Quotients of $L_p(0, 1)$ for $0 \leq p < 1$

by

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Abstract. One of the main results of this paper is a lifting theorem for operators from $L_p$, $0 \leq p < 1$, into a quotient space $L_p(N)$. (The theorem is developed separately for $L_0$ and for $L_p$, $0 < p < 1$; the hypotheses on $N$ are different in the two cases.) A corollary is that if $N$ is a non-trivial finite-dimensional subspace of $L_p$, $0 < p < 1$, then $L_p(N)$ is not isomorphic to $L_0$. Several similar results are obtained; at the end of the paper, the idea of a $K$-space ($K_0$-space) is introduced and studied in connection with the lifting theorems.

1. Introduction. Let $L_p = L_p[0, 1]$ be the space of all real (or complex) measurable functions on $[0, 1]$ with the topology of convergence in measure. A. Pełczyński has asked whether the quotient of $L_0$ by a non-trivial finite-dimensional subspace is isomorphic to $L_0$. In this paper we prove a lifting theorem for operators on $L_0$; using this theorem, we can show that if $B$ is a non-trivial closed subspace of $L_0$ which is either locally bounded or which admits a continuous linear functional, then $L_0/B \nsubseteq L_0$. Parallel results are developed for the spaces $L_0$ ($0 < p < 1$), where again we have that the quotient of $L_0$ by a non-trivial finite-dimensional subspace cannot be isomorphic to $L_0$ (contrasting of course with the case $p = 1$).

In Section 2, we show from certain general considerations that for $0 \leq p < 1$, $L_p/V \cong L_p/W$ whenever $\dim V = \dim W < \infty$. This enables us to define $(L_0/n)$ to be the (unique) space obtained by forming the quotient of $L_0$ by a subspace of dimension $n$. In Section 3 we prove our main lifting theorems and in Section 4 we apply them to show that $(L_0/n) \cong (L_0/m)$ if and only if $m = n$. We conclude Section 4 by giving an example of two isomorphic locally bounded subspaces of $L_0$, $B_1$ and $B_1$, such that $L_0/B_1 \nsubseteq L_0/B_1$.

In Section 5 we develop the idea of a $K$-space; this is an $F$-space $X$ such that every short exact sequence of $F$-spaces $0 \rightarrow R \rightarrow Y \rightarrow X \rightarrow 0$ splits. Using this notion we show that $L_0(N) \cong L_0$ implies that $N$ has no non-zero continuous linear functionals. Similar ideas for $p$-Banach spaces are also developed.

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Throughout this paper an $\mathcal{F}$-space will mean a complete metric topological vector space. An $F$-norm $x \mapsto \|x\|$ on a space $X$ is a mapping from $X$ to $\mathbb{R}_+$ such that

(a) $\|x + y\| \leq \|x\| + \|y\|$ if $x, y \in X$,

(b) $\|\lambda x\| = |\lambda| \|x\|$ if $|\lambda| \leq 1$ and if $x \in X$,

(c) $\|x\| \to 0$, as $\lambda \to 0$ for each $x \in X$,

(d) $\|x\| = 0$ if and only if $x = 0$.

For $0 < p \leq 1$, a $p$-Banach space is an $F$-space with an $F$-norm $\|\cdot\|$ such that

(e) $\|x\| = \|\|x\||p\|$ for all $x$ and $p \geq 1$.

If $X$ and $Y$ are $p$-Banach spaces and if $S : X \to Y$ is a continuous linear operator, we define

$\|S\| = \sup \{\|Sx\| : \|x\| \leq 1\}$.

We denote by $\mathcal{L}(X)$ the space of all linear operators on $X$. If $X$ is a $p$-Banach space, then so is $\mathcal{L}(X)$; if $X$ is an $F$-space, then $\mathcal{L}(X)$ has, in general, no convenient $F$-norm topology. Unless otherwise stated, “linear map” and “linear operator” always refer to continuous maps.

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2. Transitive $F$-spaces. In this section we show that if $V$ and $W$ are two subspaces of $L_p (0 < p < 1)$ of the same finite dimension, then $L_p (V) \simeq L_p (W)$. We approach this result through some general results on transitive $F$-spaces. An $F$-space is said to be transitive if given $x, y \in X$ with $x \neq 0$, there exists $T \in \mathcal{L}(X)$ with $T(x) = y$. We shall say that $X$ is strictly transitive, if for any $k \in X$, $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in X$ such that $[x_1, \ldots, x_k]$ is linearly independent, there exists $T \in \mathcal{L}(X)$ with $T(x_k) = y_k$.

