchy distribution (i.e., with probability density function

$$
f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \quad -\infty < x < \infty.
$$
Then \(|g_n|; n \in N\) is bounded in \(L(1, \infty)\) and so if \(a_n \geq 0, \sum a_n|g_n|\) converges in \(L(1, \infty)\) if
\[
\sum_{n=1}^{\infty} a_n \left(1 + \log \frac{1}{a_n}\right) < \infty
\]
and then
\[
\sum_{n=1}^{\infty} a_n|g_n(t)| < \infty \text{ a.e.}
\]

L. Schwartz [18] shows that (3.4.1) is equivalent to (3.4.2). See Kahane [4] p. 27 for a similar example.

We now give another characterization of logconvex spaces; for this we require the following lemma.

**Lemma 3.5.** Suppose \(s > 0\) and

\[
C_s = \log \left(\sum_{k=1}^{n} \frac{1}{k^{1+s}} \right) \quad = \log \zeta(1+s).
\]

If \(\xi_1 \geq \xi_2 \geq \ldots \geq \xi_n \geq 0\) and

\[
\xi_1 + \xi_2 + \ldots + \xi_n = 1
\]

then

\[
\sum_{k=1}^{n} \xi_k \log k \leq \sum_{k=1}^{n} \xi_k \log \frac{1}{\xi_k} \leq (1+s) \sum_{k=1}^{n} \xi_k \log k + C_s.
\]

**Proof.** Since \(\xi_k \leq 1/k\) the first inequality is clear. To prove the second, fix \(n\) and let \(C_n\) be the maximum of

\[
F(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{k=1}^{n} \xi_k \log \frac{1}{\xi_k} - (1+s) \sum_{k=1}^{n} \xi_k \log k
\]

subject to \(\xi_1 \geq \ldots \geq \xi_n \geq 0\) and \(\xi_1 + \ldots + \xi_n = 1\). Then for some \((u_1, \ldots, u_n), F(u_1, \ldots, u_n) = C_n\).

We claim first that \(u_1 > u_2 > u_3 > \ldots > u_n > 0\). For if \(1 \leq i \leq n\) in the first index such that \(u_i = 0\) then a small increase in \(u_i\) decreases \(u_{i+1}\) increases \(F\); a similar argument shows that \(u_i \neq u_j\) if \(i \neq j\). It follows that \((u_1, \ldots, u_n)\) is a local maximum of \(F\) subject to the single condition \(\xi_1 + \ldots + \xi_n = 1\). Hence there is a Lagrange multiplier \(\lambda\) such that

\[
\frac{\partial F}{\partial \xi_k}(u_1, \ldots, u_n) = \lambda, \quad k = 1, 2, \ldots, n,
\]

i.e.,

\[
\log \frac{1}{u_k} - (1+s) \log k = \lambda + 1.
\]

Here \(u_n = e^{\lambda+1} \left(\frac{1}{k}\right)^{1+s} \) and so

\[
\sum_{k=1}^{n} \left(\frac{1}{k}\right)^{1+s} \leq e^{\lambda+1}, \quad F(u_1, \ldots, u_n) = (\lambda + 1) = C_n,
\]

and hence \(C_n \leq C_s\) and the result is proved.

**Theorem 3.6.** A quasi-Banach space \(X\) is logconvex if and only if for some \(C < \infty\), whenever \(x_1, \ldots, x_n \in X\)

\[
\|x_1 + \ldots + x_n\| \leq C \sum_{k=1}^{n} \|x_k\|(1+\log k).
\]

**Proof.** This follows immediately from the preceding lemma and Theorem 3.3.

**Remark.** This theorem essentially means that the Orlicz space \(L_p\)
is identical to the Lorentz space of all sequences \((a_n)\) such that \(\sum a_n^p (1+\log n) < \infty\) where \((a_n^p)\) is decreasing for rearrangement of \((|a_n|)\). See [11] for similar results for convex Orlicz spaces and Lorentz spaces.

4. **Type in quasi-Banach spaces.**

**Theorem 4.1.** Suppose \(1 < p < 2\); then a quasi-Banach space of type \(p\) is convex.

**Proof.** Clearly if

\[
b_n = \sup_{\|x\|=1} \inf_{\|y\|=1} \|x_1 + \sigma_1 x_2 + \ldots + \sigma_n x_n\|,
\]

then \(b_n = o(n)\), and the result follows from Theorem 2.5 of [6].

**Theorem 4.2.** Suppose \(0 < p < 1\); then a quasi-Banach space \(X\) of type \(p\) is \(p\)-convex.

**Proof.** We can and do suppose \(X\) is an \(r\)-Banach space where \(0 < r < p\). For each \(n \in N\), let \(d_n\) be the best constant such that

\[
\|x_1 + \ldots + x_n\| \leq d_n \|x_1\|^p + \ldots + \|x_n\|^p \|
\]

for \(x_1, \ldots, x_n \in X\). Suppose for any \(n\)

\[
\left(\int_{0}^{1} \sum_{k=1}^{n} x_k(t) dx \right)^{1/p} \leq C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}.
\]

Then for any \(x_1, \ldots, x_n \in X\) there exists \(\sigma_i = \pm 1\) (1 \(\leq i \leq n\)) such that

\[
\|\sigma_1 x_1 + \ldots + \sigma_n x_n\| \leq C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}.
\]
We may suppose that if \( F = \{ i : \sigma_i = -1 \} \) then \( \sum_{i \in F} |x_i|^p \leq \frac{1}{2} \sum_{i = 1}^n |x_i|^p \).

Then

\[
\left\| \sum_{i \in F} x_i \right\| \leq 2^{-1/p} \delta_n \left( \sum_{i = 1}^n |x_i|^p \right)^{1/p},
\]

and hence

\[
\left\| \sum_{i = 1}^n x_i \right\| \leq \left\| \sum_{i = 1}^n \sigma_i x_i \right\| + 2^p \left\| \sum_{i \in F} x_i \right\| \leq (C + 2^{p(1 - 1/p)} \delta_n) \left( \sum_{i = 1}^n |x_i|^p \right)^{1/p}.
\]

Thus

\[
d_n \leq C + 2^{p(1 - 1/p)} \delta_n,
\]

so that

\[
d_n \leq \frac{C}{(1 - \frac{1}{p}) \delta_n^{(1 - 1/p)}}.
\]

As \( \{ \delta_n \} \) is bounded, \( X \) is \( p \)-convex.

Remark. As is easily seen the hypothesis actually used in the proof is that \( X \) satisfies

\[
\min_{\sigma_i = -1} \| \sigma_i x_1 + \ldots + \sigma_n x_n \| \leq C (\| x_1 \|^p + \ldots + \| x_n \|^p)^{1/p}.
\]

The same argument shows that if \( b_n(x) = O(n^{1/p}) \) (\( p < 1 \)) then \( a_n(X) = O(n^{1/p}) \) where

\[
b_n(x) = \sup_{\| x \| < 1} \min_{\sigma_i = -1} \| \sigma_i x_1 + \ldots + \sigma_n x_n \|
\]

and

\[
a_n(x) = \sup_{\| x \| < 1} \| x_1 + \ldots + x_n \|.
\]

We do not know, however, if \( a_n = O(n^{1/p}) \) implies \( X \) is \( p \)-convex when \( p < 1 \).

When \( p = 1 \) the above proof breaks down and we shall see later that a type 1 space need not be convex. It is tempting to conjecture that a type 1 space must at least be logconvex in view of the following theorem (the converse is clearly false consider \( L_0 \)).

**Theorem 4.3.** Let \( X \) be a type 1 quasi-Banach space isomorphic to a subspace of \( L_0 \). Then \( X \) is log convex.

**Proof.** By Nikolski's theorem ([17], [14]), \( X \) embeds in \( L(1, \infty) \). Now apply Theorem 3.5.

We have not been able to substantiate this conjecture and have only the following, whose proof we omit. It depends on rather more delicate handling of the argument in Theorem 4.2.

**Theorem 4.4.** Suppose \( X \) is a type 1 quasi-Banach space, then for some \( C < \infty \) and any \( x_1, \ldots, x_n \in X \)

\[
\| x_1 + \ldots + x_n \| \leq C (1 + \log n) (\| x_1 \| + \ldots + \| x_n \|).
\]

In view of the above results we shall only consider type when \( 1 \leq p \leq 2 \).

5. Twisted sums. Suppose \( X \) and \( Y \) are quasi-Banach spaces and \( Z \) is a twisted sum of \( X \) and \( Y \), so that \( Z \) has a subspace isomorphic to \( Y \) such that \( Z/Y \cong X \). Then (cf. [10], [8]) there is a map \( F : X \rightarrow X \) satisfying

\[
F(tx) = tF(x), \quad t \in \mathbb{R}, \quad x \in X,
\]

(5.0.1) \( \| F(x_1 + x_2) - F(x_1) - F(x_2) \| \leq K (\| x_1 \| + \| x_2 \|) \), \( x_1, x_2 \in X \),

where \( K \) is independent of \( x_1, x_2 \), such that \( Z \) is isomorphic to the space \( Y \otimes_X X \), i.e., the Cartesian sum \( Y \otimes_X X \) quasilinear normed by

\[
\| (y, z) \| = \| y + F(z) \| + \| z \|.
\]

Conversely, given any such quasilinear map \( F \) satisfying (5.0.1) and (5.0.2), then \( Y \otimes_X X \) is a twisted sum of \( Y \) and \( X \).

Suppose then \( F : X \rightarrow Y \) is a fixed quasilinear map. We define for a finite subset \( \{ x_1, \ldots, x_n \} \) of \( X \)

\[
A(x_1, \ldots, x_n) = F(x_1) + \ldots + F(x_n) - \sum_{i=1}^n F(x_i).
\]

We now state the properties of \( A \).

**Lemma 5.1.** (1) If \( A_1, A_2, \ldots, A_m \) are disjoint subsets of \( \{1, 2, \ldots, n\} \) such that \( A_1 \cup \ldots \cup A_m = \{1, 2, \ldots, n\} \) and

\[
u_i = \sum_{i=1}^n x_i
\]

then

\[
A(x_1, \ldots, x_n) = A(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_m}) + \sum_{i=1}^m A(x_i ; j \in A_i).
\]

(2) \( A(x) = 0 \).

(3) \( \| A(x_1, x_2) \| \leq K (\| x_1 \| + \| x_2 \|) \).

