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The Maharam Problem

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1. Introduction. The Maharam problem for submeasures is a classical problem in measure theory dating from 1947, which has a very simple and natural formulation but seems curiously elusive. The problem is quite fascinating in its own right but has achieved some special significance in functional analysis since its reappearance as the Control Measure Problem. In fact it is related to a number of fundamental questions in the theory of F-spaces (general non-locally convex complete metric linear spaces).

In this survey, we will give a somewhat personalized view of the evolution of the problem and its ramifications. As the reader will see, these sometimes wander a little distance from the original formulation.

We would like to take this opportunity to acknowledge our debt in understanding the history of the Maharam problem to the review of [29] by Z. Lipecki [33]. Another valuable reference is the Commentary by Maharam on the original von Neumann problem in the recent edition of the Scottish Book [37] p. 242.

2. The Maharam problem for Boolean algebras. Suppose $\mathcal{A}$ is a Boolean algebra. The $\mathcal{A}$ is called a *measure algebra* if it is complete and there is a strictly positive measure ϕ on $\mathcal{A}$, i.e. a map $\phi : \mathcal{A} \to \mathbf{R}$ such that:

(M1) $\phi(a) > 0$ if and only if $a \neq 0$.
(M2) $\phi(a+b) = \phi(a) + \phi(b)$ if $ab = 0$.
(M3) $\phi(a_n) \downarrow 0$ whenever $a_n \downarrow 0$.

On 4 July, 1937, von Neumann asked in the Scottish Book ([41], Problem 163) for an internal characterization of measure algebras. He noted that any measure algebra has the Countable Chain Condition:

(CCC) If $(a_i : i \in I)$ is a disjoint family of non-zero elements then I is countable.

A first guess might be that a complete Boolean algebra with (CCC) is a measure algebra. But this is false: as von Neumann notes the algebra of regular open subsets of $[0,1]$ is a counterexample. This example may also be described as the σ–algebra of Borel sets in $[0,1]$ reduced modulo the ideal of sets of first (Baire) category. Thus von Neumann suggested adding another condition, the weak distributivity law:

(WDL) If $(a_{mn})_{m,n\in\mathbf{N}}$ is such that for each m the sequence $(a_{mn})_{n=1}^{\infty}$ is increasing then

$$\inf_m \sup_n a_{mn} = \sup_{n(m)} \inf_m a_{m,n(m)}$$

where the second sup is taken over all maps $m \to n(m)$.

Shortly after this (in 1941-2) these problems were studied by Maharam who published her work in 1947 ([36]). Her viewpoint was topological. Given a complete Boolean algebra, order-convergence of sequences induces a convergence structure on $\mathcal{A}$. This structure is given by a topology precisely when (WDL) holds. For a measure algebra it is given by a (complete) metric: $d(a,b) = \phi(a+b)$.

These considerations lead her to consider what we might now call *submeasure algebras.* $\mathcal{A}$ is a submeasure algebra if it is complete and there is a map $\phi : \mathcal{A} \to \mathbf{R}$ (a *Maharam submeasure*) satisfying (M1), (M3) and (M4):

(M4) $\phi(a) \leq \phi(a+b) \leq \phi(a) + \phi(b)$ whenever $ab = 0$.

Then $\mathcal{A}$ is a complete metrizable topological group under the metric $d(a,b) = \phi(a+b)$. Further $\mathcal{A}$ has both (CCC) and (WDL). Further Maharam shows that if (CCC) and (WDL) imply that $\mathcal{A}$ is a submeasure algebra then the *Souslin hypothesis* holds. Since the Souslin hypothesis is independent of (ZFC) ([16], [55]) this, in a certain sense, resolves the question of von Neumann. It also leads to:

THE MAHARAM PROBLEM. *Is every submeasure algebra a measure algebra.*

The von Neumann problem of identifying measure algebras internally was again studied by Kelley ([31]). Using the Hahn-Banach theorem he was able to establish criteria for a complete Boolean algebra to be a measure algebra. However, these results do not provide any immediate information on the Maharam problem.

