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Unconditional and normalised bases

by

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1. Introduction. A Schauder basis (x_n) of a locally convex space E is unconditional if, whenever $\sum_{i=1}^{\infty} a_i x_i$ converges, the convergence is unconditional. In [16], Pełczyński and Singer proved that every Banach space with a basis possesses a conditional (i.e. not unconditional) basis. In this paper I shall generalise this theorem using the concept of normalisation introduced in [12].

A sequence (x_n) is *regular* if there is a neighbourhood V of zero with $x_n \notin V$ for all n; a regular bounded sequence is said to be *normalised*. If there exists a scalar sequence (a_n) with $(a_n x_n)$ normalised, then (x_n) is said to be *normal*; otherwise (x_n) is abnormal.

If (x_n) is a Schauder basis of E, then (f_n) will always denote its dual sequence in E'; if $(f_n)_{n=1}^{\infty}$ is equicontinuous, then (x_n) is *equi-regular*, and hence regular; if E is barrelled, then any regular basis is equi-regular. The sequence space of all a such that $\sum_{i=1}^{\infty} a_i x_i$ converges will be denoted by λ_x, and μ_x is the sequence space $\{(f(x_n))_{n=1}^{\infty};\ f \in E'\}$. If E is sequentially complete, then (x_n) is unconditional if and only if λ_x is solid (see [4]), that is if $a \in \lambda_x$ and $|\theta_n| \leqslant 1$ for all n, then $(\theta_n a_n) \in \lambda_x$. If E is also barrelled, it can be shown that the topology on E may be given by a collection of solid semi-norms p such that

$$p(x) = \sup_{|\theta_i| \leqslant 1} p\Big(\sum_{i=1}^{\infty} \theta_i f_i(x) x_i\Big).$$

A sequentially complete barrelled space with a Schauder basis is complete (see [10]); in this paper I shall restrict attention almost exclusively to complete barrelled spaces.

2. Reflexivity and unconditional bases. A Schauder basis (x_n) is γ-complete or boundedly-complete if whenever $\big(\sum_{i=1}^{n} a_i x_i;\ n = 1, 2 \ldots\big)$ is bounded, then $\sum_{i=1}^{\infty} a_i x_i$ converges; it is shrinking if (f_n) is a basis for E' in its strong topology. It is shown in [20] that E is semi-reflexive if and only if (x_n) is γ-complete and shrinking. The following results generalise those of [9] and [19]:

THEOREM 2.1. *Let E be a complete barrelled space with an unconditional Schauder basis (x_n); if (x_n) is not γ-complete, then E contains a complemented subspace G isomorphic to c_0.*

There exists a sequence (a_a) such that $(\sum_{i=1}^{n} a_i x_i)_{n=1}^{\infty}$ is bounded, but does not converge; thus there exists an increasing sequence (n_j) with $n_0 = 0$, and a neighbourhood V of zero such that, if

$$y_j = \sum_{n_{j-1}+1}^{n_j} a_i x_i,$$

then $y_j \notin V$. Let p be a solid semi-norm, then

$$p\Big(\sum_{i=1}^{n} \beta_i y_i\Big) \leqslant \|\beta\|_\infty p\Big(\sum_{i=1}^{n} y_i\Big),$$

where $\|\beta\|_\infty = \sup_n |\beta_n|$. However, $\big(\sum_{i=1}^{n} y_i;\ n = 1, 2, \ldots\big)$ is bounded, and thus

$$p\Big(\sum_{i=1}^{n} \beta_i y_i\Big) \leqslant K\|\beta\|_\infty.$$

Therefore $c_0 \subset \lambda_y$; but as (y_n) is regular $\lambda_y \subset c_0$, so that $\lambda_y = c_0$.