(4) For some \( s, 0 < s \leq 1 \) and \( M < \infty \)

\[
\| A(x_1, \ldots, x_n) \| \leq M (\| x_1 \|^s + \ldots + \| x_n \|^s)
\]

for \( x_1, \ldots, x_n \in X \).

**Proof.** (1)-(3) are obvious and (4) is shown in [9].

Now suppose that \( W \) is any quasi-Banach space. We define \( d_n = d_n(W) \) to be a least constant such that

\[
\| x_1 + \ldots + x_n \| \leq d_n (\| x_1 \| + \ldots + \| x_n \|)
\]
whenever \( w_1, \ldots, w_n \in W \). We also define \( \delta_n = \delta_n(W) \) to be the least constant such that
\[
\left\{ \int_0^1 \left( \sum_{i=1}^n \| e_i w_i \|_i \right)^{1/2} \, dt \right\}^{1/2} \leq \delta_n \left( \sum_{i=1}^n \| w_i \|_i^2 \right)^{1/2}.
\]

The sequence \( \{ \delta_n \} \) has been studied for Banach space in [1], [2] and [13]. It is easy enough to see that both sequences \( \{ \delta_n \} \) and \( \{ \delta_n \} \) are submultiplicative (\( \delta_n \leq \delta_n \delta_m \), \( \delta_n \leq \delta_n \delta_m \)).

For a quasilinear map \( F : X \to Y \) we define \( c_n = c_n(F) \) to be the least constant such that
\[
\| A(w_1, \ldots, w_n) \| \leq c_n (\| w_1 \| + \cdots + \| w_n \|), \quad w_1, \ldots, w_n \in X,
\]
and \( \gamma_n = \gamma_n(F) \) to be the least constant such that
\[
\left\{ \int_0^1 \left( \sum_{i=1}^n \| e_i w_i \|_i \right)^{1/2} \, dt \right\}^{1/2} \leq \gamma_n (\| w_1 \| + \cdots + \| w_n \|), \quad w_1, \ldots, w_n \in X.
\]

Our first result is simply a generalization of a result of Enflo, Lindenstrauss and Pisier.

**Theorem 5.2.** Suppose \( 0 < r \leq 1 \) and \( X \) is an \( r \)-Banach space. For \( n, m \in \mathbb{N} \)
\[
(5.2.1) \quad c_{nm} \leq c_n d_n(X) + c_m d_m(Y),
\]
\[
(5.2.2) \quad \gamma_{nm} \leq \gamma_n \delta_n(X) + \gamma_m \delta_m(Y).
\]

**Proof.** (5.2.1). Suppose \( x_1, x_2, \ldots, x_m, x_n \in X \). Let
\[
\nu_i = \sum_{i=1}^m \| x_i \|, \quad i = 1, 2, \ldots, m.
\]
Then
\[
\| A(x_1, \ldots, x_n) \| \leq c_n \sum_{i=1}^m \| x_i \| \leq c_n d_n(X) \sum_{i=1}^m \| x_i \|
\]
and
\[
\left\| \sum_{i=1}^m A(x_j; (i-1)n < j \leq in) \right\| \leq \gamma_n \delta_n(X) \sum_{i=1}^m \left\| A(x_j; (i-1)n < j \leq in) \right\|
\]
and (5.2.1) follows from Lemma 5.1.

(5.2.2). Let
\[
u_i(t) = \sum_{i=1}^m e_i(t) x_i.
\]
Then
\[
\int_0^1 \left\| A(\nu_1(t), \nu_2(t), \ldots, \nu_m(t)) \right\|^{1/2} \, dt = \int_0^1 \left\| A(\nu_1(t), \nu_2(t), \ldots, \nu_m(t)) \right\| \, dt
\]
(by symmetry)
\[
\leq \int_0^1 \gamma_n \sum_{i=1}^m \| \nu_i(t) \| \, dt \leq \gamma_n \delta_n(X) \sum_{i=1}^m \| x_i \|
\]
and (5.2.2) now follows from Lemma 5.1 and the convexity of the \( L_{\infty} \)-norm.

**Lemma 5.3.** If \( p > 0 \), there exists \( \alpha = \alpha(p) > 0 \) and \( C = C(p) \) such that for \( x_1, \ldots, x_n \in X \)
\[
(5.3.1) \quad \| A(x_1, \ldots, x_n) \| \leq C \left( \sum_{i=1}^n \| x_i \|^{2/p} \right)^{1/p}
\]

**Proof.** By Lemma 5.1 we can find \( \varepsilon > 0 \) and \( M < \infty \) such that
\[
\| A(x_1, \ldots, x_n) \| \leq M \left( \sum_{k=1}^n \| x_k \|^{2/p} \right)^{1/p}
\]
Thus for \( 0 < p < q \), \( \varepsilon = 0 \) will suffice. Now suppose \( s < p < \infty \), and choose \( \theta > 1 \). Let \( c = (\sum_{k=1}^n k^{-q})^{-1} \). Then \( \sum_{k=1}^n k^{-\theta} = 1 \) and hence
\[
\left( \sum_{k=1}^n \| x_k \|^{\theta} \right)^{1/\theta} = (\sum k^{-\theta} \| x_k \|^{\theta})^{1/\theta} \leq (\sum k^{-\theta} \| x_k \|^{\theta})^{1/\theta} \leq \left( \sum_{k=1}^n \| x_k \|^{\theta} \right)^{1/\theta}
\]
and hence \( \alpha = \theta(p/s-1) \) will suffice.

6. Twisted sums with unequal convexity.

**Lemma 6.1.** Suppose \( \mu > r > 0 \), and \( Y \) is an \( r \)-Banach space.
\[
(6.1.1) \quad \text{If } d_n(Y) \asymp O(n^\mu) \text{ and } d_n(X) \asymp O(n^r), \text{ then } c_n \asymp O(n^\mu).
\]
\[
(6.1.2) \quad \text{If } d_n(Y) \asymp O(n^\mu) \text{ and } d_n(X) \asymp O(n^r), \text{ then } \gamma_n \asymp O(n^\mu).
\]

**Remark.** The roles of \( X \) and \( Y \) may be interchanged in this lemma.

**Proof.** Suppose \( d_n(Y) \leq c_n \) and \( d_n(X) \leq b_n \). Select \( N \) so that \( b^{N+1} < 1 \). Let \( d_n = (c_n, b_n) \). Then
\[
(6.1.3) \quad c_{nm} \leq c_n d_m(X) + b_n d_{n-m+1}(Y)
\]

\[
\int_0^1 \left\| A(\nu_1(t), \nu_2(t), \ldots, \nu_m(t)) \right\|^{1/2} \, dt = \int_0^1 \left\| A(\nu_1(t), \nu_2(t), \ldots, \nu_m(t)) \right\| \, dt
\]
(by symmetry)
\[
\leq \int_0^1 \gamma_n \sum_{i=1}^m \| \nu_i(t) \| \, dt \leq \gamma_n \delta_n(X) \sum_{i=1}^m \| x_i \|
\]
and (5.2.2) now follows from Lemma 5.1 and the convexity of the \( L_{\infty} \)-norm.

**Lemma 5.3.** If \( p > 0 \), there exists \( \alpha = \alpha(p) > 0 \) and \( C = C(p) \) such that for \( x_1, \ldots, x_n \in X \)
\[
(5.3.1) \quad \| A(x_1, \ldots, x_n) \| \leq C \left( \sum_{i=1}^n \| x_i \|^{2/p} \right)^{1/p}
\]

**Proof.** By Lemma 5.1 we can find \( \varepsilon > 0 \) and \( M < \infty \) such that
\[
\| A(x_1, \ldots, x_n) \| \leq M \left( \sum_{k=1}^n \| x_k \|^{2/p} \right)^{1/p}
\]
Thus for \( 0 < p < q \), \( \varepsilon = 0 \) will suffice. Now suppose \( s < p < \infty \), and choose \( \theta > 1 \). Let \( c = (\sum_{k=1}^n k^{-q})^{-1} \). Then \( \sum_{k=1}^n k^{-\theta} = 1 \) and hence
\[
\left( \sum_{k=1}^n \| x_k \|^{\theta} \right)^{1/\theta} = (\sum k^{-\theta} \| x_k \|^{\theta})^{1/\theta} \leq (\sum k^{-\theta} \| x_k \|^{\theta})^{1/\theta} \leq \left( \sum_{k=1}^n \| x_k \|^{\theta} \right)^{1/\theta}
\]
and hence \( \alpha = \theta(p/s-1) \) will suffice.

6. Twisted sums with unequal convexity.

**Lemma 6.1.** Suppose \( \mu > r > 0 \), and \( Y \) is an \( r \)-Banach space.
\[
(6.1.1) \quad \text{If } d_n(Y) \asymp O(n^\mu) \text{ and } d_n(X) \asymp O(n^r), \text{ then } c_n \asymp O(n^\mu).
\]
\[
(6.1.2) \quad \text{If } d_n(Y) \asymp O(n^\mu) \text{ and } d_n(X) \asymp O(n^r), \text{ then } \gamma_n \asymp O(n^\mu).
\]

**Remark.** The roles of \( X \) and \( Y \) may be interchanged in this lemma.

**Proof.** Suppose \( d_n(Y) \leq c_n \) and \( d_n(X) \leq b_n \). Select \( N \) so that \( b^{N+1} < 1 \). Let \( d_n = (c_n, b_n) \). Then
\[
(6.1.3) \quad c_{nm} \leq c_n d_m(X) + b_n d_{n-m+1}(Y)
\]
so that
\[ \theta_n \leq (hN^{-r})^p \theta_{n-1} + a \theta_1 \]
and hence \( \{ \theta_n \} \) is bounded. Hence \( c_n = O(n^p) \).

(6.1.2) has a similar proof.

**Theorem 6.2.** Suppose that \( 0 < p, q \leq 1 \) and \( p \neq q \), and that \( X \) is a \( p \)-convex quasi-Banach space and \( Y \) is a \( q \)-convex quasi-Banach space. Then any twisted sum \( Y \oplus_p X \) of \( Y \) and \( X \) is min\( (p, q) \)-convex.

**Proof.** The case \( q > p \) is proved in [6]. We therefore assume that \( q < p \) and that \( Y \) is a \( q \)-Banach space and \( X \) is a \( p \)-Banach space. Then
\[ d_n(Y) \leq n^{1/q-1}, \quad d_n(X) \leq n^{1/p-1}, \]
and hence for any quasinilinear map \( \varphi : X \to Y \)
\[ c_n(\varphi) \leq C n^{1/q-1} \]
for some \( C \).

