Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1221

Probability and Banach Spaces

Proceedings of a Conference held in Zaragoza, Spain
June 17–21, 1985

Edited by J. Bastero and M. San Miguel

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): 46-XX, 60-XX

ISBN 3-540-17186-X Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-17186-X Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

DIFFERENTIABILITY PROPERTIES OF VECTOR VALUED FUNCTIONS

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1. Introduction

The theme of this paper is to study the notions of analytic convexity (A-convexity) and the analytic Radon-Nikodyn Property (ARNP) in complex quasi-Banach spaces and produce some characterizations analogous to the known work concerning local convexity and the Radon-Nikodyn Property in the real case.

The basic result on A-convexity is Theorem 3.5 which shows that in A-convex spaces there is a Mean-Value Theorem for certain types of Lipschitz functions (functions of analytic type). A similar Theorem for all Lipschitz functions is equivalent to local convexity.

The analytic Radon-Nikodyn Property was first studied by Bukhvalov and Danilovich [2] and later, in a non-locally convex setting, by Dowling [4] (see [5]). We show here that (Theorem 5.7) ARNP is equivalent to the differentiability (a.e.) of Lipschitz functions of analytic type; again a similar property for all Lipschitz functions is equivalent to the RNP. We also characterize precisely those quasi-Banach lattices with ARNP.

In our final section we relate this work to recent work of Krotov [10] on the differentiability functions in H_p for $0 \leq p < 1$. We extend Krotov's results to a wide range of spaces including locally bounded Hardy-Orlicz spaces, Hardy-Lorentz spaces, etc. The point here is that these results are now seen as part of a general theory of vector-valued analytic functions.

2. Notation

Our notation is essentially as in [9]. We shall suppose throughout that X is complex quasi-Banach space equipped with a continuous quasi-norm (this avoids certain measurability problems). Where appropriate we shall assume X is p-normable for some p, $0 < p < 1$ i.e. for some constant C,

$$\|x_1 + \dots + x_n\| \leq C(\|x_1\|^p + \dots + \|x_n\|^p)^{1/p}$$

for $x_1, \dots, x_n \in X$, or p-normed i.e.

$$\|x_1 + \dots + x_n\| \leq (\|x_1\|^p + \dots + \|x_n\|^p)^{1/p}$$

for $x_1, \dots, x_n \in X$.

We shall frequently use the symbol C for a constant which depends on X alone (unless otherwise stated) but may vary from line to line.

We recall that if U is an open subset of $\mathbb{C}$ a function $f : U \to X$ is analytic if for each $z_0 \in U$ there exists $\delta > 0$ and $x_0, x_1, \dots \in X$ so that if $|z - z_0| < \delta$ then $z \in U$ and

$$f(z) = \sum_{n=0}^{\infty} x_n z^n$$

Of course in this case $x_n = f^{(n)}(z)/n!$

If Δ is the open unit disk then the set of all functions $f : \bar{\Delta} \to X$ which are continuous and analytic on Δ is denoted by $A_0(X)$.

One important principle we shall use is that if U, V are open subsets of $\mathbb{C}$, $\phi : U \to V$ is an analytic function and $f : V \to X$ is analytic then $f \circ \phi$ is analytic. This is a computation with power series which can also be argued as follows. If $z_0 \in U$, pick $\delta > 0$ so that if $|w - \phi(z_0)| < \delta$ then

$$f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\phi(z_0))}{n!} (w - \phi(z_0))^n$$

Pick $\rho > 0$ so that $|z - z_0| < \rho$ imples $|\phi(z) - \phi(z_0)| < \frac{\delta}{2}$. Then if $|z - z_0| < \rho$

$$f(\phi(z)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\phi(z_0))}{n!} (\phi(z) - \phi(z_0))^n$$

and convergence is uniform. Thus $f \circ \phi$ is the uniform of analytic functions on $\{z : |z - z_0| < \rho\}$ and so is analytic (cf. [7]).

We shall use this remark to switch from the upper half-plane to the disk.

We recall that a quasi-Banach X is A-convex if (equivalently) either X has an equivalent plurisubharmonic quasi-norm or there exists some C so that if $f \in A_0(X)$

$$\|f(0)\| \leq C \int_0^{2\pi} \|f(e^{i\theta})\| \frac{d\theta}{2\pi}$$

A-convex spaces have also been called locally holomorphic [1] or locally pseudo-convex [13]. Plurisubharmonic quasi-norms are called PL-convex in [3].

3. A-convexity and Mean Value Properties

The following characterization of local convexity is easily established (and implicitly well-known). However it serves as a motivation for the results of this section.

Theorem 3.1. *Let X be a quasi-Banach space. Then X is locally convex (.i.e. a Banach space) if and only if for some constant C we have that whenever $F : \mathbb{R} \to X$ is a Lipschitz map so that $F'(t)$ exists almost everywhere then for* $s, t \in \mathbb{R}$

$$\|F(t) - F(s)\| \leq CM\,|t - s|$$

where

$$M = \sup_t \|F'(t)\|$$

Remarks on the proof. Of course for a Banach space we may take $C = 1$ and this is a form of the Mean Value Theorem. Conversely if the condition is satisfied for some C then by taking F as the integral of a simple function (with finite-dimensional range) we quickly obtain

$$\|x_1 + \dots + x_n\| \le Cn$$

for $\|x_i\| \le 1$, and this implies local convexity.

Now we turn to the complex version of this result. We say that a continuous map $F : \mathbb{R} \to X$ is an analytic polynomial if

$$F(t) = \sum_{k=1}^{N} x_k e^{i\lambda_k t}$$

where $\lambda_k \ge 0$. We say F is of analytic type if there is a sequence of analytic polynomials F_n so that $F_n(t) \to F(t)$ uniformly.

Lemma 3.2. Let X be an A-convex quasi-Banach space and suppose $F : \mathbb{R} \to X$ is of analytic type.

(i) If F is 2π-periodic there exists a unique $f \in A_0(X)$ with $f(e^{i\theta}) = F(\theta)$ for $-\infty < \theta < \infty$.

(ii) In general there exists a unique bounded continuous function $\tilde{F} : \{z : \operatorname{Im} z \ge 0\} \to X$ so that

(a) $\tilde{F}(t) = F(t)$ $\qquad -\infty < t < \infty$

(b) $\tilde{F}$ is analytic for $\operatorname{Im} z > 0$

(c) There exists $x \in X$ so that

$$\lim_{s\to\infty} F(t + is) = x \text{ uniformly in } t .$$

Proof. Let us first prove (ii). Let F_n be the sequence of analytic polynomials converging uniformly to F.

If

$$F_n(t) = \sum_{k=1}^{N} x_k e^{i\lambda_k t}$$

set

$$\tilde{F}_n(z) = \sum_{k=1}^{N} x_k e^{i\lambda_k z} \qquad \operatorname{Im} z \ge 0$$

It is clear that $\tilde{F}_n$ satisfies (b),(c). Furthermore by the Maximum Modulus Principle

$$\|\tilde{F}_m(z) - \tilde{F}_n(z)\| \le C \sup_{t\in\mathbb{R}} \|F_m(t) - F_n(t)\|$$

where C depends only on X. Hence $\tilde{F}_n(z)$ converges uniformly to some $\tilde{F}$ which satisfies (a),(b) and (c). If G is any other function satisfying (a),(b) and (c) then $G(z) - \tilde{F}(z)$ is bounded and continuous for $\operatorname{Im} z \ge 0$, and vanishes on the axis. Let $\phi = G - \tilde{F}$ and define $\psi : \Delta \to X$ by

$$\psi(z) = \phi\left(i\,\frac{1+z}{1-z}\right)$$

Then ψ is analytic and bounded on Δ and so since X is A-convex if $|z| < 1$,

$$\|\psi(z)\| \le C \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \|\psi(re^{i\theta})\|\, d\theta$$

$$\le C \int_0^{2\pi} \|\phi(\tfrac{1}{2} \cot \tfrac{\theta}{2})\|\, d\theta$$

$$= 0$$

Thus $G = \tilde{F}$.