Let q be a solid continuous semi-norm on E such that $q(y_j) \geqslant 1$ for all j; if $E_j = \operatorname{lin}(x_{n_{j-1}+1}, \ldots, x_{n_j})$ then there exists a linear functional h_j on E_j such that $h_j(y_j) = 1$, and $|h_j(x)| \leqslant q(x)$ for $x \in E_j$. Define $g_j \in E'$ by

$$g_j(x) = h_j\Big(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\Big);$$

then $|g_j(x)| \leqslant q(x)$. Then $(g_j)_{j=1}^{\infty}$ is equicontinuous, and so possesses a $\sigma(E', E)$ cluster point g; obviously $g(x_j) = 0$ for all j, and so $g = 0$. As zero is the sole cluster point of (g_j) it follows that $\lim_{j\to\infty} g_j = 0$ weakly.

Let

$$T_k x = \sum_{i=1}^{k} g_i(x) y_i;$$

then each T_k is continuous and $\lim_{k\to\infty} T_k x = Tx$ exists for each x. Therefore by the Banach-Steinhaus Theorem for barrelled spaces, T is a continuous projection of E onto $G = \overline{\operatorname{lin}}(y_j)$. As G is complemented in E, G is barrelled; and in (y_j) is a Schauder basis of G with $\lambda_y = c_0$, it follows that $\mu_y = c_0^\beta = l_1$, and so $\mathscr{G} \cong c_0$.

THEOREM 2.2. *Let E be a complete barrelled space with an unconditional Schauder basis (x_n); if (x_n) is not shrinking, then E has a complemented subspace $G \cong l^1$.*

As (x_n) is not shrinking, there exists $f \in E'$, and a bounded block basic sequence (y_j) such that $f(y_j) = 1$ (see [12], Theorem 5.4). If $\sum_{j=1}^{\infty} a_j y_j$ converges, then $\sum_{j=1}^{\infty} |a_j|$ converges, and as (y_j) is bounded $\lambda_y = l^1$. Let $G = \overline{\operatorname{lin}}(y_n)$, and define the norm p on G by

$$p\Big(\sum_{i=1}^{\infty} a_i y_i\Big) = \sum_{i=1}^{\infty} |a_i|,$$

then

$$p\Big(\sum_{i=1}^{\infty} a_i y_i\Big) = \sum_{i=1}^{\infty} |a_i f(y_i)| \leqslant \sum_{i=1}^{\infty} |f_i(y) f(x_i)|,$$

where $y = \sum_{i=1}^{\infty} a_i y_i$. As the topology on E may be given by solid semi-norms if

$$q(x) = \sum_{i=1}^{\infty} |f_i(x) f(x_i)|,$$

then q is continuous; thus p is continuous, and $G \cong l^1$.

For $x \in E$, let

$$Tx = \sum_{j=1}^{\infty} f\Big(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\Big) y_j, \quad \text{where } y_j = \sum_{n_{j-1}+1}^{n_j} \beta_i x_i;$$

for

$$\sum_{i=1}^{\infty} \Big| f\Big(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\Big)\Big| \leqslant q(x).$$

Then T is a projection of E onto G, and $p(Tx) \leqslant q(x)$ so that T is continuous.

As E is semi-reflexive if and only if (x_n) is shrinking and γ-complete, the following theorem is immediate:

THEOREM 2.3. *If E is a complete barrelled space with an unconditional Schauder basis, then E is reflexive if and only if E possesses no complemented subspace isomorphic to c_0 or l^1.*

3. Symmetric bases. Two basic sequences (x_n) and (y_n) are said to be *equivalent* if $\lambda_x = \lambda_y$; it can easily be seen that if (x_n) is a Schauder basis of E, and (y_n) is a Schauder basis of F, and both E and F are barrelled, then E and F are isomorphic. Suppose (x_n) is a Schauder basis of E, such that for every permutation σ of the positive integers Z, $(x_{\sigma(n)})$ is a Schauder basis of E equivalent to (x_n); then (x_n) is said to be *symmetric*. Symmetric bases of Banach spaces were introduced and studied by Singer [17] and [18]; in locally convex spaces they have been studied by Garling [6] and [7]. The definition here corresponds to condition SB_3 of [7] and [18].

