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SOME APPLICATIONS OF VECTOR-VALUED ANALYTIC AND HARMONIC FUNCTIONS

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1. Introduction

In these notes we describe some developments of ideas from [10] on the theory of analytic functions taking values in a general non-locally convex space. Our main results concern the class of functions $C_\sigma(\mathbb{T}, X)$ where X is a p-Banach space and $\sigma > \frac{1}{p} - 1$. The class C_σ was studied in [10]; a function $f : \mathbb{R} \to X$ is said to be in $C_\sigma(\mathbb{R}, X)$ if f is 2π-periodic and for some constant C and for every interval $I \subset \mathbb{R}$ there is an X-valued polynomial ϕ with degree $\phi \le \sigma$ so that

$$\|f(t) - \phi(t)\| \le C|I|^\sigma \qquad t \in I$$

We show for example in Theorem 6.3 that the class $C_\sigma(\mathbb{T}, X)$ is self-conjugate if $\sigma > \frac{1}{p} - 1$. Precisely we can define vector-valued Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$$

(where the integrals are Turpin-Waelbroeck integrals as described in [10] and [13]) and we show that there is a function $g \in C_\sigma(\mathbb{T}, X)$ with

$$\hat{g}(n) = \hat{f}(n) \qquad n \ge 0$$
$$= 0 \qquad n < 0$$

Thus the Riesz projection acts boundedly on the class $C_\sigma(\mathbb{T}, X)$. This of course generalizes a well-known scalar result (cf. Zygmund [15] p.121). The technique for proving the result leans heavily on the atomic theory of H_p-spaces (cf. [3], [5], [12], [14]).

We then utilize these ideas to characterize certain spaces generated by special atoms studied by de Souza [6] (cf. also [7],[8]). For $\frac{1}{2} < p \le 1$, de Souza considers the space C_p of all distributions f on $\mathbb{T}$ of the form

$$f = \sum_{n=1}^{\infty} c_n b_n$$

where $\Sigma|c_n|^p < \infty$ and each b_n is a special atom, i.e. either $b_n = 1$ or $b_n = |I|^{-1/p}(1_L - 1_R)$ where I is an interval in $\mathbb{T} = (-\pi,\pi)$ and L and R are its left- and right- halves.

In the case $p = 1$, de Souza and Sampson [8] identify this space with the boundary values of the real parts of the functions g in the Banach space S of all analytic $g : \Delta \to \mathbb{C}$ so that

$$\|g\|_S = |g(0)| + \int_\Delta |g'(w)|\, dm(w) < \infty$$

where m is planar Lebesgue measure. Thus $f \in C_1$ if and only if

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t)\, dt$$

is in S. An alternative way to view this result is that the complexification of C_1 (which we denote $aC_{1,1,1}$ later) is identified with the space of harmonic functions $S + \bar{S}_0$ where $\bar{S}_0 = \{g : \Delta \to \mathbb{C} : g(0) = 0,\ \bar{g} \in S\}$. We refer to this as the harmonic completion of S. In this form we recover the result later.

More generally we show that if $\frac{1}{2} < p < 1$ then $aC_{p,1/p,1}$ (the complexification of C_p) is the harmonic completion of the space of analytic functions $\phi : \Delta \to \mathbb{C}$ so that

$$\|\phi\| = |\phi(0)| + \{\int_\Delta |\phi(w)|^p(1-|w|^2)^{p-1}\, dm(w)\}^{1/p} < \infty$$

which was studied in [10], also under the name C_p!

Although at least some of these results appear new, they are clearly closely related to certain results on the atomic theory of Bergman spaces given by Aleksandrov [2] using "standard" q-atoms (cf. Theorem 3.1 of [2]). Aleksandrov's results and techniques can in all probability be used to prove the results we obtain; however we believe the approach here has some independent merit.

2. Basic notation

In this paper we will deal, except when otherwise stated, with complex quasi-Banach spaces. A p-Banach space X $(0 < p \le 1)$ is a complex vector space equipped with a quasi-norm $x \to \|x\|$ satisfying

(i) $\|x\| > 0, \quad x \neq 0$

(ii) $\|\alpha x\| = |\alpha|\,\|x\| \qquad \alpha \in \mathbb{C},\ x \in X$

(iii) $\|x_1 + x_2\|^p \le \|x_1\|^p + \|x_2\|^p \qquad x_1, x_2 \in X$

As usual H_p $(0 < p < \infty)$ denotes the space of functions ϕ analytic on the open unit disk Δ and such that

$$\|\phi\|^p = \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})|^p\, d\theta < \infty$$

H_p can be identified via its boundary values with a closed linear subspace of $L_p = L_p(\mathbb{T})$. $\bar{H}_p$ denotes those functions $\phi : \Delta \to \mathbb{C}$ so that $\overline{\phi(z)}$ is in H_p, and $\bar{H}_{p,0}$ denotes the space of functions $\phi \in \bar{H}_p$ such that $\phi(0) = 0$. In Section 5 we also study $J_{p,0} = H_p \cap \bar{H}_{p,0}$ when both spaces are identified as subspaces of $L_p(\mathbb{T})$. Note that $J_{p,0} = \{0\}$ if $p \ge 1$.

If X is a q-Banach space and $0 < q < p \le 1$, then we may define a (pseudo-) quasi-norm

$$\|x\|_{(p)} = \inf \{(\sum_{i=1}^n \|x_i\|^p)^{1/p} : x_1 + \ldots + x_n = x\}$$

If $\|\ \|_{(p)}$ is a quasi-norm then the completion of $(X, \|\ \|_{(p)})$ is called the p-Banach evelope of X. For H_q the p-Banach envelope has been identified as the Bergman space $B_{q,p}$ of all functions ϕ analytic on Δ and such that

$$\|\phi\|^p_{q,p} = \int_\Delta |\phi(w)|^p\, (1-|w|^2)^{p/q\,-2}\, dm(w) < \infty$$

where m denotes planar Lebesque measure (see [2],[4]).