3. The Control Measure Problem. Let us now change viewpoint slightly from Boolean algebras to set functions. let Σ be an algebra of subsets of some set Ω. We say that a map $\phi : \Sigma \to \mathbf{R}$ is a *submeasure* if

(S1) $\phi(\emptyset) = 0$

(S2) $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ for $A, B \in \Sigma$.

If, additionally, Σ is a $\sigma-$algebra we say that ϕ is continuous or a *Maharam* submeasure if:

(S3) $\phi(A_n) \downarrow 0$ whenever $A_n \downarrow \emptyset$.

If ϕ is a Maharam submeasure, then $\Sigma_0 = \{A : \phi(A) = 0\}$ is a $\sigma-$ideal and Σ/Σ_0 is a submeasure algebra. Thus Mahram's problem may be reformulated as:

THE MAHARAM PROBLEM (II). *If ϕ is a Maharam submeasure on a $\sigma-$algebra Σ, does there exist a (countably additive, positive finite) measure λ on Σ equivalent to ϕ, i.e. such that $\lambda(A) = 0$ if and only if $\phi(A) = 0$.*

In this formulation, it is possible to see connections with functional analysis. Maharam submeasures arise naturally in Banach space theory without perhaps being so identified. Let us suppose X is a Banach space and $F : \Sigma \to X$ is a (countably additive) vector measure. The semi-variation of F is defined by

$$\|F\|(A) = \sup(\|F(B)\| : B \subset A,\ B \in \Sigma).$$

The semi-variation $\|F\|$ is then a Maharam submeasure. The Maharam problem for this submeasure is answered by a classical early result on vector measures, the Bartle-Dunford-Schwartz theorem ([7], p. 14 or [3]). This asserts that $\|F\|$ is equivalent to a measure λ. This was taken further by Rybakov [54] who shows that we can actually pick $x^* \in X^*$ so that $|x^* \circ F|$ is equivalent to $\|F\|$.

Thus, the Bartle-Dunford-Schwartz theorem is a partial answer to the Maharam problem for certain special submeasures. Rybakov's theorem hints at the importance of the local convexity of X, so that we can use the Hahn-Banach theorem to obtain a very rich dual space. If we instead assume that X is merely an F-space (a complete metric linear space) and that $x \to \|x\|$ is an F-norm (see [27]) then we can still define the semi-variation of an X-valued measure F as above and the validity of the Bartle-Dunford-Schwartz theorem becomes an open problem:

THE CONTROL MEASURE PROBLEM. *let X be an F-space and suppose $F : \Sigma \to X$ is a vector measure. Does there exist a measure λ on Σ which controls F, i.e. $\lambda(A) = 0$ implies $F(A) = 0$.*

The Control Measure Problem is completely equivalent to the Maharam Problem. This requires two elementary observations. First, if a Maharam submeasure ϕ has a control measure λ (i.e. $\lambda(A) = 0$ implies $\phi(A) = 0$) then it also has an equivalent measure μ. To see this let B be a set of maximal $\lambda-$ measure so that $\phi(B) = 0$ and set $\mu(A) = \lambda(a \setminus B)$. Secondly if ϕ is a Maharam measure on Σ we can topologize the space $L_0(\phi)$ of all $\Sigma-$measurable functions (modulo sets of $\phi-$measure zero) with the topology of convergence in $\phi-$measure. Then $L_0(\phi)$ is an F-space with F-norm

$$\|f\| = \inf\{\epsilon : \phi(|f| > \epsilon) < \epsilon\}.$$

The vector measure $F(A) = \chi_A$ then has a control measure if and only if ϕ has a control measure.