A symmetric basis is necessarily unconditional, as for all $x \in E$, and permutations σ, $\sum_{i=1}^{\infty} f_{\sigma(i)}(x) x_{\sigma(i)}$ converges to x. The following lemma is essentially established in [3]:

LEMMA 3.1. *If (x_n) is a symmetric Schauder basis of E, then either (x_n) is bounded, or (x_n) is a Hamel basis of E.*

THEOREM 3.2. *If E is a complete barrelled space with a symmetric Schauder basis (x_n), then either $E \cong \omega$, or $E \cong \varphi$ or (x_n) is normalised.*

(ω is the space of all sequences, and φ is the dual sequence space of all sequences eventually equal to zero; ω has the topology $\beta(\omega, \varphi)$ $(= \sigma(\omega, \varphi))$ and φ has the topology $\beta(\varphi, \omega)$).

Let σ be a permutation of Z, and let $\tau = \sigma^{-1}$; suppose $\sum_{i=1}^{\infty} a_i f_i$ converges weakly to f. Let

$$g_n = \sum_{i=1}^{\infty} a_{\tau(i)} f_i;$$

then

$$g_n(x) = \sum_{i=1}^{n} f(x_{\tau(i)}) f_i(x) = f\Big(\sum_{i=1}^{n} f_i(x) x_{\tau(i)}\Big);$$

as (x_n) is symmetric, $\lim_{n\to\infty} g_n(x) = g(x)$ exists, and by the Banach-Steinhaus Theorem, g is continuous. Obviously $g(x_i) = a_{\tau(i)}$, and so

$$g = \sum_{i=1}^{\infty} a_{\tau(i)} f_i;$$

as (f_n) is an unconditional basis of $\big(E', \sigma(E', E)\big)$,

$$g = \sum_{i=1}^{\infty} a_i f_{\sigma(i)}$$

and so (f_n) is a symmetric basis of E' in its weak topology.

By Lemma 3.1, either (f_n) is a Hamel basis of E' or is bounded; in the latter case (x_n) is equi-regular. If E' has countable dimension, then $E \cong \varphi$; if E has countable dimension $E \cong \varphi$; otherwise (x_n) is regular and bounded, i.e. normalised.

Eliminating the cases of ω and φ, symmetric bases of complete barrelled spaces may be treated much like symmetric bases of Banach spaces.

The following theorem follows from the results of Cac [3]:

THEOREM 3.3. *If E is a complete barrelled space with a normalised symmetric Schauder basis (x_n), then the topology on E may be given by a collection of symmetric norms p, satisfying*

$$p(x) = \sup_{|\theta_i|\leqslant 1} \sup_{\tau\in\Pi} p\Big(\sum_{i=1}^{\infty} \theta_i f_i(x) x_{\tau(i)}\Big),$$

where π is the group of all permutations of Z.

Cac's result is essentially that for fixed x

$$\Big\{\sum_{i=1}^{\infty} \theta_i f_i(x) x_{\tau(i)};\ \tau\in\Pi,\ |\theta_i|\leqslant 1\Big\}$$

is a bounded set. For each sequence (θ_i) with $|\theta_i| \leqslant 1$ and $\tau \in \Pi$, the map

$$x \to \sum_{i=1}^{\infty} \theta_i f_i(x) x_{\tau(i)}$$

is continuous by an application of the Banach-Steinhaus theorem, and so this collection of maps is equicontinuous (E is barrelled). The result then follows at once.

If (x_n) is a symmetric Schauder basis of E, where E is complete and barrelled, then a *k-block* is an element $u(K) = \sum_{i\in K} x_i$, where K is a subset of Z with k members; two blocks $u(K_1)$ and $u(K_2)$ are disjoint if $K_1 \cap K_2 = \emptyset$. Then the following theorems generalise results of Lindenstrauss and Zippin [15]:

THEOREM 3.4. *If $(u(K_n))$ is a sequence of disjoint k_n-blocks, the averaging projection*

$$Tx = \sum_{n=1}^{\infty} \frac{1}{k_n}\Big\{\sum_{i\in K_n} f_i(x)\Big\} u(K_n)$$

is a well-defined continuous operator on E.

