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Bases and basic sequences in F-spaces

by

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Abstract. This paper is concerned with the theory of Schauder bases in non-locally convex F-spaces. We first give some results on the existence problem for basic sequences, extending work of the first author (*Basic sequences in F-spaces and their applications*, Proc. Edinburgh Math. Soc. to appear). In particular it is shown that the existence of a basic sequence in every infinite-dimensional closed linear subspace of an F-space is equivalent to an extension property for linear functionals. Then we introduce two new classes of F-spaces, which we call pseudo-Fréchet and pseudo-reflexive spaces. For example, an F-space is pseudo-reflexive if every bounded set is relatively compact in the weak topology of its closed linear span. We give criteria for spaces with bases to be pseudo-Fréchet and pseudo-reflexive and hence are able to give non-locally convex examples. Using these examples we show the existence of non-locally convex F-spaces on which there exist strictly weaker vector topologies which define the same closed subspaces as the original topology.

1. Introduction. In this paper we continue the study begun by the first author in [1] of basic sequences in F-spaces, with the emphasis on non-locally convex spaces. In §2 we restate in a more accurate form the main result from [1] on constructing basic sequences, and derive some variations on this result. It is not known if every F-space contains a basic sequence. §3 contains some contributions to this existence problem. The last section of the paper treats two new classes of F-spaces. We call an F-space *pseudo-Fréchet* if the weak topology of each linear subspace coincides on bounded sets with the weak topology of the whole space. We call an F-space *pseudo-reflexive* if the weak topology is Hausdorff, and every bounded subset is relatively compact in the weak topology of its closed linear span. It turns out that every pseudo-reflexive F-space is pseudo-Fréchet, and a Fréchet space (locally convex F-space) is pseudo-reflexive if and only if it is reflexive. We give criteria for spaces to be pseudo-Fréchet or pseudo-reflexive which involve shrinking and boundedly complete basic sequences; and we use these results to construct

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examples of non-locally convex pseudo-Fréchet and pseudo-reflexive spaces. Finally, we show that the bounded weak topology of a locally bounded, pseudo-reflexive F-space is compatible with (i.e., has the same closed subspaces as) the original topology. This provides examples of non-locally convex F-spaces which have topologies strictly weaker than, yet compatible with, the original ones; and gives non-locally convex applications of some of the results in § 5 of [1].

We wish to thank Professor A. Pełczyński for suggesting to us the examples used in § 4.

2. Basic results. First we recall some definitions. Let (E, τ) be an F-space; then a sequence $\{x_n\}$ is *semi-basic* ([1]) if for each n, we have $x_n \notin \overline{\mathrm{lin}}(x_{n+1}, x_{n+2}, \ldots)$. As observed in [1] we can then define continuous linear functionals $\{f_n\}$ on the space $E_0 = \overline{\mathrm{lin}}(x_n\colon n \in \mathcal{N})$ such that $f_i(x_j) = \delta_{ij}$. If we further have that for $x \in E_0$, $f_n(x) = 0$ for all $n \in \mathcal{N}$ implies that $x = 0$, then (x_n) is a *Markushevich basis* of E_0, and we shall then say that (x_n) is an M-*basic sequence* in E. Finally, if for $x \in E_0$, $x = \sum_{n=1}^{\infty} f_n(x) x_n$ then (x_n) is a *basic sequence* in E.

If ϱ is another Hausdorff vector topology on E we shall say that τ is *ϱ-polar* if τ has a base of ϱ-closed neighbourhoods of 0, and τ is *ϱ-compatible* if every τ-closed linear subspace of E is also ϱ-closed. It is shown in [1] that if τ is ϱ-compatible then τ is ϱ-polar. A net $(y_a\colon a \in A)$ in E is *τ-regular* if there is a neighbourhood V of 0 such that $y_a \notin V$ for all $a \in A$.

In Theorem 2.1 below we restate the main existence theorem for basic sequences from [1]. Part (ii) is a more accurate formulation of Corollary 3.4 of [1], for it asserts the existence of an M-basic sequence rather than simply a semi-basic sequence. It is clear that the proof of Corollary 3.4 yields this extra information, as the sequence obtained is a basic sequence for a topology on E which is weaker than τ.

THEOREM 2.1. *Let (E, τ) be an F-space and let ϱ be a Hausdorff vector topology on E with $\varrho < \tau$. Suppose $(x_a\colon a \in A)$ is a τ-regular net which converges to 0 in ϱ and suppose $z_1 \in E$ with $z_1 \neq 0$.*

(i) *Suppose τ is ϱ-polar. Then there is an increasing sequence $(a(n)\colon n \geqslant 2)$ such that if $z_n = x_{a(n)}$, $n \geqslant 2$, then $(z_n\colon n \in \mathcal{N})$ is a basic sequence in (E, τ).*

(ii) *In general, if τ is not ϱ-polar, there is an increasing sequence $(a(n)\colon n \geqslant 2)$ such that if $z_n = x_{a(n)}$, $n \geqslant 2$ then $(z_n\colon n \in \mathcal{N})$ is an M-basic sequence in (E, τ).*

In this section we modify this result by giving another condition under which basic sequences may be constructed. A sequence (x_n) is of *type P^** if there is a continuous linear functional φ on $\overline{\mathrm{lin}}(x_n\colon n \in \mathcal{N})$ such that $\varphi(x_n) = 1$ for all n.