Now for (i) if F is 2π-periodic, then $\tilde{F}$ is also 2π-periodic, since $\tilde{F}(z - 2\pi) = F(z)$ by the uniqueness part of (ii).

For $|z| < 1$ we define

$$f(z) = \tilde{F}(-i \log z)$$

This definition is unambiguous and define a function f analytic on Δ . Now as $z \to e^{i\theta}$, $\tilde{F}(- i \log z) \to F(\theta)$ and so (i) follows.

Proposition 3.3. Let X be a quasi-Banach space. Then there is a constant $C = C(X)$ so that if D is any open disk in the complex plane and $f : D \to X$ is an analytic function then for $z_1, z_2 \in D$ we have

$$\|f(z_1) - f(z_2)\| \le C|z_1 - z_2| \sup_{z \; D} \|f'(z)\|$$

Proof. First suppose $D = \Delta$ the unit disk, and suppose

$$B = \sup_{z \; \Delta} \|f'(z)\| < \infty$$

Then if X is a p-Banach space, we choose $n \in \mathbb{N}$ and $0 < q < p$ so that $n = [\frac{1}{q}] < \frac{1}{q}$. Let

$$F(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{(n+k-1)!} z^{n+k-1} \qquad z \in \Delta$$

Then $F^{(n)} = f'$ is bounded and so [7] there is an operator $T : H_q \to X$ with

$$T((1 - wz)^{-1}) = F(z) \qquad z \in \Delta$$

and

$$\|T\| \le C \|F\|_{1/q -1}$$

$$\le CB$$

If $z \in \Delta$ then

$$f(z) - f(0) = (n - 1)! \, T(\frac{w^{n-1}}{(1-wz)^n})$$

so that

$$\|f(z) - f(0)\| \le CB$$

since $nq < 1$.

We conclude that if $D \subset \Delta$ is any disk (open or closed) with center z_1 then for any $z_2 \in D$

$$\|f(z_2) - f(z_1)\| \le CBr(D)$$

where $r(D)$ is the radius of D .

For fixed $z_1, z_2 \in \Delta$, let $d = |z_1 - z_2|$. If $d \ge 1$ then

$$\|f(z_2) - f(z_1)\| \le CB$$

$$\le CBd$$

If $d < 1$, let $\xi_1 = (1 - d)z_1$ and $\xi_2 = (1 - d)z_2$. Clearly

$$\|f(z_1) - f(\xi_1)\| \le CBd|z_1|$$

$$\le CBd$$

and

$$\|f(z_2) - f(\xi_2)\| \leq CBd$$

Consider the closed disk D centre ξ_1 and radius $d(1-d)$. Then $\xi_2 \in D$. Further $D \subset \Delta$ since

$$|\xi_1| + d(1-d) = (1-d)(d + |z_1|)$$
$$< (1-d)(1+d)$$
$$< 1$$

Hence

$$\|f(\xi_1) - f(\xi_2)\| \leq CBd$$

and so

$$\|f(z_1) - f(z_2)\| \leq CBd$$

The result now follows for arbitrary disks. Since the half-plane is an increasing union of disks we also obtain:

<u>Proposition 3.4</u>. <u>If</u> X <u>is a quasi-Banach space then there exists</u> $C = C(X)$ <u>so that if</u> f <u>is an analytic function on the upper half-plane satisfying</u>

$$\|f'(z)\| \leq M \qquad \text{Im} z < 0$$

<u>then for</u> z_1, z_2 <u>in the upper half-plane</u>

$$\|f(z_1) - f(z_2)\| \leq CM|z_1 - z_2|$$

<u>Theorem 3.5</u>. <u>Let</u> X <u>be a quasi-Banach space. Then</u> X <u>is A-convex if and only if there exists a constant</u> C <u>so that whenever</u> $F : \mathbb{R} \to X$ <u>is a Lipschitz map of analytic type such that</u> $F'(t)$ <u>exists a.e. and</u>

$$\|F'(t)\| \leq M \qquad \text{a.e.}$$

<u>then</u>

$$\|F(s) - F(t)\| \leq CM|t - s|$$

<u>Proof</u>. If the condition holds, consider any polynomial $\phi(z) = x_0 + x_1 z + \ldots + x_n z^n$. Let

$$F_m(t) = \sum_{k=0}^{n} \frac{x_k}{2km+1} e^{(2km+1)it} \qquad t \in \mathbb{R}$$

F_m is an analytic polynomial and

$$F_m'(t) = \sum_{k=0}^{n} x_k e^{(2km+1)it}$$
$$= e^{it} \phi(e^{2mit})$$

so that

$$\|F_m'(t)\| \leq \max_{|z|=1} \|\phi(z)\|$$

Now

$$F_m(2\pi) - F_m(\pi) = 2 \sum_{k=0}^{n} \frac{x_k}{2km+1}$$

and hence letting $m \to \infty$ we obtain

$$\|x_0\| = \|\phi(0)\| \leq C \max_{|z|=1} \|\phi(z)\|$$

so that X is A-convex.

Conversely let us suppose X is A-convex and that $F : \mathbb{R} \to X$ is a Lipschitz map of analytic type, such that $\|F'(t)\| \leq M$ a.e. We suppose X has a plurisubharmonic quasi-norm.

Denote also by F the analytic extension of F to the upper-half plane given by Lemma 3.2. Then for $h > 0$

$$u(z) = \|\frac{1}{h}(F(z+h) - F(z))\|$$

is subharmonic and bounded on $\text{Im} z > 0$ and extends continuously to the real axis. Furthermore $\lim_{s\to\infty} u(t+is) = 0$ uniformly on t. Thus

$$u(x+iy) \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} u(t)\, dt$$

Letting $h \to 0$ we obtain

$$\|F'(z)\| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \|F'(t)\|\, dt$$

by the Bounded Convergence Theorem. Thus

$$\|F'(z)\| \leq M \qquad \text{Im} z > 0$$

and so if $\text{Im} z_1 > 0$, $\text{Im} z_2 > 0$

$$\|F(z_1) - F(z_2)\| \leq CM|z_1 - z_2|$$

This inequality extends by continuity to the real axis.

4. The Analytic Radon-Nikodym Property

The following definition is due to Dowling [4] and extends to non-locally convex spaces a concept introduced by Bukhvalov and Danilevich [2].

Definition 4.1. A quasi-Banach space X has the analytic Radon-Nikodym property (ARNP) if whenever $f : \Delta \to X$ is a bounded analytic function then $\lim_{r\to 1} f(re^{i\theta})$ exists θ - a.e.

Lemma 4.2. Suppose X is p-normed and fails to be A-convex. Then given $\delta > 0$ there exists an analytic function $f : \Delta \to X$ so that

(i) $\|f(z)\| \leq (1+\delta^p)^{1/p}$ $\qquad z \in \Delta$

(ii) $\limsup_{r\to 1} \inf_{0\leq\theta\leq 2\pi} \|f(re^{i\theta})\| \leq (1-\delta^p)^{1/p}$

(iii) $\liminf_{r\to 1} \sup_{0\leq\theta\leq 2\pi} \|f(re^{i\theta})\| \leq \delta$

Proof. First we establish:

Claim 1: Given $\varepsilon > 0$ there exists $\phi \in A_0(X)$ so that

$$\|\phi(0)\| = \max_{|z|\leq 1} \|\phi(z)\| = 1$$

$$\|\phi(z)\| \leq \varepsilon \qquad z$$

To prove the claim, observe that by results of [8], there exists an analytic function $g \in A_0(X)$ so that

$$\max_{|z|\leq 1} \|g(z)\| = 1$$

$$\|g(z)\| \leq \varepsilon \qquad z \in \mathbb{T}$$

Suppose g assumes its maximum at w. Then put $\phi(z) = g(\frac{w-z}{1-\bar{w}z})$

Next we establish:

<u>Claim 2</u>: <u>Given</u> $\varepsilon > 0$ <u>and</u> $0 < \rho < 1$ <u>there exists</u> $\psi \in A_0(X)$ <u>and</u> r <u>with</u> $0 < \rho < r < 1$ <u>so that</u>

$$\|\psi(z)\| \leq \varepsilon \qquad |z| \leq \rho$$

$$\|\psi(z)\| \geq 1 - \varepsilon \qquad |z| = r$$

$$\|\psi(z)\| \leq \varepsilon \qquad |z| = 1$$

$$\|\psi(z)\| \leq 1 \qquad |z| \leq 1$$

Proof of Claim 2: Pick ϕ according to Claim 1. Next pick ℓ so that $\rho^{\ell} \leq \varepsilon$. We shall set

$$\psi(z) = z^{\ell}\phi(z^m)$$

for suitable m. Note that the conditions for $|z| \leq \rho$, $|z| \leq 1$, and $|z| = 1$ are automatically satisfied.

