DIFFERENTIAL GAMES OF SURVIVAL

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1. INTRODUCTION

In these lectures we study differential games which are not of fixed duration. For such games the value does not depend in a nice "continuous" way on the control variables selected by the players, and our results are therefore less precise than those for games of fixed duration. Nevertheless it is important for practical applications to develop a theory which goes beyond this basic type of game. We shall start by giving examples of the main types of differential game.

Suppose Y and Z are compact metric spaces and $f : \mathbb{R} \times \mathbb{R}^m \times Y \times Z \to \mathbb{R}^m$ is a continuous function which is Lipschitz in the first two variables (these conditions can be weakened). We suppose that two competing players J_Y (the maximizer) and J_Z (the minimizer) select control functions $y : [t_0,\infty) \to Y$ and $z : [t_0,\infty) \to Z$ which are Lebesgue measurable. The precise method of selection is discussed in the next section. The resulting control functions $y(t)$, $z(t)$ determine a trajectory given by

$$\dot{x} = f(t,x,y(t),z(t)) \quad \text{a.e.} \quad t_0 \le t$$

subject to an initial condition,

$$x(t_0) = x_0 .$$

Corresponding to the trajectory $x(t)$, and the control functions $y(t)$ and $z(t)$, we calculate a pay-off $P[y(\cdot),z(\cdot)]$

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which J_Y aims to maximize and J_Z aims to minimize. In a differential game of *survival* (Friedman [6] Ch.5) the pay-off takes the form

$$P = \int_{t_0}^{t_F} h(t,x(t),y(t),z(t))dt + g(t_F,x(t_F)) \qquad (1)$$

where $h: R \times R^m \times Y \times Z \to R$, and $g: R \times R^m \to R$ are continuous and t_F is the first time for which $(t,x(t)) \in F$, where F is a given closed subset of $R \times R^m$.

If $g \equiv 0$ and $h \equiv 1$, then $P = t_F - t_0$. In this case we call the game, a *pursuit-evasion* game ([6] Ch.3). If $g \equiv 0$ and $h \geq 0$ then the game is called a *generalized pursuit-evasion* game.

The other possible type of pay-off which we consider is a differential game of *optional stopping* ([9]) where

$$P = \min_{t_0 \leq t \leq t_F} g(t,x(t))$$

In all these examples we shall impose a finiteness condition on the game, that for some $T > t_0$, $F \supset [T,\infty) \times R^m$. Thus we will always have $t_F \leq T$.

Two important types of game will not be treated. In [6] Ch.6, Friedman studies games with restricted phase co-ordinates, while in [1] Bensoussan and Friedman have studied a generalization of optional stopping games in which both players (rather than just the minimizer as above) have the option of halting the game at any time $t_0 \leq t \leq t_F$.

2. THE DEFINITION OF VALUE

We now turn to the problem of defining the upper and lower value of a differential game. There is an abundance of different definitions in the literature; in this section we shall discuss some of the ideas involved. Let $\mathcal{M}_Y$ and $\mathcal{M}_Z$ denote the spaces of equivalence classes of Lebesgue measurable functions $y: [t_0,T] \to Y$ and $z: [t_0,T] \to Z$ where functions equal almost everywhere are identified. The pay-off P is defined on $\mathcal{M}_Y \times \mathcal{M}_Z$ (this abstract setting, without reference to the underlying differential equation, we call an *evolutionary* game [9]).

The most mathematically attractive definition of value was proposed by Varaiya (see Roxin [10] or Varaiya-Lin [11]). A *strategy* for J_Z is a map $\beta: \mathcal{M}_Y \to \mathcal{M}_Z$ satisfying the condition that if

$$y_1(t) = y_2(t) \quad \text{a.e.} \quad t_0 \le t \le \tau,$$

then

$$\beta y_1(t) = \beta y_2(t) \qquad t_0 \le t \le \tau.$$

The value of β to J_Z will then be

$$v(\beta) = \sup_y P(y, \beta y)$$

and clearly the value of the game to J_Z is $\inf_\beta v(\beta)$. Similarly a strategy for J_Y is a map $\alpha: \mathcal{M}_Z \to \mathcal{M}_Y$ satisfying the same condition, and the value $v(\alpha)$ of that strategy to J_Y is

$$u(\alpha) = \inf_z P(\alpha z, z).$$

The value of the game to J_Y is $\sup_\alpha u(\alpha)$.