LEMMA 2.2. *Let (x_n) be a sequence of type P^* and suppose that $u \notin \overline{\mathrm{lin}}(x_n\colon n \in \mathcal{N})$. If (x_n) is basic (resp. M-basic; resp. semi-basic), then $(u + x_n)$ is basic (resp. M-basic; resp. semi-basic).*

Proof. Suppose (x_n) is semi-basic, and that there are continuous linear functionals (f_n) on $\overline{\mathrm{lin}}(x_n)$ such that $f_i(x_j) = \delta_{ij}$. Let φ be a continuous linear functional on $\overline{\mathrm{lin}}(x_n)$ such that $\varphi(x_n) = 1$ for all n. Let X be the space $\overline{\mathrm{lin}}(x_n; u)$, and extend f_n and φ to continuous linear functionals $\bar{f}_n$ and $\bar{\varphi}$ defined on X such that $\bar{f}_n(u) = \bar{\varphi}(u) = 0$. Also let ψ be the linear functional defined on X such that $\psi(u) = 1$ and $\psi\big(\overline{\mathrm{lin}}(x_n)\big) = 0$. Then ψ is a continuous function as $\psi^{-1}(0)$ is closed.

Now $\bar{f}_i(u + x_j) = \delta_{ij}$, so that $(u + x_n)$ is semi-basic. Suppose, in addition, (x_n) is M-basic and $x \in \overline{\mathrm{lin}}(u + x_n)$. Then $\psi(u + x_n) = \bar{\varphi}(u + x_n)$ for all n, and hence $\psi(x) = \bar{\varphi}(x)$. However, $x = au + y$ where $y \in \overline{\mathrm{lin}}(x_n)$ and so $a = \varphi(y)$. If $\bar{f}_n(x) = 0$ for all $n \in \mathcal{N}$, then $f_n(y) = 0$ for all n and hence $y = 0$. Therefore $a = 0$ and $x = 0$, i.e., $(u + x_n)$ is M-basic.

Finally, if (x_n) is basic then

$$x - \sum_{i=1}^{n} \bar{f}_i(x)\,(u + x_i) = \Big[a - \sum_{i=1}^{n} f_i(y)\Big]\, u + \Big(y - \sum_{i=1}^{n} f_i(y)\, x_i\Big).$$

Then

$$\sum_{i=1}^{\infty} f_i(y) = \varphi(y) = a$$

since $y \in \overline{\mathrm{lin}}(x_n)$ and (x_n) is a basic sequence. Also $\sum f_i(y) x_i = y$. Hence

$$\sum_{i=1}^{\infty} \bar{f}_i(x)\,(u + x_i) = x$$

and $(u + x_n)$ is a basic sequence.

THEOREM 2.3. *Let (E, τ) be an F-space and suppose $\varrho < \tau$ is a Hausdorff vector topology on E. Suppose $(x_a\colon a \in A)$ is a ϱ-Cauchy net in E. Suppose either that (x_a) converges in ϱ to some $u \notin \overline{\mathrm{lin}}(x_a\colon a \in A)$ or that (x_a) does not converge in ϱ. Then*

(i) *there is an increasing sequence $a(n)$ such that $(x_{a(n)})$ is an M-basic sequence;*

(ii) *if τ is ϱ-polar, there is an increasing sequence such that $(x_{a(n)})$ is a basic sequence.*

Proof. First suppose (x_a) converges to some $u \notin \overline{\mathrm{lin}}(x_a\colon a \in A)$. Then by Theorem 2.1 there is an increasing sequence $(a(n)\colon n \geqslant 2)$ such that if $z_n = x_{a(n)} - u$ for $n \geqslant 1$ then (z_n) is M-basic (or basic if τ is ϱ-polar). There is a continuous linear functional φ on $\overline{\mathrm{lin}}(x_a; u)$ such that $\varphi(u) = -1$

but $\varphi(x_a) = 0$ for $a \in A$. Then $\varphi(z_n) = 1$ for $n \geqslant 1$ and so (z_n) is of type P^*. By Lemma 2.2, $u + z_n = x_{a(n)}$ is M-basic (or basic if τ is ϱ-polar).

Next suppose $(x_a\colon a \in A)$ does not converge. Let $(\hat{E}, \hat{\varrho})$ be the completion of (E, ϱ) and $Y \subset \hat{E}$ be the linear span of E and $u = \lim\limits_a x_a$. For (i) we extend τ to a topology $\hat{\tau}$ on Y so that $Y = E \oplus \operatorname{lin}(y)$, and apply the preceding proof. For (ii) suppose (V_n) is a base of balanced ϱ-closed τ-neighbourhoods of 0 satisfying $V_{n+1} + V_{n+1} \subset V_n$. Let W_n be the closure of V_n in $(Y, \hat{\varrho})$. If each W_n is absorbent in Y, then (W_n) defines a $\hat{\varrho}$-polar topology $\hat{\tau}$ on Y which extends τ (cf. Theorem 5.7 of [1]) and again we may apply the earlier proof. Otherwise $\bigcup\limits_{\lambda>0} \lambda W_n \neq Y$, and then since $W_{n+1} + W_{n+1} \subset W_n$, $\bigcup\limits_{\lambda>0} \lambda W_{n+1} = E$, in particular $W_{n+1} \subset E$. Thus $W_m = V_m$ for $m > n$. If we define $U_m = W_m + \{\lambda u\colon |\lambda| \leqslant 2^{-m}\}$, then (U_m) defines a topology $\hat{\tau}$ on Y which is $\hat{\varrho}$-polar and $\hat{\tau} = \tau$ on E (since for $k > n$, $U_k \cap E = V_k$). Again we apply the earlier proof.