If X is a quasi-Banach space a function $f : \Delta \to X$ is analytic if

$$f(z) = \sum_{n=0}^\infty x_n z^n \qquad z \in \Delta$$

For $\sigma > 0$ we say $f \in A_\sigma(X)$ if for some constant C we have

$$\|f^{(\nu+1)}(z)\| \le C(1-|z|^2)^{\sigma-\nu-1}$$

where $\nu = [\sigma]$ (cf. [10]). It is shown in [10] that if $\sigma > \frac{1}{p} - 1$ there is a natural correspondence between $A_\sigma(X)$ and the space of linear operators $\mathcal{L}(H_p, X)$ given by

$$T(w^n) = x_n \qquad n \ge 0$$

where $f(z) = \sum_{n=0}^\infty x_n z^n$ or

$$T((1-wz)^{-1}) = f(z)$$

(see [10], Theorem 5.1).

If $\sigma > 0$ and $f \in A_\sigma(X)$ then f extends continuously to $\Delta \cup \mathbb{T}$ and the boundary function $\tilde{f} : \mathbb{T} \to X$ is in class $C_\sigma(\mathbb{T},X)$ which we discuss in the next section ([10], Theorem 5.3). We refer the reader to [10] for a complete discussion of these ideas.

A function $f : \Delta \to X$ is called <u>harmonic</u> if $f(z) = f_1(z) + f_2(\bar{z})$ where f_1 and f_2 are analytic. Harmonic functions are discussed in [11].

In Section 3 we give an account of the class $C_\sigma(\mathbb{T},X)$ leading to a new characterization of this class. Section 4 reviews standard material from the atomic theory of H_p-spaces from perhaps a slightly different viewpoint. In Section 5 we prove a theorem identifying operators on $H_q \oplus \bar{H}_{q,0}$ with the class $C_\sigma(\mathbb{T},X)$ when $\sigma = \frac{1}{q} - 1$, which permits us to obtain our main results on spaces generated by special atoms in Section 6.

Throughout the paper C will be used for a constant which may vary from line to line and depend on parameters p,q,σ,β etc. but is independent of f,g,ϕ,x etc.

3. <u>The class</u> C_σ

Suppose $\sigma > 0$ and that $\nu = [\sigma]$. Let $f : \mathbb{R} \to X$ be any map. We define for any interval $I \subset \mathbb{R}$, $\gamma_{\sigma,I}(f)$ to be the least constant γ, possibly $+\infty$, so that if $J \subset I$ is any bounded interval there is a polynomial $\phi : J \to X$ with degree at most ν so that

$$\|f(t) - \phi(t)\| \leq \gamma |J|^\sigma \qquad t \in J$$

In the case $I = \mathbb{R}$ we write $\gamma_\sigma(f) = \gamma_{\sigma,\mathbb{R}}(f)$. We set $C_\sigma = C_\sigma(\mathbb{R},X)$ to be the class of all functions so that $\gamma_\sigma(f) < \infty$.

In each C_σ we identify the space $C_\sigma(\mathbb{T},X)$ as the subspace of all 2π-periodic functions, which can be interpreted as functions defined on the unit circle in the complex plane under the identification $\theta \to e^{i\theta}$. For $f \in C_\sigma(\mathbb{T},X)$ we set $\|f\|_\sigma = \gamma_\sigma(f) + \max_{t \in \mathbb{R}} \|f(t)\|$.

Our first objective is to reinterpret the class $C_\sigma(\mathbb{R},X)$. For convenience later we consider functions defined only on a dense subset of $\mathbb{R}$. Let D consist of all dyadic rational multiples of π. For $f : D \to X$ we define, for $\sigma > 0$ and $\beta \in \mathbb{N}$, $\delta(\sigma,\beta,f)$ to be the least constant δ so that if $t,h \in D$ then

$$\left\| \sum_{k=0}^{\beta} (-1)^k \binom{\beta}{k} f(t + kh) \right\| \leq \delta |h|^\sigma$$

Let us start with a simple observation.

<u>Lemma 3.1</u>. <u>Suppose</u> $0 \leq t_1 < t_2 < \dots < t_\beta \leq 1$ <u>are fixed. Then there is a constant</u> $C = C(\beta,p)$ <u>so that</u> X <u>is a p-normed quasi-Banach space and</u> $\phi :\ \to X$ <u>is a polynomial of degree at most</u> $\beta - 1$ <u>then</u>

$$\|\phi(t)\| \leq C \max_{1 \leq j \leq \beta} \|\phi(t_j)\| \qquad 0 \leq t \leq 1$$

<u>Proof</u>. Let u_j $(1 \leq j \leq \beta)$ be real polynomials of degree at most $\beta - 1$ so that

$$u_j(t_k) = \delta_{jk}$$

Then

$$\phi(t) = \sum_{k=1}^{\beta} \phi(t_k)\, u_k(t)$$

and the result follows.