Now pick r so that $r^{\ell} > 1 - \varepsilon$. Then if $|z| = r$

$$\|\psi(z)\| \geq r^{\ell} \inf_{|z|=r} \|\phi(z^m)\|$$

$$= r^{\ell} \inf_{|z|=r^m} \|\phi(z)\|$$

$$\to r^{\ell}$$

as $m \to \infty$. Hence for suitable m the claim is satisfied.

We now proceed to the proof of the lemma. Suppose $\varepsilon_n > 0$ are such that $\Sigma\varepsilon_n^p < \delta^p$. We select by induction a sequence $f_n \in A_0(X)(n \geq 0)$ and increasing sequences $\rho_n (n \geq 0)$, $r_n (n \geq 1)$ where $0 \leq \rho_n \uparrow 1$, $0 \leq r_n \uparrow 1$. For convenience we set $\rho_0 = 0$, $f_0 = 0$.

We select (f_n), (ρ_n), (r_n) so that

$$(1) \quad \|f_n(z)\| < (1 + \delta^p)^{1/p} \qquad |z| \leq 1$$

$$(2) \quad \|f_n(z) - f_{n-1}(z)\| < \varepsilon_n \qquad |z| \leq \rho_{n-1}$$

$$(3) \quad \|f_n(z)\| < \delta \qquad |z| = \rho_k, \quad 0 \leq k \leq n$$

$$(4) \quad \|f_n(z)\| > (1 - \delta^p)^{1/p} \qquad |z| = r_k : 1 \leq k \leq n$$

$$(5) \quad \tfrac{1}{2}(1 + \rho_{n-1}) < r_n < \rho_n$$

$$(6) \quad \|f_n(z)\| < \delta \qquad |z| \geq \rho_n$$

Suppose f_k, ρ_k, r_k have been selected for $k \leq n - 1$. Select η so small that $\eta < \varepsilon_n$,

$$\eta^p + \max_{|z|\leq 1} \|f_{n-1}(z)\|^p < 1 + \delta^p$$

$$\eta^p + \max_{|z|=\rho_k} \|f_{n-1}(z)\|^p < \delta^p \qquad 0 \le k \le n-1$$

$$\min_{|z|=r_k} \|f_{n-1}(z)\|^p - \eta^p > 1 - \delta^p \qquad 0 \le k \le n-1$$

$$\eta^p + \max_{|z|\ge\rho_{n-1}} \|f_{n-1}(z)\|^p < \delta^p$$

Now use Claim 2 to pick $\psi \in A_0(X)$ so that for some $r_n > \frac{1}{2}(1 + \rho_{n-1})$

$$\|\psi(z)\| \le \eta \qquad |z| \le \rho_{n-1}$$

$$\|\psi(z)\| \ge (1 - \eta^p)^{1/p} \qquad |z| = r_n$$

$$\|\psi(z)\| \le \eta \qquad |z| = 1$$

$$\|\psi(z)\| \le 1 \qquad |z| \le 1$$

Let $f_n(z) = f_{n-1}(z) + \psi(z)$. If $|z| \le \rho_{n-1}$ $\|f_n(z) - f_{n-1}(z)\| < \eta$ so that (2) holds and

$$\|f_n(z)\| < 1 + \delta^p$$

If $|z| \ge \rho_{n-1}$,

$$\|f_n(z)\| < \delta^p + \|\psi(z)\|^p$$

$$< 1 + \delta^p$$

Hence (1) holds. (3) holds easily for $0 \le k \le n-1$. Clearly $\|f_n(z)\| < \delta$ for $|z| = 1$ so we may pick $\rho_n > r_n$ so that (3) and (6) hold.

For (4) note if $|z| = r_n$

$$\|f_n(z)\|^p > 1 - \eta^p - \|f_{n-1}(z)\|^p$$

$$> 1 - \eta^p - \max_{|z|\ge\rho_{n-1}} \|f_{n-1}(z)\|^p$$

$$> 1 - \delta^p$$

If $|z| = r_k$ where $k \le n - 1$,

$$\|f_n(z)\|^p > 1 - \delta^p$$

by choice of η . Clearly (5) holds by construction.

Now f_n converges uniformly on compacta to a function $f : \Delta \to X$. f is analytic (cf. [7]) and satisfies the conclusions of the lemma.

Thus we have:

<u>Theorem 4.3</u>. <u>If</u> X <u>has ARNP then</u> X <u>is A-convex</u>.

It is shown by Bukhvalov and Danilevich [2] that c_0 fails ARNP and that a Banach lattice has ARNP if and only if it fails to contain a copy of c_0. Dowling[4] showed that L_p $(0 < p < 1)$ has ARNP.

Let us suppose X is a separable quasi-Banach lattice. Then by using standard representation arguments we can represent X as a space of (equivalence classes of) Borel functions on some compact metric space Ω . Thus X is a sublattice of the lattice of all Borel functions on Ω containing $C(\Omega)$ and satisfying

(i) If $f \in C(\Omega)$ then $\|f\|_X \le \|f\|_{C(\Omega)}$

(ii) If f is any Borel function and g X with $|f| \le |g|$ then f X with $\|f\|_X \le \|g\|_X$.

It is clear that a quasi-Banach lattice with ARNP must have two properties: it cannot contain c_0 and it must be A-convex. In [8] it is shown that a quasi-Banach lattice X is A-convex if and only if it is L-convex or equivalently there exists $\alpha > 0$, $C < \infty$ so that if $x_1, \ldots, x_n \in X$

$$(*) \qquad \|(\sum_{i=1}^{n} |x_i|^{\alpha})^{1/\alpha}\| \leq C(\sum_{i=1}^{n} \|x_i\|^{\alpha})^{1/\alpha}$$

Now X fails to contain c_0 if and only if we have the further assumption:

(iii) If $f_n \in X$ $0 \leq f_n \uparrow f$ and $\sup_n \|f_n\| < \infty$ then $f \in X$ and $\|f_n - f\|_X \to 0$.

If X is L-convex and so satisfies (*) then we can define

$$B = \{f : f \text{ a Borel function with } |f|^{\frac{1}{\alpha}} \in X\}$$

and impose

$$\|f\|_B = \| |f|^{\frac{1}{\alpha}} \|_X^{\alpha}$$

Assumption (*) means that B can be renormed as a separable Banach lattice (the $1/\alpha$ -convexification of X, cf. [11]). We assume for convenience that $\| \ \|_B$ is already a norm and we define

$$\|f\|_X = \| |f|^{\alpha} \|_B^{\frac{1}{\alpha}}$$

B also fails to contain c_0 since (iii) holds for B.
We can find on B a strictly positive linear functional with $\phi(1_\Omega) = 1$. Then ϕ can be represented by a probability measure on Ω, μ say. Then we have

$B \subset L_1(\Omega,\mu)$ and

$$\int_\Omega |f| \, d\mu \leq \gamma \|f\|_B \qquad f \in B$$

for some $\gamma > 0$.