However there are problems with this definition. The use of a strategy β allows the minimizer to instantaneously anticipate his opponent's choice of control function; it allows J_Z too much of an advantage. The appropriate definition of the upper value or the value of the game to the minimizer, in which we suppose the minimizer instantaneously at a disadvantage is

$$U^+ = \sup_\alpha u(\alpha)$$

and conversely

$$U^- = \inf_\beta v(\beta).$$

Even with this definition it is, however, unknown whether $U^+ \ge U^-$ in general (for fixed time differential games this is the case).

These difficulties are circumvented by Varaiya and Lin [11] and Friedman [6], [7], by a partition method. Let $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[t_0, T]$ where

$t_0 < t_1 < \dots < t_n = T$. Corresponding to the partition $\mathcal{P}$ they consider a game $G_{\mathcal{P}}$ in which the players J_Y and J_Z alternately choose their control functions on the intervals $[t_0, t_1), [t_1, t_2), \dots, [t_{n-1}, t_n]$. If at each step, J_Z chooses his control function first then the game G has value denoted by $V_{\mathcal{P}}^+$; conversely if J_Y plays first the game has value $V_{\mathcal{P}}^-$ where $V_{\mathcal{P}}^+ \geq V_{\mathcal{P}}^-$. It is then easy to show that the nets $(V_{\mathcal{P}}^+ : \mathcal{P}$ a partition of $[t_0, \tau))$ and $(V_{\mathcal{P}}^-)$ are monotonic and we define the "Friedman upper value" and the "Friedman lower value" of the game by

$$V_F^+ = \lim_{\mathcal{P}} V_{\mathcal{P}}^+$$

$$V_F^- = \lim_{\mathcal{P}} V_{\mathcal{P}}^- .$$

It then follows that $V_F^+ \geq V_F^-$. We note that we may express $V_{\mathcal{P}}^+$ and $V_{\mathcal{P}}^-$ in the language used above, thus

$$V_{\mathcal{P}}^+ = \inf v(\beta)$$

where β runs over all strategies for J_Z which satisfy that if

$$y_1(t) = y_2(t) \quad \text{a.e.} \quad t \leq t_j$$

then

$$\beta y_1(t) = \beta y_2(t) \quad \text{a.e.} \quad t \leq t_{j+1}$$

for $j < n$.

This definition has the added attraction that the approximating process mirrors the physical constraints on the players. It is a reasonable restriction to impose as in $G_{\mathcal{P}}$ that J_Z may only make decisions at a finite number of times $t_0, t_1, \dots, t_n$. However, it seems to the author that if we are attempting to make the definition fit a physical model, we should also consider reaction times. In $G_{\mathcal{P}}$, with J_Z playing first at each step, J_Z is still allowed at the times $t_1, t_2, \dots, t_n$ to be aware of the full history of the game and to act instantaneously on that knowledge. In a real situation there would be a slight delay before J_Z could act, i.e. he would respond to the position at time t_j at a time $t_j + \sigma$ where σ is a reaction time. If we attempt to incorporate this idea then we reach a definition of value which can be made without reference to a partition ([3]).

We shall say that a strategy β for J_Z is an s-delay strategy (where $s > 0$) if whenever

$$y_1(t) = y_2(t) \quad \text{a.e.} \quad t_0 \le t \le \tau$$

then

$$\beta y_1(t) = \beta y_2(t) \quad \text{a.e.} \quad t_0 \le t \le \min(T, \tau + s).$$

We define

$$V_s^+ = \inf v(\beta)$$

where β runs over s-delay strategies for J_Z. Then the upper value V^+ is given by $V^+ = \lim_{s \to 0} V_s^+$.

We can similarly define V^- and we have $V^+ \ge V^-$.