<u>Lemma 3.2</u>. <u>There is a constant</u> $C = C(\sigma,\beta,p)$ <u>so that if</u> X <u>is a p-normed quasi-Banach space</u>, $f : D \to X$ <u>is any function and</u> $a,b \in D$ <u>with</u> $a < b$ <u>then there is a polynomial</u> $\phi : \mathbb{R} \to X$ <u>with degree</u> $\phi \leq \beta - 1$ <u>and</u>

$$\|f(t) - \phi(t)\| \leq C(b - a)^{\sigma}\ \delta(\sigma,\beta,f)$$

for $t \in D$ with $a \leq t \leq b$.

Proof. For convenience of exposition let us reduce first to the case $a = 0$. Let 2^N be the least power of two so that $2^N \geq \beta$. For $m \geq 0$ and $1 \leq k \leq 2^m$ set $g_{m,k}$ to be the unique X-valued polynomial of degree at most $\beta - 1$ so that

$$g_{m,k}(t_j) = f(t_j)$$

for $0 \leq j \leq \beta - 1$ where

$$t_j = (\frac{k-1}{2^m} + \frac{j}{2^{m+N}})b$$

We shall set $g_{0,1} = \phi$.

Since

$$\sum_{\ell=0}^{\beta} (-1)^{\ell}\binom{\beta}{\ell}\ g_{m,k}\ (t_{j+\ell}) = 0$$

for $0 \leq j \leq 2^N - \beta$ we can use induction to show

$$\|f(t_j) - g_{m,k}(t_j)\| \leq C\ 2^{-m\sigma}\ b^{\sigma}\delta$$

where $\delta = \delta(\sigma,\beta,f)$ and C depends only on σ,β and p .

Now let

$$\tau_j = (\frac{k-1}{2^m} + \frac{j}{2^{N+m+1}})\ b$$

Then

$$\|g_{m,k}\ (\tau_{2j}) - g_{m+1,2k-1}(\tau_{2j})\| \leq C\ 2^{-m\sigma}\ b^{\sigma}\delta$$

for $0 \leq j \leq \beta - 1$ and so by Lemma 3.1 we can deduce

$$\|g_{m,k}(t) - g_{m+1,2k-1}(t)\| \leq C\ 2^{-m\sigma}\ b^{\sigma}\delta$$

for $\frac{k-1}{2^m} \leq \frac{t}{b} \leq \frac{2k-1}{2^{m+1}}$. Similarly

$$\|g_{m,k}(t) - g_{m+1,2k}(t)\| \leq C\ 2^{-m\sigma}\ b^{\sigma}\delta$$

for $\frac{2k-1}{2^{m+1}} \leq \frac{t}{b} \leq \frac{k}{2^m}$. Now by an induction proof

$$\|\phi(t) - g_{m,k}(t)\| \leq C\ b^{\sigma}\delta$$

for all m,k and $\frac{k-1}{2^m} \leq \frac{t}{d} \leq \frac{k}{2^m}$. Hence we conclude

$$\|\phi(t) - f(t)\| \leq C\ b^{\sigma}\delta$$

for $0 \leq t \leq b$ with $t \in D$.

Theorem 3.3. Let M be a subset of $\mathbb{R}$ containing D and suppose $f : M \to X$ is a map whose graph is closed in $\mathbb{R} \times X$. Then $M = \mathbb{R}$ and $f \in C_{\sigma}(\mathbb{R},X)$ if and only if $\delta(\sigma,\beta,f) < \infty$, for some (and hence for all) $\beta \geq \sigma$.

Proof. If $f \in C_{\sigma}(\mathbb{R},X)$ then $\delta(\sigma,\beta,f) < \infty$; this is an easy consequence of the polynomial approximation.

Conversely suppose $b > a$ with $b,a \in D$. Suppose $m \geq 0$ and $1 \leq k \leq 2^m$ and let $J(m,k) = [a + \frac{k-1}{2^m}(b - a),\ a + \frac{k}{2^m}(b - a)]$. Then there is a polynomial

$g_{m,k} : \mathbb{R} \to X$ with

$$\|f(t) - g_{m,k}(t)\| \le C\, 2^{-m\sigma} (b-a)^{\sigma}\delta \qquad t \in J(m,k)$$

If $\delta < \infty$ we conclude that, since each $g_{m,k}$ is continuous, the set $f(D \cap [a,b])$ is relatively compact. If $a \le t \le b$ then there is a sequence $d_n \in D$ with $d_n \to t$ and $f(d_n)$ converging. Hence $t \in M$ and so $M = \mathbb{R}$ and f is continuous on $\mathbb{R}$.

The conclusion now follows easily from Lemma 3.2.

4. Some remarks on integration theory

Now let us suppose X is a p-normed quasi-Banach space and $\sigma > \frac{1}{p} - 1$. In this case (cf. [10],[13]) it is possible to develop a theory of integration on the class C_σ. Let μ be a regular Borel measure supported on a closed bounded interval $I \subset \mathbb{R}$. Then it is possible to define (cf. [10]) the Turpin-Waelbroeck integral

$$\int f(t)\, d\mu = L_\mu(f)$$

for $f \in C_\sigma$ in such a way that L_μ is a linear map $L_\mu : C_\sigma \to X$ which agrees with standard integration if $\dim f(I) < \infty$ (f is of finite-rank on I) and

$$\left\| \int f(t)\, d\mu \right\| \le C\|\mu\| \left(\max_I \|f(t)\| + |I|^{\sigma}\gamma_{\sigma,I}(f)\right)$$

where $C = C(\sigma,p)$. See [10] for details.