Using all these remarks we show:

<u>Proposition 4.4</u>. <u>Let</u> X <u>be a separable L-convex quasi-Banach lattice not containing</u> c_0. <u>Then</u> X <u>has ARNP</u>.

<u>Proof</u>. Suppose ϕ is analytic and $\|\phi(z)\| \leq 1$ for $|z| < 1$. Thus we can write

$$\phi(z) = \sum_{n=0}^{\infty} f_n z_n$$

where $f_n \in X$ and $\limsup_{n\to\infty} \|f_n\|^{1/n} \leq 1$. It follows that $\Sigma|f_n|r^n$ converges in X for $0 < r < 1$ and hence also μ-a.e. Thus we can determine a Borel subset $\Omega_1 \subset \Omega$ with $\mu(\Omega_1) = 1$ so that

$$\phi(z,w) = \sum_{n=0}^{\infty} f_n(w) z^n$$

is defined for $w \in \Omega_1$, $z \in \Delta$. Now

$$\int_\Omega \{ \int_0^{2\pi} |\phi(re^{i\theta}, w)|^\alpha \frac{d\theta}{2\pi} \} d\mu = \int_0^{2\pi} (\int_\Omega |\phi(re^{i\theta}, w)|^\alpha d\mu) \frac{d\theta}{2\pi}$$

$$\leq \gamma \int_0^{2\pi} \| |\phi(re^{i\theta})|^\alpha \|_B \frac{d\theta}{2\pi}$$

$$\leq \gamma$$

The integrand is an increasing function of r so that we conclude that for a Borel set $\Omega_2 \subset \Omega_1$ with $\mu(\Omega_2) = 1$, we have $\phi(\cdot, w) \in H_\alpha$. If $w \in \Omega_2$ there exists a Borel subset V_w of $[0,2\pi]$ with $mV_w = 2\pi$ so that

$$\lim_{r\to 1} \phi(re^{i\theta}, w) = \phi^*(e^{i\theta}, w)$$

exists for $\theta \in V_w$. By application of Fubini's theorem if $V = \{\theta : \theta \in V_w$ μ-a.e.$\}$ then $mV = 2\pi$; thus we can define $\phi^*(e^{i\theta}) \in L_0$ and $\lim_{r\to 1} \phi(re^{i\theta}) = \phi^*(e^{i\theta})$ for $\theta \in V$.

Now if $\theta \in V$

$$\lim_{r\to 1} |\phi(re^{i\theta})| = |\phi^*(e^{i\theta})|$$

a.e. and so

$$\lim_{r\to 1} |\phi(re^{i\theta})| \wedge n\, 1_\Omega = |\phi^*(e^{i\theta})| \wedge n\, 1_\Omega$$

a.e.

It follows easily from property (iii) of X that

$$\| |\phi^*(e^{i\theta})| \wedge n\, 1_\Omega \| \leq \liminf_{r\to 1} \|\phi(re^{i\theta})\|_X$$

$$\leq 1$$

Thus also from (iii), $\phi^*(re^{i\theta}) \in X$ and

$$\|\phi^*(e^{i\theta})\|_X \leq 1$$

It remains only to show that

$$\lim_{r\to 1} \|\phi^*(e^{i\theta}) - \phi(re^{i\theta})\|_X = 0$$

a.e.

For $|z| < 1$ and $w \in \Omega_2$ set

$$G(z,w) = \exp\left\{\alpha \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |\phi^*(e^{i\theta}, w)| \frac{d\theta}{2\pi}\right\}$$

If $w \in \Omega_2$

$$\lim_{r\to 1} G(re^{i\theta}, w) = G^*(e^{i\theta}, w)$$

exists θ-a.e. and

$$|G^*(e^{i\theta}, w)| = |\phi^*(e^{i\theta}, w)|^\alpha$$

(θ-a.e.). Thus $G^*(e^{i\theta}) \in B$ and $\|G^*(e^{i\theta})\|_B \leq 1$

It is easy to see that the map $\theta \to G^*(e^{i\theta})$ is measurable into L_0, and hence also into B (since B is separable). We have that for $w \in \Omega_2$

$$G(re^{i\theta}, w) = \int P(r, \theta - \phi)\, G^*(e^{i\phi}, w) \frac{d\phi}{2\pi}$$

where $P(r, \theta)$ is the Poisson kernel. This imples the Bochner integral in B

$$G(re^{i\theta}) = \int_0^{2\pi} P(r, \theta - \phi)\, G^*(e^{i\phi}) \frac{d\phi}{2\pi}$$

Similarly

$$\int_0^{2\pi} e^{in\phi}\, G^*(e^{i\phi}) \frac{d\phi}{2\pi} = 0 \qquad n = 1,2, \ldots$$

so that G is analytic into B. Further

$$\|G(re^{i\theta})\| \leq 1$$

for $0 < r < 1$. Hence as B as ARNP

$$\lim_{r\to 1} \|G^*(e^{i\theta}) - G(re^{i\theta})\|_B = 0$$

for $\theta \in V_1$ with $mV_1 = 2\pi$

Now suppose $\theta \in V \cap V_1$. If $0 < r < 1$

$$|\phi(re^{i\theta}, w)| \leq |G(re^{i\theta}, w)|^{1/\alpha}$$

for $w \in \Omega_2$. Thus

$$|\phi(re^{i\theta})| \leq 2^{1/\alpha -1} (|G^*(e^{i\theta})|^{1/\alpha} + |G^*(e^{i\theta}) - G(re^{i\theta})|^{1/\alpha})$$

so that we can write

$$\phi(re^{i\theta}) = \phi_1(r) + \phi_2(r)$$

where

$$|\phi_1(r)| \leq 2^{1/\alpha -1} |G^*(e^{i\theta})|^{1/\alpha}$$

$$|\phi_2(r)| \leq 2^{1/\alpha -1} |G^*(e^{i\theta}) - G(re^{i\theta})|^{1/\alpha}$$

Now $\|\phi_2(r)\|_X = \| |\phi_2(r)|^\alpha\|_B^{1/\alpha} \to 0$ so that $\phi_2(r) \to 0$ in L_0 as $r \to 1$. Hence $\phi_1(r) \to \phi^*(e^{i\theta})$ in L_0 and so

$$\|\phi_1(r) - \phi^*(e^{i\theta})\| = \| |\phi_1(r) - \phi^*(e^{i\theta})|^\alpha\|_B^{1/\alpha}$$

$$\to 0$$

since

$$|\phi_1(r) - \phi^*(e^{i\theta})|^\alpha \leq (2^{1-\alpha} + 1) \ |G^*(e^{i\theta})|$$

We conclude that

$$\|\phi(re^{i\theta}) - \phi^*(e^{i\theta})\|_X \to 0$$

as required.

Now since ARNP is plainly determined by separable subspaces we have:

Theorem 4.5. A complex quasi-Banach lattice has ARNP if and only if it is L-convex and does not contain c_0 .

Thus there are many non-locally convex examples of ARNP spaces, e.g. the Lorentz spaces $L(p,q)$ where $0 < p < 1$, $0 < q < \infty$ and $p = 1$, $q \neq 1,\infty$, or the Orlicz spaces L_ϕ where ϕ satisfies the Δ_2-condition (cf. [6]).

Problem. (cf. [8]). Does the Schatter class S_p has ARNP when $p < 1$?