We do not know whether a transitive $F$-space is strictly transitive; however, it is possible to generalize standard arguments in Banach algebra theory (cf. Rickart [5], pp. 60-62 to derive the following):}

**Proposition 2.1.** Let $X$ be a transitive $F$-space; suppose that

(a) $X$ is separable,
(b) the centre of $\mathcal{L}(X)$ consists only of scalar multiples of the identity operator.

Then $X$ is strictly transitive. If $X$ is a $p$-Banach space, condition (a) may be omitted; if $X$ is a complex $p$-Banach space, then conditions (a) and (b) may be omitted.

**Proof.** By [5], Lemma 2.4.3, it is enough to show that given two linearly independent elements $e, w \in X$, there exists $T \in \mathcal{L}(X)$ such that $Te = 0$ and $Tw \neq 0$. If not, we may define $a$ (not necessarily continuous) operator $D$ on $X$ by $Dx = Tw$ when $Te = x$ (cf. [5], Theorem 2.4.6). Then $D$ commutes with each $T \in \mathcal{L}(X)$.

It is necessary to show that $D \in \mathcal{L}(X)$; at this point we require condition (a) in general. Consider $\mathcal{L}(X)$ with the topology of pointwise convergence. Then $\mathcal{L}(X)$ is a Banach space and by the Open Mapping Theorem, the map $e : \mathcal{L}(X) \to X$ defined by $e(T) = Tw$ is open. Since $D \circ e = T$, it follows that $D \in \mathcal{L}(X)$. If $X$ is a $p$-Banach space, then so is $\mathcal{L}(X)$ with its usual topology and the Open Mapping Theorem may be applied to this topology.

Now by condition (b), $D$ is a multiple of $I$ and we have a contradiction.

If $X$ is a complex $p$-Banach space, then it may be shown that the centre of $\mathcal{L}(X)$ is a field and by Zelinski's extension of the Gelfand-Mazur theorem [11], condition (b) must hold.

It is easy to check that each of the spaces $L_p (p \geq 0)$ satisfies conditions (a) and (b) of the proposition (use an argument similar to [7], pp. 253-254; see also [5]), and hence is strictly transitive.

**Proposition 2.2.** Suppose $X$ is a strictly transitive $F$-space and $X \simeq X \times X$; then if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two linearly independent sets in $X$, there exists an invertible $T \in \mathcal{L}(X)$ such that $Te_i = y_i$, $1 \leq i \leq n$.

**Proof.** We prove that there exists a projection $P \in \mathcal{L}(X)$ such that $P(X) \cong X$, $(I - P)(X) \cong X$ and $(P \circ P)(X) = (I - P)(X)$. Let $F$ be the linear span of $(x_1, \ldots, x_n)$; then we may choose a projection $P$ so that $P(X) = F(X)$, $(I - P)(X) \cong X$ and $\dim P(X)$ is maximal. Since $(I - P)(X) \cong X \cong X \times X$, there exists a projection $Q \in \mathcal{L}(X)$ such that $FQ = QF = 0$ and $Q(X) \cong (I - P)(X)$ and $X$. Then since $(P \circ Q)(X) \cong (I - P)(X)$ and $X$, we have $\dim (P \circ Q)(X) = \dim P(X)$. Hence $P$ is one-to-one on $(P \circ Q)(X)$ and if we have $\sum a_i x_i = 0$, then $\sum a_i (P \circ Q)(X) = 0$. Similarly we have $\sum a_i (I - P)(X) = 0$; combining, $\sum a_i P(X) = 0$ and $\sum a_i = 0$. Hence $P(X) = (I - P)(X)$.

Now pick projections $F_1$ and $F_2$ in $\mathcal{L}(X)$ so that $P(X) \cong (I - P)(X)$ and $(P \circ F_1)(X) \cong (I - P)(X)$ and $(P \circ F_2)(X) \cong F(X)$, $(I - P)(X) \cong P(X)$, $(P \circ F_2)(X) \cong P(X)$, $(P \circ F_1)(X) \cong (I - P)(X)$ and $(P \circ F_1)(X) \cong (I - P)(X)$. Then $F_1$ is invertible on $X$ and $F_2$ is invertible on $X$ and $(P \circ F_1)(X) \cong (I - P)(X)$ and $(P \circ F_2)(X) \cong P(X)$, $(I - P)(X) \cong P(X)$, $(P \circ F_1)(X) \cong (I - P)(X)$ and $(P \circ F_2)(X) \cong P(X)$, $(I - P)(X) \cong P(X)$.