For future use let us make the following observation:

Lemma 4.1. *Let μ be a regular Borel measure supported on I and such that*

$$\int t^j\, d\mu = 0 \qquad j = 0,1,2,\dots,\nu$$

Then for $f \in C_\sigma$

$$\left\| \int f\, d\mu \right\| \le C|I|^{\sigma}\|\mu\|\,\gamma_\sigma(f)$$

Proof. Let g be a polynomial of degree at most ν so that

$$\|f(t) - g(t)\| \le \gamma_{\sigma,I}(f)\, |I|^{\sigma}$$

for $t \in I$. Then $\gamma_{\sigma,I}(f - g) \le \gamma_\sigma(f)$ and so the lemma follows.

Now consider the space $C_\sigma(\mathbb{T},X)$. The next Proposition summarizes a number of results from [10]. In particular we use Lemma 3.2 [10] and the following remark and Theorem 3.4 [10].

Proposition 4.2. *Suppose $\sigma > \frac{1}{p} - 1$. Then there is a constant $C = C(\sigma,p)$ so that if X is a p-normed space and $f \in C_\sigma(\mathbb{T},X)$ there is a sequence g_m in $C_\sigma(\mathbb{T},X)$ with*

(i) rank $g_m (= \dim g_m(\mathbb{R})) < \infty$

(ii) $g_m \in C^{\infty}(\mathbb{R},X)$

(iii) $\|f(t) - g_m(t)\| \le Cm^{-\sigma}\|f\|_\sigma$

(iv) $\|g_m\|_\sigma \le C\|f\|_\sigma$

(v) *For any regular Borel measure on* $[-\pi,\pi]$

$$\lim_{m\to\infty} \int g_m(t)\, d\mu(t) = \int f(t)\, d\mu(t)$$

Now if $\sigma > \frac{1}{p} - 1$ and $f \in C_\sigma(\mathbb{T},X)$ we define the 'Fourier coefficients' $\hat{f}(n)$ $(n \in \mathbb{Z})$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$$

By [10] Lemma 3.3, essentially quoted above,

$$\|\hat{f}(n)\| \leq C\|f\|_\sigma$$

The formal Fourier series of f is the series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{in\theta}$$

In fact if $0 \leq r < 1$ we can define a function

$$f(re^{i\theta}) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, r^{|n|} e^{in\theta}$$

and f is harmonic on the open unit disk Δ.
By summing the series we see that

$$f(re^{i\theta}) = \int_{-\pi}^{\pi} P(r, \theta - \phi)\, f(e^{i\phi})\, d\phi$$

where P is the Poisson kernel

$$P(r,t) = \frac{1-r^2}{1-2r\cos t + r^2}$$

In the weak*-topology on $M[-\pi,\pi]$ we have $P(r,\theta-\phi)\,d\phi \to \delta(\theta_0)$ as $re^{i\theta} \to e^{i\theta_0} \in \mathbb{T}$. Hence by Theorem 3.4 of [10]

$$\lim_{z\to e^{i\theta}} f^*(z) = f(e^{i\theta})$$

We summarize these remarks in:

<u>Proposition 4.3</u>. <u>Suppose</u> $\sigma > \frac{1}{p} - 1$ <u>and</u> $f \in C_\sigma(\mathbb{T},X)$ <u>then</u>

(i) <u>If</u> $\hat{f}(n) = 0,\ n \in \mathbb{Z}$, <u>then</u> $f = 0$

(ii) <u>The function</u>

$$h(z) = f^*(z) \qquad z \in \Delta$$
$$\quad\;\; = f(e^{i\theta}) \qquad z = e^{i\theta}$$

<u>is harmonic on</u> Δ <u>and continuous on</u> $\bar{\Delta}$ (<u>and hence solves the Dirichlet problem for</u> f).

5. <u>Distributions and the space</u> aH_p

Let $C^\infty(\mathbb{T})$ be the Frechet space of all 2π-periodic C^∞-functions on $\mathbb{R}$. The dual of $C^\infty(\mathbb{T})$ can be interpreted as the space of all distributions on $\mathbb{T}$, but we shall identify it with the space $\mathcal{H}$ of all harmonic functions $h : \Delta \to \mathbb{C}$ so that

$$h(re^{i\theta}) = \sum_{n\in\mathbb{Z}} \hat{h}(n)\, r^{|n|} e^{in\theta}$$

where for some C,α we have

$$|\hat{h}(n)| \leq C(|n| + 1)^\alpha \qquad n \in \mathbb{Z}$$

The duality is given by

$$\langle\phi,h\rangle = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta)\, h(re^{i\theta})\, d\theta$$

In $\mathcal{H}$ we identify the Poisson kernel $P(z) = P_z$ for $|z| \leq 1$ by

$$P_z(re^{i\theta}) = \sum_{n\geq 0} z^n r^n e^{-in\theta} + \sum_{n>0} \bar{z}^n r^n e^{in\theta}$$

Note that if $z = e^{it}$ then

$$\langle \phi, P_z \rangle = \phi(t)$$

so that $P(e^{it})$ identified with the Dirac measure δ_t.