3. The existence problem. An F-space E is called *minimal* if there is no strictly weaker Hausdorff vector topology on E. It is shown in [1] that ω, the space of all sequences, is a minimal space; however it is not known whether there are other examples. This problem is central to the problem of finding basic sequences in any F-space. In [5], Peck considers the space $M[0,1]$ of measurable functions on $[0,1]$ with the F-norm

$$\|x\| = \int\limits_0^1 \frac{|x(t)|}{1+|x(t)|}\, dt$$

and shows that $M[0,1]$ is not a minimal space. His method of proof yields the following result:

PROPOSITION 3.1. *Let (E, τ) be a minimal F-space and suppose (x_n) is an M-basic sequence in E. Then (x_n) is a basic sequence equivalent to the usual basis of ω.*

[Two basic sequences (x_n) and (y_n) are *equivalent* if $\sum a_n x_n$ converges if and only if $\sum a_n y_n$ converges.]

Proof. Let $L_n = \overline{\operatorname{lin}}(x_k\colon k \geqslant n)$ and define a vector topology λ with a base of neighbourhoods of 0 of the form $L_n + U$, where U is a τ-neighbourhood of 0. Then

$$\bigcap_{n,U} (L_n + U) = \bigcap_n \bigcap_U (L_n + U) = \bigcap_n L_n = \{0\}$$

since (x_n) is M-basic. Therefore λ is Hausdorff and as $\lambda \leqslant \tau$ we conclude $\lambda = \tau$. For any sequence (t_n) of scalars we have

$$\sum_{n+1}^{m} t_k x_k \in L_n + U$$

for any τ-neighbourhood U. Hence $\sum t_k x_k$ converges for any scalar sequence, and it follows that (x_n) is a basic sequence equivalent to the usual basis of ω.

An M-basic sequence will be called *strongly regular* if for $x \in \overline{\operatorname{lin}}(x_n)$, $\lim\limits_{n\to\infty} f_n(x) = 0$, where (f_n) is the biorthogonal sequence of linear functionals.

THEOREM 3.2. *Let E be an F-space; then the following are equivalent:*

(i) *E is non-minimal,*

(ii) *E contains a regular M-basic sequence,*

(iii) *E contains a strongly regular M-basic sequence,*

(iv) *E contains a regular basic sequence.*

Proof. (iv) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (i). By Proposition 3.1, since the usual basis of ω is not regular.

(i) $\Rightarrow$ (iii). By the proof of Theorem 2.1 (ii) ($=$ Corollary 3.4 of [1]) E contains a sequence (x_n) which is regular and basic in a weaker metrizable topology μ. If $x \in \overline{\operatorname{lin}}(x_n)$, then x is also in the μ-closed linear span of (x_n) and therefore

$$(\mu) \qquad x = \sum_{n=1}^{\infty} f_n(x) x_n.$$

Since (x_n) is μ-regular, $\lim\limits_{n\to\infty} f_n(x) = 0$ and (x_n) is strongly regular.

(iii) $\Rightarrow$ (iv). Let $E_0 = \overline{\operatorname{lin}}(x_n)$, where (x_n) is a strongly regular M-basic sequence. Let (f_n) be the biorthogonal sequence of linear functionals on E_0. Since the topology induced by the functionals (f_n) is strictly weaker than the original topology on E_0, E_0 is non-minimal and contains a basic sequence (y_n) by Theorem 4.2 of [1]. For $x \in E_0$, $\sup\limits_n |f_n(x)| < \infty$, and so by the Baire Category Theorem, the norm

$$\|x\| = \sup_n |f_n(x)|$$

is continuous on E_0. The sequence $\|y_n\|^{-1} y_n$ is a regular basic sequence in E.

COROLLARY 3.3. *If E is an F-space, then E contains a basic sequence if and only if E contains a closed infinite-dimensional subspace Y with a total family of continuous linear functionals.*

Proof. One direction is trivial. Suppose Y is a closed infinite-dimensional subspace and possesses a total family of continuous linear functionals. If Y is minimal, then the weak topology on Y is the original topology and so $Y \cong \omega$. If Y is non-minimal, then Y contains a basic sequence.

Remark. Corollary 3.3 shows that the existence question for basic sequences is equivalent to Problem IV.2.4, p. 114, of Rolewicz [6].

In our last two results of this section we attempt to classify F-spaces in which every closed subspace contains a basic sequence. An F-space E (of infinite dimension) will be said to have the *Restricted Hahn–Banach Extension Property* (RHBEP) if whenever $L \subset E$ is an infinite-dimensional closed subspace and $0 \neq x \in L$, then there is an infinite-dimensional closed subspace M of L with $x \notin M$.

PROPOSITION 3.4. *Let E be an infinite-dimensional F-space; the following are equivalent:*

(i) *If L is an infinite-dimensional closed subspace of E and G is a finite-dimensional subspace of L, then there is an infinite-dimensional subspace M of L with $M \cap G = \{0\}$.*

(ii) *Let L be an infinite-dimensional closed subspace of E and G a finite-dimensional subspace of L. If φ is a linear functional on G, there is an infinite-dimensional closed subspace K of L containing G, and a continuous linear functional ψ on K extending φ.*

(iii) *E has* RHBEP.