We now have:

**Theorem 2.3.** If $0 < p < 1$ and $V$ and $W$ are two subspaces of $L_p$ and $\dim V = \dim W < \infty$, then $L_p (V) \cong L_p (W)$. 
Proof. This is immediate, since there is an invertible operator $T$ on $L_p$ such that $T(V) = V$.

Let us denote now by $(L_p/m)$ the quotient of $L_p$ by an $n$-dimensional subspace; of course $(L_p/0) = L_p$. Theorem 2.3 guarantees that $(L_p/m)$ is well defined.

**Theorem 2.4.** For $0 < p < 1$, $(L_p/[(m+n)]) \cong (L_p/m) \oplus (L_p/n)$ when $m, n > 0$.

**Proof.** Let $V$ be a subspace of $L_p[0,1]$ of dimension $m$ (embedded in $L_p$ in the obvious way) and $W$ a subspace of $L_p[1,2]$ of dimension $n$. Then

$$L_p/[(m+n)] \cong L_p[0,1]/V \oplus L_p[1,2]/W \cong (L_p/m) \oplus (L_p/n).$$

3. **Lifting theorems.** Let $X$ be a $p$-Banach space ($0 < p \leq 1$) and $N$ a closed subspace of $X$. It is easy to see that any linear operator $S$: $L_p \to X/N$ may be lifted to a linear operator $\hat{S}$: $L_p \to X$, so that $\pi \hat{S} = S$, where $\pi$: $X \to X/N$ is the quotient map. In the case $p = 1$, a similar lifting property holds if $L_p$ is replaced by any $L_p$-space and $N$ is isomorphic to a complemented subspace of a dual space (this is effectively proved by Lindenstrauss [4]). Not surprisingly there is a corresponding result for the case $p < 1$. We say that a $p$-Banach space $X$ is an $L_p$-space ($0 < p < 1$) if there is an increasing net $(X_n; \alpha \in A)$ of finite-dimensional subspaces of $X$ such that $\bigcup \{X_n; \alpha \in A\}$ is dense in $X$ and there exist linear maps $S_n$: $X_n \to \ell^p$ and $T_n$: $\ell^p \to X_n$ (where $\alpha_n = \dim X_n$) such that $\sup \|S_n\| \|T_n\| < \infty$ and $T_n S_n = 1$ on $X_n$. Clearly $L_p$ is an $L_p$-space.

We shall also call a $p$-Banach space $Z$ pseudo-dual if there is a Hausdorff vector topology $\mathcal{G}$ on $Z$ such that the unit ball is relatively compact for $\mathcal{G}$. The space $L_p$ is not pseudo-dual (see [1]), but the spaces $L_p$ and $L_\infty$ ($0 < p < 1$) are pseudo-dual.

**Theorem 3.1.** Let $Y$ be an $L_p$-space ($0 < p \leq 1$) and $X$ a $p$-Banach space. Let $N$ be a closed subspace of $X$ and suppose $N$ is isomorphic to a complemented subspace of a pseudo-dual $p$-Banach space $Z$. Then any operator $S$: $Y \to X/N$ may be lifted to an operator $\hat{S}$: $Y \to X$.

If $Y = L_p$, then the lifting is unique.

**Proof.** We observe that the unit ball of $Z$ may be supposed to be $\mathcal{G}$-compact (by [1], Lemma 1). Then the argument is a straightforward imitaiton of the Lemma of [4]. We omit the details.

In the case $Y = L_p$, suppose $T$ is any other lifting. Then $T - S$ maps $L_p$ into $N$ and there is a non-zero operator from $L_p$ into $Z$. The induced map into $(Z, \mathcal{G})$ is compact, contradicting the results of [2].

We now give another result of a similar type for the space $L_p$.

**Theorem 3.2.** Let $X$ be a $p$-Banach space ($0 < p < 1$) and let $N$ be a closed subspace of $X$ which is isomorphic to a $q$-Banach space, where $p < q \leq 1$. Then any linear operator $S$: $L_p \to X/N$ may be lifted uniquely to a linear operator $\hat{S}$: $L_p \to X$.