If $a \in L_1(-\infty,\infty)$ then a induces an harmonic function $\tilde{a} \in \mathcal{H}$ by setting

$$a(re^{i\theta}) = \sum_{n \in \mathbb{Z}} \alpha_n r^{|n|} e^{in\theta}$$

where

$$\alpha_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\theta) e^{-in\theta} \, d\theta$$

Then if $\phi \in C^\infty(\mathbb{T})$

$$\langle \phi, a \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t)\, a(t)\, dt$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \phi(t) \Big(\sum_{n \in \mathbb{Z}} a(t+2n\pi) \Big) dt$$

We define aH_p to be the subspace $H_p \oplus \bar{H}_{p,0}$ of $\mathcal{H}$ quasi-normed by

$$\|h\| = \max(\|h_1\|_{H_p}, \|h_2\|_{H_p})$$

where

$$h_1(w) = \sum_{n \geq 0} \hat{h}(n) w^n$$

$$h_2(w) = \sum_{n > 0} \hat{h}(-n) w^n$$

We think of aH_p as the "harmonic completion" of H_p. aH_p can be described atomically (cf. [3],[12],[14]). Let $\nu = [\frac{1}{p}]$ and let $A_p \subset L_1(\mathbb{R})$ consist of functions either of the form $1_{[-\pi,\pi]}$ or functions a such that for some interval I we have

(i) $\operatorname{supp} a \subset I$

(ii) $$\int_I |a(t)|^2 \, dt \leq |I|^{1-2/p}$$

(iii) $$\int_I t^j a(t)\, dt = 0 \qquad 1 \leq j \leq \nu$$

Then $h \in aH_p$ if and only if there exist $a_n \in A_p$, $c_n \in \mathbb{C}$ with $\Sigma|c_n|^p < \infty$ and

$$h = \sum_{n=1}^{\infty} c_n a_n$$

(where we identify $\mathcal{H}$ and $\mathcal{D}$ as described above), for the weak*-topology. Furthermore

$$\|h\| \sim \inf ((\Sigma|c_n|^p)^{1/p})$$

over all such representations.

<u>Proposition 5.1</u>. <u>The Poisson map</u> $P : \bar{\Delta} \to aH_p$ $(0 < p < 1)$ <u>is harmonic on</u> Δ <u>and continuous on</u> $\bar{\Delta}$. <u>Furthermore the map</u> $t \to P(e^{it})$ <u>is in</u> $C_\sigma(\mathbb{T}, aH_p)$ <u>where</u> $\sigma = \frac{1}{p} - 1$.

<u>Proof</u>. Set $u : \Delta \to H_p$ to be

$$u(z) = (1 - wz)^{-1}$$

(where $w = re^{i\theta}$), and $v : \bar{\Delta} \to \bar{H}_p$ to be

$$v(z) = (1 - \bar{w}z)^{-1}$$

Then $u, v \in A_\sigma(aH_p)$ and their boundary values $u(e^{i\theta}), v(e^{i\theta})$ are in C_σ (cf. [10]). Note

$$P(z) = u(\bar{z}) + v(z) - 1 \qquad |z| < 1$$

From Proposition 5.1 it is clear that if $T : aH_p \to X$ is a bounded operator then $T(P(z))$ is an harmonic X-valued function whose boundary values $T(P(e^{i\theta}))$ are in class C_σ where $\sigma = \frac{1}{p} - 1$.

In the next section we turn to the converse of this remark and we shall see that a converse is available provided X is "more convex" than aH_p. This result hinges critically on the Turpin-Waelbroeck integral which forms the basis of our results, and also on the atomic theory of Hardy spaces. The reader is refered to [10] for a discussion of the representation of operators on H_p by an analytic X-valued function.

6. Operators on aH_q

Theorem 6.1. Suppose $0 < q < p \le 1$, and that $\sigma = \frac{1}{q} - 1$. Then there is a constant $C = C(p,q)$ so that if X is a p-normed space and $f \in C_\sigma(\mathbb{T}, X)$ then there is a bounded linear operator $T : aH_q \to X$ such that

(i) $T(P(z)) = f^*(z)$ $\qquad |z| \le 1$

(ii) $T(\bar{w}^n) = \hat{f}(n)$ $\qquad n \in \mathbb{Z}$

(iii) $\|T\| \le C\|f\|_\sigma$

(where $f^*(z)$ and $\hat{f}(n)$ are, respectively, the harmonic extension and Fourier coefficients of f given by Proposition 4.3).

Proof. First suppose $f \in C^\infty$ and f is finite-rank. We define $T : \mathcal{H} \to X$ by

$$Th = \langle f,h\rangle$$

$$= \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, h(re^{i\theta})\, d\theta$$

T is certainly bounded on aH_q and verifies (i) - (ii). Suppose now $a \in A_q$. Clearly if $a = 1_{(-\pi,\pi]}$ $(0 \le j \le \nu)$,

$$\|T\tilde{a}\| = \left\| \int_{-\pi}^{\pi} f(\theta) \frac{d\theta}{2\pi} \right\|$$

$$\le C\|f\|_\sigma$$

Now suppose I is any interval and $a \in L^1(\mathbb{R})$ satisfies supp $a \subset I$,

$$\int_I t^j a(t)\, dt = 0 \qquad 0 \le j \le \nu$$

and

$$\int_I |a(t)|^2\, dt \le |I|^{1-2/q}$$

Then

$$Ta = \int_I a(t)\, f(t)\, \frac{dt}{2\pi}$$

By Lemma 4.1

$$\|Ta\| \le C|I|^{\sigma} \left(\int_I |a(t)|\, dt\right) \|f\|_{\sigma}$$

$$\le C|I|^{\sigma+\frac{1}{2}} \left\{\int_I |a(t)|^2\, dt\right\}^{\frac{1}{2}} \|f\|_{\sigma}$$

$$\le C|I|^{\sigma+1-1/q} \|f\|_{\sigma}$$

$$= \|C\|f_{\sigma}$$

It follows from the atomic formulation of aH_q that $\|T\| \le C\|f\|_{\sigma}$.
For the general case we use the approximation procedure of Proposition 4.2. There exist $g_m \in C_{\sigma}(\mathbb{T},X)$ which are C^{∞}, of finite-rank, and satisfy

$$\|g_m\|_{\sigma} \le C\|f\|_{\sigma}$$

$$\int g_m\, d\mu \to \int f\, d\mu$$

for any Borel measure μ supported on $[-\pi,\pi]$. If we associated operators T_m to each g_m then

$$\|T_m\| \le C\|g_m\|_{\sigma} \le C\|f\|_{\sigma}$$

and $T_m h$ converges for a dense subset of $h \in aH_q$. Thus we can define

$$Th = \lim_{m\to\infty} T_m h$$

for $h \in aH_q$ and $\|T\| \le C\|f\|_{\sigma}$. It is easy to see that T has the desired properties.

