VALUES OF POSITIONAL GAMES

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1. INTRODUCTION

This paper is a summary of results to be published in detail elsewhere ([6]).

Many situations in the theory of two person games can be described in broad terms as follows. Two players play a game possibly involving random moves whose result is a point s of a topological space S. The pay-off, which one player seeks to maximize and the other to minimize, is given by $f(s)$ where f is a continuous real-valued function on S. Thus the game has two distinct phases; the first phase, which includes the actual mechanics of the game, being independent of the second phase which is the evaluation of the pay-off. Of course, the particular function f will influence the behaviour of the players in the first phase. Such a situation we will call a *positional game*.

Our main motivation for the study of positional games is the particular example of a differential game of survival with purely terminal pay-off. In fact any differential game of survival can be reduced to one with purely terminal pay-off. In this case S can be taken as ∂F the boundary of the terminal set F. If we admit the possibility of random moves we can also treat stochastic differential games governed by a stochastic differential equation of the type

$$dx = f(t,x,y,z)dt + \sigma(t,x)dw$$

where w is an n-dimensional Brownian motion and $\sigma(t,x)$ is an $n \times n$-matrix; see Friedman [5] for a discussion of the problems involved in such games.

J. D. Grote (ed.), The Theory and Application of Differential Games. 293-300.

Even a two-person matrix game can be described as a positional game with S a finite set with mn elements, where m and n are the numbers of pure strategies available to each player.

Suppose S is compact and Hausdorff and $f \in C(S)$. Let $V(f)$ be any notion of value associated with the game with pay-off f. It is clear that $V: C(S) \to R$ must have certain properties to be reasonable as a definition of value

(i) $V(f) \geq V(g)$ whenever $f \geq g$,

(ii) $V(\alpha f) = \alpha V(f)$ whenever $\alpha \in R$, with $\alpha \geq 0$,

(iii) $V(f+\alpha) = V(f) + \alpha$ whenever $\alpha \in R$.

Such a functional on $C(S)$ we shall call a *gamonic functional.*

For example, consider a differential game of fixed duration; the terminal set may be taken to be compact (since the set of points attainable at time T is relatively compact). Then both the upper and lower values (see [3]) considered as functions of the pay-off are gamonic. Also the Danskin σ-value for $0 \leq \sigma \leq 1$ (see [1] and [4]) and the value with relaxed controls ([2]) are gamonic.

Consider a positional game from the point of view of the minimizer. If he adopts a given strategy Σ then according to his opponent's choice of strategy the result of the game (taking into account random moves) may be any point in a set $A(\Sigma)$ of regular probability measures on S. Thus the value or upper value to the minimizer takes the form

$$V(f) = \inf_{\Sigma} \sup_{A(\Sigma)} \int f \, d\mu .$$

The main result of this note is that any gamonic functional on $C(S)$ can be represented in this form; precisely

$$V(f) = \min_{C \in \mathscr{C}} \max_{\mu \in C} \int f \, d\mu$$

where $\mathscr{C}$ is a collection of weak*-closed convex subsets of $P(S)$ the regular probability measures on S. Thus any gamonic functional on $C(S)$ can be realized in a certain sense as the upper value of a positional game. Of course it follows there is equally a representation in the form

$$V(f) = \max_{D \in \mathscr{D}} \min_{\mu \in D} \int f \, d\mu .$$

Similar results can be obtained for functionals on $C_0(S)$ where S is locally compact. Here condition (iii) must be replaced by $V(f) - V(g) \leq \|f - g\|$, and a further condition is required to ensure that V does not depend too much on points near infinity (see [6]).

Of course if a positional game is purely deterministic (i.e. involves no random moves) then we may expect a representation in the form

$$V(f) = \min_{E\in\mathcal{E}} \max_{s\in E} f(s)$$

where $\mathcal{E}$ is a collection of closed subsets of S. We obtain an external characterization of such functionals in §4. Let Φ be a 'utility transformation', i.e. a continuous increasing map $\Phi: \mathbb{R} \to \mathbb{R}$. Then if V is deterministic (i.e. represents a deterministic game)

$$\Phi[V(f)] = V(\Phi \circ f).$$

Thus a change in utility does not affect the optimum strategy of the minimizer. We show in §4 that if V is gamonic and satisfies this equation for some non-linear utility transformation then V has a representation

$$V(f) = \min_{E\in\mathcal{E}} \max_{s\in E} f(s).$$

As noted at the beginning of this section, these results are proved in detail in a forthcoming note [6]. One of the aims of this line of research is to study an abstract theory of differential games, akin to modern potential theory [7]