**Proof.** For $x \in N$, let $Y_x$ be the linear span of the functions $x^n_k$ ($1 \leq k \leq n^x$), where $x^n_k$ is the characteristic function of $|k-1|^{3^{-a}}$, $k \geq 1$. Each $Y_x$ is isometric to $\ell^p$. Let $S_x$: $Y_x \to X$ be a lift of $S$: $L_p \to X/N$ with $\|S_x\| \leq 2 \|S\|$. We shall show that for $x \in \bigcup Y_x$, lim $S_x(y) - S(y)$ exists. Then clearly $\|\hat{S}\| \leq 2 \|S\|$ and $\hat{S}$ may be extended to $L_p$ by continuity and is then a lift of $S$.

Since $N$ is isomorphic to a $q$-Banach space, there exists a constant $C$ such that the $q$-convex hull of the unit ball of $N$ is bounded by $C$. Hence, if $x_1, \ldots, x_n$ are in $N$, then

$$\left\| \sum_{i=1}^n x_i \right\| \leq C \left( \sum_{i=1}^n \|x_i\|^{q/p} \right)^{1/q}.$$

Now suppose $j \leq m \leq n$, and $1 \leq k \leq 2^j$; then

$$\|S_x y_k - S_y y_k\| = \|S_x - S_y\| \left\| \sum_{i=k}^{2^j} x^n_{i(2^j-k+1)} \right\|$$

$$\leq (\sum_{i=k}^{2^j} \|S_x - S_y\| \|x^n_{i(2^j-k+1)}\|)^{1/p}$$

$$\leq 4 C \|S \| \left( \sum_{i=k}^{2^j} \|x^n_{i(2^j-k+1)}\| \right)^{1/p}$$

$$= 4 C \|S \| \left( 2^{-j \cdot (q/p - 1)} \right)^{1/p}$$

and so $(S_x y_k; m \geq j)$ is a Cauchy sequence. Hence we may find the lift $\hat{S}$. As in Theorem 3.1, $\hat{S}$ must be unique, since there are no non-zero operators from $L_p$ into $N$ (see Théorème 3.4.5 of [10] or Proposition 2 of [9], or use the argument above).

We now examine the case $p = 0$, which is rather different. Suppose $X$ is an $F$-space with $F$-norm $\|x\|$. For $x \in X$ we define $s$: $X \to R \cup \{\infty\}$ by

$$s(x) = \sup \{r : x \leq r\}.$$

In the case of $L_p$ with the $F$-norm

$$\|x\| = \frac{1}{n} \|x(n)\|_p d\mu(x),$$

we have that $s(x) = \mu(supp x)$. 

In general, note that \( \sigma(ax) = a\sigma(x) \) if \( a \neq 0 \) and that \( \sigma(x + y) \leq \sigma(x) + \sigma(y) \). If \( L \) is a linear subspace of \( X \), we define \( \sigma(L) = \sup \{ \sigma(x) : x \in L \} \).

We shall say that \( X \) admits \( L_x \)-structure, if for any \( a > 0 \) there exist \( n = n(a) \) and subspaces \( X_1, \ldots, X_n \) of \( X \) such that \( X = X_1 \oplus \cdots \oplus X_n \) and \( \sigma(X_i) \leq a, i = 1, 2, \ldots, n \). In addition to the obvious example of \( L_x \) itself, any space of the type \( L_n(X) \) (all measurable functions from \([0, 1]\) into an \( F \)-space \( Z \)) admits \( L_x \)-structure.

The following proposition is trivial.

**Proposition 3.3.** Suppose \( X \) admits \( L_x \)-structure and \( B \) is a locally bounded space. If \( T : X \to B \) is continuous, then \( T = 0 \).

We next prove two lemmas before giving the main lifting theorem.

**Lemma 3.4.** Suppose \( X \) is an \( F \)-space and \( B \) is a closed locally bounded subspace of \( X \); let \( \pi : X \to X/B \) be the quotient map. Let \( \delta \) be chosen so that the set \( \{ b \in B : ||b|| \leq \delta \} \) is bounded.

Then if \( \xi \in X/B \) and \( \sigma(\xi) \leq \delta/\beta \), there is a unique \( x \in X \) such that \( \pi x = \xi \) and \( \sigma(x) \leq \delta/\beta \). For this \( x \), \( \sigma(x) = \sigma(\xi) \).