Now let us consider $J_{q,0}$, which is the subspace of $L_q(\mathbb{T})$ given by $J_{q,0} = H_q \cap \bar{H}_{q,0}$.

<u>Theorem 6.2</u>. <u>Suppose</u> $0 < q < p$ <u>and</u> $\sigma = \frac{1}{q} - 1$. <u>If</u> X <u>is a p-normable space then there is an isomorphism between</u> $\mathcal{L}(J_{q,0}, X)$ <u>and</u> $C_{\sigma}(\mathbb{T},X)$ <u>given by</u> $T \leftrightarrow f$ <u>where</u>

$$T(u(e^{-i\theta})) = f(\theta) \qquad 0 \le \theta \le 2\pi$$

<u>where</u> $u(w) = (1 - wz)^{-1}$

<u>Proof</u>. Since $u(e^{i\theta}) \in C_{\sigma}(\mathbb{T}, J_{q,0})$, it is clear that if $T \in \mathcal{L}(J_{q,0}, X)$ then

$$f(\theta) = T(u(e^{-i\theta}))$$

is in class C_{σ} and $\|f\|_{\sigma} \le C\|T\|$.
Conversely suppose $f \in C_{\sigma}(\mathbb{T},X)$. Then there is an operator $T_0 : aH_q \to X$ so that

$$T_0 P(e^{i\theta}) = f(\theta)$$

Define $S : aJ_{q,0} \to aH_q$ as follows. If $\phi \in J_{q,0}$ then there exist unique $\psi_1 \in H_q$, $\psi_2 \in \bar{H}_{q,0}$ with

$$\phi(w) = \psi_1(w) = \psi_2(w) \qquad w \in \mathbb{T}$$

Set $S\phi = \psi_1 - \psi_2 \in aH_q$ (here $S\phi$ is a harmonic function but also induces a boundary distribution). Let $T = T_0 S$. Then

$$T(u(e^{-i\theta})) = T_0(u(e^{-i\theta})) + T_0(v(e^{i\theta})) - T_0(1)$$

where

$$v(e^{i\theta}) = (1 - e^{i\theta}\bar{w})^{-1}$$

Hence

$$T(u(e^{-i\theta})) = T_0 P(e^{i\theta})$$
$$= f(\theta)$$

and $\|T\| \le \|T_0\| \; \|S\| \le C\|f\|_\sigma$.

Remarks. The case $X = \mathbb{C}$ is due to Aleksandrov [1]. See also [9] for more abstract reasoning that $\mathcal{L}(aH_q, X)$ and $\mathcal{L}(J_{q,0}, X)$ are isomorphic.

Theorem 6.3. (The Riesz projection on $C_\sigma(\mathbb{T},X)$). Suppose $\sigma > \frac{1}{p} - 1$ and X is p-normable. Then if $f \in C_\sigma(\mathbb{T},X)$, then there exist $F \in A_\sigma(X)$ and $g \in C_\sigma(\mathbb{T},X)$ so that

(i) $F(z) = \sum_{n\ge 0} \hat{f}(n) z^n$

(ii) $\hat{g}(n) = \hat{f}(n) \qquad n \ge 0$

$\qquad = 0 \qquad n < 0$

Furthermore $\|F\|_{A_\sigma} \le C\|f\|_\sigma$ and $\|g\|_\sigma \le C\|f\|_\sigma$, where $C = C(\sigma,X)$.

Proof. By Theorem 6.1 there exists $T \in \mathcal{L}(aH_q, X)$ (where $\sigma = \frac{1}{q} - 1$) so that

$$T(P(e^{i\theta})) = f(\theta) \qquad 0 \le \theta \le 2\pi$$

$$\|T\| \le C\|f\|_\sigma$$

Let $F(z) = T(v(z))$ where $v(z) = (1 - z\bar{w})^{-1}$. Then

$$F(z) = \sum_{n\ge 0} f(n) z^n$$

and $F \in A_\sigma(X)$ with $\|F\|_{A_\sigma} \le C\|f\|_\sigma$. Hence [10] F extends continuously to $\bar{\Delta}$ and determines $g : \mathbb{T} \to X$ with $g \in C_\sigma(\mathbb{T},X)$. Clearly $\|g\|_\sigma \le C\|f\|_\sigma$ as required.

7. An application to spaces with special atoms

In this section, we consider certain examples of spaces generated by atoms studied by de Souza [6]. Suppose $\sigma > 0$ and $\beta \in \mathbb{N} \cup \{0\}$ with $\beta \ge \sigma - 1$. We let $B_{\sigma,\beta}$ consist of all functions $a : \mathbb{R} \to \mathbb{R}$ so that either $a = 1_{(-\pi,\pi]}$ or there exists an interval $I \subset \mathbb{R}$ with $I = [\alpha, \alpha + (\beta+1)h]$ so that

$$a(t) = h^{-\sigma} \binom{\beta}{k} (-1)^k$$

for $\alpha + kh < t \le \alpha + (k+1)h$, and $0 \le k \le \beta$, and $a(t) = 0$ if $t \notin I$. The functions $\{\tilde{a} : a \in B_{\sigma,\beta}\}$ form a bounded set in $\mathcal{H}$ (with the weak*-topology from $C^\infty(\mathbb{T})$). We define $aC_{p,\sigma,\beta}$ to be the space of all $h \in \mathcal{H}$ of the form

$$h = \sum_{n=1}^{\infty} c_n \tilde{a}_n$$

(in $\mathcal{D}$) where $\Sigma|c_n|^p < \infty$ and $a_n \in B_{\sigma,\beta}$. $aC_{p,\sigma,\beta}$ is quasi-normed by

$$\|h\| = \inf (\Sigma|c_n|^p)^{1/p}$$

over all such representations.