Proof. (iii) $\Rightarrow$ (i). We prove (i) by induction on $\dim G$. Certainly (i) is true for $\dim G = 1$. Now suppose it is true for $\dim G = k$. Suppose $\dim G = k+1$, and let G_0 be any subspace of G of dimension k. Choose a closed infinite-dimensional subspace M_0 of L such that $G_0 \cap M_0 = \{0\}$. Let $L_0 = M_0 + G_0$ and suppose $x \in G \setminus G_0$. Then there is an infinite-dimensional closed subspace N of L_0 with $x \notin N$. Let $M = N \cap M_0$; then $\dim N/M \leqslant \dim L_0/M_0 = k$ so that $\dim M = \infty$, and clearly $M \cap G = \{0\}$.

(i) $\Rightarrow$ (ii). Choose M as in (i) and let $K = M+G$; we extend φ by $\psi(x) = \varphi(x)$, $x \in G$, and $\psi(x) = 0$, $x \in M$. Then $\psi^{-1}(0) = \varphi^{-1}(0) + G$ is closed, and so ψ is continuous.

(ii) $\Rightarrow$ (i). Suppose $x \in L$; let $G = \operatorname{lin}\{x\}$ and define $\varphi(\lambda x) = \lambda$. Extending φ as in (ii) we take $M = \psi^{-1}(0)$.

THEOREM 3.5. *An F-space E has* RHBEP *if and only if every closed infinite-dimensional subspace contains a basic sequence.*

Proof. Suppose E has RHBEP and let E_0 be a closed infinite-dimensional subspace of E; we may suppose E_0 separable. We may determine a collection $\mathscr{L}$ of closed infinite-dimensional subspaces of E_0 maximal with respect to the property that any finite subcollection has infinite-dimensional intersection. Let $G = \bigcap \mathscr{L}$. If $\dim G = \infty$, then $G \in \mathscr{L}$ by maximality; however, by RHBEP, G contains a proper closed infinite-dimensional subspace G_0 and $G_0 \in \mathscr{L}$ by the maximality of $\mathscr{L}$. Hence $\dim G < \infty$. Then $E_0 \setminus G = \bigcup (E_0 \setminus L;\ L \in \mathscr{L})$, and as $E_0 \setminus G$ is a Lindelöf space, there is a countable subset $(L_n\colon n \in \mathscr{N})$ of $\mathscr{L}$ such that $\bigcup (E_0 \setminus L_n) = E_0 \setminus G$, i.e., $G = \bigcap_n L_n$. Letting $M_n = L_1 \cap \ldots \cap L_n$ we have $\dim M_n = \infty$ and $\bigcap_n M_n = G_0$. We may select a subsequence $M_{p(n)}$ such that $M_{p(n)} \neq M_{p(n+1)}$ for $n \geqslant 1$. Suppose $z_n \in M_{p(n)} \setminus M_{p(n+1)}$ for all n, and let $K_n = \operatorname{lin}(z_n, z_{n+1}, \ldots)$. Then K_n is a strictly decreasing sequence of closed infinite-dimensional subspaces and $\bigcap K_n = H \subset G$.

By RHBEP, pick an infinite-dimensional subspace J of K_1 so that $J \cap H = \{0\}$. For each n $\dim J/J \cap K_n \leqslant \dim K_1/K_n < \infty$ so that $J \cap K_n$ is infinite-dimensional and $\bigcap_n (J \cap K_n) = \{0\}$. We may find a subsequence $J \cap K_{q(n)}$ so that $J \cap K_{q(n)} \neq J \cap K_{q(n+1)}$, $n \geqslant 1$ and hence we may select $x_n \in J \cap K_{q(n)} \setminus J \cap K_{q(n+1)}$. Then (x_n) is an M-basic sequence in E_0. If E_0 is minimal, then (x_n) is already a basic sequence (3.1); otherwise E_0 contains a basic sequence (3.2).

Conversely, if $L \subset E$ is an infinite-dimensional closed subspace and $x \in L$, we may find a basic sequence (x_n) in L. Then $x \notin \overline{\operatorname{lin}}(x_n, x_{n+1}, \ldots)$ for some $n \in \mathscr{N}$.

4. Pseudo-Fréchet and pseudo-reflexive F-spaces.

Let $w(E, E')$ denote the weak topology induced on a linear topological space E by its (topological) dual E'. We call an F-space E *pseudo-Fréchet* if for each linear subspace S of E and each bounded subset B of S, the topology $w(S, S')$ coincides on B with $w(E, E')$. In general, $w(S, S')$ is at least as strong as the restriction to S of $w(E, E')$, and if E is locally convex then the Hahn–Banach theorem guarantees that the two topologies on S coincide. Thus every Fréchet space is pseudo-Fréchet. On the other hand, it follows immediately from [1], Corollary 5.3, that every non-locally convex F-space has a subspace S for which $w(S, S')$ is properly stronger than the restriction to S of $w(E, E')$. So it is not obvious that any non-locally convex pseudo-Fréchet spaces exist. Moreover, the simplest non-locally convex F-spaces — the sequence spaces l^p for $0 < p < 1$ — are not pseudo-Fréchet, as we will soon see; and neither are the Hardy spaces H^p of analytic functions for $0 < p < 1$.