During the late sixties and early seventies there was a tremendous surge of interest in the theory of vector measures in Banach spaces (see Diestel-Uhl [7]). Quite naturally, there was a corresponding move to establish the foundations of a theory in more general spaces (F-spaces or even topological groups); see [9], [10], [11], [17], or [59] for example. Strong motivation for such a theory is provided by the work of Metivier and Pellaumail [40] relating stochastic integrals to vector measures valued in the space of measurable functions, $L_0(0,1)$. In fact the development of vector measure theory in this framework has proved richly rewarding for the study of general F-spaces. Prior to 1970, very little had been done in this area but questions related to vector measures proved a valuable stimulus. In this context we mention generalizations of the Orlicz-Pettis theorem ([21]) or the question of boundedness of the range of a general vector measure ([26], [58] or [60]).

Thus the equivalence of the Maharam and Control Measure problems resulted in a resurgence of interest in this question. Submeasures were studied in several papers in the early seventies both from the viewpoint of Boolean algebras and, in more disguised form, from the vector measure angle (see e.g. [5], [6], [8], [9], [10], [11], [15], [46], [47], [48] and [56]).

Let us now turn to a discussion of the Control Measure problem. It is easy to show that the solution is positive if we consider an F-space X with a separating family of continuous linear functionals; this essentially reduces the problem to the Bartle-Dunford-Schwartz theorem. Thus, in the quest for a counterexample, we are naturally led to the classical examples of spaces with trivial dual, the spaces $L_p(0,1)$ where

$0 < p < 1$. However, even here the answer is positive although it lies much deeper. The key is a factorization theorem of Maurey [38].

THEOREM 1. *Let Ω be a compact Hausdorff space and let $T : C(\Omega) \to L_p$ be a bounded linear operator. Then there exists $g \in L_0$ with $g(s) > 0$ a.e. so that T_0 maps $C(\Omega)$ into L_2 where $T_0 f = g.Tf$.*

Now if $F : \Sigma \to L_p$ is a vector measure, we may take Ω to be the Stone space of Σ and define the operator $T : C(\Omega) \to L_p$ so that if $A \in \Sigma$ then $T(\chi_A) = F(A)$. (Here we identify Σ with the algebra of clopen sets of Ω.) Now applying the Bartle-Dunford-Schwartz theorem to the vector measure $F_0(A) = T_0(\chi_A)$ gives the result. So we have:

COROLLARY 2. *If $F : \Sigma \to L_p$ is a vector measure then F has a control measure.*

A key step in the above proof is that T can be well-defined; this depends on the easily verified fact that the convex hull of the range of F, $\operatorname{co} F(\Sigma)$ is bounded in L_p. If we try to extend the result to L_0 this is more difficult. In fact Maurey and Pisier [39] and Kashin [30] showed that if $F(\Sigma)$ is bounded in L_0 then so is $\operatorname{co} F(\Sigma)$. However Turpin [60] showed that there are vector measures whose range $F(\Sigma)$ is unbounded. Whether this could happen in L_0 remained open until Talagrand [58] and the author, Peck and Roberts [26] established that every L_0-valued vector measure is bounded. This then enables us to state:

THEOREM 3. *If $F : \Sigma \to L_0$ is a vector measure, then F has a control measure.*

The conclusion of this discussion is that if the Control Measure Problem has a negative solution, it will require a new space outside the most obvious examples.

4. Pathological submeasures and pathological spaces. Let us suppose that ϕ is a submeasure defined on some algebra of sets Σ. We may, by a standard identification, regard Σ as consisting of the algebra of clopen subsets of some totally disconnected compact Hausdorff space Ω. It is then possible to describe precisely the conditions under which ϕ may be extended to a Maharam submeasure on the σ-algebra of Borel subsets of Ω. We require that:

(S4): If $A_n \in \Sigma$ is a disjoint sequence then $\lim_{n\to\infty} \phi(A_n) = 0$.