There is one question however which arises in this definition. In selecting an s-delay strategy J_Z chooses a fixed control function $z(t)$ to operate on the interval $[t_0, t_0 + s]$. The choice of this control depends on exact knowledge of the starting conditions (t_0, x_0). There may be situations in which it is unrealistic to suppose that J_Z is able to act on a precise knowledge of the starting conditions until after a delay s. This also turns out to have mathematical importance in the next section.

For $s > 0$ and $z_0(\cdot)$ a measurable Z-valued function on $[t_0, t_0 + s]$ we define

$$Q_s^+[z_0] = \inf v(\beta)$$

where β runs over all s-delay strategies such that $\beta y(t) = z_0(t)$ for $t_0 \le t \le t_0 + s$. Then

$$Q_s^+ = \sup(Q_s^+[z_0]) \quad (\text{over all } z_0(\cdot))$$

and $Q^+ = \lim_{s \to 0} Q_s^+$. We similarly define Q_s^- and Q^- ([5]).

We have thus defined four different notions of upper value which are related by the inequalities $Q^+ \ge V^+ \ge V_F^+ \ge U^+$. Similarly for the lower values we have $Q^- \le V^- \le V_F^- \le U^-$; and we have $V_F^- \le V_F^+$ although it is not known whether $U^- \le U^+$. Of course in fixed time games all these concepts reduce to two; however in games of survival it is by no means clear that they are equivalent.

We know of no example to distinguish between V^+, V_F^+ and U^+, although there seems no reason to believe that such an example does not exist. However it is very easy to illustrate the difference between Q^+ and V^+ by a control problem [5] (i.e. a trivial differential game). Let $Z \subset R^2$ be the set of all $z = (\xi_1, \xi_2)$ where $\xi_1^2 + \xi_2^2 \leq 1$ and $\xi_1 \geq 0$. Consider the game in R^2 with dynamics

$$\dot{x} = z$$

initial condition $$x(0) = (0, \tfrac{1}{4})$$

and terminal set $F = \{(t,\xi_1,\xi_2) : \xi_1 \leq 0 \text{ and } \xi_2 \leq 0\} \cup \{(t,\xi_1,\xi_2)\ t \geq 1\}$. Suppose the pay-off is given by $P = t_F$.

J_Z's optional strategy is to adopt $z(t) \equiv (0,-1)$ and so $V^+ = \frac{1}{4}$. However it is easy to see that $Q_s^+ = 1$ for all $s > 0$ and so $Q^+ = 1$.

3. DYNAMIC PROGRAMMING

We now specialize to differential games of survival, although the same techniques can be applied equally to optional stopping games ([9]); all the results of §§3-5 are taken from [5]. Consider a game then with pay-off defined by (1). As we allow the initial conditions to vary we obtain functions $U^+(t,x)$, $V^+(t,x)$, $Q_s^+(t,x)$, etc. Suppose now that τ is any function $\tau: \mathcal{M}_Y \times \mathcal{M}_Z \to [t_0, T]$ such that if

$$y_1(t) = y_2(t) \quad \text{a.e.} \quad t \leq \tau(y_1, z_1)$$
$$z_1(t) = z_2(t) \quad \text{a.e.} \quad t \leq \tau(y_1, z_1)$$

then $\tau(y_1,z_1) = \tau(y_2,z_2)$. We call such a map τ a non-anticipating stopping time; an example is $\tau \equiv t_F$, for the trajectory corresponding to y_1, z_1. Consider now the game $G(\tau;\varphi)$, where φ is some real-valued function on $R \times R^m$, in which the pay-off corresponding to a pair of controls (y,z) is given by

$$P = \int_{t_0}^{\tau} h(t,x(t),y(t),z(t))dt + \varphi(\tau, x(\tau)).$$

$G(\tau;\varphi)$ is an evolutionary game, and therefore has values as defined in the preceding section $U^{+}_{\tau,\varphi}$, etc. By a dynamic programming theorem we shall mean a result relating the values of $G(\tau;\varphi)$ where φ is itself a value function to the values of the original game.