THEOREM 3.5. *If (k_n) is a sequence with $k_n > 1$ for all n, then E possesses an unconditional Schauder basis (y_n) with a subsequence (y_{n_j}) of disjoint k_j-blocks.*

The proofs of both these theorems are almost identical to the proofs of the original results for Banach spaces in [15]; all the calculations may be carried out with individual symmetric norms.

One further property of symmetric norms will be required. Let E be a 2^n-dimensional vector space and let $(x_i)_{i=1}^{2^n}$ be a basis of E; then following Pełczyński and Singer [16], one may define the Haar system $(y_i)_{i=1}^{2^n}$ of (x_i) by

$$y_1 = \sum_{i=1}^{2^n} x_i, \quad y_{2^k+s} = \sum_{i=1}^{2^n} \beta_i(k,s) x_i,$$

where

$$\beta_i(k,s) = \begin{cases} 1 & \text{if } (2s-2)2^{n-k-1}+1 \leqslant i \leqslant (2s-1)2^{n-k-1}, \\ -1 & \text{if } (2s-1)2^{n-k-1}+1 \leqslant i \leqslant 2s\cdot 2^{n-k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The Rademacher system $(z_i)_{i=1}^{n}$ is given by

$$z_k = \sum_{s=1}^{2^{k-1}} y_{2^{k-1}+s}.$$

The following results are proved in [16]:

PROPOSITION 3.6. (i) *$(y_i)_{i=1}^{2^n}$ is a basis of E, and for any norm p which is symmetric with respect to (x_i), and any sequence $(\alpha_i)_{i=1}^{2^n}$*

$$p\Big(\sum_{i=1}^{k} \alpha_i y_i\Big) \leqslant p\Big(\sum_{i=1}^{2^n} \alpha_i y_i\Big) \quad \text{for } k \leqslant 2^n.$$

(ii) *$(z_i)_{i=1}^{n}$ is a block basic sequence with respect to $(y_i)_{i=1}^{2^n}$, and for any symmetric p, and sequence $(\alpha_i)_{i=1}^{n}$*

$$p\Big(\sum_{i=1}^{n} \alpha_i \frac{z_i}{p(z_i)}\Big) \geqslant \frac{1}{8}\Big(\sum_{i=1}^{n} |\alpha_i|^2\Big)^{1/2}.$$

4. The existence of conditional bases. For convenience, I shall define a J-space as a complete barrelled space with a normalised Schauder basis, and which has the property that any two normalised Schauder bases are equivalent. I do not know whether any J-space exists, although it has been shown in [16] that there is no Banach J-space. The techniques employed in this section stem largely from those of [16].

PROPOSITION 4.1. *If E is a J-space, then:*

(i) *any normalised Schauder basis of E is symmetric,*

(ii) *E is reflexive,*

(iii) *E' is a J-space in its strong topology.*

(i) If (x_n) is a normalised Schauder basis of E, and θ_n is any sequence with $|\theta_n| = 1$, then $(\theta_n x_n)$ is a normalised Schauder basis equivalent to (x_n); thus $\alpha \in \lambda_x$ if and only if $(\theta_n \alpha_n) \in \lambda_x$, so that λ_x is solid. Thus (x_n) is unconditional, and for any π, a permutation of Z, $(x_{\pi(n)})$ is a basis equivalent to (x_n).

(ii) This follows from Theorem 2.3, as otherwise $E = c_0 \oplus F \cong c_0 \oplus c_0 \oplus F \cong c_0 \oplus E$ or $E \cong l^1 \oplus E$, so that by combining a conditional normalised basis of c_0 or l^1 (see [8] and [14]) with a normalised Schauder basis of E, one obtains a conditional normalised Schauder basis of E, contradicting (i).

(iii) Let (f_n) be a normalised Schauder basis of E', with dual (x_n) in E; then (x_n) is also normalised (see [12], Theorem 3.4). If (g_n) is any other normalised Schauder basis of E' with dual (y_n) then $\lambda_x = \lambda_y$, and so $\lambda_f = \mu_x = \lambda_x^\beta = \lambda_y^\beta = \mu_y = \lambda_g$ as E is barrelled.