For the case $\beta = 1$ and $\frac{1}{2} < p \le 1$, these spaces have been studied by de Souza, Sampson and O'Neil. In [8] they identify the real version of $aC_{1,1,1}$. In our language above their result is equivalent to the result that $aC_{1,1,1} \equiv S \oplus \bar{S}_0$ where S is the Banach space of all analytic functions $g : \Delta \to \mathbb{C}$ such that

$$\|g\|_S = |g(0)| + \int_\Delta |g'(w)| \, dm(w) < \infty$$

(where m is planar Lebesgue measure). Here $\bar{S}_0 = \{f \in \mathcal{H} : \bar{f} \in S \text{ and } f(0) = 0\}$.
The case $p = \beta = 1$ and $\sigma < 1$ is studied in [7]. In this case it is similarly shown that $aC_{1,\sigma,1} = S_\sigma + \bar{S}_{\sigma,0}$ where S_σ is the Banach space of analytic $g : \Delta \to \mathbb{C}$ so that

$$\|g\| = |g(0)| + \int_\Delta |g'(w)|(1 - |w^2|)^{\sigma-1} \, dm(w) < \infty$$

Let us define $\Lambda_0 : \mathbb{R} \to L_1(\mathbb{R})$ by

$$\Lambda_0(t) = 1_{[0,t]} - \frac{t}{2\pi} 1_{[0,2\pi]}$$

Let

$$\Gamma_0(\beta,t,h) = \sum_{k=0}^{\beta+1} (-1)^k \binom{\beta+1}{k} \Lambda_0(t + kh)$$

$$= \sum_{k=0}^{\beta+1} (-1)^k \binom{\beta}{k} 1_{[t+kh,\ t+(k+1)h]}$$

Let $\Lambda(t) \in \mathcal{H}$ be defined by $\Lambda(t) = \widetilde{\Lambda_0(t)}$. Then Λ is periodic and if

$$\Gamma(\beta,t,h) = \sum_{k=0}^{\beta+1} (-1)^k \binom{\beta+1}{k} \Lambda(t + kh)$$

then

$$\Gamma(\beta,t,h) = \widetilde{\Gamma_0(\beta,t,h)}$$

Hence $\Gamma(\beta,t,h) \in aC_{p,\sigma,\beta}$ and

$$\|\Gamma(\beta,t,h)\| \le h^\sigma \qquad h > 0$$

<u>Lemma 7.1</u>. $\Lambda \in C_\sigma(\mathbb{T}, aC_{p,\sigma,\beta})$.

<u>Proof</u>. Let $M = \{t \in \mathbb{R} , \Lambda(t) \in aC_{p,\sigma,\beta}\}$. Then $\Lambda : M \to aC_{p,\sigma,\beta}$ has a closed graph.
Let λ be <u>any</u> linear functional on $\mathcal{H}$ vanishing on $aC_{p,\sigma,\beta}$. Then

$$\lambda(\Gamma(\beta,t,h)) = 0$$

for all t,h and so $\lambda(\Lambda(kt))$ is a polynomial in k for all t . If $t \in D$ (the dyadic rational multiples of π) this is periodic and hence constant. Thus

$$\lambda(\Lambda(t)) = \lambda(\Lambda(0)) = 0$$

Thus Λ maps D into $aC_{p,\sigma,\beta}$ i.e. $D \subset M$.
Finally

$$\|\Gamma(\beta,t,h)\| \le h^\sigma$$

implies $S(\sigma,\beta,\Lambda) < \infty$ and so by Theorem 3.3, $M = \mathbb{R}$ and $\Lambda \in C_\sigma(\mathbb{T}, aC_{p,\sigma,\beta})$.

Theorem 7.2. *Suppose* $0 < p \leq 1$, $\sigma > \frac{1}{p} - 1$ *and* $\beta > \sigma - 1$. *Then* $aC_{p,\sigma,\beta}$ is *the space of harmonic functions* h *of the form* $h(z) = f(z) + g(\bar{z})$ *where* $g(0) = 0$; f,g *are analytic on* Δ *and*

$$\int_\Delta |f'(w)|^p (1 - |w|^2)^{p(\sigma+1)-2}\, dm(w) < \infty$$

$$\int_\Delta |g'(w)|^p (1 - |w|^2)^{p(\sigma+1)-2}\, dm(w) < \infty$$

The quasi-norm on $aC_{p,\sigma,\beta}$ *is equivalent to*

$$\|h\| \sim |h(0)| + \{\int_\Delta (|f'(w)|^p + |g'(w)|^p)\,(1 - |w|^2)^{p(\sigma+1)-2} dm(w)\}^{1/p}$$

(m is planar Lebesgue measure $dm = r\,dr\,d\theta$).