Theorem 3.1

$$U^{+}_{\tau,U^{+}} = U^{+}(t_0,x_0) \quad \text{and} \quad U^{-}_{\tau,U^{-}} = U^{-}(t_0,x_0).$$

This is a very nice result whose only defect is the difficulty in treating U^{+} and U^{-} as discussed in the preceding section. For V^{+} and V^{-} the results are less fortunate.

Theorem 3.2

$$V^{+}_{\tau,V^{+}} \leq V^{+}(t_0,x_0) \quad \text{and} \quad V^{-}_{\tau,V^{-}} \geq V^{-}(t_0,x_0).$$

However we can now justify the need for the Q-values by providing the reverse inequalities

Theorem 3.3 If $0 < s \leq \tau$ everywhere then

$$Q_{s,\tau,Q^{+}_{s}} \geq Q^{+}_{s}(t_0,x_0) \quad \text{and} \quad Q_{s,\tau,Q^{-}_{s}} \leq Q^{-}_{s}(t_0,x_0).$$

Furthermore

$$Q_{s,\tau,U^{+}} \leq Q^{+}_{s}(t_0,x_0) \quad \text{and} \quad Q_{s,\tau,U^{-}} \geq Q^{-}_{s}(t_0,x_0).$$

These dynamic programming results are the basic tools for the study of differential games. We do not know a result of the type of Theorem 3.3 for the Q-value.

One immediate application of Theorem 3.1 can be given

Theorem 3.4 Let (t_0,x_0) be a point of differentiability of the function U^{+}. Then at (t_0,x_0)

$$\frac{\partial U^{+}}{\partial t} + \min_{z}\max_{y}\left(\nabla U^{+}.f + h\right) = 0.$$

i.e. the Isaacs-Bellman equation is satisfied.

4. REGULARITY OF THE BOUNDARY

We now introduce some ideas inspired by similar concepts in potential theory. We denote by $\overline{Q}_s^+$ the upper semi-continuous regularization of Q_s^+ given by

$$\overline{Q}_s^+(t,x) = \limsup_{(\tau,\xi)\to(t,x)} Q_s^+(\tau,\xi).$$

We shall consider Q_s^+ and $\overline{Q}_s^+$ as defined on all $R \times R^m$; of course if $(t,x) \in F$ then $Q_s^+(t,x) = g(t,x)$. Suppose now $(t,x) \in \partial F$; we shall say that (t,x) is *Q^+-regular* if

$$\left.\begin{aligned} &\text{(i)} \quad \lim_{s\to 0} \overline{Q}_s^+(t,x) = g(t,x) \\ &\text{and} \\ &\text{(ii)} \quad \lim_{(\tau,\xi)\to(t,x)} U^+(\tau,\xi) = g(t,x) \end{aligned}\right\} \tag{2}$$

Similarly we shall say that (t,x) is *Q^--regular* if

$$\left.\begin{aligned} &\text{(i)} \quad \lim_{s\to 0} \tilde{Q}_s^-(t,x) = g(t,x) \\ &\text{(ii)} \quad \lim_{(\tau,\xi)\to(t,x)} U^-(t,x) = g(t,x) \end{aligned}\right\} \tag{3}$$

where

$$\tilde{Q}_s^-(t,x) = \liminf_{(\tau,\xi)\to(t,x)} Q_s^-(\tau,\xi),$$

is the lower semi-continuous regularization of Q_s^-.

These conditions seem rather complicated, but in the next section we give some sufficient criteria for regularity. In this section, however, we shall simply list some applications.

Theorem 4.1 Suppose every point of ∂F is Q^+-regular. Then for any $(t_0,x_0) \in R \times R^m$, $Q^+(t_0,x_0) = V^+(t_0,x_0) = U^+(t_0,x_0)$. Furthermore the function Q^+ is continuous in (t,x), and

$$\lim_{s\to 0} Q_s^+(t,x) = Q^+(t,x)$$

uniformly on compact sets.

[Remark: this result depends on the Lipschitz conditions assumed on the function $f(t,x,y,z)$; see §1.]