If a J-space E exists, then certainly E' is not isomorphic to E, for if it were, $\lambda_x = \mu_x$ for any normalised basis (x_n) of E; thus $\lambda_x = \lambda_x^\beta$ and so $\lambda_x = l^2$, and E is a separable Hilbert space, and this is not a J-space [1].

THEOREM 4.2. *Let E be a complete barrelled space with a normalised Schauder basis; then E is a J-space if and only if every normalised Schauder basis of E is unconditional.*

Certainly, by Proposition 4.1, every normalised basis of a J-space is unconditional. Conversely suppose (x_n) is a normalised Schauder basis of E, and let $E_1 = \overline{\text{lin}}\,(x_{2n-1})_{n=1}^{\infty}$ and $E_2 = \overline{\text{lin}}\,(x_{2n})_{n=1}^{\infty}$; as (x_n) is unconditional, by an application of the Banach-Steinhaus theorem $E = E_1 \oplus E_2$.

Let (y_n) be any normalised Schauder basis of E_1; then if $z_{2n-1} = y_n$ and $z_{2n} = x_{2n}$, (z_n) is a normalised Schauder basis of E, and is thus unconditional. Hence (y_n) is unconditional.

Let $u_{2n-1} = z_{2n-1}$, and $u_{2n} = z_{2n} + z_{2n-1}$; then (u_n) is a block perturbation of (z_n) (see Lemma 4.4 of [13]), and is also normalised, as (z_n) is equi-regular and bounded. Similarly, if $v_{2n} = z_{2n}$ and $v_{2n-1} = z_{2n-1} + z_{2n}$, then (v_n) is a normalised Schauder basis of E.

The maps $Q\colon E \to E$ and $R\colon E \to E$ given by

$$Q\Big(\sum_{i=1}^{\infty} \alpha_i u_i\Big) = \sum_{i=1}^{\infty} \alpha_{2i-1} u_{2i-1} \quad \text{and} \quad R\Big(\sum_{i=1}^{\infty} \alpha_i v_i\Big) = \sum_{i=1}^{\infty} \alpha_{2i} v_{2i}$$

are continuous (the Banach-Steinhaus theorem),

$$-Q\Big(\sum_{i=1}^{\infty} \alpha_i x_{2i}\Big) = -Q\Big(\sum_{i=1}^{\infty} \alpha_i z_{2i}\Big) = \sum_{i=1}^{\infty} \alpha_i (u_{2i-1} - u_{2i})$$

$$= \sum_{i=1}^{\infty} \alpha_i u_{2i-1} = \sum_{i=1}^{\infty} \alpha_i y_i,$$

while

$$-R\Big(\sum_{i=1}^{\infty} a_i y_i\Big) = R\Big(\sum_{i=1}^{\infty} a_i(v_{2i}-v_{2i-1})\Big) = \sum_{i=1}^{\infty} a_i v_{2i} = \sum_{i=1}^{\infty} a_i x_{2i},$$

so that (x_{2n}) is equivalent to (y_n); as E_1 and E_2 are barrelled it follows that $E_1 \cong E_2$, and also that E_1 is a J-space. Further, by the Theorem 3.3, it follows that, as (y_n) is symmetric, (y_n) is equivalent to both (y_{2n-1}) and (y_{2n}); for if p is a symmetric norm with respect to (y_n)

$$p\Big(\sum_{i=1}^{k} a_i y_i\Big) = p\Big(\sum_{i=1}^{k} a_i y_{2i}\Big).$$

Thus $E_1 \cong E_1 \oplus E_1 \cong E_1 \oplus E_2 \cong E$, and so E is a J-space.