Now suppose \( x_1, \ldots, x_n \in X \) is non-zero and
\[ \|x_1\|^q + \ldots + \|x_n\|^q = 1. \]
Let \( A_n = \{ i : 2^{-m} < \|x_i\| \leq 2^{-m} \} \), \( m = 1, 2, 3, \ldots \) Then for some \( N, A_1, \ldots, A_N \) partitions \( \{1, 2, \ldots, n\} \). Let
\[ a_m = \sum_{i \in A_m} x_i. \]
Then, if we make the convention \( A(0) = 0 \) and \( \sum x_i = 0 \),
\[ \|A(x_1, \ldots, x_n)\|^q \leq \|A(x_1, \ldots, x_N)\|^q + \sum_{i \in A_{N+1}} \|A(x_1, \ldots, x_i)\|^q. \]
Now
\[ \|A(x_1, \ldots, x_i)\| \leq C \|A_i\|^{1/q-1} \sum_{j \in A_i} \|x_j\| \leq 2C \|A_i\|^{1/q-2} \]
so that
\[ \|A(x_1, \ldots, x_i)\| \leq (2C)^q \|A_i\|^{2/q-q} \leq (2C)^q \sum_{j \in A_i} \|x_j\|^q. \]
Hence
\[ \sum_{i=1}^N \|A(x_1, \ldots, x_i)\|^q \leq (2C)^q \sum_{j \in A_i} \|x_j\|^q = (2C)^q \sum_{i=1}^N \|x_i\|^q. \]
Now by Lemma 5.3 there is a constant \( M \) and \( \alpha > 0 \) so that
\[ \|A(x_1, \ldots, x_i)\|^p \leq M \sum_{i=1}^N k^p \|x_i\|^p, \quad x_1, \ldots, x_i \in X. \]

**Hence**
\[ ||A(x_1, \ldots, x_N)||^p \leq M \sum_{i=1}^N k^p \|x_i\|^p \leq M \sum_{i=1}^N k^p \sum_{j \in A_i} \|x_j\|^p \]
\[ \leq M^* \sum_{i=1}^N \|x_i\|^p \left( \log \frac{2}{\|x_i\|} \right)^n \]
where \( M^* = M \log 2 \).

Now map \( \xi^* (\log(2/\xi))^n = 0 \) \( \leq \infty \) and hence if \( M^{**} = \theta M^* \)
\[ ||A(x_1, \ldots, x_N)||^p \leq M^{**} \sum_{i=1}^N \|x_i\|^p = M^{**}. \]

**Hence**
\[ ||A(x_1, \ldots, x_n)||^p \leq (M^{**})^p + (2C)^p \]
where both \( M^{**} \) and \( C \) are independent of \( x_1, \ldots, x_n \). We conclude that for any \( x_1, \ldots, x_n \)
\[ ||A(x_1, \ldots, x_n)||^p \leq D \sum_{i=1}^n \|x_i\|^p. \]

Now suppose \( (y_1, x_1) \in Y \oplus_p X \). Then
\[ \left( \sum \|y_i - F(x_1)\| \right)^p \leq \left( \sum \|y_i - F(x_1)\|\right)^p \]
\[ \leq \left( \sum \|y_i - F(x_1)\| \right)^p + \left( \sum \|x_i\| \right)^p \]
\[ \leq 2D \left( \sum \|y_i - F(x_1)\| \right)^p + \left( \sum \|x_i\| \right)^p \]
\[ \leq 2^{1/q-1} D \sum \|x_i\|^p. \]
and so \( Y \oplus_p X \) is \( p \)-convex.

**Theorem 6.4.** Suppose that \( X \) is a quasi-Banach space of type \( p \) (\( 1 \leq p \leq 2 \)) and \( Y \) is a quasi-Banach space of type \( q \) (\( 1 \leq q \leq 2 \)). Then if \( q < p \), any twisted sum \( Y \oplus_p X \) is of type \( q \).

**Proof.** This proof mimics Theorem 6.2. We assume that \( X \) is an \( r \)-Banach space where \( r \leq 1 \); of course, if \( q > 1 \), we may take \( r = 1 \). We suppose that if \( x_1, \ldots, x_n \in X \) and \( y_1, \ldots, y_n \in Y \), then
\[ \left\{ \frac{1}{n} \sum_{i=1}^n \|x_i(t)\|^p dt \right\}^{1/p} \leq a \left( \sum \|x_i\|^p \right)^{1/p}, \]
\[ \left\{ \frac{1}{n} \sum_{i=1}^n \|y_i(t)\|^q dt \right\}^{1/q} \leq a \left( \sum \|x_i\|^q \right)^{1/q}, \]
and that
\[ \|A(x_1, \ldots, x_n)\| \leq M \left( \sum_{k=1}^{n} k^\alpha \|x_k\|^p \right)^{1/p} \]
as in Lemma 5.3.
We have \( \delta_n(Y) = O(n^{1/2 - \beta}) \) and \( \delta_n(X) = O(n^{1/\alpha - \beta}) \) and hence
\[ \gamma_n(F) \leq Cn^{\frac{1}{2} - \frac{1}{p} - \frac{1}{q}} \quad \forall n \in \mathbb{N}. \]
Suppose \( x_1, \ldots, x_n \in X \) are non-zero and
\[ \|x_1\| + \cdots + \|x_n\|^p = 1. \]
Let \( A_m = \{ i : 2^{-m} \leq \|x_i\| \leq 2 \cdot 2^{-m} \} \) and suppose \( A_1, \ldots, A_M \) positions \( \{1, 2, \ldots, N\} \). As before we make the convention \( A(0) = 0 \) and \( \sum x_i = 0 \).
Then
\[ u_i(t) = \sum_{j \in A_i} \epsilon_j(t) x_j, \quad 0 \leq t \leq 1, \]
\[ A(x_1, \ldots, x_n) = A(u_1(t), \ldots, u_N(t)) + \sum_{j \in A} A(\epsilon_j(t)x_j : j \in A). \]
Now by symmetry
\[ \left\| \sum_{i=1}^{N} A(\epsilon_j(t)x_j : j \in A_i) \right\|^p dt = \frac{1}{M} \left\| \sum_{i=1}^{N} A(\epsilon_j(t)x_j : j \in A_i) \right\|^p dt \]
\[ \leq C \left\| \sum_{i=1}^{N} \|A(\epsilon_j(t)x_j : j \in A_i)\|^p dt \right\| \]
\[ \leq C \|A\|_p^p \left( \sum_{i=1}^{N} \|x_i\|^p \right)^{1/p} \]
\[ \leq 2^{\alpha} \|A\|_p^p \left( \sum_{i=1}^{N} \|x_i\|^p \right)^{1/p} \]
\[ \leq 2^{\alpha} \|A\|_p^p \sum_{i=1}^{N} \|x_i\|^{1/p} \leq 2^{\alpha} \|A\|_p^p. \]
Also
\[ \left\| \sum_{i=1}^{N} A(\epsilon_j(t)x_j : j \in A_i) \right\|^p dt \leq M \left( \sum_{k=1}^{n} k^\alpha \|u_k\|^p \right)^{1/p} \]
\[ \leq M^* \left( \sum_{k=1}^{n} k^\alpha \|x_k\|^p \right)^{1/p} \]
as in Theorem 6.2, where \( M^* \) is independent of \( x_1, \ldots, x_n \). Hence
\[ \left\{ \int_0^1 \left\| A(\epsilon_j(t)x_j : j \in A_i) \right\|^p dt \right\}^{1/p} \leq \left\{ \int_0^1 \left\| A(u_1(t), \ldots, u_N(t)) \right\|^p dt \right\}^{1/p} \]
\[ + \left\{ \int_0^1 \left\| \sum_{j \in A} A(\epsilon_j(t)x_j : j \in A_i) \right\|^p dt \right\}^{1/p} \]
\[ \leq (M^*)^p + 2^{\alpha} \|A\|_p^p. \]
It follows easily that for any \( x_1, \ldots, x_n \)
\[ \left\{ \int_0^1 \left\| A(\epsilon_j(t)x_j : j \in A_i) \right\|^p dt \right\}^{1/p} \leq D \left( \sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}. \]
Now suppose \( (y_1, x_j) \in Y \cap_\alpha X \) \( 1 = i = n \). Then
\[ \left\{ \int_0^1 \left\| \sum_{i=1}^{N} \epsilon_i(t)(y_i, x_i) \right\|^p dt \right\}^{1/p} \]
\[ \leq \left\{ \int_0^1 \left\| \sum_{i=1}^{N} \epsilon_i(t)y_i \right\|^p dt + \left\| \sum_{i=1}^{N} \epsilon_i(t)x_i \right\|^p dt \right\}^{1/p} \]
\[ \leq \left\{ \int_0^1 \left\| \sum_{i=1}^{N} \epsilon_i(t)y_i - F \left( \sum_{i=1}^{N} \epsilon_i(t)x_i \right) \right\|^p dt \right\}^{1/p} \]
\[ + \left\{ \int_0^1 \left\| \sum_{i=1}^{N} \epsilon_i(t)x_i - F \left( \sum_{i=1}^{N} \epsilon_i(t)x_i \right) \right\|^p dt \right\}^{1/p} \]
\[ \leq 2^{1/\alpha} \left\{ \int_0^1 \left\| \sum_{i=1}^{N} \epsilon_i(t)y_i \right\|^p dt \right\}^{1/p} \]
\[ + \left\{ \int_0^1 \left\| \sum_{i=1}^{N} \epsilon_i(t)x_i \right\|^p dt \right\}^{1/p} \]
So
\[ \left\{ \int_0^1 \left\| \sum_{i=1}^{N} \epsilon_i(t)(y_i, x_i) \right\|^p dt \right\}^{1/p} \]
\[ \leq 2^{1/\alpha - \alpha} \left\{ \sum_{i=1}^{N} \|y - F(x_i)\|^p + \left( 2^{1/\alpha - \alpha} D + 2^{1/\alpha - 1} \right) \sum_{i=1}^{N} \|x_i\|^p \right\}^{1/p} \]
\[ \leq 2^{1/\alpha - \alpha} \left( \sum_{i=1}^{N} \|y - F(x_i)\|^p + \left( 2^{1/\alpha - \alpha} D + 2^{1/\alpha - 1} \right) \sum_{i=1}^{N} \|x_i\|^p \right)^{1/p} \]
\[ \leq K \left( \sum_{i=1}^{N} \|y_i, x_i\|^p \right)^{1/p}, \]
i.e., \( X \oplus_\alpha Y \) is type \( q \).
We now show that a type \( 1 \) space need not be convex.
In [9], [15] and [11] it is shown that one can construct a non-convex, twisted sum of \( K \) and a Banach space. This is type \( 1 \) by the next theorem.