The proof of Theorem 4.1 uses the dynamic programming theorems of §3. For example to establish continuity of Q^+ we consider two initial positions (t_1,x_1) and (t_2,x_2) and a stopping time $\tau(y,z)$ which is the first t at which one of the trajectories $(t,x_1(t))$, $(t+t_2-t_1,x_2(t+t_2-t_1))$ enters F where

$$\dot{x}_1(t) = f(t,x_1(t),y(t),z(t))$$

and

$$x_1(t_1) = x_1$$

while

$$\dot{x}_2(t) = f(t,x_2(t),y(t+t_1-t_2),z(t+t_1-t_2))$$

with

$$x_2(t_2) = x_2.$$

In the case of fixed time games, the Isaacs condition

$$\min_z \max_y (p.f+h) = \max_y \min_z (p.f+h) \tag{4}$$

for all $p \in R^m$, guarantees that the upper and lower values of the game coincide. For general games of survival this result may be extended in the presence of regularity conditions.

Theorem 4.2 Suppose every point of ∂F is both Q^+- and Q^--regular and that the Isaacs condition (4) is fulfilled. Then for any (t_0,x_0), $Q^+(t_0,x_0) = Q^-(t_0,x_0)$.

Again the proof uses dynamic programming arguments.

5. CRITERIA FOR REGULARITY

For a differentiable function u defined on an open subset of $R \times R^m$ we define

$$L^+u = \frac{\partial u}{\partial t} + \min_{z\in Z} \max_{y\in Y} (\nabla u.f+h)$$

$$L^-u = \frac{\partial u}{\partial t} + \max_{y\in Y} \min_{z\in Z} (\nabla u.f+h)$$

$$L_0^+u = \frac{\partial u}{\partial t} + \min_{z\in Z}\max_{y\in Y} \nabla u.f$$

$$L_0^-u = \frac{\partial u}{\partial t} + \max_{y\in Y}\min_{z\in Z} \nabla u.f .$$

Thus L^+ and L^- are the differential operators associated with the Isaacs-Bellman equations, while L_0^+ and L_0^- are the corresponding operators for the game with h replaced by 0. If the Isaacs condition is satisfied then $L^+ \equiv L^-$ and $L_0^+ \equiv L_0^-$. We shall show in this section how regularity of the boundary is related to the operators L^+, L^-, L_0^+ and L_0^-.

Theorem 5.1 Suppose θ_1, θ_2 are two functions on $R \times R^m$ which are C^1 on $R \times R^m$-int F and such that $\theta_1 = \theta_2 = g$ on ∂F with

$$L^+\theta_1 \leq 0 \leq L^+\theta_2 \quad \text{on} \quad R \times R^m - \text{int} F.$$

Then every point of ∂F is Q^+-regular. Furthermore for any $(t_0,x_0) \in R \times R^m$- int F

$$\theta_1(t_0,x_0) \geq Q^+(t_0,x_0) \geq \theta_2(t_0,x_0) .$$

Corollary 5.2 If θ is a C^1-solution of the Isaacs-Bellman equation $L^+\theta = 0$ subject to $\theta(t,x) = g(t,x)$ for $(t,x) \in \partial F$, then $\theta = Q^+ = U^+$.

Corollary 5.3 Suppose the Isaacs condition (4) is fulfilled and there exist C^1-functions θ_1, θ_2 on $R \times R^m$-int F such that $\theta_1 = \theta_2 = g$ on ∂F and

$$L\theta_1 \leq 0 \leq L\theta_2 \quad \text{on} \quad R \times R^m - \text{int} F$$

(where $L \equiv L^+ \equiv L^-$). Then for any (t_0,x_0) $Q^+(t_0,x_0) = Q^-(t_0,x_0)$.

Corollary 5.4 In general suppose there exist gunctions θ_1, θ_2 which are C^1 on $R \times R^m$- int F and such that $\theta_1 = \theta_2 = g$ on ∂F with

$$L^+\theta_1 \leq 0 \leq L^-\theta_2 .$$

Then every point of ∂F is both Q^+-regular and Q^--regular.

We now turn to criteria involving L_0^+ and L_0^-. Here we are able to give a condition under which a single point of ∂F is Q^+-regular.