A submeasure satisfying (S4) is called an exhaustive submeasure. We can now reformulate the Maharam problem in yet another way:

THE MAHARAM PROBLEM (III). *If ϕ is an exhaustive submeasure on an algebra Σ does there exist a finitely-additive measure λ on Σ equivalent to ϕ in the sense that $\lim \phi(A_n) = 0$ if and only if $\lim \lambda(A_n) = 0$.*

Let us remark here that a finitely additive vector measure induces an exhaustive submeasure if and only if it is strongly bounded ([7]) or exhaustive.

In [6], Christensen and Herer went on to reformulate the problem in terms of *pathological submeasures.* They defined ϕ to be pathological if for every finitely-additive measure λ on Σ with $0 \leq \lambda \leq \phi$ we have $\lambda = 0$. They then showed that the existence of a Maharam submeasure non-equivalent to measure implies the existence of a non-trivial pathological Maharam submeasure, or equivalently the existence of a pathological exhaustive submeasure defined on some algebra. Since the converse statements are essentially trivial, we can rephrase the question:

THE MAHARAM PROBLEM (IV). *Does there exist a non-trivial exhaustive pathological submeasure?*

So we may ask the simpler question of whether pathological submeasures exist at all. Several authors (Popov [46], [47], Christensen-Herer [6] and Talagrand [57]) showed that there are nontrivial pathological submeasures. The problem is to build in the extra condition of exhaustivity.

Let us now describe an example (essentially that of Talagrand [57], although it was independently discovered by Roberts). We start by making a finite algebra $\Sigma = 2^\Omega$ and a normalized submeasure ϕ so that if λ is a measure on Ω with $0 \leq \lambda \leq \phi$ then $\lambda(\Omega)$ is small. Let $n \in \mathrm{N}$ and let Ω consists of all subsets of $\{1, 2, \dots, 2n\}$ of cardinality n. Thus $|\Omega| = \binom{2n}{n}$. For $1 \leq i \leq 2n$ set $C_i = \{\omega : i \in \omega\}$. Then $\{C_1, \dots, C_{2n}\}$ is a family of subsets of Ω with the property that any covering of Ω by a subcollection uses at least $n+1$ sets. If $A \subset \Omega$ we set $\phi(A) = \frac{k}{n+1}$ where k is the cardinality of a minimal covering of A by sets C_j. In the space of functions on Ω we have

$$\chi_\Omega = \frac{1}{n} \sum_{j=1}^{2n} \chi_{C_j}$$

and hence if λ is a measure with $0 \leq \lambda \leq \phi$ we have

$$\lambda(\Omega) \leq \frac{1}{n} \sum_{j=1}^{2n} \phi(C_j) \leq \frac{2}{n+1}.$$

Now using this we construct a sequence of finite sets Ω_n and normalized submeasures ϕ_n on 2^{Ω_n} such that if $0 \leq \lambda \leq \phi_n$ then $\lambda(\Omega_n) \leq \epsilon_n$ where $\lim \epsilon_n = 0$. Consider the product space $\Omega = \prod_{n=1}^{\infty} \Omega_n$ and let Σ be the algebra generated by all cylinder sets, i.e. sets of the form $A \times \prod_{j \neq n} \Omega_j$, where $A \subset \Omega_n$. We define

$$\phi_0(A \times \prod_{j \neq n} \Omega_j) = \phi_n(A)$$

and then, if $B \in \Sigma$ we define

$$\phi(B) = \inf \sum_{k=1}^{N} \phi_0(C_k)$$

where the infimum is taken over all finite coverings $\{C_1, \dots, C_N\}$ of B by cylinder sets. We can check that $\phi(\Omega) = 1$ and if $0 \leq \lambda \leq \phi$ is a finitely additive measure then

$$\lambda(A \times \prod_{j \neq n} \Omega_j) \leq \phi_n(A).$$

for all $A \subset \Omega_n$. Hence $\lambda(\Omega) \leq \epsilon_n$ for all n and the result follows.

As can be seen, the above construction is quite elementary but closer inspection suggests that it is too simple to work to construct an exhaustive pathological submeasure.