**Theorem 6.5.** \( X \) is a type \( p \) quasi-Banach space and \( Y \) is a type \( q \) Banach space where \( q > p \). Then any twisted sum \( X \oplus_\alpha Y \) is type \( p \).

**Proof.** Here our techniques are rather different. We use a result of Kahane [4] that there is a constant \( K = K(p, q) \) such that for any
elements \( u_1, \ldots, u_n \) of \( X \)

\[
\left\{ \frac{1}{\alpha} \left\| \sum e_i u_i \right\|^p dt \right\}^{1/p} \leq K \left\{ \frac{1}{\alpha} \left\| \sum e_i u_i \right\|^q dt \right\}^{1/q}.
\]

(For the case \( p = 1 \), we apply Theorem 2.1.)

Suppose \( a \) has the same meaning as in Theorem 6.4. Let \( \theta_n \) be the least constant such that

\[
\left\{ \frac{1}{\alpha} \left\| \sum e_i x_i + a e_i \right\|^q dt \right\}^{1/q} \leq \theta_n \left\{ \sum e_i x_i \right\}^{1/p}.
\]

Suppose \( \|x_1\|^p + \cdots + \|x_n\|^p = 1 \) and \( \|x_i\|^p \leq 1/N \). Then

\[
A(e_i x_i, \ldots, e_i x_i) = A(u(t), e_i x_i) + A(e_i x_i, \ldots, e_i x_i - 1)
\]

where \( u(t) = \sum e_i x_i \). Now

\[
\left\{ \frac{1}{\alpha} \left\| A(u(t), e_i x_i) \right\|^q dt \right\}^{1/q} \leq \theta_n^p \left\{ \frac{1}{\alpha} \left\| u(t) \right\|^p dt \right\}^{1/p} + \left\| x_i \right\|^p
\]

by the usual symmetrization argument. Hence

\[
\left\{ \frac{1}{\alpha} \left\| A(u(t), e_i x_i) \right\|^q dt \right\}^{1/q} \leq \theta_n^p \left\{ \frac{1}{\alpha} \left\| u(t) \right\|^p dt \right\}^{1/p} + \left\| x_i \right\|^p
\]

\[
\leq \theta_n^p \left[ K^p p^{p-1} \sum_{i=1}^{\infty} \|x_i\|^p \right] \leq K^p \theta_n^p.
\]

Hence if \( \|x_1\|^p + \cdots + \|x_n\|^p = 1 \) and \( \max \|x_i\|^p \geq N^{-1} \),

\[
\left\{ \frac{1}{\alpha} \left\| A(e_i x_i, \ldots, e_i x_i) \right\|^q dt \right\}^{1/q} \leq \theta_n (1 - 1/N)^{1/p} + K \theta_n.
\]

Now suppose \( \|x_1\|^p + \cdots + \|x_n\|^p = 1 \) and \( \max \|x_i\|^p < N^{-1} \). Then it is possible to subdivide \( \{1, 2, \ldots, n\} \) into \( N \) sets \( A_1, \ldots, A_N \) such that

\[
\sum_{x_i \in A_j} \|x_i\|^p \leq 2/N, \quad j = 1, 2, \ldots, N.
\]

Then let

\[
A(e_i x_i, \ldots, e_i x_i) = A(u(t), \ldots, u_N(t)) + \sum_{t=1}^{\infty} A(e_i x_i, i \in A_j)
\]

and by symmetrization

\[
\left\{ \frac{1}{\alpha} \left\| A(u(t), \ldots, u_N(t)) \right\|^q dt \right\}^{1/q} \leq \theta_n^q \left\{ \frac{1}{\alpha} \left\| u(t) \right\|^p dt \right\}^{1/p}
\]

\[
\leq \theta_n^q \left\{ \sum_{t=1}^{\infty} \left\| u(t) \right\|^p dt \right\}^{1/p}
\]

\[
\leq \theta_n^q \left( \sum_{t=1}^{\infty} \left\| u(t) \right\|^p dt \right)^{1/p}
\]

(by symmetrization)

\[
\leq \theta_n \left( \sum_{t=1}^{\infty} \left( \sum_{x_i \in A_j} \|x_i\|^p \right)^{1/q} \right)^{1/l}
\]

\[
\leq \theta_n \left( \sum_{t=1}^{\infty} \left( \sum_{x_i \in A_j} \|x_i\|^p \right)^{1/q} \right)^{1/l}
\]

\[
\leq \left( \frac{2}{N} \right)^{1/p-1/q} \theta_n.
\]

Hence

\[
\left\{ \frac{1}{\alpha} \left\| A(e_i x_i, \ldots, e_i x_i) \right\|^q dt \right\}^{1/q} \leq \theta_n \left( \frac{2}{N} \right)^{1/p-1/q} \theta_n.
\]

Thus

\[
\theta_n \leq \max \left( \theta_n \left( \frac{1}{2} \right)^{1/p} + K \theta_n \right)
\]

\[
+ \left( \frac{2}{N} \right)^{1/p-1/q} \theta_n.
\]

If we choose \( N \) so that

\[
\left( \frac{2}{N} \right)^{1/p-1/q} \theta_n < 1,
\]

this implies a bound on \( \theta_n \).

The fact that \( \theta_n \) is bounded implies that \( Y \oplus_p X \) is type \( p \) in the usual way, as in Theorem 6.4.

7. Twisted sums with equal convexity. Since the twisted sum of two Banach spaces may not be convex we may ask what class does belong to.

It turns out that we can give a complete answer to this. We require first the following lemma. We use the notation of Section 5.
Lemma 7.1. Suppose $\mu \geq 0$ and $X$ is a Banach space.

(7.1.1) If $d_n(X) = 1$ and $d_n(Y) = 1$, then $c_n = O(\log n)$.

(7.1.2) If $d_n(X) \leq n^\alpha$ and $d_n(Y) = O(n^\alpha)$ (or $d_n(X) = O(n^\alpha)$
and $d_n(Y) \leq n^\alpha$), then $\gamma_n = O(n^\alpha \log n)$.

Proof. We prove only (7.1.2) (as the same argument then proves (7.1.1)). If $d_n(X) \leq n^\alpha$ and $d_n(Y) \leq Cn^\alpha$, then by Theorem 5.2

$$\frac{\gamma_{mn}}{(mn)^\alpha} \leq \frac{\gamma_m}{mn^\alpha} + C \frac{\gamma_n}{n^\alpha},$$

and hence $\gamma_{mn} n^{-\alpha} \leq C \gamma_m n^{-\alpha}$ and the result follows easily.

Theorem 7.1. Suppose $X$ and $Y$ are Banach spaces. Then any twisted sum $Y \oplus_p X$ is logconvex.

Proof. Here we have $d_n(X) = d_n(Y) = 1$ for all $n$ and hence $c_n \leq C(\log n + 1)$. Induction on $n$ as in Lemma 3.3 of [6] shows that

$$\|A(x_1, \ldots, x_n)\| \leq M \sum_{k=1}^{n} k \|x_k\|, \quad x_1, \ldots, x_n \in X,$$

in this case, for some $M$ independent of $x_1, \ldots, x_n$.

Suppose $|x_1| + |x_2| + \ldots + |x_n| = 1$ and suppose $|x_k| \geq |x_{k+1}| \geq \ldots \geq |x_n| > 0$. Let $N_k$ be the greatest suffix such that $|x_{N_k}| > 2^{-k}$ ($k = 1, 2, \ldots$) and let

$$u_k = \sum_{i=N_{k-1}+1}^{N_k} x_i, \quad k = 1, 2, \ldots$$

(where $N_0 = 0$). Suppose $N_1 = n$. Then

$$A(x_1, \ldots, x_n) = A(u_1, \ldots, u_n) + \sum_{k=1}^{n} A(x_{N_{k-1}+1}, \ldots, x_{N_k}),$$

$$\left\| \sum_{k=1}^{n} A(x_{N_{k-1}+1}, \ldots, x_{N_k}) \right\| \leq C \left( \sum_{k=1}^{n} \left( 1 + \log(N_k - N_{k-1}) \right) \right) \sum_{k=1}^{n} \|x_k\| \leq C \sum_{k=1}^{n} \|x_k\| \sum_{k=1}^{N_k} \|x_k\| \leq C \sum_{k=1}^{n} \sum_{j=1}^{N_k} \sum_{i=N_{k-1}+1}^{N_k} \|x_i\| \cdot \|x_j\| \leq C \sum_{k=1}^{n} \sum_{i=N_{k-1}+1}^{N_k} \|x_i\| \cdot \|x_j\| \leq C \sum_{k=1}^{n} \|x_k\| \sum_{i=N_{k-1}+1}^{N_k} \|x_i\| \cdot \|x_j\|,$$

and the result follows from the fact that the function

$$\Phi(\xi_1, \ldots, \xi_n) = \sum_{\ell=1}^{n} \xi_\ell + \sum_{\ell=1}^{n} \xi_\ell \log \frac{\sum_{\ell=1}^{n} \xi_\ell \cdot \xi_\ell}{\xi_\ell}, \quad \xi_1, \ldots, \xi_n \geq 0$$

is monotone in each $\xi_\ell$, and $\|x_k\| \leq \|y_k, x_k\| < \|y_k, x_k\| < \|y_k, x_k\| < \|y_k, x_k\|.$

Remark. If we take $X = Y : l_1$ and $F : l_1 \to l_1$ is given on the finitely non-zero sequences by

$$F(x) = \left( x_n \log \frac{\|x\|}{\|x_n\|} \right),$$

then $l_1 \oplus_{p_1} X$ contains $l_1 + \log(1/\|x\|)$ near zero). (See [8].) This shows that the result of Theorem 7.1 is best possible.