Theorem 5.5 Let $(t_0, x_0) \in \partial F$ and suppose N is a neighbourhood of (t_0, x_0) such that there exist C^1-functions θ_1, θ_2 on N such that

(i) $\theta_1(t_0, x_0) = \theta_2(t_0, x_0) = 0$,

(ii) $\theta_2(t,x) < 0 < \theta_1(t,x)$

for $(t,x) \in N\text{-int}F$ with $(t,x) \neq (t_0, x_0)$,

(iii) $L_0^+\theta_1 \leq 0 \leq L_0^+\theta_2$ on N-intF.

Then (t_0, x_0) is Q^+-regular.

To interpret this result suppose for convenience that the Isaacs condition is satisfied so that $L_0^+ \equiv L_0^-$. Then the conditions express the ability of both players to steer the trajectory towards (t_0, x_0) from nearby points. We can derive a global result from Theorem 5.5.

Theorem 5.6 Suppose there exist C^1-functions θ_1, θ_2 such that $\theta_1 = \theta_2 = 0$ on ∂F, $\theta_1 \geq 0 \geq \theta_2$ on $R \times R^m - F$ and $L_0^+\theta_1 < 0 < L_0^+\theta_2$ on ∂F. Then every point of ∂F is Q^+-regular.

A special case of the conditions of Theorem 5.6 is used by Friedman [6] p.81 and [8]. This illustrates the meaning of the Theorem. Suppose F has a C^2-boundary, i.e. ∂F can be expressed by the equation

$$x_i = \psi(t, x_1 \ldots x_{i-1}, x_{i+1} \ldots x_m)$$

where ψ is twice differentiable. Suppose also that any point $(t,x) \in \partial F$ we have

$$\nu_0 + \min_{z \in Z} \max_{y \in Y} \sum_{i=1}^{m} \nu_i f_i < 0 \qquad (3)$$

where $\nu = (\nu_0, \nu_1, \ldots, \nu_m)$ is the normal to ∂F at (t,x). This is called condition (F) by Friedman. Then consider $\rho(t,x)$ the distance of (t,x) from ∂F. It follows from (3) that $L_0^+\rho < 0$

on ∂F and so ρ will serve for θ_1.

Equally suppose we also have

$$\nu_0 + \min_{y\in Y} \max_{z\in Z} \sum_{i=1}^{m} \nu_i f_i < 0 \tag{4}$$

on ∂F (condition $(\tilde{F})$ of Friedman [8]). Then

$$-\nu_0 + \max_{y\in Y} \min_{z\in Z} (-\sum_{i=1}^{m} \nu_i f_i) > 0$$

from which it follows that $L_0^-(-\rho) > 0$ on ∂F. Then since $L_0^+ u \geq L_0^- u$ for any function u, it follows that every point of ∂F is both Q^+- and Q^--regular. Then Theorem 4.2 will apply under conditions (F) and $(\tilde{F})$.

The criteria of Theorems 5.1 and 5.6 can be mixed with θ_1 satisfying Theorem 5.1 and θ_2 satisfying Theorem 5.6 or vice versa.

In one special case, if the game is a generalized pursuit-evasion game, θ_2 may be taken identically zero in Theorem 5.1. Thus for example in the discussion above, condition (3) alone suffices for regularity in a generalized pursuit-evasion game.

6. OPTIONAL STOPPING GAMES

Many of the results of §§3-5 extend with suitable interpretation to optional stopping games. We shall not give detailed results here, but some typical results (see [9]). We recall that the pay-off takes the form

$$P = \min_{t_0 \leq t \leq t_F} g(t,x(t))$$

where g is a continuous function on $\mathbb{R} \times \mathbb{R}^m$. The definitions of Q^+- and Q^--regularity are as before ((2) and (3)). However it is easy to see that 2(i) is trivially satisfied since

$$Q_s^+(t,x) \leq g(t,x)$$

for any (t,x). It follows easily from this observation that if $(t,x) \in \partial F$ is Q^--regular then it must also be Q^+-regular. From this we can deduce

Theorem 6.1 Suppose the Isaacs condition

$$\min_{z\in Z}\max_{y\in Y} p.f = \max_{y\in Y}\min_{z\in Z} p.f, \quad p\in R^m \quad (t,x)\in R\times R^m\text{-int}\,F$$

is satisfied, and that every point of ∂F is Q^--regular. Then for any (t_0,x_0), $Q^+(t_0,x_0) = Q^-(t_0,x_0)$.