In order to provide non-trivial (i.e., non-locally convex) examples of pseudo-Fréchet spaces we study the notion of a shrinking basis for an F-space. We call a basis for an F-space *shrinking* if each of its bounded block bases tends weakly to zero (cf. [9], Theorem 4.2, for the Banach space case). We say a basic sequence is shrinking if it is a shrinking basis for its closed linear span. The usual Banach space arguments ([9], Theorem 4.2, or [4], Chapter III, §3), appropriately generalized, show that a basis (e_n) for an F-space E is shrinking if and only if its coordinate functionals (e_n') span a dense linear subspace of E' if and only if (e_n') is a basis for E'; where E' is given the strong topology (uniform convergence on bounded sets). We will not need these alternate characterizations in this paper, so we omit their proofs.

We call a basis (e_n) for an F-space E *hyper-shrinking* if every bounded block basis for (e_n) tends to zero *in the weak topology of its closed linear* *span*; or equivalently, if every block basis for (e_n) is shrinking. Clearly, every hyper-shrinking basis is shrinking, and the converse holds for pseudo-Fréchet spaces (it seems unlikely that the converse should hold for general F-spaces, but we have not been able to find a counter-example). In this section we show that every F-space with a hyper-shrinking basis is pseudo-Fréchet, and we use this result to construct examples of pseudo-Fréchet spaces that are not locally convex.

We call an F-space *pseudo-reflexive* if it has enough continuous linear functionals to separate points, and every bounded subset is relatively compact in the weak topology of its closed linear span. It is easy to see that every pseudo-reflexive F-space is pseudo-Fréchet. It follows from standard results ([3], §23, Sec. 5, p. 303) that a Fréchet space is pseudo-reflexive if and only if it is reflexive. We show that an F-space with a basis is pseudo-reflexive if and only if the basis is boundedly complete and hyper-shrinking (a basis (e_n) is *boundedly complete* if the series $\sum a_n e_n$ converges whenever (a_n) is a scalar sequence for which the collection of partial sums $\{\sum_{n=1}^{N} a_n e_n\colon N = 1, 2, \ldots\}$ is bounded). This generalizes a result of James for Banach spaces [4], Chapter V, §2, Theorem 2, and allows us to construct examples of non-locally convex pseudo-reflexive F-spaces.

Before getting to the proofs we note some simple properties of equivalent basic sequences. If (x_n) and (y_n) are equivalent basic sequences in F-spaces there is a linear homeomorphism $T\colon \overline{\operatorname{lin}}(x_n)\to\overline{\operatorname{lin}}(y_n)$ such that $Tx_n = y_n$. The following lemma is then immediate.

LEMMA 4.1. *Suppose (x_n) and (y_n) are equivalent basic sequences in F-spaces:*

(i) *If (x_n) tends to zero in the weak topology of its closed linear span, then so does (y_n).*

(ii) *If (x_n) is of type P^*, then so is (y_n).*

We begin our study of pseudo-Fréchet spaces with the promised non-examples.

PROPOSITION 4.2. *l^p is not a pseudo-Fréchet space for $0 < p < 1$.*

Proof. Fix $0 < p < 1$, let $(e_n)_1^\infty$ be the standard unit vector basis for l^p, and let

$$\|f\|_p^p = \sum_{n=1}^{\infty} |f(n)|^p$$

or $f = (f(n))_1^\infty \in l^p$. Since the pairing

$$\langle f, g\rangle = \sum_{n=1}^{\infty} f(n)g(n) \qquad (f\in l^p,\ g\in l^\infty)$$

identifies l^∞ as the dual of l^p, the basis (e_n) does not tend weakly to zero in l^p. Now it is easy to find a block basis (f_k) for (e_n) such that $\|f_k\|_p = 1$ for all k, but $\|f_k\|_1 \to 0$. In particular, (f_k) tends to zero weakly in l^p. But, clearly, (f_k) is a basic sequence equivalent to (e_k), so (f_k) does *not* tend to zero in the weak topology of its closed linear spans. Thus $w(l^p, l^\infty)$ does not coincide with $w(S, S')$ on the bounded subset (f_k) of S, hence l^p is not pseudo-Fréchet; and the proof is complete.

Note that every closed subspace of a pseudo-Fréchet space is again pseudo-Fréchet. In [7], Sec. 4, Prop. 1, it was observed that the Hardy spaces H^p of analytic functions in the unit disc contain a subspace isomorphic to l^p $(0 < p < 1)$. In particular, H^p is not pseudo-Fréchet for $0 < p < 1$.

In order to move toward more positive results we require two simple lemmas, both of which are known for Banach spaces.

LEMMA 4.3. *Suppose E is an F-space with basis (e_n), and let γ denote the topology induced on E by the coordinate functionals of the basis. Then (e_n) is shrinking if and only if γ coincides with $w(E, E')$ on every bounded subset of E.*

Proof. Every block basis for (e_n) is γ-convergent to zero, so certainly (e_n) is shrinking whenever γ coincides with $w(E, E')$ on bounded sets.

Conversely, suppose (e_n) is shrinking; it is enough to show that if x_n is bounded and $x_n \to 0(\gamma)$, then $x_n \to 0$, $w(E, E')$.