2. REPRESENTATION THEOREMS

We begin by considering an abstract situation. Suppose X is a real normed space and $V: X \to \mathbb{R}$ is a positively homogeneous functional. Suppose V is uniformly continuous; then it follows that V satisfies a Lipschitz condition

$$|V(x) - V(y)| \leq K\|x-y\|. \qquad x, y \in X$$

A sublinear functional p is a positively-homogeneous functional, satisfying in addition

$$p(x) + p(y) \geq p(x+y) \qquad x, y \in X.$$

It follows from the Hahn-Banach theorem that if p is a continuous sublinear functional then

$$p(x) = \max_{x^*\in C(p)} x^*(x)$$

where $C(p) = \{x^* \in X^*: x^*(x) \leq p(x), x \in X\}$ is a weak*-compact convex subset of X^*. Conversely if C is a weak*-compact convex subset of X^* then $\max_{x^*\in C} x^*(x)$ is a continuous sublinear functional.

If V is a positively homogeneous functional then we define $p_V : X \to \mathbb{R}$ by

$$p_V(x) = \sup_{y \in X} [V(x+y) - V(y)].$$

Lemma 2.1 If V is uniformly continuous then p_V is a continuous sublinear functional.

The corresponding convex subset $C(p_V)$ of X^* is called the *support* of V or $\mathrm{Supp} V$.

A weak*-compact convex subset C of X^* is *V-admissible* if $\max_{x^* \in C} x^*(x) \geq V(x)$ for $x \in X$. It is clear that $\mathrm{Supp} V$ is V-admissible. By an argument based on Zorn's Lemma it is easy to verify that

Lemma 2.2 Every V-admissible subset of X^* contains a minimal V-admissible set.

We shall denote by $\mathcal{M}(V)$ the class of minimal V-admissible subsets of X^*; from Lemma 2.2 it follows that $\mathcal{M}(V) \neq \emptyset$. We now come to our main representation theorem.

Theorem 2.3 If V is a uniformly continuous positively homogeneous functional on X then

$$V(x) = \min_{M \in \mathcal{M}(V)} \max_{x^* \in M} x^*(x).$$

Proof (Sketch). For given $y \in X$ we define

$$C(y) = \{x^* \in \mathrm{Supp} V : x^*(y) \leq V(y)\}.$$

The main part of the proof consists of showing that $C(y)$ is V-admissible (although possibly not minimal). This is done by proving by the Hahn-Banach theorem that for any $z \in X$ there exists $x^* \in \mathrm{Supp} V$ such that

$$x^*(z) \geq V(z)$$

$$x^*(y) \leq V(y).$$

This demonstrates that

$$\max_{x^* \in C(y)} x^*(z) \geq V(z) \qquad z \in X$$

and hence that $C(y)$ is V-admissible.

Now suppose $C \subset C(y)$ is minimal and V-admissible. Then clearly $V(y) = \max_{x^* \in C} x^*(y)$ so that the theorem will follow.

Corollary 2.4 $\overline{co}[\cup(C : C \in M(V))] = \text{Supp}V$.

It is now an easy matter to establish our main application, which enables us to identify certain functionals as resulting from a positional game.

Theorem 2.5 Suppose S is a compact Hausdorff space and $V : C(S) \to R$ is positively homogeneous and in addition satisfies

(i) $V(f) \geq V(g)$ whenever $f \geq g$

(ii) $V(f+\alpha) = V(f) + \alpha$ whenever $\alpha \in R$.

Then

$$V(f) = \min_{C \in M} \max_{\mu \in C} \int_S f \, d\mu$$

where M is the collection of weak*-closed convex subsets C of the set $P(S)$ of regular probability measures on S, which are minimal with respect to the condition

$$\max_{\mu \in C} \int_S f \, d\mu \geq V(f).$$

Theorem 2.5 is proved by observing that under conditions (i) and (ii), $p_V(f) \leq \max_{s \in S} f(s)$ for any $f \in C(S)$.