5. Boundary behaviour of analytic functions

Aleksandrov [1] has developed a theory of H_p-spaces $(0 < p < \infty)$ in A-convex quasi-Banach spaces mimicking that of the scalar case. We shall here use a version of his results in the case $p = \infty$. Let $f : \Delta \to X$ be an analytic function. We shall say that $f \in H_b(X)$ if f is bounded and $f \in H_\infty(X)$ if $f \in H_b(X)$ and $\lim_{r\to 1} f(re^{i\theta})$ exists θ-a.e. Thus X has ARNP if and only if $H_b(X) = H_\infty(X)$. If

$f \in H_\infty$ we denote its boundary values also by f. Further if $e^{i\theta} \in \mathbb{T}$ we let Γ_θ be the convex ball of the circle $\{z : |z| \le \frac{1}{2}\}$ and $e^{i\theta}$. Aleksandrov's results imply:

Proposition 5.1. *Suppose* X *is A-convex.*

(i) *Suppose* $f_n \in A_0(X)$ *is a uniformly bounded sequence such that* $\lim_{n\to\infty} f_n(e^{i\theta})$ *exists* a.e. *for* $0 \le \theta \le 2\pi$. *Then there exists* $g \in H_\infty(X)$ *with*

$$\lim_{n\to\infty} f_n(z) = g(z) \qquad |z| < 1$$

$$\lim_{n\to\infty} f_n(e^{i\theta}) = g(e^{i\theta}) \qquad \text{a.e. } 0 \le \theta \le 2\pi$$

(ii) *If* $f \in H_\infty(X)$ *then*

$$\lim_{\substack{z\to e^{i\theta}\\ z\in\Gamma_\theta}} f(z) = f(e^{i\theta}) \qquad \text{a.e. } 0 \le \theta \le 2\pi$$

Remarks. For (i) note that if $p < \infty$ then $f_n \to g$ in $H_p(X)$ and use Theorems 2.2.2 and 2.2.4 of [1]. For (ii) use Corollary 2.2.6.

In order to proceed further we shall need some estimates based on Proposition 3.3. Let U be a bounded open convex set in $\mathbb{C}$; we let $d = d(U)$ be the diameter of U and let $\alpha = \alpha(U)$ be the greatest number $\alpha > 0$ such that U contains an open disk of radius αd.

Proposition 5.2. *Let* X *be a p-normable quasi-Banach space. Then there is a constant* $C = C(X)$ *so that if* U *is a bounded open set with* $\alpha(U) = \alpha$ *and* $f : U \to X$ *is an analytic function satisfying*

$$\|f'(z)\| \le M \qquad z \in U$$

then for any $z_1, z_2 \in U$

$$\|f_1(z_1) - f(z_2)\| \le C\,\alpha^{1-1/p}\,Md(U)$$

Proof. Let z_0 be the centre of a disk radius d contained in U. Suppose $z_1 \in U$. Then

$$|z_1 - z_0| \le d$$

For $0 \le \lambda \le 1$ let $z_\lambda = (1-\lambda)z_0 + \lambda z_1$. Then there is a disk radius $(1-\lambda)\alpha d$ contained in U and centred at z_λ. Note that if $\mu \ge \lambda$ then $|z_\mu - z_\lambda| = (\mu-\lambda)\,|z_1 - z_0| \le (\mu-\lambda)d$. Hence if $\mu - \lambda \le (1-\lambda)\alpha$ we conclude from Proposition 3.3 that

$$\|f(z_\mu) - f(z_\lambda)\| \le CM\,(\mu-\lambda)d$$

where $C = C(X)$.

Let $\lambda_n = 1 - (1-\alpha)^n$ for $n = 0,1,2,\ldots$ and let $x_n = f(z_{\lambda_n})$. Then

$$\|x_n - x_{n-1}\| \le CM\,\alpha(1-\alpha)^{n-1}d$$

so that

$$\|f(z_1) - f(z_0)\| \le CM\,d\alpha\,\Big(\sum_{n=1}^{\infty}(1-\alpha)^{(n-1)p}\Big)^{1/p}$$

$$\le C'Md\alpha^{1-1/p}$$

where C' depends on X.

The Proposition now follows immediately since

$$\|f(z_2) - f(z_0)\| \le C'M\, d\, \alpha^{1-1/p}$$

Proposition 5.3. Let X be a p-normable quasi-Banach space and suppose $n \in \mathbb{N}$. Then there is a constant $C = C(n,X)$ so that if U is a bounded open set and $f : U \to X$ is an analytic function with

$$\|f^{(n)}(z)\| \le M \qquad z \in U$$

then for any $z_1, z_2 \in U$

$$\|f(z_2) - \sum_{k=0}^{n-1} f^{(k)}(z_1)(z_2 - z_1)^k\| \le C\, \alpha^{n(1-1/p)}\, M\, d(U)^n$$

This follows by induction from the preceding proposition.

If $f : U \to X$ is analytic let $T_n f(z,w)$ be the Taylor polynomial of degree n for f at z i.e.

$$T_n f(z,w) = \sum_{k=0}^{n} f^{(k)}(z)\,(w - z)^k$$

Proposition 5.4. Let X be a quasi-Banach space and suppose $n \in \mathbb{N}$, $\alpha > 0$. Then there is a constant $C = C(\alpha,n,X)$ so that if U is a bounded open set with $\alpha = \alpha(U)$ and $d = d(U)$ and $f : U \to X$ is an analytic function with

$$\|f^{(n)}(z)\| \le M \qquad z \in U$$

then for any $z_1, z_2 \in U$ and any w with $d(w,U) = \inf_{u \in U} \|w - u\| \le d$

$$\|T_{n-1}f(z_1,w) - T_{n-1}f(z_2,w)\| \le CMd^n$$

Proof. If $d(w,u) \le d$ then

$$|w - z_2| \le 2d$$

Note also for $0 \le k \le n-1$

$$\|f^{(k)}(z_2) - \frac{\partial^k T_{n-1}}{\partial w^k} f(z_1, z_2 - z_1)\| \le CMd^{n-k}$$

where $C = C(k,\alpha,X)$.

Hence if $|w - z_2| \le 2d$

$$\|T_{n-1}f(z_2,w) - T_{n-1}f(z_1,w)\| \le CMd^n$$

where $C = C(n,\alpha,X)$.

Theorem 5.5. Let X be an A-convex quasi-Banach space and let $f \in A_0(X)$. Define $F : \mathbb{R} \to X$ by $F(\theta) = f(e^{i\theta})$. Then the following are equivalent:

(i) $f^{(n)} \in H_b(X)$

(ii) There is a constant β so that

$$\|\sum_{k=0}^{n} \binom{n}{k} (-1)^k F(\theta + k\phi)\| \le \beta|\phi|^n$$

for $\theta, \phi \in \mathbb{R}$.

(iii) F is $(n-1)$-times continuously differentiable and for some γ

$$\|F(\theta + \phi) - \sum_{k=0}^{n-1} \frac{F^{(k)}(\theta)}{k!} \phi^k\| \le \gamma|\phi|^k$$

for $\theta, \phi \in \mathbb{R}$.

Proof. Clearly (iii) implies (ii).

(ii) $\Rightarrow$ (i). Define for $\phi > 0$

$$g_\phi(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(e^{i\phi k} z)$$

Then since X is A-convex

$$\|g_\phi(z)\| \leq C\, \beta |\phi|^n$$

Now if $|z| < 1$

$$g_\phi(z) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(0) (1 - e^{ij\phi})^n z^j$$

so that

$$i^n \lim_{\phi\to 0} \phi^{-n} g_\phi(z) = \sum_{j=0}^{\infty} \frac{j^n}{j!} f^{(j)}(0) z^j$$

$$= (z \frac{d}{dz})^n f(z)$$

Note that if $zg'(z)$ is bounded on Δ then so $g'(z)$ and hence $g(z)$. Thus $(z \frac{d}{dz})^k f \in H_b(X)$ for $0 \leq k \leq n$ and hence $z^n f^{(n)} \in H_b(X)$ which implies $f^{(n)} \in H_b(X)$.

(i) $\Rightarrow$ (iii). Observe first that $f^{(n)} \in H_b(X)$ implies $f^{(n-1)} \in A_0(X)$ and hence $f^{(k)} \in A_0(X)$ for $0 \leq k \leq n$. It is therefore possible to define $T_{n-1}f(z,w)$ for $z \in \bar{\Delta}$.