If

$$x_n = \sum_{k=1}^{\infty} t_{n,k} e_k$$

and $|\psi(x_n)| \geq \varepsilon$ for some $\psi \in E'$, then by a gliding hump argument (see [6], p. 52) we find increasing sequences m_n, p_n so that

$$\Big\| x_{m_n} - \sum_{k=p_n+1}^{p_{n+1}} t_{m_n,k} e_k \Big\| \leq \frac{1}{n}$$

(where $\|\cdot\|$ is an F-norm determining the topology on E). The sequence $\{\sum_{p_n+1}^{p_{n+1}} t_{m_n,k} e_k\}$ is a block basic sequence and is bounded since the partial-sum operators

$$S_n x = \sum_{i=1}^{n} t_i e_i, \quad \text{where} \quad \sum_{i=1}^{\infty} t_i e_i = x,$$

are equicontinuous. Hence

$$\lim_{n\to\infty} \sum_{k=p_n+1}^{p_{n+1}} t_{m_n,k} e_k = 0, \qquad w(E, E').$$

It follows that $\lim_{n\to\infty} \psi(x_{m_n}) = 0$, contrary to the assumption.

LEMMA 4.4 (cf. [8], Theorem 12.2, p. 369, for Banach spaces). *Suppose E is an F-space with a basis (e_n). Then (e_n) is hyper-shrinking if and only if no bounded block basis for (e_n) is of type P^*.*

Proof. If (e_n) is hyper-shrinking, then every bounded block basis tends to zero in the weak topology of its closed linear span, hence cannot be of type P^*. Conversely, suppose (e_n) is *not* hyper-shrinking, so there exists a bounded block basis (f_k) which does not tend to zero in the weak topology of $S = \overline{\mathrm{lin}}(f_k)$. By passing to a subsequence if necessary, we may assume that there exists $\psi \in S'$ with $\inf_k |\psi(f_k)| > 0$. Thus the vectors $f_k/\psi(f_k)$ form a bounded block basis for (e_n) of type P^*, and the proof is complete.

We now give our main criteria for an F-space to be pseudo-Fréchet.

THEOREM 4.5. *Every F-space with a hyper-shrinking basis is pseudo-Fréchet.*

Proof. Suppose E is an F-space with hyper-shrinking basis (e_n), S is a subspace of E, and B is a bounded subset of S. We want to show that $w(E, E')$ coincides on B with $w(S, S')$. Suppose otherwise, i.e., suppose $w(S, S')$ is properly stronger on B than $w(E, E')$. By Lemma 4.3 the coordinate topology γ agrees on B with $w(E, E')$, and is therefore properly weaker than $w(S, S')$. Since γ is metrizable, it follows that there is a γ-convergent sequence in B that is not $w(S, S')$-convergent. After translating this sequence by its γ-limit (which by definition lies in B, hence in S) we arrive at a bounded sequence in S which is γ-convergent to zero but not $w(S, S')$-convergent. By passing to a subsequence if necessary we may further assume that our sequence is $w(S, S')$-regular, hence regular for the original topology of E. By Theorem 2.1, this sequence contains an M-basic subsequence (b_n): thus (b_n) is a bounded M-basic sequence in S that is γ-convergent to zero, but $w(S, S')$-regular.

By a gliding hump argument ([6], p. 52) there is a subsequence (b_{n_k}) and a block basis (x_k) for (e_n) such that $\sum \|b_{n_k} - x_k\| < \infty$, where $\|\cdot\|$ is an F-norm inducing the topology of E. According to [1], Lemma 4.3 and its proof, (b_{n_k}) is therefore a basic sequence equivalent to (x_k). Thus Lemma 4.1 and the remarks preceding it show that (x_k) is bounded but not convergent to zero in the weak topology of its closed linear span, which contradicts the fact that (e_n) is hyper-shrinking. Thus $w(S, S')$ coincides on B with $w(E, E')$, and the proof is complete.

We can at last give examples of non-locally convex pseudo-Fréchet spaces. For $0 < p < q$ and $f = (f(n))_1^\infty$ a complex sequence, let

$$\|f\|_{p,q} = \Big\{ \sum_{n=0}^{\infty} \Big\{ \sum_{2^n \leqslant k < 2^{n+1}} |f(k)|^p \Big\}^{q/p} \Big\}^{1/q}$$

when $q < \infty$, and let

$$\|f\|_{p,\infty} = \sup_n \Big\{ \sum_{2^n \leqslant k < 2^{n+1}} |f(k)|^p \Big\}^{1/p}.$$

Define $l^q(p)$ to be the collection of sequences f such that $\|f\|_{p,q} < \infty$, and let $c_0(p)$ denote those members of $l^\infty(p)$ for which

$$\lim_n \sum_{2^n \leqslant k < 2^{n+1}} |f(k)|^p = 0.$$

For $p \geqslant 1$ the functional $\|\cdot\|_{p,q}$ is a norm which makes $l^q(p)$ into a Banach space. For $0 < p < 1$, $\|\cdot\|_{p,q}$ is a quasi-norm in the sense of [3], p. 159, and the sets

$$\{f \in l^q(p) \colon \|f\|_{p,q} < \varepsilon\} \quad (\varepsilon > 0)$$

form a local base for a complete, Hausdorff, locally bounded topology on $l^q(p)$. So in any case $l^q(p)$ is a locally bounded F-space in the topology induced by $\|\cdot\|_{p,q}$. Now $c_0(p)$ is easily seen to be a closed subspace of $l^\infty(p)$, so it is also a locally bounded F-space.