Proof. For fixed t, we identify $\Lambda(t)$ in $\mathcal{H}$ by

$$\Lambda(t) = \sum_{n>0} \frac{1-e^{-int}}{in} w^n + \sum_{n>0} \frac{1-e^{int}}{-in} \bar{w}^n$$

If $\sigma > \frac{1}{p} - 1$ we can compute in $aC_{p,\sigma,\beta}$

$$\hat{\Lambda}(n) = \frac{1}{2\pi} \int_0^{2\pi} \Lambda(t)\, e^{-int}\, dt$$

We can perform this integration in $\mathcal{H}$ to find for $n > 0$

$$\hat{\Lambda}(n) = \frac{1}{in} \bar{w}^n$$

$$\hat{\Lambda}(-n) = -\frac{1}{in} w^n$$

while

$$\Lambda(0) = \sum_{n>0} \frac{w^n}{in} - \sum_{n>0} \frac{\bar{w}^n}{in}$$

In particular each of these functions is in $aC_{p,\sigma,\beta}$.

Since $\Lambda \in C_\sigma$, $F \in A_\sigma(aC_{p,\sigma,\beta})$ where

$$F(z) = \sum_{n>0} \hat{\Lambda}(-n) z^n \qquad |z| < 1$$

and its boundary values are in $C_\sigma(aC_{p,\sigma,\beta})$. These boundary values are given by

$$g(t) = -\sum_{n>0} \frac{1}{in} e^{int} w^n$$

(and these functions are in $aC_{p,\sigma,\beta}$). See Theorem 6.3.

Now let $R : \mathcal{H} \to \mathcal{H}$ be the Riesz projection i.e.

$$Rh = \sum_{n\geq 0} h(n) w^n$$

Then

$$R(\Lambda(t)) = \sum_{n>0} \frac{1-e^{-int}}{in} w^n$$

$$= g(t) - g(0)$$

Thus $R \circ \Lambda \in C_\sigma(\mathbb{T}, aC(p,\sigma,\beta))$ and hence

$$\|R(\Gamma(\beta,t,h))\| \leq Ch^\sigma$$

It follows from the atomic definition of $aC_{p,\sigma,\beta}$ that R maps $aC_{p,\sigma,\beta}$

boundedly into itself.

Now suppose $\sigma = \frac{1}{q} - 1$ and $0 < q < p$. Since $F \in A_\sigma$, $z^{-1}F \in A_\sigma$ ([10] Lemma 5.4) and there is a bounded linear operator $T : H_q \to aC_{p,\sigma,\beta}$ so that

$$Tw^n = -i\Lambda(-(n+1)) \qquad n \geq 0$$

$$= \frac{w^{n+1}}{n+1}$$

(cf. [10] Theorem 5.1). T extends to the p-Banach envelope of H_q i.e. the Bergman space $B(q,p)$ of all functions ϕ analytic on Δ so that

$$\|\phi\|_{q,p}^p = \int_\Delta |\phi(w)|^p (1 - |w|^2)^{p/q-2}\, dm(w) < \infty$$

([2],[4]).

If ϕ is analytic, $\phi \in aC_{p,\sigma,\beta}$ and $\phi' \in B(q,p)$ then

$$\|\phi\| \leq \|T\| \quad \|\phi'\|_{q,p}$$

Finally set $S : aC_{p,\sigma,\beta} \to \mathcal{H}$ to be the map $S\phi = (R\phi)'$ i.e.

$$S\phi = \sum_{n>0} n\phi(n)\, w^{n-1}$$

Then

$$S(\Lambda(t)) = (-i) \sum_{n\geq 1} (1 - e^{-int})\, w^{n-1}$$

$$= (-i)(v(1) - v(e^{-it}))$$

where

$$v(z) = z(1 - wz)^{-1}$$

In $B(q,p)$, $v \in A_\sigma$ and its boundary values belong to C_σ. Hence $S(\Lambda(t))$ $C_\sigma(\mathbb{T}, B(q,p))$ and so

$$\|S(\Gamma(\beta,t,h))\|_{q,p} \leq Ch^\sigma$$

It follows that S maps $aC_{p,\sigma,\beta}$ boundedly into $B(q,p)$. If ϕ is analytic in $aC_{p,\sigma,\beta}$ then $S\phi = \phi' \in B_{q,p}$ and

$$\|\phi'\|_{q,p} \leq C\|\phi\|$$

Combining we see that if ϕ is analytic, $\phi \in aC_{p,\sigma,\beta}$ if and only if $\phi' \in B(q,p)$. The boundedness of the Riesz projection R on $aC_{p,\sigma,\beta}$ implies that $aC_{p,\sigma,\beta}$ splits as a direct sum of an analytic and a conjugate-analytic part. In view of the fact that the defining atoms in $B_{\sigma,\beta}$ are real functions it is clear that $\phi \in aC_{p,\sigma,\beta}$ if and only if $\bar{\phi} \in aC_{p,\sigma,\beta}$. Thus the statement of the theorem follows quickly.

Remark. Note $aC_{p,\sigma,\beta}$ is independent of $\beta \geq \sigma - 1$. In the case $\frac{1}{2} < p < 1$, $\sigma = \frac{1}{p}$, the spaces C_p of de Souza now are represented as the harmonic completions of the spaces of analytic functions $\phi : \Delta \to \mathbb{C}$ such that

$$\int_\Delta |\phi'(w)|^p (1 - |w|^2)^{p-1}\, dm(w) < \infty$$

If $p = 1$ and $\sigma < 1$, $aC_{p,\sigma,\beta}$ is the harmonic completion of the space of all $\phi : \Delta \to \mathbb{C}$ so that

$$\int_\Delta |\phi'(w)| (1 - |w|^2)^{\sigma-1}\, dm(w) < \infty$$

(cf. [7]).

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