**Proof.** If \( \xi \in X/B \) and \( \sigma(\xi) \leq \delta/\beta \), then we may assume that \( \sigma(\xi) = \sigma(\zeta) \) for some \( \zeta \in X \). Let \( \nu = \pi x = \xi \) and \( \sigma(\nu) = \sigma(\xi) \).

Let \( \nu_n = \nu_n - \nu \) (\( n \in \mathbb{N} \)). Then \( \nu_n \to \nu \) and \( ||\nu_n|| \to ||\nu|| \).

Thus \( ||\nu_n|| \leq \frac{1 + n}{n} ||\nu|| \), \( n \in \mathbb{N} \).

Let \( \nu_n = \pi x_n - \pi x \) (\( n \in \mathbb{N} \)); then \( \nu_n \to \nu \) and \( ||\nu_n|| \to ||\nu|| \).

Then \( ||\nu_n|| \leq \frac{1 + n}{n} ||\nu|| \), \( n \in \mathbb{N} \).

This implies that \( \{ \nu_n \} \) is a Cauchy sequence and hence so is \( \{ \pi x_n \} \). If \( x = \lim \pi x_n \), then \( \sigma(x) = \sigma(\xi) \leq \delta/\beta \). If \( x \) is any other point satisfying \( \pi x = \xi \) and \( \sigma(\xi) \leq \delta/\beta \), then \( x - y \in B \) and \( \sigma(x - y) \leq \frac{3}{2} \delta/\beta \); this implies \( x - y = 0 \).

**Lemma 3.5.** Under the assumptions of Lemma 3.4, let \( Y \) be a linear subspace of \( X/B \) with \( \sigma(Y) \leq \delta/\beta \). Then there is a linear operator \( h : Y \to X \) such that \( \pi \circ h(\xi) = \xi \) for \( \xi \in X \).

**Proof.** Fix \( \xi \in X \). Define \( h(\xi) \) to be the unique \( x \in X \) such that \( \pi x = \xi \) and \( \sigma(x) = \sigma(\xi) \). If \( \alpha, \beta \in \mathbb{R} \) and \( \delta, \gamma, \eta \in X \), then

\[
\sigma(\alpha h(\xi) + \beta h(\eta)) \leq \alpha \sigma(h(\xi)) + \beta \sigma(h(\eta)) \leq \delta/\beta.
\]

Thus

\[
h(\alpha \xi + \beta \eta) = \alpha h(\xi) + \beta h(\eta),
\]

and \( h \) is linear.

Now suppose \( \xi_n \to 0 \) in \( Y \); choose \( x_n \in X \) such that \( \pi x_n = \xi_n \) and \( ||x_n|| \leq 2 ||\xi_n|| \). Then \( x_n - h(x_n) \to 0 \). If \( x_n - h(x_n) \to 0 \), we may assume, by passing to a subsequence, that for some \( a > 0 \) we have

\[
||x_n - h(x_n)|| \geq \delta
\]

(since the set \( \{ b \in B : ||b|| \leq \delta \} \) is bounded).

Then

\[
||x_n|| > \delta - ||h(x_n)|| \geq \delta - \frac{1}{2} \delta = \frac{1}{2} \delta.
\]

This is a contradiction since \( ||x_n|| \to 0 \). Hence we have \( x_n - h(x_n) \to 0 \) and so \( h(\xi_n) \to 0 \).

**Theorem 3.6.** Suppose \( X \) admits \( L_x \)-structure, \( Y \) is an \( F \)-space, and \( B \) is a closed locally bounded subspace of \( Y \). Then if \( \pi : X \to X/B \) is a linear operator, there is a unique linear operator \( \tilde{S} : X \to X \) such that \( \pi \circ \tilde{S} = \pi \).

**Proof.** Choose \( \delta > 0 \) so that \( \{ b \in B : ||b|| \leq \delta \} \) is bounded, and then \( \delta > 0 \) so that \( ||\xi|| \leq \delta \). Then \( ||\xi|| \leq \delta/\beta \). Let \( X_1, \ldots, X_n \) be closed subspaces of \( X \) such that \( X = X_1 \oplus \cdots \oplus X_n \) and \( \sigma(X_i) \leq \delta/\beta \).

Then \( \sigma(\pi X_i) \leq \delta/\beta \), and so there exist linear operators \( h_i : \pi X_i \to Y \), such that \( h_i(\xi) = \tilde{S}(X_i) \). If we define \( \tilde{S} : X \to Y \) by

\[
\tilde{S}(x_1 + \cdots + x_n) = \sum_{i=1}^{n} h_i x_i,
\]

then \( \tilde{S} \) is the required lifting of \( S \).