PROPOSITION 4.6. *$l^q(p)$ and $c_0(p)$ are not locally convex if $0 < p < 1$.*

Proof. We need only find a bounded set whose convex hull is unbounded. Define f_k by

$$f_k(n) = \begin{cases} 2^{-k} & \text{if } \quad 2^k \leqslant n < 2^{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

for $k = 0, 1, 2, \ldots$ Then each f_k is a convex combination of the standard unit vectors (e_n), where

$$e_n(m) = \delta_{nm}$$

for $n = 1, 2, \ldots$ Moreover, for $0 < p < q \leqslant \infty$:

$$\|f_k\|_{p,q} = 2^{k(1/p-1)} \quad (k = 0, 1, 2, \ldots)$$

and $\|e_n\|_{p,q} = 1$ for all n. Thus (e_n) is bounded in $l^q(p)$ and $c_0(p)$, but when $0 < p < 1$, its convex hull is not. This completes the proof.

Note that the standard unit vectors (e_n) defined in the above proof form a *basis* for $l^q(p)$ and $c_0(p)$ when $q < \infty$.

THEOREM 4.7. *(e_n) is a hyper-shrinking basis for $l^q(p)$ $(1 < q < \infty)$ and $c_0(p)$.*

Proof. Suppose (f_k) is a block basis for (e_n), say

$$f_k = \sum_{n_k \leqslant n < n_{k+1}} a_n e_n,$$

where $1 \leqslant n_1 < n_2 < \ldots$, and (a_n) is a scalar sequence. Choose integers

$0 \leqslant p_1 < p_2 < \ldots$ and a subsequence (k_j) such that

$$2^{p_j} \leqslant n_{k_j} < n_{k_{j+1}} < 2^{p_{j+1}}.$$

Then the vectors $g_j = f_{k_j}$ $(j = 1, 2, \ldots)$ form a block basis for (e_n), the jth member of which is "supported" on the integers $2^{p_j} \leqslant n < 2^{p_j+1}$, i.e.,

$$g_j = \sum_{2^{p_j} \leqslant n < 2^{p_{j+1}}} b_n e_n \quad (j = 1, 2, \ldots)$$

for an appropriate scalar sequence (b_n). Now if $q < \infty$ and (t_j) is a scalar sequence, then letting $q_j = p_{j+1} - p_j - 1$ we have

$$\begin{aligned}\Big\| \sum_j t_j g_j \Big\|_{p,q}^q &= \Big\| \sum_j \sum_{2^{p_j} \leqslant n < 2^{p_{j+1}}} b_n e_n \Big\|_{p,q}^q \\ &= \sum_j \sum_{0 \leqslant k \leqslant q_j} \Big\{ \sum_{2^{p_j+k} \leqslant n < 2^{p_j+k+1}} |t_j b_n|^p \Big\}^{q/p} \\ &= \sum_j |t_j|^q \sum_{0 \leqslant k \leqslant q_j} \Big\{ \sum_{2^{p_j+k} \leqslant n < 2^{p_j+k+1}} |b_n|^p \Big\}^{q/p} \\ &= \sum_j |t_j|^q \|g_j\|_{p,q}.\end{aligned}$$

In particular, if (f_k) is regular and bounded, then (g_j) is equivalent to the standard unit vector basis of l^q. Now since $1 < q < \infty$, this latter basis tends weakly to zero in l^q, hence by Lemma 4.1, (g_j) tends to zero in the weak topology of its closed linear span. Thus every bounded block basis for (e_n) tends to zero in the weak topology of its closed linear span, hence (e_n) is a hyper-shrinking basis for $l^q(p)$.

For $c_0(p)$ a calculation similar to the one above shows that every bounded regular block basis for (e_n) has a subsequence equivalent to the standard unit vector basis of c_0, which is a shrinking basis. By the argument just given, (e_n) is a hyper-shrinking basis for $c_0(p)$, and the proof is complete.

Corollary 4.8. *$l^q(p)$ is a non-locally convex pseudo-Fréchet space for $0 < p < 1 < q < \infty$. The same is true of $c_0(p)$ for $0 < p < 1$.*

Proof. The result follows immediately from Theorem 4.5, Proposition 4.6, and Theorem 4.7.

We next turn to pseudo-reflexive F-spaces. To set the stage for our main result recall that a Banach space with a basis is reflexive if and only if the basis is boundedly complete and shrinking ([4], Chapter V, §2, Theorem 2).

Theorem 4.9. *An F-space with a basis is pseudo-reflexive if and only if the basis is boundedly complete and hyper-shrinking.*

Proof. Let E be an F-space with basis (e_n), and let γ denote the topology induced by the coordinate functionals for this basis. It is not difficult to see that (e_n) is boundedly complete if and only if every bounded subset of E is relatively γ-compact (see [7], Lemma 1, p. 1081, for a proof).

Now suppose E is pseudo-reflexive. We will show that (e_n) is boundedly complete. Let B be a bounded subset of E, and let S be the closed linear span of B. Then the $w(S, S')$-closure C of B is $w(S, S')$-compact, hence γ-compact since γ is Hausdorff and $\leqslant w(S, S')$ on S. It follows easily that C is also the γ-closure of B, so B is relatively γ-compact, hence (e_n) is boundedly complete. Note for future reference that $\gamma = w(S, S')$ on C, hence on B.

To see that (e_n) is hyper-shrinking suppose that (f_k) is a bounded block basis for (e_n) and let $S = \overline{\operatorname{lin}}(f_k)$. By the above remark, $\gamma = w(S, S')$ on (f_k). Clearly, (f_k) is γ-convergent to zero, hence $w(S, S')$-convergent to zero, so (e_n) is hyper-shrinking.