Theorem 7.2. A quasi-Banach space is logconvex if and only if it is the quotient of a subspace of a twisted sum of two Banach spaces.
Proof. By Theorem 7.1 such a space must be logconvex. The above example generalized to \( L_p(I) \) for arbitrary index sets \( I \) enables one to obtain \( L_p(I) \) as a subspace of a twisted sum of Banach spaces and hence every logconvex space as a quotient (cf. [19]) or the method of Theorem 3.3.

**Definition 7.3.** We say a quasi-Banach space \( X \) is of logtype \( p \) (\( 1 < p \leq 2 \)) if for some constant \( C < \infty \), we have

\[
\left( \int_0^1 \left( \sum_{i=1}^n \varepsilon_i(t) x_i \right)^p dt \right)^{1/p} \leq C \left( \sum_{i=1}^n \|x_i\|^p \left( \log \frac{1}{\|x_i\|} \right)^p \right)^{1/p}
\]

whenever \( \|x_1\|^p + \ldots + \|x_n\|^p = 1 \).

**Remark.** In order that \( X \) is of logtype \( p \) it is sufficient that

\[
\left( \int_0^1 \left( \sum_{i=1}^n \varepsilon_i(t) x_i \right)^p dt \right)^{1/p} \leq C \left( \sum_{i=1}^n \|x_i\|^p(1 + \log k)^p \right)^{1/p}.
\]

To see this arrange \( x_k \) so that \( \|x_k\| \) decreases. Then if \( \sum \|x_i\|^p = 1 \), \( \|x_k\|^p \leq k^{-1} \) and hence

\[
\log \frac{1}{\|x_k\|} \geq \frac{1}{p} \log k.
\]

We will see later that (7.3.1) and (7.3.2) are equivalent; of course, for \( p = 1 \) this is immediate from Section 3, and for \( 1 < p < 2 \) could be established directly in a similar manner. However, our indirect methods also establish this result without difficulty.

**Definition 7.4.** A Banach space \( X \) is of exact type \( p \) if

\[
\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \leq 2 \left( \|x_1\|^p + \|x_2\|^p \right), \quad x_1, x_2 \in X.
\]

**Remarks.** If \( p = 1 \), this is automatic. If \( p = 2 \), it implies that \( X \) is a Hilbert space, for in this case

\[
\|2x_1\|^p + \|2x_2\|^p \leq 2 \left( \|x_1\|^p + \|x_2\|^p + 2 \|x_1 + x_2\|^p \right)
\]

and hence the parallelogram law holds; then apply a result of Jordan and von Neumann [3]. For \( 1 < p < 2 \) it is sufficient that \( X \) is a quotient of a subspace of an \( L_p \)-space.

**Theorem 7.5.** Suppose \( X \) and \( Y \) are Banach spaces of type \( p \) where \( 1 \leq p \leq 2 \). Suppose that either \( X \) or \( Y \) is of exact type \( p \). Then any twisted sum \( Z = Y \oplus_p X \) satisfies

\[
\left( \int_0^1 \left( \sum_{i=1}^n \varepsilon_i(t) x_i \right)^p dt \right)^{1/p} \leq C \left( \sum_{i=1}^n \|x_i\|^p + \sum \|x_i\|^p(\log k)^p \right)^{1/p}
\]

and hence is of logtype \( p \).

**Proof.** If \( X \) is of exact type \( p \), then

\[
\left( \int_0^1 \sum_{i=1}^n \varepsilon_i(t) x_i \right)^p dt \leq \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p}
\]

for \( x_1, \ldots, x_n \) so that \( \delta_n(x) \leq \kappa^{-p} \). Hence Lemma 7.1 implies that

\[
\gamma_n \leq B_n\kappa^{-1/2} (\log n + 1).
\]

By Lemma 5.3 there exists \( a > 0 \) and \( M < \infty \) such that

\[
\|A(x_1, \ldots, x_n)\| \leq M \left( \sum_{i=1}^n k_i \|x_i\|^p \right)^{1/p}
\]

for \( x_1, \ldots, x_n \in X \).

Now suppose \( \|x_1\|^p, \ldots, \|x_n\|^p = 1 \) and \( \|x_1\| \geq \ldots \geq \|x_n\| > 0 \). Let \( N \) be the greatest suffix such that \( \|x_N\| > 2^{-a} \) and let

\[
u_k(t) = \sum_{N_{k-1}+1}^{N_k} \varepsilon_i(t) x_i, \quad k = 1, 2, \ldots
\]

Suppose \( N_k = n \). Then

\[
A(x_1, \ldots, x_n) = A(u_1(t), \ldots, u_N(t)) + \sum_{k=1}^{N_k} A(e_{N_k-N_{k-1}+1} x_{N_k-N_{k-1}+1}, \ldots, e_{N_k} x_{N_k}).
\]

Now let

\[
a \leq \left( \int_0^1 \left( \sum_{k=1}^N \sum_{i=N_{k-1}+1}^{N_k} A(e_{N_k-N_{k-1}+1} x_{N_k-N_{k-1}+1}, \ldots, e_{N_k} x_{N_k}) \right)^p dt \right)^{1/p}
\]

\[
\leq B_n \left( \sum_{k=1}^N \left( \sum_{i=N_{k-1}+1}^{N_k} A(e_{N_k-N_{k-1}+1} x_{N_k-N_{k-1}+1}, \ldots, e_{N_k} x_{N_k}) \right)^p dt \right)^{1/p}
\]

by the symmetrization argument, where \( B_n \) is the type \( p \) constant of \( Y \). Hence

\[
a \leq BB_n \left( \sum_{N_{k-1}+1}^{N_k} (N_k-N_{k-1}-1)^{-2/p} \left( \log(N_k-N_{k-1}-1) \right)^2 \right) \sum_{i=N_{k-1}+1}^{N_k} \|x_i\|^p \right)^{1/p}
\]

\[
\leq BB_n \left( \sum_{N_{k-1}+1}^{N_k} \log(N_k-N_{k-1}-1) + 1 \right)^p \sum_{i=N_{k-1}+1}^{N_k} \|x_i\|^p \right)^{1/p}.
\]

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Now observe
\[
\sum_{m+1}^n (1 + \log k)^p \geq \int_m^n (1 + \log x)^p \, dx = (n-m)(1 + \log n)^p - p \int_m^n \frac{\log x}{x} (1 + \log x)^{p-1} \, dx \\
\geq (n-m)(1 + \log n)^p - p (n-m)(1 + \log n)^{p-1} \\
\geq \frac{1}{2} (n-m)(1 + \log n)^p
\]
provided \( n \geq n_0 \) where \( n_0 \) depends on \( p \).
Hence there is a constant \( c > 0 \) such that \( c > 0 \) and
\[
\sum_{k=m+1}^n (1 + \log k)^p \geq c (n-m)(1 + \log n)^p
\]
for all \( n, m \). Thus we have
\[
\sum_{N_{k-1}+1}^{N_k} (1 + \log j)^p \|x_j\|^p \geq 2^{-m} \sum_{N_{k-1}+1}^{N_k} (1 + \log j)^p \\
\geq 2^{-m} (N_n - N_{k-1})(1 + \log N_k)^p \\
\geq 2^{-m} (1 + \log N_k)^p \sum_{k=m+1}^n \|x_k\|^p.
\]
Thus
\[
a \leq 2e^{-m}BB_2 \left( \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right)^{1/p}.
\]
Now we shall show that if
\[
b = \left( \int_0^1 \left\| \frac{d}{dt} (u_1(t), \ldots, u_n(t)) \right\|^p \, dt \right)^{1/p},
\]
then
\[
b \leq D \left( \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right)^{1/p}
\]
for some \( D \) independent of \( x_1, \ldots, x_n \). We have
\[
(7.5.1) \quad b \leq M \left( \int_0^1 \left( \sum_{k=1}^n \|u_k(t)\|^p \right)^{1/p} dt \right)^{1/p} \leq MB_3 \left( \sum_{k=1}^n \sum_{k=1}^{N_k-1} \|x_k\|^p \right)^{1/p}
\]
where \( B_3 \) is the type \( p \) constant of \( X \). Hence
\[
(7.5.2) \quad b \leq M_1 \left( \sum_{k=1}^n \|x_k\|^p \left( \log \frac{2}{\|x_k\|} \right) \right)^{1/p}
\]
(for some constant \( M_1 \))
\[
\quad \leq M_1 \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p}
\]
where \( \frac{1}{2} p < q < p \). Let \( \theta = q/(p-q) > 1 \). Then
\[
b \leq M_1 \left( \sum_{k=1}^n \|x_k\|^p \|\|x_k\|^\theta \right)^{1/p} \\
\leq M_1 \left( \sum_{k=1}^n \|x_k\|^p \|x_k\|^{\theta q} \right)^{1/p} \leq M_1 \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p}.
\]
Combining
\[
a + b \leq M_4 \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p}
\]
so that
\[
\left\{ \int \|D(e_{1,n}, \ldots, e_{1,n})\|^p \, dt \right\}^{1/p} \leq M_4 \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p}
\]
and this must hold for any \( x_1, \ldots, x_n \in X \). Returning to (7.5.1) it is clear that (after a symmetrization argument) we may now take \( a = 1 \), and in (7.5.2)
\[
b \leq M_4 \left( \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right)^{1/p}
\]
By Lemma 3.5
\[
b \leq M_6 \left( \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right)^{1/p}
\]
and combining we now have the estimate
\[
\left\{ \int_0^1 \left\| \frac{d}{dt} (e_{1,1}, \ldots, e_{1,1}) \right\|^p \, dt \right\}^{1/p} \leq C \left( \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right)^{1/p}.
\]
We omit the verification this implies the desired property of \( Z = Y \otimes_p X \).