If \( T \) is any other lifting, then \( \tilde{S} - T \) maps \( X \) into \( B \) and hence \( \tilde{S} = T \) by Proposition 3.3.

4. **Quotient spaces of \( L_\rho \) (0 < \rho < 1).** In this section we treat the case \( \rho = 0 \) first and in more detail than the case \( \rho > 0 \); the arguments are analogous.

**Theorem 4.1.** Suppose \( X_1 \) and \( X_2 \) are two \( F \)-spaces with \( L_\rho \)-structure. Suppose \( B_1 \) and \( B_2 \) are closed locally bounded subspaces of \( X_1 \) and \( X_2 \), respectively. Then \( X_1/B_1 \cong X_2/B_2 \) if and only if there is an isomorphism \( \phi : X_1/B_1 \to X_2/B_2 \) mapping \( X_1 \) onto \( X_2 \) and such that \( V(B_1) \to B_2 \).

**Proof.** The “if” part is clear. For the “only if” part, let \( S : X_1/B_1 \to X_1/B_2 \) be an isomorphism, and let \( \pi_1, \pi_2 \) be the quotient maps. Then
by Theorem 3.6, there exist lifts \( V, U \) of \( S_{n_1} : X_0 \to X_0/B_0 \) and \( S^{-1} n_1 : X_1 \to X_1/B_1 \).

\[
\begin{array}{c|c|c}
X_0 & U & X_1 \\
\hline
X_0/B_0 & V & X_1/B_1 \\
\end{array}
\]

Then \( UV : X_0 \to X_1 \) is a lift of \( n_1 : X_0 \to X_1/B_1 \). By the uniqueness, \( UV = \iota_{X_1} \); similarly \( VU = \iota_{X_0} \), so \( V \) is an isomorphism of \( X_0 \) onto \( X_1 \).

Clearly \( V(B_0) = B_1 \) and \( U(B_1) = V^{-1}(B_1) = B_1 \); hence \( V(B_0) = B_1 \).

**Corollary 4.3.** If \( X \) admits \( L_p \)-structure and \( B \in X \) is a locally bounded subspace, then \( X/B \) admits \( L_p \)-structure if and only if \( B = \{0\} \).

Theorem 2.3 and Theorem 4.2 give

**Theorem 4.3.** If \( B_1 \) and \( B_2 \) are locally bounded subspaces of \( L_p \),

then \( L_p[B_1] \equiv L_p[B_2] \) if and only if there is an invertible operator \( T : L_p \to L_p \) such that \( T(B_1) = B_2 \).

In particular, \( (L_p[m]) \equiv (L_p[n]) \) if and only if \( m = n \), and \( (L_p[1]) \equiv L_p \).

This solves the problem of Pelszthicki (see Introduction). Also in this section we shall illustrate this corollary by showing that \( B_1 \equiv B_2 \) does not imply \( L_p[B_1] = L_p[B_2] \). First however, we state the corresponding theorem for \( p > 0 \); the proofs are similar.

**Theorem 4.4.** Suppose \( B_1 \) and \( B_2 \) are two closed subspaces of \( L_p \), each of which is either isomorphic to a complemented subspace of a pseudo-dual \( p \)-Banach space or to a \( q \)-singular Banach space where \( p < q \leq 1 \).

Then \( L_p[B_1] \equiv L_p[B_2] \) if and only if there is an invertible operator \( T : L_p \to L_p \) such that \( T(B_1) = B_2 \).

In particular, \( (L_p[m]) \equiv (L_p[n]) \) if and only if \( m = n \), and \( (L_p[1]) \equiv L_p \).

**Theorem 4.5.** If \( B \subset L_p \) is isomorphic to a complemented subspace of a pseudo-dual \( p \)-Banach space and \( B \neq \{0\} \), then \( L_p[B] \) is not a \( L_p \)-space.

**Proof.** If \( L_p[B] \) is an \( L_p \)-space, then the identity map \( I : L_p[B] \to L_p[B] \) may be lifted to a map \( J : L_p[B] \to L_p[B] \). Then on \( L_p[I = J \) maps \( L_p[B] \) into \( B \). Hence, by applying the results of [2] as in Theorem 3.1, \( I = J \) and so \( B = \{0\} \).