In their paper [6], Christensen and Herer used a different more involved existence proof, and then went on to use pathological submeasures to build some examples of F-spaces (or, more precisely, topological groups). If ϕ is a pathological submeasure on some algebra Σ then the space $L_0(\phi)$ can be defined as the completion of the simple functions with respect to the F-norm $\|f\| = \inf\{\epsilon : \phi(|f| > \epsilon) < \epsilon\}$. The basic idea of Christensen and Herer is that if ϕ is pathological then any continuous linear operator $T : L_0(\phi) \to L_0(\mu)$, where μ is a measure, vanishes. Motivated by these ideas, the author and Roberts [28] defined a point x in an F-space X to be *pathological* if whenever $T : X \to L_0(\mu)$ is continuous, and μ is a measure, then $Tx = 0$. A space X is *pathological* if every $x \in X$ is pathological. Thus the Christensen-Herer result shows that from pathological submeasures we can build pathological F-spaces. The converse route can also be followed. Indeed, suppose $x \in X$ is patholigical and non-zero. Let $X^\sharp$ be the algebraic dual and set $\Omega = \{f \in X^\sharp : f(x) = 1\}$. Let $\Sigma = 2^\Omega$ and define for $A \in \Sigma$,

$$\phi(A) = \inf \sum_{i=1}^{n} \|x_i\|$$

where the infimum is taken over all finite sets $x_1, \ldots, x_n$ such that

$$\sum_{i=1}^{n} |f(x_i)| \geq 1$$

whener $f \in A$. Then it may be shown that ϕ is pathological.

There are more elementary examples of pathological spaces, however. For $0 < p < 1$ consider the space $L_p(\mathbf{T})$ of all complex-valued Borel functions on the unit circle $\mathbf{T}$ with the quasi-norm:

$$\|f\|_p = \left(\int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}}.$$

Let H_p be the closed subspace generated by the functions $\{e^{in\theta} : n \geq 0\}$. Then the quotient space L_p/H_p is pathological. The author showed this in [20] via a representation theorem for operators on L_p, but an alternative neat proof has been given by Aleksandrov [1]. We begin with the observation that by virtue of a celebrated theorem of Nikishin [42], it will suffice to show that if $0 < q < p$ and $T : L_p \to L_q$ is a bounded operator with $T(H_p) = \{0\}$ then $T = 0$. Define a map h on the unit disk Δ with $h : \Delta \to L_p$ by

$$h(z) = (1 - ze^{-i\theta})^{-1}.$$

Then h is analytic on Δ (cf. [1]) in the sense that it has a power-series expansion. Further it extends continuously to $\mathbf{T}$ and $h(z) \in H_p$ if $|z| = 1$. Thus $T \circ h$ is also analytic and $T \circ h(z) = 0$ if $|z| = 1$. Using the fact that the standard quasi-norm on L_p is plurisubharmonic this implies that $T \circ h(z) = 0$ for all $z \in \Delta$. It can then easily be argued (for example by considering the power series for $T \circ h$) that this forces $T = 0$.

Let us remark that although the example given is complex there is no problem in then showing that the underlying real space is also pathological.

This is a suitable moment to mention another very fundamental problem for F-spaces which seems to have some connection with the Maharam problem.

THE ATOMIC SPACE PROBLEM. *Does every infinite-dimensional F-space have a proper closed infinite-dimensional subspace?*

It is remarkable that this simply stated problem, which has been around for a number of years in some form or other (cf. [32]), should be apparently so difficult. We call a space atomic if it has no proper closed infinite-dimensional subspace. It seems that the problem should be split into two cases, depending on whether we require the space to be locally bounded (a quasi-Banach space) or not.