Theorem 6.1 above can be deduced from a generalization of Lemma 7.1 and Theorem 6.2 of [9] (taking $h \equiv 0$).

A special type of optional stopping game is a *restricted fuel* game in which we introduce an extra space co-ordinate ξ governed by

$$\dot{\xi} = h(t,x,y,z)$$

where $h: R\times R^m \times Y\times Z$ is continuous (note that h does not depend on ξ). We suppose that the pay-off takes the form

$$P = \min_{t_0\le t\le t_F} g(t,x(t),\xi(t))$$

where g is monotonically increasing in ξ, and F is of the form $F = \{(t,x,\xi): t\ge T \text{ or } \xi\ge\Lambda\}$. This describes a game in which J_Z seeks to reach a point of minimum g without exhausting his resources either of time t or of energy ξ. If $h\ge 0$ everywhere we can think of ξ as measuring fuel consumption, and Λ as a fuel constraint. In this case we obtain

Theorem 6.2 Suppose every point of ∂F is Q^--regular and the Isaacs condition

$$\min_{z\in Z}\max_{y\in Y}(p.f+h) = \max_{y\in Y}\min_{z\in Z}(p.f+h) \quad p\in R^m \; (t,x,\xi)\in R\times R^m\times R-\text{int}\,F$$

holds; then for every (t_0,x_0,ξ_0),

$$Q^+(t_0,x_0,\xi_0) = Q^-(t_0,x_0,\xi_0).$$

Note that this is stronger than simply Theorem 6.1 applied to restricted fuel games in that the Isaacs condition is not homogeneous. We remark that in [9] Theorem 7.2 the hypothesis of regularity should be included.

Theorem 6.3 Suppose in a restricted fuel game we also have

(i) $g(t,x,\xi)\equiv\xi$

(ii) $\max_{y\in Y}\min_{z\in Z} h(t,x,y,z) = \varepsilon > 0$ for all $(t,x)\in R\times R^m - F$.

Then every point of ∂F is Q^{-}-regular and so the conclusion of Theorem 6.2 is valid.

7. EXTENDED VALUES

An alternative definition of value has been suggested particularly with reference to games of pursuit and evasion. (Friedman [6] p.78.) We shall describe here a definition of extended value in the context of games of survival which differs from Friedman's definition, but has the same underlying idea; however we do not know whether the two definitions coincide in general.

We shall suppose that the pay-off is of the form

$$P = \int_{t_0}^{t_F} h(t,x(t),y(t),z(t))dt + g(t_F,x(t_F))$$

as before. We then introduce a 'dummy' space-variable ξ governed by

$$\xi = \int_{t_0}^{t} h(s,x(s),y(s),z(s))ds$$

and write the pay-off P as

$$P = g(t_F,x(t_F)) + \xi(t_F).$$

Now an *approximate strategy* **A** for J_Y is a sequence (α_n) where each α_n is a delay-strategy. Similarly an approximate strategy $B = (\beta_n)$ for J_Z is a sequence of delay-strategies. Each pair (α_n, β_n) induces a unique pair of controls $(y_n(\cdot),z_n(\cdot))$ satisfying

$$\alpha_n z_n = y_n,$$

$$\beta_n y_n = z_n,$$

and the controls induce trajectories $(x_n(t),\xi_n(t))$ in R^{m+1}. This sequence of trajectories is relatively compact in the topology of uniform convergence on $[t_0,T]$. Let $(\bar{x}(t),\bar{\xi}(t))$ be any limit point; then let

$$P_{\bar{x},\bar{\xi}} = g(t_F,\bar{x}(t_F)) + \bar{\xi}(t_F)$$

where t_F is the time of entry of $(t,x(t))$ into F. Then $P[A,B]$ is the set of such $P_{\bar{x},\bar{\xi}}$ and we define the extended upper and lower values by

$$V_e^+ = \inf_B \sup_A \sup P[A,B]$$

$$V_e^- = \sup_A \inf_B \inf P[A,B] .$$

It is clear that $V_e^+ \geq V_e^-$. We say the game has extended value if $V_e^+ = V_e^-$.