**Example.** Let \( B_1 \subset L_p \) be the closed linear span of the Rademacher functions \( r_n \) on \([0,1]\)

\[
(r_n(t)) = \text{sgn} (\sin (2^n \pi t))
\]

and let \( B_2 \) be the closed linear span of a sequence of independent random variables normally distributed with mean zero and variance one. Then \( B_1 \equiv B_2 \equiv L_2 \); we shall show, however, that \( L_p[B_1] \not\equiv L_p[B_2] \).

For suppose \( L_p[B_1] \equiv L_p[B_2] \); then there is an invertible linear operator \( T : L_p \to L_p \) such that \( T(B_1) = B_2 \). By Kwapień's Representation Theorem [3], \( T \) takes the form

\[
T_t(t) = \sum \frac{t}{n} \eta_n(t) = (\Phi(t)) \;
\]

where

1. \( \eta_n \in L_p \) \( n \geq 1 \),
2. \( m(t; \eta_n(t)) \neq 0 \) for infinitely many \( n \) = 0,
3. \( \sigma_0 \) maps \([0,1] \) into \([0,1] \); if \( A \) is measurable, then \( \sigma_0[A] \) is measurable; if \( m(A) = 0 \), then \( m(\sigma_0[A] \cap \text{Supp} \eta_n) = 0 \).

Thus for almost every \( t \in [0,1] \), the sequence \( \{T_t(n)\} \) assumes only finitely many values. Hence for some \( j \), with \( j \neq k \), we must have \( m(t; T_t(n) - T_k(0)) = 0 \). However \( T_t(t) - T_k(t) \) is normally distributed and hence \( T_t(t) - T_k(t) \); thus \( T \) is not injective, and we have a contradiction.

**Remarks.** For \( p > 0 \), let \( x \in L_p \) be non-zero and let \( Y \) be the linear span of \( x \). Let \( L_p(Y) \) and \( L_p(Y') \) be the \( p \)-Banach algebras of all bounded linear operators on \( L_p \) and \( L_p[V] \), respectively. If \( S \in L_p(Y') \), let \( \tilde{S} : L_p \to L_p \) be the unique lift of \( S \circ \sigma_0 \). Then the map \( \tilde{S} \to \tilde{S} \) is an algebra homomorphism, and in fact an embedding of \( L_p(Y') \) into \( L_p(Y) \). Thus \( L_p(Y') \) is isomorphic to the closed subalgebra of \( L_p(Y) \) consisting of all \( T \in L_p(Y) \) such that \( T(x) \in Y \). We may define a multiplicative linear functional \( \varphi \) on \( L_p(Y') \) by

\[
\varphi(Sx) = \tilde{S}x
\]

5. **K-spaces.** In this section, we abstract a particular property of the space \( L_p \) and consider it in more generality. We restrict to the real case, but the complex case is identical.

If \( X \) is an \( F \)-space, we shall say that \( X \) is a \( K \)-space if every short exact sequence \( 0 \to R \to Y \to X \to 0 \), with \( Y \) an \( F \)-space, splits. Alternatively, if \( Y : Y \to X \) is onto and \( \dim S^{-1}(0) = 1 \), then there exists an operator \( T : Y \to Y \) such that \( ST = I_Y \).

If \( X \) is a \( p \)-Banach space (\( 0 < p \leq 1 \)), we shall say that \( X \) is a \( K_p \)-space if every short exact sequence \( 0 \to R \to Y \to X \to 0 \), with \( Y \) a \( p \)-Banach space, splits.

**Theorem 5.1.** An \( F \)-space \( X \) is a \( K \)-space \( (K_p \)-space) if and only if whenever \( Y \) and \( Z \) are \( F \)-spaces \( (p \)-Banach spaces) and \( S : Y \to Z \) is a surjective operator with \( \dim S^{-1}(0) = 1 \), then each linear operator \( X \to Z \) may be lifted to an operator \( T : X \to X \) such that \( ST = T \).

**Proof.** We prove the statement for \( K \)-spaces. Suppose \( X \) is a \( K \)-space. Let \( V = X \oplus Y \) be the subspace of all \((x,y)\) such that \( Tw = Sy \),
and define $P: V \to X$ by $P(a, y) = a$. Then $P: V \to X$ is surjective, and $\dim P^{-1}(0) = 1$. Hence there exists a linear operator $R: X \to V$ such that $PR = I_X$. Then $Rv = (v, T_0v)$; clearly $ST = T$.

For the converse take $Z = X$ and $T$ to be the identity.