Now suppose $0 \leq \theta \leq 2\pi$ and $|\phi| \leq \frac{\pi}{4}$. Let $\delta = |e^{i\theta} - e^{i(\theta+\phi)}| = 2|\sin \frac{\phi}{2}|$.

Let z_0 be the point interior to Δ so that z_0, $e^{i\theta}$, $e^{i(\theta+\phi)}$ form an equilateral triangle. Let U be the intersection of Δ and the circle centre z_0 radius δ. Then $d(U) \leq 2\delta$ while U certainly contains the circle centre z_0 radius $\frac{\delta\sqrt{3}}{2}$ so that $\alpha \geq \frac{1}{4}\sqrt{3}$.

We conclude that if $z_1, z_2 \in U$ and $d(w,U) \leq 2d(U)$ then

$$\|T_{n-1}f(z_1,w) - T_{n-1}f(z_2,w)\| \leq \gamma_0 d^n$$

where γ_0 is some constant.

By continuity

$$\|T_{n-1}f(e^{i(\theta+\phi)}, w) - T_{n-1}f(e^{i\theta}, w)\| \leq \gamma_0 d^n$$

and so if $w = e^{i(\theta+\phi)}$

$$\|f(e^{i(\theta+\phi)}) - T_{n-1}f(e^{i\theta}, e^{i(\theta+\phi)})\| \leq \gamma_1 |\phi|^n$$

for some constant γ_1 independent of θ.

Now

$$T_{n-1}f(e^{i\theta}, e^{i(\theta+\phi)}) = \sum_{k=0}^{n-1} \frac{f^{(k)}(e^{i\theta})}{k!} e^{ik\theta} (e^{i\phi} - 1)^k$$

and since $f^{(k)}(e^{i\theta})$, $0 \leq k \leq n-1$, $0 \leq \theta \leq 2\pi$ is bounded it follows easily that for each θ there is a polynomial P_θ of degree n-1 so that

$$\|T_{n-1}f(e^{i\theta}, e^{i(\theta+\phi)}) - P_\theta(\phi)\| \leq \gamma_2 |\phi|^n$$

where γ_2 is independent of θ.

Since the range of this function is contained in a finite-dimensional space we can compute

$$P_\theta^{(k)}(0) = \frac{\partial^k}{\partial\phi^k} T_{n-1}f(e^{i\theta}, e^{i(\theta+\phi)})\Big|_{\phi=0}$$

for $0 \le k \le n-1$. Thus

$$P_\theta^{(k)}(0) = (iz\frac{d}{dz})^k T_{n-1}f(e^{i\theta}, e^{i(\theta+\phi)})\Big|_{\phi=0}$$

$$= (iz\frac{d}{dz})^k f(z)\Big|_{z=e^{i\theta}}$$

$$= \frac{\partial^k}{\partial\phi^k} F(\theta+\phi)\Big|_{\phi=0}$$

Hence

$$\|F(\theta+\phi) - \sum_{k=0}^{n-1} \frac{F^{(k)}(\theta)}{k!}\phi^k\| \le \gamma|\phi|^n$$

<u>Theorem 5.6</u>. <u>Let</u> X <u>be an A-convex quasi-Banach space and suppose</u> $f \in A_0(X)$. <u>Let</u> $F(\theta) = f(e^{i\theta})$. <u>Suppose further that</u> $f^{(n)} \in H_b(X)$; <u>then the following conditions are equivalent</u>

(i) $f^{(n)} \in H_\infty(X)$

(ii) $\lim_{\phi\to 0} \phi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^n F(\theta+k\phi)$

exists a.e.

(iii) $F^{(n-1)}$ <u>is differentiable a.e.</u>

(iv) $F^{(n-1)}$ <u>is differentiable a.e. and for a.e.</u> θ

$$\|F(\theta+\phi) - \sum_{k=0}^{n} \frac{F^{(k)}(\theta)}{k!}\phi^k\| = O(|\phi|^n)$$

<u>Proof</u>. (i)$\Rightarrow$(iv). Since $f^{(n)} \in H_\infty(X)$ there is a subset J of $[0,2\pi]$ with $mJ = 2\pi$ such that

$$\lim_{\substack{z\to e^{i\theta}\\ z\in\Gamma_\theta}} f^{(n)}(z) = f^{(n)}(e^{i\theta})$$

For $\rho \le \frac{1}{2}$ and $\eta > 0$ let

$$J(\rho,\eta) = \{\theta : \|f^{(n)}(z) - f^{(n)}(e^{i\theta})\| < \eta \text{ for } z\in\Gamma_\theta, |z - e^{i\theta}| \le \rho\}$$

Suppose $\theta \in J(\rho,\eta)$. Let

$$g(z) = f(z) - f^{(n)}(e^{i\theta})\frac{(z - e^{i\theta})^n}{n!}$$

so that

$$g^{(n)}(z) = f^{(n)}(z) - f^{(n)}(e^{i\theta})$$

Note all the sets $U(\theta,\rho) = \{z : |z - e^{i\theta}| < \rho, z \in \Gamma_\theta\}$ are similar for $\rho \le \frac{1}{2}$ so that $\alpha(U(\theta,\rho))$ is independent of θ, ρ. By Proposition 5.4 there is a constant C depending only on n and X so that if z_1, z_2 $U(\theta,\rho)$ and $d(w,U(\theta,\rho)) \le \rho$ then

$$\|T_n g(z_1,w) - T_n g(z_2,w)\| \le C\eta\rho^n$$

and hence

$$\|T_nf(z_1,w) - T_nf(z_2,w)\| \leq C\eta\ \rho^n$$

Now there is a constant $\gamma < 1$ so that $|\phi| \leq \gamma\rho$ implies that there exists $z \in \Gamma_\theta \cap \Gamma_{\theta+\phi}$ with $|z - e^{i\theta}| = |z - e^{i(\theta+\phi)}| < \rho$.

Suppose now $\theta,\ \theta + \phi \in J(\rho,\eta)$ and that $|\phi| \leq \gamma\rho$. Then there exists $z \in \Gamma_\theta \cap \Gamma_{\theta+\phi}$ with $|z - e^{i\theta}| = |z - e^{i(\theta+\phi)}| < \gamma^{-1}|\phi|$. Since θ, $\theta + \phi$ $J(\gamma^{-1}|\phi|,\eta)$ we conclude that

$$\|T_nf(e^{i\theta},w) - T_nf(z,w)\| \leq C\eta|\phi|^n$$

for $|w - e^{i\theta}| < \gamma^{-1}|\phi|$, and

$$\|T_nf(e^{i(\theta+\phi)}, w) - T_nf(z,w)\| \leq C\eta|\phi|^n$$

for $|w - e^{i(\theta+\phi)}| < \gamma^{-1}|\phi|$.

Now if $|\psi| < (\gamma^{-1} - 1)|\phi|$

$$\|T_nf(e^{i(\theta+\phi)}, e^{i(\theta+\phi+\psi)}) - T_nf(e^{i\theta}, e^{i(\theta+\phi+\psi)})\| \leq C\eta|\phi|^n$$

By Theorem 5.5 (and its proof) we have

$$\|f(e^{i(\theta+\phi+\psi)}) - T_{n-1}f(e^{i(\theta+\phi)}, e^{i(\theta+\phi+\psi)})\| \leq C|\psi|^n$$

(where C is independent of $\theta,\ \phi$), and since

$$\|f^{(n)}(e^{i(\theta+\phi)})\| \leq C$$

we conclude

$$\|f(\theta + \phi + \psi) - T_nf(e^{i\theta}, e^{i(\theta+\phi+\psi)})\| \leq C\eta|\phi|^n + C|\psi|^n$$

where C depends only on X and the function f, and not on θ, ϕ, η or ρ.

Next suppose θ is a Lebesgue point of $J(\rho,\eta)$. Then there exists $\delta > 0$ so that if $|\phi| < \delta$ then $m[(\theta - |\phi|,\ \theta + |\phi|)\ \ J(\rho,\eta)] \geq 2|\phi|(1 - \eta)$.