Let us now restrict to the special case when $g \equiv 0$. We can consider an associated restricted fuel game G_Λ^* with dynamics

$$\dot{x} = f(t,x,y,z)$$

$$\dot{\xi} = h(t,x,y,z)$$

subject to $x(t_0) = x_0$ and $\xi(t_0) = 0$, and pay-off

$$P = \min_{t_0 \leq t \leq t_{F_\Lambda^*}} \rho(t,x(t))$$

where $\rho(t,x)$ is the distance of (t,x) from F and $F_\Lambda^* = \{(t,x,\xi) : t \geq T \text{ or } \xi \geq \Lambda\}$. Obviously if this restricted fuel game can be shown to have value zero then this means that, in some sense, the original game of survival reaches the terminal set F with $\xi \lneq \Lambda$. This argument can be made precise to give the following result using Theorem 6.3.

Theorem 7.1 Suppose in a game of survival $g \equiv 0$, and

$$\max_{y \in Y} \min_{z \in Z} h(t,x,y,z) \geq 0 \text{ for } (t,x) \in R \times R^m - F.$$

Then if the Isaacs condition (4) is satisfied we have

$$V_e^+(t_0,x_0) = V_e^-(t_0,x_0) .$$

This result is given in [9] (Theorem 8.3); the corresponding result for Friedman's definition of extended value is obtained by Elliott and Friedman in [2]. We call a game satisfying the conditions of Theorem 7.1 a *quasi-pursuit-evasion* game. Obviously the theorem applies to generalized pursuit-evasion games (see [4]).

Finally we can extend the result to general games of survival, and our result is interesting to compare with Corollary 5.3.

Theorem 7.2 Suppose in a game of survival the Isaacs condition (4) holds and there is a C^1-function θ on $R \times R^m$ such that

$$L\theta = \frac{\partial\theta}{\partial t} + \min_{z\in Z}\max_{y\in Y} (f.\nabla\theta + h) \geq 0$$

for $(t,x) \in R \times R^m - F$, and $\theta = g$ on ∂F. Then for any (t_0,x_0), $V_e^+(t_0,x_0) = V_e^-(t_0,x_0)$.

To prove Theorem 7.2 we simply consider the game of survival with pay-off

$$P = \int_{t_0}^{t_F} h_1(t,x(t),y(t),z(t))dt$$

where $h_1(t,x,y,z) = h(t,x,y,z) + \frac{\partial\theta}{\partial t} + f.\nabla\theta$.

The new 'dummy' variable η then satisfies

$$\eta(t) = \int_{t_0}^{t} h_1(t,x,y(t),z(t))dt$$

$$= \xi(t) + \theta(t,x(t)) - \theta(t_0,x_0)$$

along any trajectory.

It follows that in the definition of $P[A,B]$, to any limit trajectory $(\bar{x}(\cdot),\bar{\xi}(\cdot))$ there corresponds a limit trajectory $(\bar{x}(\cdot),\bar{\eta}(\cdot))$ where

$$\bar{\eta}(t) = \bar{\xi}(t) + \theta(t,\bar{x}(t)) - \theta(t_0,x_0)$$

and so

$$\bar{\eta}(t_F) = \bar{\xi}(t_F) + g(t_F,\bar{x}(t_F)) - \theta(t_0,x_0).$$

From this it follows that if $V_{e,1}^+$ and $V_{e,1}^-$ denote the extended values of the new game then

$$V_{e,1}^+ = V_e^+ - \theta(t_0,x_0),$$

$$V_{e,1}^- = V_e^- - \theta(t_0,x_0)$$

and then we can apply Theorem 7.1.

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