THEOREM 4.3. *Let E be a J-space, and let (x_n) be a normalised Schauder basis of E; then if (y_n) is a normalised block basic sequence with respect to (x_n), $\lambda_y = \lambda_x$ and $\overline{\operatorname{lin}}(y_n)$ is complemented in E and isomorphic to E.*

Let

$$y_j = \sum_{n_{j-1}+1}^{n_j} a_i x_i,$$

and let $u_j = y_j - a_{n_j} x_{n_j}$. As E is barrelled, (x_n) is a simple (see [11]) Schauder basis, and the set (u_j) is bounded. Thus one can define the block perturbation (z_n) by $z_i = x_i$, $i \neq n_j$ and $z_i = x_i + u_j$ for $i = n_j$; (z_n) is a normalised Schauder basis of E. Suppose $\beta \in \lambda_x = \lambda_z$; then as (z_i) is equivalent to (z_{n_i}), $\sum_{i=1}^{\infty} \beta_i z_{n_i}$ converges and similarly $\sum_{i=1}^{\infty} \beta_i x_{n_i}$ converges; thus $\sum_{i=1}^{\infty} \beta_i u_i$ converges. As (y_j) is bounded, and (f_n) is equicontinuous, $\sup_j |a_{n_j}| < \infty$, and so $\sum_{i=1}^{\infty} a_{n_i}\beta_i x_{n_i}$ converges; therefore $\sum_{i=1}^{\infty} \beta_i y_i$ converges and so $\lambda_x \subset \lambda_y$.

As (y_n) is regular, there is a continuous symmetric norm p such that $p(y_j) \geqslant 1$ for all j. Let h_j be a linear functional on lin $(x_{n_{j-1}+1}, \ldots x_{n_j})$ such that $h_j(y_j) = 1$ and $|h_j(x)| \leqslant p(x)$; let $g_j \in E'$ be defined by

$$g_j(x) = h_j\Big(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\Big).$$

Then $|g_j(x)| \leqslant p(x)$, and so the set (g_j) is equicontinuous, and thus strongly bounded in E'; as (y_j) is bounded (g_j) is regular, and hence is a normalised block basic sequence with respect to (f_n). As E' is a J-space (Proposition 4.1), $\lambda_g \subset \lambda_f$; however, $\lambda_g \subset \lambda_y^{\beta} \subset \lambda_x^{\beta} = \lambda_f$ and hence $\lambda_f = \lambda_g$. Therefore $\lambda_x = \lambda_f^{\beta} = \lambda_y^{\beta\beta} \supset \lambda_y$ and so $\lambda_x = \lambda_y$.

If $f \in E'$, $f = \sum_{i=1}^{\infty} f(x_i) f_i$ in E', and as $\lambda_f = \lambda_g$, $\sum_{i=1}^{\infty} f(x_i) g_i$ converges. For $x \in E$, $\sum_{i=1}^{\infty} f(x_i) g_i(x)$ converges; as E is reflexive, E is weakly sequentially complete and so $\sum_{i=1}^{\infty} g_i(x) x_i$ converges. As $\lambda_x = \lambda_y$, $\sum_{i=1}^{\infty} g_i(x) y_i$ converges.

Let $T(x) = \sum_{i=1}^{\infty} g_i(x) y_i$; then by the Banach-Steinhaus theorem, T is a continuous projection of E onto $\overline{\operatorname{lin}}(y_n)$.

LEMMA 4.4. *Let E be a J-space, and (x_n) be a normalised Schauder basis of E; then any sequence (v_n) of 2^n-blocks is abnormal.*

Suppose (v_n) is a normal sequence of 2^n-blocks, and suppose (a_i) is a sequence such that $\sum |a_i|^2 = \infty$. Then there is a sequence n_j with $n_0 = 0$ such that

$$\sum_{n_{j-1}+1}^{n_j} |a_i|^2 > 1 \quad \text{for all } j;$$

let $m_0 = 0$ and $m_j = \sum_{i=1}^{j} 2^{n_j - n_{i-1}}$. Let $E_k = \operatorname{lin}(x_{m_{k-1}+1}, \ldots, x_{m_k})$, and let $(y_j)_{m_{k-1}+1}^{m_k}$ and $(z_j)_{n_{k-1}+1}^{n_k}$ be the Haar and Rademacher systems of E_k.