If X is a quasi-Banach space which is atomic then it is pathological. To see this we first observe that L_p has no atomic subspaces. In fact, the recently developed theory of stable Banach spaces has been extended by Bastero [4] to show that every subspace of L_p contains a subspace isomorphic to ℓ_q for some $p \le q \le 2$. Then Nikishin's theorem allows us to show that if X is not pathological then there exists $0 < p < 1$ and a non-zero operator $T : X \to L_p$. We may assume T is not of finite rank and then show that there is a proper closed subspace $Y \sim \ell_q$ of the closure of $T(X)$ which intersects $T(X)$ in an infinite-dimensional subspace. Then $T^{-1}(Y)$ is a proper closed subspace of X.

In the non-locally bounded case we do not know if L_0 has any atomic subspaces. Recently, Reese [49] has constructed an F-space which is very nearly atomic. More precisely she constructs a space X with a sequence V_n of finite-dimensional subspaces such that dim $V_n \to \infty$ and whenever we pick $x_n \in V_n$ such that an infinite number of the x_n's are non-zero then $\{x_n\}$ spans a dense subspace of X. It is unknown whether a similar example can be made which is also locally bounded.

It seems to the author that there is some underlying relationship between the Maharam Problem and the Atomic Space Problem, although apart from the above discussion of pathological spaces there is no really concrete evidence.

We also note here that most of the quasi-Banach spaces that arise naturally in analysis are very far from being pathological. The author has introduced the class of natural spaces which have "sufficiently many" operators into some $L_0(\mu)$ space, and most of the spaces of functions commonly studied are natural (see [23], [24]). A closely related concept for complex spaces is that of A-convexity; a quasi-Banach space is A-convex if it has an equivalent plurisubharmonic quasi-norm. Natural spaces are A-convex but the converse is false [24]; the argument used by Aleksandrov above for the pathology of L_p/H_p essentially proves that this space admits no non-zero bounded operators into an A-convex space. It may be shown by using a recent argument due to Ghoussoub and Maurey [13] that A-convex spaces cannot be atomic, and must contain basic sequences.

5. Uniformly exhaustive submeasures. If a submeasure is to be equivalent to a finitely-additive measure then it must satisfy a uniform version of exhaustivitiy. We say that a submeasure ϕ is ***uniformly exhaustive*** if, given $\epsilon > 0$ there exists $N = N(\epsilon)$ so that for any N disjoint sets $A_1, \ldots, A_N \in \Sigma$,

$$\min_{1 \le i \le N} \phi(A_i) < \epsilon.$$

Uniformly exhaustive submeasures appear to have been introduced in [56] and have been studied in [35], [48] and [57]. In this section, we consider a reduced form of Maharam's Problem (see Talagrand [57]).

REDUCED MAHARAM PROBLEM. *Is every uniformly exhaustive submeasure equivalent to a measure.*

From the point of view of vector measures this is an entirely natural restriction. In fact if $F : \Sigma \to X$ is a vector measure with relatively compact range then its semivariation $\|F\|$ is uniformly exhausitive. In the late seventies the author had become interested in precisely this situation. It will perhaps be of some interest to describe the chain of ideas in some detail.

An important open question in the early seventies was to resolve whether every compact convex subset of an F-space has extreme points; clearly, a closely related problem was whether the set has a locally convex topology. The author achieved certain positive results by considering the following question: if X is a Banach space and Y is an F-space and $T : X \to Y$ is a compact operator, under what assumptions on X can one conclude that the closure of the set $T(B_X)$ is locally convex. In fact it was shown in [18] that it is sufficient that X be reflexive; for more recent results see Godefroy-Kalton [14]. The same techniques then yielded:

THEOREM 4 [18]. *Let X be an F-space and let $F : \Sigma \to X$ be a countably additive vector measure such that $\operatorname{co} F(\Sigma)$ is relatively compact. Then $\overline{\operatorname{co}} F(\Sigma)$ is locally convex if and only if F has a control measure.*

We may remark further that a version of Liapunoff's theorem is true here, namely that if F is nonatomic and $F(\Sigma)$ is relatively compact then $\operatorname{co} F(\Sigma) \subset \overline{F(\Sigma)}$. If Σ is any algebra and $F : \Sigma \to X$ is an additive map we shall say that a set of the form $\operatorname{co} F(\Sigma)$ is a *zonoid*. The author was then led to the following version of Maharam's problem.