Remark. We observe that Theorem 7.5 is best possible, in the sense that each \( p, 1 \leq p \leq 2 \), there is a twisted sum \( Z_p \) of \( L_p \) and itself and a constant \( c > 0 \) such that if \( \varepsilon_1 + \ldots + \varepsilon_n = 1 \) there are \( x_i \in Z_p \), \( i = 1, 2, \ldots, n \) with \( \|x_i\| = \varepsilon_i \) and
\[
\left\{ \int_0^1 \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \, dt \right\}^{1/p} \geq c \left( 1 + \sum_{i=1}^n \|x_i\|^p \left( \log \frac{1}{\|x_i\|} \right) \right)^{1/p}.
\]
Indeed, consider the spaces $Z_p$ of [9]. Then $Z_p = l_p \oplus l_p$ where $\mathbb{R}^p \rightarrow \ell_p$ is defined by

$$F(x) = \left( \sum_{n=1}^{\infty} x_n \log \frac{|x_n|}{|x_n|} \right).$$

If $e_n$ is the $n$th basis vector of $l_p$, then

$$\left\| \left( 0, \sum \pm \xi_n e_n \right) \right\| = \left\| \left( \sum \pm \xi_n e_n \right) \right\| + \left\| \sum \pm \xi_n e_n \right\| = \left( \sum \xi_n^p \left( \log \frac{1}{\xi_n^p} \right)^{1/p} \right)^{1/p} + \left( \sum \xi_n^p \left( \log \frac{1}{\xi_n^p} \right)^{1/p} \right)^{1/p} \geq \left( 1 + \sum \xi_n^p \left( \log \frac{1}{\xi_n^p} \right) \right)^{1/p}.$$  

Similarly this implies

$$\left( 1 + \sum \xi_n^p \left( \log \frac{1}{\xi_n^p} \right) \right)^{1/p} \leq C \left( \sum \xi_n^p (1 + \log n)^p \right)^{1/p}$$

so that (7.3.1) and (7.3.3) are equivalent.

3. Twisted sums of $l_1$ and $R$. We now recall two ways of forming a twisted sum $R \oplus_{\ell_1} l_1$. One method due to the author is by defining $F_1: \ell_1 \rightarrow R$ by

$$F_1(x) = \sum_{n=1}^{\infty} \tilde{x}_n \log n, \quad x \geq 0,$$

where $(\tilde{x}_n)$ is the decreasing rearrangement of $x$, and

$$F_1(x) = F_1(x^+) - F_1(x^-)$$

where $x = x^+ - x^-$ where $x^+ \geq 0$, $x^- \geq 0$ and $|x^+| + |x^-| = 0$; see [6]. The other functional due to Ribe [15] is $F_2: \ell_1 \rightarrow R$ given by

$$F_2(x) = \sum_{n=1}^{\infty} x_n \log \frac{|x_n|}{|x_n|}.$$

(Actually Ribe uses the equivalent functional

$$F_2(x) = \sum_{n=1}^{\infty} x_n \log \frac{1}{\xi_n} + \left( \sum_{n=1}^{\infty} x_n \right) \log \left( \sum_{n=1}^{\infty} x_n \right).$$

$\| \cdot \|$ and $\| \cdot \|$ denote the induced norms on $R \oplus l_1$. Then if $e_n$ is the $n$th basis vector of $l_1$ and $t_n \geq 0$ ($n = 1, 2, \ldots, N$),

$$\left\| \left( 0, \sum t_n e_n \right) \right\| = \sum_{n=1}^{N} t_n \log n$$

and

$$\left\| \left( 0, \sum_{n=1}^{N} t_n e_n \right) \right\| = \sum_{n=1}^{N} t_n \log \frac{N}{t_n}.$$  

Thus we see for both $F_1$ and $F_2$ we have examples to show

**Theorem 8.1.** There is a twisted sum of $R$ and $l_1$ where $\ell_1$ is $l_1$.

In view of this we remark that these two twisted sums are not projectively equivalent ([9]) in the sense that there is no isomorphism $S: R \oplus_{\ell_1} l_1 \rightarrow R \oplus l_1$ such that the diagram

$$R \oplus_{\ell_1} l_1 \rightarrow R \oplus_{\ell_1} l_1 \rightarrow$$

commutes where $\alpha \neq 0$. For (see [9]), projective equivalence implies the existence of $\alpha \neq 0$, $N < \infty$ and a linear map $t: l_1 \rightarrow R$ so that

$$|F_1(ax) - F_1(x) - t(x)| \leq N \|x\|, \quad x \in l_1.$$

Since $F_1(e_n) = F_2(e_n) = 0$ for all $n$, this would imply $t(e_n)$ bounded so that $t$ is continuous and

$$|F_1(ax) - F_1(x)| \leq (N + \|t\|) \|x\|, \quad x \in l_1.$$

Now by Lemma 3.5 we see the only possible value of $\alpha$ is $\alpha = 1$.

Thus

$$|F_1(x) - F_2(x)| \leq (N + \|t\|) \|x\|.$$  

Now let $x_N = \sum_{n=1}^{N} \frac{1}{n} e_n$.

$$F_1(x_N) = \sum_{n=1}^{N} \frac{\log n}{n},$$

$$F_2(x_N) = \sum_{n=1}^{N} \frac{1}{n} \log n + \log S_N$$

where $S_N = \sum_{n=1}^{N} \frac{1}{n}$. Hence

$$F_2(x_N) - F_1(x_N) = S_N \log S_N$$

while $\|x_N\| = S_N \rightarrow \infty$ and so we have a contradiction.
To conclude this short section we consider the following:

**Definition 8.2.** An operator \( T : l_1 \to l_1 \) is liftable if for every twisted sum \( R \oplus \mathbb{P} l_1 \) there is a map \( \tilde{T} : l_1 \to R \oplus \mathbb{P} l_1 \) such that the diagram

\[
\begin{array}{ccc}
R \oplus \mathbb{P} l_1 & \xrightarrow{\tilde{T}} & R \\
\downarrow & & \downarrow \\
l_1 & \xrightarrow{T} & l_1
\end{array}
\]

commutes.

**Theorem 8.3.** Let \( T : l_1 \to l_1 \) be given by \( \text{Te}_n = d_n e_n \) where \( \|T\| = \sup |d_n| < \infty \). Then the following are equivalent:

(a) \( T \) is liftable.

(b) For some \( \tau < \infty \), \( \sum_{n=1}^{\infty} \exp(-\tau|d_n|) < \infty \).

(c) \( \{d_n\} \in c_0 \) and if \( d_n^* \) is the decreasing rearrangement of \( \{d_n\} \), then \( \{d_n d_n^* \log n\} \) is bounded.

(d) \( T(l_1) \subseteq l_1 \).

(e) For any log-convex space \( X \) and quotient map \( q : X \to l_1 \) there is an operator \( S : l_1 \to X \) such that \( qS = T \).

**Proof.** It clearly suffices to consider the case \( d_n \geq 0 \). Note first that if \( d_n \to 0 \), then there is a projection \( P \) onto a subspace isomorphic to \( l_1 \) such that \( P = ST \) for some bounded \( S \). Then \( T \) is liftable so is \( P \) and this clearly contradicts the fact that \( \{R(l_1) \} \) does not split. Here we may assume \( d_n \to 0 \) and then we may suppose \( \{d_n\} \) decreasing.

(a) \( \Rightarrow \) (c). Consider

\[
\begin{array}{ccc}
R \oplus \mathbb{P} l_1 & \xrightarrow{\tilde{T}} & R \\
\downarrow & & \downarrow \\
l_1 & \xrightarrow{T} & l_1
\end{array}
\]

Suppose \( \tilde{T}e_n = (a_n, d_n e_n) \). Then

\[ \|\tilde{T}e_n\| = |a_n| + |d_n| \leq \|\tilde{T}\| \]

and

\[ \|\tilde{T}(e_1 + \ldots + e_n)\| = \sum_{k=1}^{n} d_k + \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} d_k \log k \]

\[ \geq \sum_{k=1}^{n} d_k + \sum_{k=1}^{n} d_k \log k - \sum_{k=1}^{n} c_k \]

\[ \geq nd_n \log n - n\|T\|. \]

Hence

\[ d_n \log n \leq 2\|\tilde{T}\|. \]

(c) \( \Rightarrow \) (b). If \( d_n \leq b \log n \), \( n \geq 2 \), then if \( \tau > b \)

\[ \exp\left(-\frac{\tau}{d_n}\right) \leq n^{-\tau/2b}. \]

(b) \( \Rightarrow \) (d). Suppose \( |l_1| + \ldots + |l_n| \leq 1 \). Then

\[ \|T(l_1 e_1 + \ldots + l_n e_n)\| = \sum_{i=1}^{n} d_i |l_i| + \sum_{i=1}^{n} d_i |l_i| \log \frac{\sum_{i=1}^{n} d_i |l_i|}{d_i |l_i|} \]

\[ = S + S \log S - \sum_{i=1}^{n} d_i |l_i| \log d_i \]

\[ - \sum_{i=1}^{n} d_i |l_i| \log |l_i| \]

where \( S = \sum_{i=1}^{n} d_i |l_i| \leq \|T\| \). Also \( -d_i \log d_i \leq -e \). Hence

\[ \|T(l_1 e_1 + \ldots + l_n e_n)\| \leq \|T\| + \|T\| \log \|T\| + e^{-1} + \sum_{i=1}^{n} d_i |l_i| \log \frac{1}{|l_i|}. \]

Now suppose \( \xi_1, \ldots, \xi_n \geq 0 \) are chosen to maximize \( \psi(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} d_i \xi_i \log \frac{1}{\xi_i} \) subject to \( \xi_1 + \ldots + \xi_n = 1 \).

Then there is a Lagrange multiplier \( \lambda \) such that if \( \xi_i \neq 0 \),

\[ d_i \log \frac{1}{\xi_i} - d_i = \lambda, \]

i.e.,

\[ \log \frac{1}{\xi_i} = 1 + \frac{\lambda}{d_i}, \]

so that

\[ \xi_i = e^{1+\lambda/d_i}. \]

Let \( A = \{i : \xi_i > 0\} \). Then

\[ \sum_{i \in A} e^{1+\lambda/d_i} = 1, \]

\[ \psi(\xi_1, \ldots, \xi_n) = \sum_{i \in A} d_i e^{1+\lambda/d_i} (1 + \lambda/d_i) \leq \|T\| + \lambda. \]

Now

\[ \sum_{i \in A} e^{-\lambda/d_i} = \epsilon. \]
Hence
\[ \sum_{i=1}^{\infty} e^{-\lambda_i t_i} \leq \varepsilon. \]

Since for some \( r < \infty \)
\[ \sum_{i=1}^{\infty} e^{-\lambda_i t_i} < \infty, \]
there exists \( \lambda_0 \) such that \( \lambda > \lambda_0 \) implies
\[ \sum_{i=1}^{\infty} e^{-\lambda_i t_i} < \varepsilon. \]

Thus \( \gamma \leq \lambda_0 \) and so
\[ \|T(t_1 e_1 + \ldots + t_n e_n)\|_{l_p} \leq \|T\| (2 + \log \|T\|) + \sigma^{-1} + \lambda_0. \]

Hence \( T \) maps \( l_1 \) into \( l_p \).