Suppose $|\phi| < \delta$ and $|\phi| < \gamma^{-1}\rho$. Then there exists $\phi_0 \in (\theta - |\phi|,\ \theta + |\phi|)$ such that $\phi_0 \in J(\rho,\eta)$ and $|\phi - \phi_0| \leq 2|\phi|\eta$. Now

$$\|f(\theta + \phi) - T_nf(e^{i\theta}, e^{i(\theta+\phi)})\| \leq C\eta|\phi|^n$$

by taking $\psi = \phi - \phi_0$ and replacing ϕ by ϕ_0 in the preceding inequality.

For $\eta > 0$, almost every θ is a Lebesgue point in some $J(\rho,\eta)$. Hence almost everywhere we have

$$\|F(\theta + \phi) - T_nf(e^{i\theta}, e^{i(\theta+\phi)})\| = 0(|\phi|^n)$$

The same argument can be applied to $F^{(n-1)}(\theta) = (iz\frac{d}{dz})^{n-1} f(e^{i\theta})$, so that almost everywhere we also have

$$\|F^{(n-1)}(\theta + \phi) - F^{(n-1)}(\theta) - x(e^{i\phi} - 1)\ e^{i\theta}\| = 0(|\phi|)$$

for some $x \in X$. In fact

$$x = \frac{d}{dz}\ (iz\ \frac{d}{dz})^{n-1}\ f(e^{i\theta})$$

so that $F^{(n-1)}$ is differentiable at θ and

$$F^{(n)}(\theta) = (iz\ \frac{d}{dz})^n\ f(e^{i\theta})$$

Now arguing as in the preceding theorem

$$\|T_n f(e^{i\theta}, e^{i(\theta+\phi)}) - \sum_{k=0}^{n} \frac{F^{(k)}(\theta)}{k!} \phi^k\| = O(|\phi|^n)$$

and we obtain (iv).

(iv) $\Rightarrow$ (ii) and (iii) is obvious. That (iii) $\Rightarrow$ (i) is essentially the same as (ii) $\Rightarrow$ (i) for the case $n = 1$. We thus prove (ii) $\Rightarrow$ (i).

Consider the functions

$$g_\phi(z) = \phi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^n f(e^{ik\phi} z)$$

By Theorem 5.5 and the A-convexity of X these functions are in $A_0(X)$ uniformly. Hence by Proposition 5.1

$$\lim_{\phi \to 0} g_\phi(z) = g(z) \in H_\infty(X)$$

Thus $(z \frac{d}{dz})^n f(z) \in H_\infty(X)$. Clearly $(z \frac{d}{dz})^k f \in H_\infty(X)$ for $0 \le k \le n-1$ and hence $z^n f^{(n)} \in H_\infty(X)$ so that $f^{(n)} \in H_\infty(X)$.

Theorem 5.7. Let X be a quasi-Banach space. The following conditions on X are equivalent:

(i) X has ARNP

(ii) If $F : \mathbb{R} \to X$ is of analytic type, Lipschitz and 2π-periodic then F is differentiable a.e.

(iii) If $F : \mathbb{R} \to X$ is of analytic type and Lipschitz then F is differentiable a.e.

Proof. Clearly (iii) $\Rightarrow$ (ii). Let us prove (i) $\Rightarrow$ (iii). Let $\tilde{F}$ be the analytic extension of F given by Lemma 3.2. Then set

$$G(z) = \tilde{F}(i(\tfrac{1+z}{1-z}))\ (1 - z)^2$$

Since X has ARNP, X is A-convex and so $\tilde{F}'$ is bounded. Now G is analytic on Δ and G' is also bounded. Thus $G(e^{i\theta})$ is differentiable a.e. by the preceding theorem. This implies that F is differentiable a.e.

(ii) $\Rightarrow$ (i). We show first that X is A-convex; suppose X fails to be A-convex. We shall assume X to be p-normed. Then by Lemma 4.2 and an approximation argument we can produce, for given $\eta > 0$, a polynomial $\psi(z) = x_0 + x_1 z + \dots + x_n z^n$ so that $\|\psi(0)\| = 1$, $\|\psi(z)\| \le 2$ for $|z| \le 1$ and $\|\psi(z)\| \le \eta$ for $|z| = 1$. Arguing as in Theorem 3.5, if we set for m

$$F_m(t) = \tfrac{1}{2} \sum_{k=0}^{n} \frac{x_k}{2km+1} e^{(2km+1)it}$$

then F_m is an analytic polynomial and

$$\lim_{m\to\infty} \|F_m(\pi) - F_m(0)\| = 1$$

Further

$$F_m(t) = \phi_m(e^{it})$$

where $\phi_m'(z) = \tfrac{1}{2}\psi(z^m)$

so that by Proposition 3.3, for $z, z_2 \in \Delta$,

$$\|\phi_m(z_1) - \phi_m(z_2)\| \le C|z_1 - z_2|$$

where $C = C(X)$. Thus by a limiting procedure

$$\|F_m(t) - F_m(s)\| \le C|t - s|$$

where $C = C(X)$.

Combining these remarks we see that if $\eta_n \to 0$ we can construct a sequence G_n of 2π-periodic analytic polynomials so that $G_n(0) = 0$ and

(a) $\|G_n(\pi)\| = 1$

(b) $\|G_n'(t)\| \le \eta_n$ $\qquad t \in \mathbb{R}$

(c) $\|G_n(t) - G_n(s)\| \le L|t - s|$ $\qquad t,s \in \mathbb{R}$

where L is independent of n .

Now let k_n be a sequence of natural numbers with $k_n \to \infty$, and let $H_n(t) = k_n^{-1} G_n(k_n t)$. We further set

$$\rho_n(h) = \sup_t \left\| \frac{H_n(t+h) - H_n(t)}{h} \right\|$$

Since H_n is 2π-periodic and $\|H_n'(t)\| \le \eta_n$ for all t , we see that 1 , by a compactness argument,

$$\lim_{h\to 0} \sup \rho_n(h) \le \eta_n$$

Further

$$\rho_n(h) \le L \qquad h > 0$$

and since

$$\|G_n(t) - G_n(s)\| \le 2\pi L \qquad t \ne s$$

$$\rho_n(h) \le \frac{2\pi L}{k_n h} \qquad h > 0$$

Combining these statements we may determine a subsequence, which we still label G_n, H_n, k_n, ρ_n so that

$$\left(\sum_{n=1}^{\infty} \rho_n(h)^p\right)^{1/p} \le 2L \qquad h > 0$$

and

$$\sum \frac{1}{k_n^{\,p}} < \infty$$

Then let ε_n be a sequence of independent random variables on some probability space (Ω,Σ,P) so that $P(\varepsilon_n = 1) = P(\varepsilon_n = 0) = \frac{1}{2}$. For every $w \in \Omega$, the function

$$\phi_w(t) = \sum_{n=1}^{\infty} \varepsilon_n(w)\, H_n(t)$$

is 2π-periodic, of analytic type and Lipschitz. By assumption ϕ_w is a.e. differentiable. Now by Furbini's theorem there exists $t_0 \in \mathbb{R}$ so ϕ_w is differentiable at t_0 for a.e. w . By Egoroff's theorem we can further find $A \in \Sigma$ with $P(A) > \frac{1}{2}$ so that

$$\lim_{h\to 0} \left\| \frac{\phi_w(t_0+h) - \phi_w(t_0)}{h} - \phi_w'(t_0)\right\| = 0$$

uniformly for $w \in A$.

For each n , there exist w_1 , w_0 A with $\varepsilon_j(w_1) = \varepsilon_j(w_0)$ for $j \ne n$ and $\varepsilon_n(w_1) = 1$, $\varepsilon_n(w_0) = 0$. Thus $H_n(t) = \phi_{w_1}(t) - \phi_{w_0}(t)$ and hence

$$\lim_{h\to 0} \left\| \frac{H_n(t_0+h) - H_n(t_0)}{h} - H_n'(t_0) \right\| = 0$$

uniformly in n.