We remark now that if $X$ admits $L_0$-structure, then $X$ is a $K$-space (Theorem 3.6), and that an $L_p$-space is a $K_p$-space.

**Theorem 5.2.** If $X$ is an $F$-space (or Banach space) and $N$ is a closed subspace of $X$ such that $X/N$ is a $K$-space ($K_p$-space), then $X$ has the Hahn-Banach Extension Property in $X$.

**Proof.** Again we restrict to the $K$-space case. Suppose $x \in N$ is non-zero; let $M = q^{-1}(0) \subset N$. Consider the natural quotient map $\pi: X/M \to X/N$; then there is a map $S: X/N \to X/M$ such that $\pi S = I$ on $X/N$. Then $S(X/N)$ is a closed subspace of co-dimension one in $X/M$ and so there exists $y \in (X/M)^*$ such that $y(x) = 0$. Since $S(X/N)$ is the natural quotient map, then $yq \in X$. If $u \in N$, then $yq(u) = 0$ if and only if $q(u) = 0$. Thus $(yq)^{-1}(0)$ is a suitable multiple of $yq$.

There is also a converse to Theorem 5.2.

**Theorem 5.3.** If $X$ is a $K$-space ($K_p$-space) and $N \subset X$ is a closed subspace with HBEF, then $X/N$ is a $K$-space ($K_p$-space).

**Proof.** Suppose we have a short exact sequence

$$0 \to R \to Z \to X/N \to 0,$$

and let $\pi: X \to X/N$ be the quotient map. Then there is a lifting of $\pi$, $S: X \to Z$ so that $\pi S = \pi$ (by Theorem 5.1). Suppose first $S$ is not surjective; then $S(X)$ has co-dimension one in $Z$ and $\pi S(X)$ is one-one. Define $R: X/N \to Z$ by $R = 0$ where $s \in S(X)$ and $\pi s = 0$. If $\xi \in 0$ in $X/N$, then there exists a sequence $n_i$ in $N$ such that $n_i \to 0$ and $\pi n_i = 0$. Since $S_0 \to 0$ and $S_0 - R_{\pi 0} \to 0$, i.e. $R$ is continuous.

Now suppose $S$ is surjective; then $S^{-1}(0)$ is a non-zero linear functional on $N$. Let $\varphi \in N^*$ be a non-zero linear functional with kernel $S^{-1}(0)$. Then $\varphi$ may be extended to $\varphi \in X$. Now define $g: X \to Z$ by $g(x) = S(x - \varphi(x))$ where $x \in N$ is chosen so that $\varphi(x) = 0$. Then $g\varphi = \pi(x - \varphi(x)) = 0$ for $x \in N$. Hence $g(x) = S(x - \varphi(x)) = 0$. Thus $g$ is one-one on $S(X)$ and $S(X)$ has co-dimension one in $Z$; we can apply the previous part of the proof.

**Corollary 5.4.** $X$ is a $K_p$-space if and only if $X \cong L_p(I)/N$ where $I$ is some index set and $N \subset L_p(I)$ has the HBEF.

We remark that if $L_p(N) \cong L_p$, then $N$ has HBEF and the extension is unique, since $L_p = \{0\}$.

**Corollary 5.5.** (i) If $N$ is a closed subspace of $L_p$, then $L_p(N)$ is a $K$-space if and only if $N^* = \{0\}$. In particular, if $L_p(N)$ has $L_p$-structure, then $N^* = \{0\}$.

(ii) If $N$ is a closed subspace of $L_p$, then $L_p(N)$ is a $K_p$-space if and only if $N^* = \{0\}$. In particular, if $L_p(N)$ is an $L_p$-space, then $N^* = \{0\}$.

Note here that if for $x$ the closed linear span of a sequence of functions with disjoint supports in $L_p$, then $N \cong o$, and hence $L_p/N \not\cong L_p$. However $L_p/N \cong c_0$ (the countable product of copies of $L_p/I$); hence $c_0/L_p \not\cong L_p$.

**Problem.** Is $L_p$ or $L_p(N)$ a $K_p$-space for any $p < p_1$, or even a $K$-space? In particular, is $L_1$ (or any Banach space) a $K_p$-space for any $p < p_1$? This latter question is essentially the same as a problem of Gelles [8]: if $L_p(N)$ is locally convex, must $N$ have the HBEF?

**References**


- University College of Swansea
- Singleton Park, Swansea
- University of Illinois
- Urbana, Illinois

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