ZONOID PROBLEM. *Is every compact zonoid locally convex?*

At the time [18] was submitted I learned of the work of James Roberts [52] who had resolved the extreme point problem for compact convex sets negatively. In fact, Roberts [51] showed that when $0 < p < 1$ L_p contains a compact convex set which has no extreme points and is of course not locally convex. The naive hope that these techniques would immediately yield a non-locally convex compact zonoid (and hence a counterexample to the Maharam problem) is quickly destroyed by our observations in the previous section that every vector measure valued in L_p has a control measure, from which it follows that every compact zonoid in L_p is locally convex. Nevertheless I was optimistic that something along these lines might work; I explained my ideas to Roberts in 1977 and during the years 1980-2 (when he was visiting Missouri) we continued our investigations.

I should say immediately that Roberts was not so optimistic about my approach and preferred a more direct assault on the problem. Events were to prove him right, but nonetheless I will describe how I hoped the argument would go.

In order to construct a compact convex set without extreme points, Roberts had introduce the key idea of a *needle-point.* A point $x \in X$ is called a needle-point if for every $\epsilon > 0$ there exist $x_1, \ldots, x_n \in X$ so that $x \in \mathrm{co}\{x_1, \ldots, x_n\}$ and so that if $y \in \mathrm{co}\{x_1, \ldots, x_n\}$ then for some $0 \leq t \leq 1$ we have $\|y - tx\| < \epsilon$. In [51] he shows that every point of L_p for $0 < p < 1$ is a needle-point. Once this is established then a compact convex set without extreme points can be built by an inductive procedure.

In retrospect, needle-points can best be understood from the point of view of the three-space problem for local convexity. In [19], [50] and [53], independently, were produced three examples of a non-locally convex quasi-Banach space X and a non-zero vector $e \in X$ so that $X/[e]$ is isomorphic to a Banach space (in [19] and [50] this space is ℓ_1). Such a point e is actually a very strong form of needle-point. Let us now describe the argument used in both [19] and [50]. One first makes a real-valued functional F defined on a dense subspace E of ℓ_1 which is homogeneous and satisfies an approximate additivitiy condition:

$$|F(x+y) - F(x) - F(y)| \leq C(\|x\| + \|y\|).$$

Then consider $E \oplus \mathbf{R}$ quasi-normed by

$$\|(x, \alpha)\| = \|x\| + |F(x) - \alpha|.$$

The completion of this space is X if $e = (0,1)$ $X/[e]$ is isomorphic to ℓ_1. Finally X is locally convex if and only if there exists a linear map G defined on E so that for some constant K we have $|F(x) - G(x)| \leq K\|x\|$. In order to give an example where such a linear approximation is impossible Ribe [50] defined

$$F(x) = \sum_{n=1}^{\infty} x_n \log |x_n| - S \log S$$

where $S = \sum_{n=1}^{\infty} x_n$ and E is the subspace of ℓ_1 of all finitely non-zero sequences.

If we wish to build a non-locally convex compact zonoid, the only change in the above reasoning is that we must insist that $X/[e]$ is isomorphic to c_0 instead of ℓ_1. The construction of the nearly additive F can then be reduced to a problem on set functions. Let Σ be an algebra of sets. We say $f : \Sigma \to \mathbf{R}$ is approximately additive if whenever $A, B \in \Sigma$ with $A \cap B = \emptyset$ then

$$|f(A \cup B) - f(A) - f(B)| \leq 1.$$

(AASF) [22]. *Does there exist an absolute constant K so that for every approximately additive set function f there is an additive set function $g : \Sigma \to \mathbf{R}$ with $|f(A) - g(A)| \leq K$?*

Thus if problem (AASF) has a negative solution then one could build a non-locally convex compact zonoid and hence a vector measure with compact range and no control measure. Thus one would have a counterexample to the Maharam problem which is uniformly exhaustive.