Thus for any $\delta > 0$ there exists $h_0 > 0$ so that if $0 < h < h_0$, $n \in \mathbb{N}$

$$\left\| \frac{H_n(t_0+h) - H_n(t_0)}{h} \right\| < (\eta_n^p + \delta^p)^{1/p}$$

If $k_n h_0 > 2\pi$ then $[k_n t_0, k_n t_0 + k_n h_0]$ includes a complete period of G_n and so for all t we have

$$\|G_n(t) - G_n(k_n t_0)\| < (\eta_n^p + \delta^p)^{1/p}\,(2\pi)$$

and hence for any pair t,s

$$\|G_n(t) - G_n(s)\| < 2^{1/p+1}\,(\eta_n^p + \delta^p)^{1/p}\pi$$

In particular taking $t = \pi$, $s = 0$

$$(\eta_n^p + \delta^p)^{1/p} > 2^{-1/p-1}\,\pi^{-1}$$

for all large enough n. Thus

$$\delta > 2^{-1/p-1}\,\pi^{-1}$$

which is, of course, a contradiction.

We conclude that X is A-convex.

Now suppose $f \in H_b(X)$; let $g \in A_0(X)$ be such that $g' = f$. Let $G(\theta) = g(e^{i\theta})$. Then by Theorem 5.5 G is Lipschitz and so G is differentiable a.e. By Theorem 5.6, $f \in H_\infty(X)$ so that X has ARNP.

6. Some applications

Let us suppose X is a symmetric (or re-arrangement invariant) quasi-Banach function space on the unit circle $\mathbb{T}$ with normalised Haar measure $(2\pi)^{-1}\,d\theta = d\lambda$. By this, we mean that X is a complex vector subspace of $L_0(\mathbb{T})$ equipped with a quasi-norm $\|\ \|_X$ so that $(X, \|\ \|_X)$ is complete and

(i) $C(\mathbb{T}) \subset X$ and if $f \in C(\mathbb{T})$

$$\|f\|_X \le \beta \|f\|_{C(\mathbb{T})}$$

for some constant β.

(ii) If $\|f_n\|_X \to 0$ then $f_n \to 0$ in $L_0(\mathbb{T})$.

(iii) If $g \in X$, $f \in L_0$ and

$$\lambda(|f| > t) \le \lambda(|g| > t) \qquad t > 0$$

then $f \in X$ and $\|f\|_X \le \|g\|_X$.

It follows easily from Nikishin's theorem [12] and the symmetry of X that (ii) is equivalent to:

(ii)' There exists $r > 0$ and $\gamma < \infty$ so that if $f \in X$ then $f \in L_r(\mathbb{T})$ and

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^r\,d\theta\right)^{1/r} \le \gamma \|f\|_X$$

It follows from Section 3 that X is A-convex if and only if there exists $\alpha > 0$ and a symmetric Banach function space B so that $X = \{f \in L_0 : |f|^\alpha \in B\}$. X then has ARNP if and only if X (or B) fails to contain a copy of c_0. Examples of such spaces X with ARNP are the spaces L_p $(0 < p < \infty)$ the Lorentz spaces

$(L(p,q)$ $(0 < p,q < \infty)$ and Orlicz spaces L_ϕ where ϕ satisfies the Δ_2-condition. The weak spaces $L(p,\infty)$ are A-convex but do contain copies of c_0.
For a fixed symmetric quasi-Banach function space X let H_X be the closed linear span of the functions $e^{i\theta} \to e^{in\theta}$ for $n \geq 0$. It follows from condition (ii)' that $f \in H_X$ implies $f \in H_r$ and so each $f \in H_X$ can be identified with the boundary values of an analytic function, which we also denote f, $f : \Delta \to \mathbb{C}$. In fact

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) \qquad \text{a.e.}$$

and the linear functionals $f \to f(z)$ for $|z| < 1$ are bounded on H_X. If $r < 1$ then $f_r \in C(\mathbb{T}) \subset X$. If

$$f(z) = \Sigma\, a_n z^n \qquad |z| < 1$$

then

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq 1$$

Hence since $\|e^{in\theta}\|_X$ is constant we can define an analytic function $F : \Delta \to H_X$ associated to f by

$$F(z) = \Sigma\, a_n z^n e^{in\theta}$$

so that

$$F(z)(e^{i\theta}) = f(e^{i\theta} z)$$

Now we can relate properties of H_X to properties of vector-valued functions.

Proposition 6.1. If X is A-convex then $f \in H_X$ if and only if $F \in H_\infty(X)$.

Proof. First suppose $f \in H_X$. Then there is a sequence g_n of polynomials so that $\|g_n - f\|_X \to 0$. Let G_n be the associated vector-valued polynomials. Then by the A-convexity of X

$$\|G_n(z)\| \leq C \max_{0\leq\phi\leq 2\pi} \|G_n(e^{i\phi})\|$$

$$= C\|g_n\|_X$$

so that G_n is uniformly bounded and $G_n(e^{i\phi}) \to F(e^{i\theta})$ for all θ. Hence $F \in H_\infty(X)$.
Next suppose $F \in H_\infty(X)$. Then $F(re^{i\theta}) \to F(e^{i\theta})$ for some θ and this implies that $F(r) \to F(1)$ in X. Clearly

$$F(r) = \Sigma r^n a_n e^{in\theta} \in H_X \qquad \text{for } r < 1$$

so that $F(1) = f \in H_X$.

We can translate the results of Section 5 into this setting. The theorems we obtain are due to Krotov [10] in the special case $X = L_p$ $(H_X = H_p)$. We introduce first some notation. If $g \in X$ we say that:

(a) g has a kth global Peano differential if there exist $g_0, g_1, \ldots, g_k$ X so that

$$\|g(e^{i(\theta+\phi)}) - \sum_{\nu=0}^{k} \frac{\phi^\nu}{\nu!} g_\nu(e^{i\theta})\|_X = O(|\phi|^k)$$

(b) g has a kth global Riemann-Schwartz derivative if there exists $g_k \in X$ so that

$$\left\| \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} g(e^{i(\theta+\nu\phi)}) - g_k(e^{i\theta})\, \phi^k \right\| = O(|\phi|^k)$$

Theorem 6.2. Let $f \in H_X$ where X is A-convex and suppose $n \in \mathbb{N}$. Let $f^{(n)}$ denote the nth derivative of f as an analytic function on Δ. Then the following are equivalent:

(i) $f^{(n)} \in H_X$

(ii) f has an nth global Peano differential

(iii) f has an nth global Riemann-Schwartz derivative.

If further X has ARNP these are equivalent to:

(iv) $\displaystyle \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} f(e^{i(\theta+\nu\phi)})) \| = O(|\phi|^n)$

Proof. If we let F be the associated analytic function, then (i) implies $F^{(n)} \in H_\infty(X)$. By Theorem 5.6 there exists θ and $g_0, \ldots, g_n \in H_X$ so that

$$\left\| F(e^{i(\theta+\phi)}) - \sum_{k=0}^{n} \frac{g_k}{k!} \phi^k \right\| = O(|\phi|^n)$$

By suitable translation a similar statement is true when $\theta = 0$ and so f has a nth global Peano differential. Thus (i) => (ii) and clearly (ii) => (iii). For (iii) => (i) note that (iii) implies

$$\left\| \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} F(e^{i(\theta+\nu\phi)}) \right\| \le C|\phi|^n$$

and

$$\lim_{\phi \to 0} \phi^{-n} \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} F(e^{i(\theta+\nu\phi)})$$

exists a.e. Thus again by Theorems 5.6 and 5.5 $F^{(n)} \in H_\infty(X)$ i.e. $f^{(n)} \in H_X$.

For (iv) simply note that in this case $H_\infty(X) = H_b(X)$ and use Theorem 5.5.

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