We shall call a functional on $C(S)$ satisfying (i) and (ii) *gamonic*. Note that if V is gamonic then so is V^* where $V^*(f) = -V(-f)$. From this we obtain

Corollary 2.6 If $V : C(S) \to R$ is gamonic then

$$V(f) = \max_{c \in N} \min_{\mu \in C} \int_S f \, d\mu$$

where N is the collection of weak*-closed convex subsets of $P(S)$ minimal subject to the condition

$$\min_{\mu \in C} \int_S f \, d\mu \leq V(f) \qquad f \in C(S).$$

3. THE INDICATOR FUNCTION

It is natural to consider the extension of V to certain discontinuous functions. One might hope to obtain a natural extension to all bounded Borel functions as for the case of linear functionals on $C(S)$. This seems to be impossible, but it is possible to define an extension to (upper or lower) semi-continuous functions. Our main result in this direction is:

Theorem 3.1 Suppose $\{f_\lambda\}$ is a decreasing net of continuous functions on S and $u = \lim_\lambda f_\lambda$ pointwise is a bounded function. Then

$$\lim_\lambda V(f_\lambda) = \inf_{C\in \mathcal{M}} \sup_{\mu\in C} \int u\, d\mu.$$

The same result is true for increasing nets. Thus if u is a bounded upper-semi-continuous function we may define

$$\tilde{V}(u) = \inf_{C\in \mathcal{M}} \sup_{\mu\in C} \int u\, d\mu$$

and then expect to obtain decent results. In particular if E is a closed subset of S we define

$$\delta_V(E) = \tilde{V}(\chi_E) = \inf_{C\in \mathcal{M}} \sup_{\mu\in C} \mu(E)$$

and then δ_V is called the *indicator function* of V. The indicator function does not determine V except in one special case (see the next section).

One result we can obtain is a 'value-existence' result that

$$\delta_V(E) = \inf_{C\in \mathcal{M}} \sup_{\mu\in C} \mu(E) = \sup_{C\in \mathcal{N}} \inf_{\mu\in C} \mu(E)$$

for any closed subset E of S.

4. DETERMINISTIC FUNCTIONALS

A gamonic functional V is called *deterministic* if V has a representation

$$V(f) = \min_{E\in \mathcal{E}} \max_{s\in E} f(s)$$

where $\mathscr{E}$ is a collection of closed subsets of S. In this section we obtain some external characterizations of deterministic functionals.

Proposition 4.1 If V is a deterministic gamonic functional, then for any continuous increasing map $\Phi: \mathrm{R} \to \mathrm{R}$

$$\Phi(V(f)) = V(\Phi \circ f) \quad \text{for } f \in C(S).$$

Our main result will be a strong converse of Proposition 4.1; first we need to relate deterministic functionals with properties of the indicator functional.

Proposition 4.2 Suppose for every closed subset E of S $\delta_V(E) = 0$ or $\delta_V(E) = 1$. Then V is deterministic.

To prove 4.2 we define $\mathscr{E}$ to be the set of closed E such that $E \cap F \neq \emptyset$ whenever $\delta_V(F) = 1$. It is then possible to show that if $V(f) = \alpha$ then the set $G = \{s : f(s) \leq \alpha\}$ belongs to $\mathscr{E}$.

Theorem 4.3 Suppose $\Phi: \mathrm{R} \to \mathrm{R}$ is any non-linear continuous increasing map, and suppose that

$$\Phi(V(f)) = V(\Phi \circ f) \quad \text{for } f \in C(S).$$

Then V is deterministic.

Proof (Sketch) Φ is almost everywhere differentiable; if c is a point of differentiability then

$$\Phi[\tilde{V}(c+t\chi_E)] = \tilde{V}(\Phi(c+t\chi_E))$$

for $t > 0$ so that

$$\begin{aligned}\Phi(c+t\delta) &= \tilde{V}[\Phi(c) + (\Phi(c+t) - \Phi(c))\chi_E] \\ &= \Phi(c) + \delta(\Phi(c+t) - \Phi(c))\end{aligned}$$

where $\delta \equiv \delta_V(E)$. Hence

$$\Phi(c+t\delta) - \Phi(c) = \delta[\Phi(c+t) - \Phi(c)].$$

Hence for any natural number n,

$$\Phi(c+t\delta^n) - \Phi(c) = \delta^n[\Phi(c+t) - \Phi(c)].$$

Suppose $0 < \delta < 1$; then letting $n \to \infty$

$$t\Phi'(c) = \Phi(c+t) - \Phi(c)$$

and hence Φ is linear on $\{\xi : \xi \geq c\}$. It quickly follows that Φ is everywhere linear.

Hence $\delta = 0$ or 1 and we use 4.2.

We note that under the hypotheses of Theorem 4.3, each $C \in \mathcal{M}$ is a face of $P(S)$, i.e. if $C \in \mathcal{M}$ there exists a closed subset E of S such that

$$C = \{\mu \in P(S) : \mu(E) = 1\}.$$

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