Unfortunately, this does not work. In fact, the approach preferred by Roberts was more direct and was also ultimately successful. His idea was to relate the existence of pathological uniformly exhaustive submeasures to the non-existence of certain bipartite graphs called *linear concentrators*. If $m \in \mathbf{N}$ we write $[m] = \{1, 2, \ldots, m\}$. If $m \geq p \geq q$ and if $r \in \mathbf{N}$ then a map $R : [m] \to 2^{[p]}$ is a (m, p, q, r)−concentrator if

$$\sum_{j=1}^{m} |R(j)| \leq rm$$

and if $R[E] = \cup\{R(e) : e \in E\}$ then $|R[E]| \geq |E|$ whenever $|E| \geq q$. Such a map R may be considered to induce a bipartite graph by defining connections between $[m]$ and $[p]$. It turns out that concentrators have received considerable attention in the literature owing to potential applications in telecommunications (see [12], [44] and [45]). The object here is to make concentrators when r is fixed and small, p/m is small and q/p is relatively big. In fact if $r = 3$ and $2e^2mq \leq p^2$ then there do exist $(m, p.q, r)$−concentrators [29]. The existence of such concentrators leads, somewhat surprisingly, to the non-existence of uniformly exhaustive pathological submeasures. Thus we have a resolution of the Reduced Maharam Problem (see also Louveau [34]):

THEOREM 5. *An exhaustive submeasure is equivalent to a measure if and only if it is uniformly exhaustive.*

COROLLARY 6. *Let F be a vector measure valued in an F-space X with relatively compact range. Then F has a control measure.*

There are conditions on X which allow one to obtain this conclusion without requiring relatively compact range. For example, if X is a quasi-Banach space and ℓ_∞ is not finitely representable in X then every vector measure F has a control measure [29]. It also follows that if X is a quasi-Banach lattice in which ℓ_∞ is not finitely representable then X is natural [23].

COROLLARY 7. *Every compact zonoid is locally convex.*

We may now retrace our steps and deduce also that Problem (AASF) has an affirmative answer. Fortunately, more direct reasoning is also available.

THEOREM 8 [29]. *There exists an absolute constant K so that for every approximately additive set function f defined on an algebra Σ there exists an additive set function g with $|f(A) - g(A)| \leq K$ for all $A \in \Sigma$.*

What is the best choice of K here? We seem to be some way from this. The methods of [29] depend on the existence of certain concentrators. A result of Pippenger [45] that for every m there is a $(6m, 4m, 3m, 6)$− concentrator gives the estimate $K < 45$ but there's is no reason to believe this is close to the best possible. The natural try that $K = 1$ is false, however, by a result of Pawlik [43] that $K \geq 3/2$. The immediate corollary of Theorem 8 is, of course, that if X is a quasi-Banach space with a one-dimensional subspace L so that X/L is isomorphic to c_0 then X is locally convex (i.e. a Banach space).

At this point we may raise the question of classifying Banach spaces Y with the above property. We say that a Banach space Y is a K−space if, whenever X is a quasi-Banach space and L is a one-dimensional subspace with X/L isomorphic to Y then X is a Banach space. This can be interpreted in terms of the ability to approximate a nearly linear functional on Y by a linear functional. We know that ℓ_1 is not a K−space but c_0 is a K−space. It is also known that if Y is a K−space then ℓ_1 is finitely representable in Y [19] but of course the converse is false. It seems quite likely that Y is a K−space if and only if ℓ_∞ is finitely representable in Y^*.

The results mentioned in this section now reduce the Maharam Problem to one raised by Dobrakov [8]:

THE MAHARAM PROBLEM (V). *Is every exhaustive submeasure uniformly exhaustive?*

I have no real idea about this, but I would rather like there to be a counterexample!

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