(d) \( \Rightarrow \) (e).

\[ \begin{array}{c}
\mathbb{X} \\
\downarrow \pi_I \\
l_1 \xrightarrow{T} l_p \xrightarrow{J} l_1
\end{array} \]

\( T \) factors \( T = JT \), where \( J: l_p \to l_1 \) is the inclusion map. The existence of a lift \( S \) follows from the fact that \( X \) is logconvex.

(c) \( \Rightarrow \) (a). Theorem 7.1.

9. Orlicz sequence spaces. We recall that an \( F \)-space \( X \) is a \( X \)-space if \((R, X)\) splits ([8], [9]). In this section we classify completely those locally bounded Orlicz sequence spaces \( l_p \subset l_1 \) which are \( X \)-spaces. It is known ([6]) that \( l_2 \) is a \( X \)-space if \( 0 < p < 1 \) and fails to be a \( X \)-space if \( p = 1 \).

We shall suppose throughout that \( f \) is a twice-differentiable strictly increasing Orlicz function with \( f(1) = 1 \) such that \( af(x) \) is convex (cf. [5]); these assumptions may be made without loss of generality. We also suppose that \( f \) satisfies the \( A_2 \)-condition, i.e. for some \( K \)
\[ f(2x) \leq Kf(x), \quad 0 < x < \infty. \]

We define
\[ \alpha_f = \sup \{ x : \exists M, f(ax) \leq Ma^2f(x), \ 0 < a, x < 1 \}; \]
\[ \beta_f = \inf \{ p : \exists M, f(ax) \geq Ma^2f(x), \ 0 < a, x < 1 \}. \]

Since \( l_p \) is locally bounded, \( \alpha_f > 0 \), and the \( A_2 \)-condition implies \( \beta_f < \infty \).

Since \( \xi < l_1 \), we shall suppose
\[ f(x) \leq Mx, \quad 0 < x < 1, \]
for some \( M \).

Now let \( h: \mathbb{R} \to \mathbb{R} \) be defined by
\[ h(x) = x \int \frac{f(t)}{t^2} dt, \quad x > 0, \]
\[ h(0) = 0, \]
\[ h(x) = h(-x), \quad x < 0. \]

Lemma 9.1. \( h \) has the following properties:
(i) \( h \) is continuous, and twice differentiable for \( x \neq 0 \).
(ii) \( h''(u) \leq 0, \quad u \neq 0 \).
(iii) If \( u + e + w = 0 \),
\[ h(u) + h(v) + h(w) \leq 2f(|u|) + f(|v|) + f(|w|). \]
(iv) If \( 0 < u < 1 \) and \( x \in \mathbb{R} \),
\[ |h(ax) - ah(x)| \leq f(ax). \]

Proof. (i) For continuity at 0, observe if \( 0 < x < 1 \)
\[ h(x) \leq x \int \frac{M}{t} \frac{dt}{x} \leq Mx \log \frac{1}{x}. \]

The other assertion is clear.

(ii) \( h'(u) = \int \frac{f(t)}{t^2} dt, \quad x > 0, \)
\[ h''(u) = -\frac{f'(u)}{u} \leq 0. \]

(iii) Suppose without loss of generality \( u > 0, \ v > 0 \) and \( w = -(u + v) \).

Since \( h'' \leq 0 \),
\[ h(u + v) \leq h(u) + h(v) \leq 2h\left(\frac{1}{2}(u + v)\right), \]
so that
\[ g \leq h(u) + h(v) + h(w) \leq 2h\left(\frac{1}{2}(u + v)\right) - h(u + v), \]
\[ 2h\left(\frac{1}{2}(u + v)\right) - h(u + v) = (u + v) \int_{v}^{u} \frac{f(x)}{x^2} dx \leq 2f(u + v). \]

Hence
\[ h(u) + h(v) + h(w) \leq 2f(|w|) \leq 2f(|u|) + f(|v|) + f(|w|). \]
(iv) For \( x > 0 \),

\[
h(ax) - ah(x) = ax \int_0^x \frac{f(t)}{t^\alpha} \, dt \leq axf(x) \int_0^x \frac{1}{t^\alpha} \, dt = f(x)(1 - \alpha) \leq f(x).
\]

**Lemma 9.2.** Suppose for some \( B < \infty \) we have for \( 0 \leq x \leq 1 \)

\[
h(x) \leq Bf(x).
\]

Then \( \beta_f < 1 \).

**Proof.** Let \( C_f = C[0, 1] \) be defined by

\[
C_f = \overline{co}\{f_t : 0 < t \leq 1\}
\]

where

\[
f_t(x) = \frac{f(tx)}{f(t)}
\]

(cf. [5], [10]). Since \( l_r = l_1, \ a_r = 1 \). If \( \beta_f > 1 \), then \( x \in C_f \) ([10]).

Now

\[
\int_0^1 \frac{f(t)}{t^\alpha} \, dt \leq f(x)
\]

and if \( 0 < a \leq 1 \)

\[
\int_0^a \frac{f(t)}{t^\alpha} \, dt = \int_0^1 \frac{f(tx)}{t^\alpha} \, dt = \int_0^a \frac{af(u)}{u^\alpha f(u)} \, du \leq \frac{1}{f(1)} \int_0^a \frac{f(u)}{u^\alpha} \, du
\]

\[
\leq B \cdot \frac{a}{f(ax)} \frac{f(ax)}{ax} = Bx f(x), \quad 0 < x \leq 1.
\]

Hence, if \( g \in C_f \)

\[
\int_0^1 \frac{g(t)}{t^\alpha} \, dt \leq B \cdot \frac{g(x)}{x^\alpha}, \quad 0 < x \leq 1.
\]

In particular if \( \beta_f \geq 1 \), we may let \( g(t) = t \)

\[
\int_0^1 \frac{1}{t} \, dt \leq B, \quad 0 < x \leq 1
\]

and this contradiction shows \( \beta_f < 1 \).

**Theorem 9.3.** Suppose \( f \) is a Orlicz function satisfying the \( \Delta_2 \)-condition and that \( l_f \) is locally bounded and contained in \( l_1 \). Then \( l_f \) is a \( \mathcal{X} \)-space if and only if \( \beta_f < 1 \).

**Proof.** If \( \beta_f < 1 \), \( l_f \) is a \( \mathcal{X} \)-space [6]. Conversely, suppose \( l_f \) is a \( \mathcal{X} \)-space. We define

\[
H : l_f \to \mathbb{R}
\]

(\( l_f \) is the finitely non-zero sequences in \( l_f \)) by

\[
H(u) = \sum_{n=1}^\infty h(u_n) \text{ if } \sum_{n=1}^\infty f(|u_n|) = 1
\]

and extend so that

\[
H(ax) = aH(x), \quad a \in \mathbb{R}.
\]

We first assert that \( H : l_f \to \mathbb{R} \) is quasilinear. To see this we show that

\[
u + v + w \leq 0
\]

and

\[
||v|| + ||w|| + ||w|| \leq 1;
\]

then

\[
|H(u) + H(v) + H(w)| \leq 9.
\]

Indeed,

\[
|H(u) - \sum_{n=1}^\infty h(u_n)| \leq 1
\]

and similarly for \( v, w \) while

\[
\sum_{n=1}^\infty h(u_n) + h(v_n) + h(w_n) \leq 6.
\]

Now \( l_f \) is a \( \mathcal{X} \)-space so that there a linear map \( \psi : l_f \to \mathbb{R} \) such that

\[
\sup_{\|x\| < 1} |H(x) - \psi(x)| < \infty.
\]

Thus, \( \{\psi(\epsilon_n) : n = 1, 2, \ldots\} \) is bounded since \( H(\epsilon_1) = 0 \) and so \( \psi \) is continuous. Hence,

\[
|H(x)| \leq L||x||, \quad x \in l_f,
\]

for some \( L < \infty \).

Suppose \( 0 < \xi \leq 1 \); choose \( n \) so that \( n \in \mathbb{N} \) and \( \frac{1}{n} < n\xi \leq 1 \). Choose \( \eta \) so that \( f(\eta) = 1 - n\xi \) and let

\[
x = \xi \epsilon_1 + \ldots + \epsilon_n + \eta \epsilon_{n+1}.
\]

Then \( \|x\| = 1 \) and

\[
H(x) \geq n\xi \int_0^1 \frac{f(t)}{t^\alpha} \, dt \geq \frac{1}{2} \frac{\xi}{f(\xi)} \int_0^1 \frac{f(t)}{t^\alpha} \, dt.
\]
Hence,
\[ \int_{\xi}^{1} \frac{f(t)}{\xi} \, dt \leq 2L \frac{f(\xi)}{\xi}, \quad 0 < \xi \leq 1, \]
and so by Lemma 9.2, \( \beta_{f} < 1 \).

Suppose now \( f \) is submultiplicative at 0. Then we say \( X \) is \( f \)-convex if
\[ \sum f(||x||) < \infty \]
implies \( \sum x_{i} \) converges, i.e., \( X \) is ga蔼ly by \( l_{f} \).

**Corollary 9.4.** Suppose \( f \) is submultiplicative at 0; then every twisted sum of \( R \) and a \( f \)-convex space is also \( f \)-convex if and only if
\[ \beta_{f} = \lim_{x \to a} \frac{\log f(x)}{\log \omega} < 1. \]

**Proof.** Observe that \( L_{f}(I) \) (for an index set \( I \)) is projective among \( f \)-convex, i.e.
\[ X \]
\[ \overline{\otimes}_{R} \]
\[ L_{f}(I) \rightarrow X/N \]
Hence if every twisted sum of \( R \) and \( l_{f} \) is \( f \)-convex, then \( l_{f} \) is an \( f \)-space. Conversely, suppose \( l_{f} \) is an \( f \)-space and \( X \) is any \( f \)-convex space and \( Y = R \otimes_{X} X \) is a twisted sum of \( R \) and \( X \). Then there is a quotient map \( T: L_{f}(I) \rightarrow X \) for some index set \( I \). Now \( L_{f}(I) \) is also an \( f \)-space (this is easy to show) and so there is a lift \( \tilde{T}: L_{f}(I) \rightarrow Y \). If \( \tilde{T} \) fails to be surjective \( Y \) splits, while if \( \tilde{T} \) is surjective, \( Y \) is \( f \)-convex.