If $x \in E$,

$$\sum_{m_{k-1}+1}^{m_k} f_i(x) x_j = \sum_{m_{k-1}+1}^{m_k} g_i(x) y_i$$

with each g_i continuous on E. If $m_{k-1}+1 \leqslant s \leqslant m_k$, then for any symmetric norm p, by Proposition 3.6,

$$p\Big(\sum_{m_{k-1}+1}^{s} g_i(x) y_i\Big) \leqslant p\Big(\sum_{m_{k-1}+1}^{m_k} f_i(x) x_i\Big).$$

It follows that (y_i) is a Schauder basis of E.

For each symmetric norm p, and each $j, p(y_j) = p(v_{t_j})$ for some t_j; for $(f_k(y_j))_{k=1}^{\infty}$ takes only the values ± 1 and zero, and is non-zero on 2^{t_j} values of k. As (v_j) is normal, (y_j) is normal; similarly, (z_j) is normal. Let p be a symmetric norm such that $w_j = z_j/p(z_j)$ is normalised; then (w_j) is a normalised block basic sequence with respect to (y_j), and so $\lambda_w = \lambda_y = \lambda_z$. As p is symmetric,

$$p\Big(\sum_{n_{k-1}+1}^{n_k} a_i w_i\Big) \geqslant \frac{1}{8} \quad \text{(Proposition 3.6)}.$$

Therefore, $a \notin \lambda_x$ and $\lambda_x \subset l^2$.

Now let $g_i = (\sum f_i;\ i \in K_j)$ be a sequence of disjoint 2^j-blocks with respect to (f_n), and let $u_j = (\sum x_i;\ i \in K_j)$. Then (u_j) is normal as for any symmetric p, $p(u_j) = p(v_j)$. By Theorem 3.4, for any $x \in E$,

$$\sum_{j=1}^{\infty} \frac{1}{2^j} g_j(x) u_j$$

converges; suppose $(\beta_j u_j)$ is normalised, then it follows that

$$\lim_{j\to\infty} \frac{1}{2^j \beta_j} g_j(x) = 0$$

so that $\left((1/2^j \beta_j) g_j\right)$ is bounded in E' (weakly and strongly).

As

$$\frac{1}{2^j \beta_j} g_j(\beta_j u_j) = 1,$$

it follows that $\left((1/2^j \beta_j) g_j\right)$ is regular, and so (g_j) is normal.

By the first part applied to E', $\lambda_f \subset l^2$ so that $l^2 \subset \lambda_x \subset l^2$, i.e. $\lambda_x = l^2$. However l^2 is not a J-space, as already observed. This is the required contradiction.

THEOREM 4.5. *A complete barrelled space with a Schauder basis has either an abnormal Schauder basis or a conditional Schauder basis.*

If not E is a J-space (Theorem 4.2); however, by Lemma 4.4 and Theorem 3.4, E has an abnormal Schauder basis.

With a Fréchet space, one can go slightly further, using results established in [2] and [12].

LEMMA 4.6. *Let E be a Fréchet J-space, and let G be a closed non-Montel subspace of E; then $G \cong G \oplus E$.*

There exists in G a closed bounded set A which is not compact, and so using Theorem 10 of [5], A is not sequentially compact; however, A is weakly sequentially compact, since E is reflexive. Thus there exists in G a sequence y_n tending to zero weakly but not strongly. By Theorem 4.3 of [12], there is a subsequence (z_n) of (y_n) which is a normalised Schauder basic sequence equivalent to a block basic sequence (u_n) of (x_n), where (x_n) is any normalised basis of E. Furthermore, $\overline{\text{lin}}\ (u_n)$ is complemented (Theorem 4.3), and, as remarked in [12], $D = \overline{\text{lin}}\ (z_n)$ is complemented, and as $\lambda_z = \lambda_x$, $D \cong E$.

Therefore

$$G = D \oplus H \cong E \oplus H \cong E \oplus E \oplus H \cong E \oplus G.$$

THEOREM 4.7. *Let E be an F-space with a normalised basis; if E has an infinite-dimensional normed subspace G (i.e. G has a norm topology), then E has a conditional normalised basis.*

Otherwise E is a J-space, and using Lemma 4.6, $\bar{G} \cong E \oplus \bar{G}$, so that E is a Banach space. However, this is impossible, by the results of [16].

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