Conversely, suppose (e_k) is hyper-shrinking and boundedly complete. If E is not pseudo-reflexive, then there is a bounded subset B that is not relatively $w(S, S')$-compact, where $S = \overline{\operatorname{lin}} B$. Since (e_n) is boundedly complete, the γ-closure C of B is γ-compact. We claim that $C \nsubseteq S$. Indeed, C is bounded in E, since the original topology of E is γ-polar (this follows easily from the fact that (e_n) is a basic). Now if C were contained in S we would have $\gamma = w(S, S')$ on C because (e_n) is hyper-shrinking (4.3 and 4.5), so C would be $w(S, S')$-compact, hence B would be relatively $w(S, S')$-compact: a contradiction. Thus there exists a vector $b \in C \setminus S$, and since γ is metrizable, there is a sequence (b_n) in B that is γ-convergent to b.

Now the sequence $(b - b_n)$ is bounded, γ-convergent to zero, and regular, so it follows as in the proof of Theorem 4.5 that there is a subsequence equivalent to a block basis (s_n) for (e_n). We may as well assume this subsequence is $(b - b_n)$ itself. We claim that $(b - b_n)$ is of type P^*. To see this, define a linear functional φ on $T = \operatorname{lin}(S, b)$ by letting $\varphi = 0$ on S, and $\varphi(b) = 1$. Now T is closed in E ([3], §15, sec. 5, p. 152), hence $T \supseteq \overline{\operatorname{lin}}(b - b_n)$. Moreover, φ is continuous on T, since $\ker\varphi = S$ is closed in T; and finally, $\varphi(b - b_n) = 1$ for all n. Thus $(b - b_n)$ is a basic sequence of type P^* which is bounded in E, hence (x_n) is a bounded block basis of type P^*. By Lemma 4.4, (e_n) is not hyper-shrinking: a contradiction. Thus E is pseudo-reflexive, and the proof is complete.

Corollary 4.10. *$l^q(p)$ is pseudo-reflexive for $0 < p < q < \infty$. $c_0(p)$ is not pseudo-reflexive $(0 < p < \infty)$.*

Proof. We observed in Theorem 4.7 that the standard unit vector basis (e_n) is hyper-shrinking for all the spaces mentioned above. It is easy to see that it is also boundedly complete for $l^q(p)$, but not for $c_0(p)$. By Theorem 4.9 the proof is complete.

Since $l^q(p)$ and $c_0(p)$ are not locally convex when $0 < p < 1$, we have:

Corollary 4.11. *There exist pseudo-reflexive locally bounded F-spaces that are not locally convex. There exist non-locally convex locally bounded pseudo-Fréchet spaces that are not pseudo-reflexive.*

A number of results in [1], Sec. 5, deal with vector topologies on an F-space compatible with (i.e., having the same closed subspaces as) the original topology. The Hahn–Banach theorem guarantees that the weak topology of a locally convex space is compatible with the original one, but it follows from [1], Corollary 5.3, that this fails in *every* non-locally convex F-space. So it is not obvious that a non-locally convex F-space can have a weaker compatible vector topology.

Our next result shows that every locally bounded, pseudo-reflexive F-space does have such a topology: the bounded weak topology. The *bounded weak topology* on an F-space E is the strongest topology on E that agrees with the weak topology on bounded sets.

Theorem 4.12. *The bounded weak topology of a locally bounded, pseudo-reflexive F-space is a vector topology compatible with the original one.*

Proof. Let β denote the bounded weak topology on the locally bounded, pseudo-reflexive F-space E. Since every bounded subset of E is weakly relatively compact, it follows from [2], Proposition 3.3, or [10], Proposition 6.2, p. 48, that β is a vector topology. To see that β is compatible with the original topology of E, suppose S is a closed subspace of E: we will show that S is β-closed, that is, $S \cap B$ is relatively weakly closed in B for every bounded subset B of E. Indeed, $B \cap S$ is $w(S, S')$-relatively compact, so its $w(S, S')$-closure C is $w(S, S')$-compact, hence $w(E, E')$-compact. Recall that every pseudo-reflexive space has, by definition, a Hausdorff weak topology; so C is $w(E, E')$-closed. Since $B \cap S = B \cap C$, we see that $B \cap S$ is $w(E, E')$-closed in B, which completes the proof.

We remark that the bounded weak topology on a Hausdorff locally bounded space coincides with the original topology only when the space is finite dimensional. For if the two topologies coincide, then the space has a compact neighbourhood of zero, and must therefore be finite dimensional ([3], §15, Sec. 7, p. 155). In particular, the bounded weak topology on the space $l^q(p)$ for $0 < p < 1 < q < \infty$ is strictly weaker than, yet compatible with, the original topology.

We close with an application of Theorem 4.12 to basis theory. In [1], Theorem 5.5, it is shown that if a sequence in an F-space is a basis for a weaker vector topology compatible with the original one, then it is also a basis for the original topology. This, along with Theorem 4.12, yields the following "bounded weak basis theorem":

Corollary 4.13. *In a locally bounded, pseudo-reflexive F-space every bounded weak basis is a basis.*

This result contrasts sharply with the main result of [7] which states that if a locally bounded, non-locally convex F-space has a weak basis, then it has a weak basis that is not a basis.

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