**Remark.** In this case \( \beta_{f} < 1 \) implies \( f(x) \gg \omega^{\delta} \) for some \( p < 1 \), and \( \delta > 0 \) for all \( 0 < \omega \leq 1 \).

**10. Locally convex \( f \)-spaces.** Let \( X \) be a metrizable locally convex space. Let \( ||.||_{n} \) be a sequence of semi-norms on \( X \) which define the topology of \( X \) and such that
\[ ||x||_{n} \leq ||x||_{n+1}, \quad n \in \mathbb{N}. \]
Define a map \( F: X \rightarrow R \) be quasi-linear if for some \( n \in \mathbb{N}, X < \infty \)
\[ F(tx) = tF(x), \quad t \in R, \quad x \in X, \]
\[ |F(x+y) - F(x) - F(y)| \leq \varepsilon ||x||_{n} + ||y||_{n} \]
if \( F \) is quasi-linear we define \( R \otimes_{X} X \) to be the space \( R \otimes_{X} X \) equipped with the quasi-semi-norms
\[ ||(t, x)||_{m}^{n} = ||tF(x)|| + ||x||_{m}, \quad m, n \geq 0. \]

Then if \( q: R \otimes_{X} X \rightarrow X \) is given by
\[ q(t, x) = a, \]
\( q \) is a quotient map and \( q^{-1}(0) = \{(t, 0), t \in R\} \). Thus we have a short exact sequence \( 0 \rightarrow R \rightarrow R \otimes_{X} X \rightarrow X \rightarrow 0 \) and \( R \otimes_{X} X \) is a twisted sum of \( R \) and \( X \). It is easy to show that if \( X \) is complete, then so is \( R \otimes_{X} X \) (since it is such a twisted sum).

The twisted sum \( R \otimes_{X} X \) will split if and only if there is a linear map \( \psi: X \rightarrow R \) such that
\[ ||F(x) - \psi(x)|| < M ||x||_{n}, \quad x \in X, \]
for some \( M \in \mathbb{N} \) and \( M < \infty \).

**Theorem 10.1.** Let \( X \) be a Fréchet space (complete metrizable locally convex space) and suppose an \( F \)-space \( X \) is a twisted sum of \( R \) and \( X \). Then there exists a quasi-linear map \( F: X \rightarrow R \) such that \( X \) is isomorphic to \( R \otimes_{X} X \) (as a twisted sum).

**Proof.** It is convenient to write \( Y = R \otimes_{X} X \) algebraically so that the quotient map \( q: X \rightarrow Y \) is given by \( q(t, x) = a \).

Let \( \{V_{n}, n \in \mathbb{N}\} \) be a base of balanced neighborhoods of 0 such that \( V_{n+1} + V_{n+1} \subset V_{n} \) for all \( n \) and \( V_{n} \cap Re \) is bounded where \( e = (1, 0) \). Let \( L_{f}(I) \) be the Minkowski functional of \( V_{n} \). Then we have
\[ (10.1.1) \quad (t, x), n \leq (t, x), n+1, \quad t \in R, x \in X, \]
\[ (10.1.2) \quad 1(x, t, x+y), n \leq 1(x, t, x+y), n+1 \]
\[ (10.1.3) \quad 1(x, 0), n \leq a, \quad a > 0 \text{ and } a \uparrow \infty. \]
Also since \( q \) is open, there exist increasing sequences \( \{m(n)\} \) and \( \{\beta_{n}\} \) such that
\[ (10.1.4) \quad \text{For } x \in X \text{ there exists } t_{x} \in R \]
\[ ||(t_{x}, x)||_{m} \leq \beta_{n} ||x||_{m(n)}, \]
In view of (10.1.4) there is a map \( F_{n}: X \rightarrow R \) such that \( F(tx) = tF(x), t \in R \) and
\[ ||F_{n}(x), x||_{n} \leq \beta_{n} ||x||_{m(n)}. \]
Now if \( n > p > 1 \),
\[ |F_{n}(x) - F_{p}(x)| = a_{n-p-1} \left(|F_{n}(x) - F_{p}(x), 0|^{p-1}ight) \]
\[ \leq a_{n-p-1} \left(|F_{n}(x), x| + |F_{p}(x), x|^{p}ight) \leq a_{n-p-1} \left(\beta_{n} + \beta_{p} ||x||_{m(n)}\right). \]
Also if \( n \geq 3 \) and \( x, y \in X \)

\[
(10.1.6) \quad \| (F_n(x) + F_n(y)) + (x + y) \|_{n-1} \leq \beta_n \left( \|x\|_{m(n)} + \|y\|_{m(n)} \right).
\]

\[
(10.1.7) \quad \| (F_{n-1}(x+y), x+y) \|_{n-1} \leq \beta_{n-1} \left( \|x\|_{m(n)} + \|y\|_{m(n)} \right).
\]

Hence combining (10.1.6) and (10.1.7) with (10.1.2)

\[
\alpha_{n-2} F_n(x) + F_n(y) - F_n(x+y) \leq (\beta_n + \beta_{n-1}) \left( \|x\|_{m(n)} + \|y\|_{m(n)} \right).
\]

Thus

\[
\| F_n(x) + F_n(y) - F_n(x+y) \| \leq C_n \left( \|x\|_{m(n)} + \|y\|_{m(n)} \right).
\]

In particular \( F = F_3 \) in quasilinear, and for \( n \geq 3 \)

\[
\| F_n(x) - F(x) \| \leq D_n \|x\|_{m(n)}.
\]

Thus

\[
1(u, 1)_n \leq \alpha_{n+1} \|u - F_n(x)\| + \beta_{n+1} \|x\|_{m(n+1)}
\]

\[
\leq \alpha_{n+1} \|u - F(x)\| + \left( \alpha_n \|D_n - F_n\| + \beta_{n+1} \|x\|_{m(n+1)} \right)
\]

\[
\leq \Lambda_n \left( \|u - F(x)\| + \|x\|_{m(n+1)} \right).
\]

Hence the identity \( \iota: R \oplus pX \to Y \) is continuous. By the closed graph theorem \( \iota \) is an isomorphism.

**Theorem 10.5.** Any nuclear Fréchet space is a \( \mathcal{K} \)-space.

**Proof.** If \( X \) is a nuclear Fréchet space and \( F: X \to R \) is quasilinear, then there is a Hilbert semi-norm \( \| . \|_n \) on \( X \) such that

\[
\| F(x+y) - F(x) - F(y) \| \leq \Xi \|x\|_n + \|y\|_n.
\]

Since a Hilbert space is a \( \mathcal{K} \)-space ([6]), there is a linear map \( \psi: X \to R \) such that

\[
\| F(x) - \psi(x) \| \leq \Xi \|x\|_n.
\]

Let \( (a_{mn}) \) be a matrix with non-negative entries such that \( a_{n+1,n+1} \geq a_{m,n} \) for all \( n \) and for each \( n \) there exists \( m \) with \( a_{m,n} > 0 \). Then the Köthe sequence space \( l_1[a_{mn}] \) is the space of sequences \( (x_n) \) such that

\[
\|x\|_n = \sum_{m=1}^{n} |a_{mn}| |x_m| < \infty, \quad m = 1, 2, \ldots
\]

**Theorem 10.3.** \( l_1[a_{mn}] \) is a \( \mathcal{K} \)-space if and only if given \( m \in N \) there exists \( \tau < \infty \) and \( \tau > m \) with

\[
\sum_{n=1}^{\infty} \exp \left( -\tau \frac{a_{mn}}{a_{m,n}} \right) < \infty \quad (0/0 = \infty).
\]

**Proof.** If (10.3.1) fails, there exists \( m \) such that for all \( \tau > m \) and \( \tau < \infty \)

\[
\sum_{n=1}^{\infty} \exp \left( -\frac{a_{mn}}{a_{m,n}} \right) = \infty.
\]

Define \( F: l_1[a_{mn}] \to R \) by

\[
F(x) = \sum_{n=1}^{\infty} a_{mn} x_n \log \frac{\|x\|_n}{a_{m,n}|x_n|}
\]

(for finitely non-zero sequences). If \( R \oplus p[l_1[a_{mn}]] \) splits then there is a linear map \( \psi: l_1[a_{mn}] \to R \) such that

\[
\| F(x) - \psi(x) \| \leq \Xi \|x\|_n
\]

for some \( \tau > m \). This implies

\[
\| \psi(x) \| \leq K \|x\|_n
\]

and so

\[
\| \psi(x) \| \leq K \|x\|_n
\]

so that

\[
\| F(x) \| \leq 2K \|x\|_n
\]

Hence

\[
\sum_{n=1}^{\infty} a_{mn} |x_n| \log \frac{\|x\|_n}{a_{m,n}|x_n|} \leq 2K \sum_{n=1}^{\infty} a_{mn} |x_n|.
\]

This means the diagonal map \( \{x_n\} \to \{d_n x_n\} \) maps into \( l_\tau \) where

\[
d_n = \frac{a_{mn}}{a_{m,n}} \quad (= 0 \text{ if } a_{mn} = 0).
\]

Hence by Theorem 8.3 we have contradicted (10.3.2).

Conversely if (10.3.1) holds and \( F \) is quasilinear, then if

\[
\| F(x+y) - F(x) - F(y) \| \leq K(\|x\|_m + \|y\|_m)
\]

choose \( \tau > m \) to satisfy (10.3.1). Let

\[
d_n = \frac{a_{mn}}{a_{m,n}}
\]

Then \( D: \{x_n\} \to \{d_n x_n\} \) is liftable. If we define \( G: l_1 \to R \) by

\[
G(x) = F(\{a_{mn} x_n\}) \quad (1/0 = 0).
\]
$G$ is quasinlinear on $l_1$ and hence there is a linear map $\tilde{D}$

\[
\begin{array}{c}
\mathbb{R} \oplus l_1 \\
\tilde{D} \\
l_1 \\
\end{array}
\]

If $\tilde{D}x = (\psi(x), Dx)$, then

\[|\psi(x) - G(Dx)| \leq C\|x\|, \quad x \in l_1.
\]

Hence

\[|\psi(a_m a_n) - F(x)| \leq C\|x\|, \quad x \in l_1[a_m]
\]

so that $\mathbb{R} \oplus l_1[a_m]$ splits.

Remark. Condition (10.3.1) is thus a topological invariant of $l_1[a_m]$.

References