Embedding $l^n_\infty$-cubes in finite-dimensional $l$-subsymmetric spaces

JESÚS BASTERO*, JULIO BERNUES** and NIGEL KALTON***

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ABSTRACT. In this paper we prove that the $l^n_\infty$-cube can be $(1+\varepsilon)$-embedded into any $1$-subsymmetric $C(\varepsilon)n$-dimensional normed space.

Marcus and Pisier in [5] initiated the study of the geometry of finite metric spaces. Bourgain, Milman and Wolfson introduced a new notion of metric type and developed the non-linear theory of Banach spaces (see [2] and [7]). All these themes have been studied more intensively over the last years.

Johnson and Lindenstrauss proved that, given $N$ points in the Euclidean space, they can be $(1+\varepsilon)$-embedded into a subspace of dimension $K(\varepsilon) \log N$ (see lemma 1 in [3]). The method they use is based in the isoperimetric inequality of P. Levy. Another proof of the same fact was given by Pisier, using Gaussian processes ([8]). Bourgain, Milman and Wolfson, in the paper before mentioned, studied the $l^p_\ell$-cubes and their $(1+\varepsilon)$-embeddings in finite metric spaces. More recently, Schechtman obtained estimates for $(1+\varepsilon)$-embeddings of finite subsets of $L^r$ into $l^p_\ell$-spaces (see [9]).

In this paper we will study $(1+\varepsilon)$-embeddings of the $l^n_\infty$-cube in finite-dimensional subsymmetric spaces. The result we prove for the $l_p$-case $1 \leq p \leq 2$, can be deduced from Johnson and Lindenstrauss's lemma plus a refinement of Dvoretzky's theorem (see for instance [7], Theorem 3.9), but, as far as we know, it is new in other cases. The method we use is in essence of probabilistic nature and the main tool is a well known deviation inequality.

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We begin by recalling some definitions. Given two metric spaces \((M, d)\) and \((M', d')\), we say that \((M, d)\) \((1 + \varepsilon)\)-embeds into \((M', d')\) if there is a one-to-one map \(f\) from \(M\) into \(M'\) such that \(\|f\|_{L_p} \|f^{-1}\|_{L_p} \leq 1 + \varepsilon\), where

\[
\|f\|_{L_p} = \sup_{x,y} \frac{d(f(x), f(y))}{d(x, y)}
\]

The \(l_\infty^n\)-cube is the metric space \((C^n_\infty, \rho_\infty)\) where \(C^n_\infty = \{ -1, +1 \}^n\) and \(\rho_\infty(\varepsilon, \varepsilon') = \max_{1 \leq i \leq n} |\varepsilon_i - \varepsilon'_i|\), for any pair of elements \(\varepsilon, \varepsilon'\) belonging to \(C^n_\infty\).

Since \(\rho_\infty(\varepsilon, \varepsilon') = 2\), whenever \(\varepsilon \neq \varepsilon'\), the problem we are considering may be related with the sphere-packing problem, i.e., how many balls, with radius \(\frac{1-\varepsilon}{2}\), can be packed into the unit ball of a finite dimensional Banach space, in an asymptotic way? (See the paper by Ball [1] for infinite dimensional sphere-packing problem)

In the sequel \(E_n\) will denote a finite-dimensional Banach space with a \(1\)-subsymmetric normalized basis \(\{e_1, \ldots, e_n\}\). We use standard Banach space theory notation as may be found in [4].

The theorem we will prove here is the following

**Theorem.**—There exists a numerical constant \(C > 0\) such that, for any \(\varepsilon > 0\) we can find a subset of \(N\) points \(\{x_1, \ldots, x_N\}\) in \(E_n\) verifying

\[
1 - \varepsilon \leq \|x_i - x_j\| \leq 1 + \varepsilon, \quad i \neq j
\]

provided that

\[
n > \frac{C}{\varepsilon^2} \log N
\]

**Proof.**—Let \(\varepsilon\) a given positive number verifying \(0 < \varepsilon < 1\). Let \(n\) be a natural number to be determined after. Consider the function \(\psi\) defined by

\[
\psi \left( -\frac{m}{n} \right) = \frac{\| \sum_{i=1}^{m} e_i \|}{\| \sum_{i=1}^{n} e_i \|}, \quad \text{if } 0 \leq m \leq n,
\]

and by a nondecreasing continuous extension in the other points of the unit interval \([0,1]\). The function \(\psi\) depends on \(n\), but in some particular cases we can choose the same fixed function for all \(n\). This happens, for instance, in the \(l^p\)-spaces where we may define \(\psi(t) = t^{\frac{1}{p}}\), \(0 \leq t \leq 1\).
We note that function \( \psi \) verifies \( \psi(0) = 0, \psi(1) = 1 \) and

\[
\psi(2^{-i}) \geq 2^{-\nu+n}, j = 0, 1. \tag{*}
\]

Indeed, if \( \frac{m}{n} \leq \frac{j}{2^n} \leq \frac{m+1}{n} \) we have

\[
\|\sum_{j} e_i\| \leq \|\sum_{j} 2^{(m+1)} e_i\| \leq 2\|\sum_{j} e_i\| \leq 2^{m+1} \|\sum_{j} e_i\|
\]

In general we don't know the behaviour of the derivative of \( \psi \) in \( [0,1] \), but, by averaging in the interval \( [1/4, 1/2] \), given \( \delta = \varepsilon/128 \)

\[
\int_{1/4}^{1/2} [\psi(t+\delta) - \psi(t-\delta)] dt = \int_{1/2}^{1/2+\delta} \psi(t) - \int_{1/2-\delta}^{1/2} \psi(t) \leq \psi \leq 2\delta
\]

and then, we can pick a number \( a \) in the interval \( (1/4, 1/2) \) such that \( \psi(a+\delta) - \psi(a-\psi) \leq 8\delta \). Hence, for every \( x,y \in [a-\delta,a+\delta] \), we have

\[
|\psi(x) - \psi(y)| \leq 8\delta = \varepsilon/16. \tag{**}
\]

Let \( k \) be the integer part of \( 2an, (k \leq 2an < k+1) \). Then, by (*)

\[
\psi(-\frac{k}{2n}) \geq \psi(-\frac{l}{8}) \geq \frac{l}{16} \quad \text{if} \ n \geq 4 \tag{***}
\]

We now define \( X \) a random \( E_n \)-valued vector by \( X(\omega) = \sum_{j} e_i(\omega) e_i \), where

\( \{e_i\} \) is an i.i.d. sequence of symmetric \( \{+1,-1\} \)-valued random variables defined in some probability space. If \( Y \) is another i.i.d. copy of \( X \), it is clear that the two random variables \( \|X - Y\| \) and \( 2\|\sum_{j} \eta_i e_i\| \) (where \( \{\eta_i\} \) is an i.i.d. sequence of random variables uniformly distributed on the set \( \{0,1\} \)) have the same distribution. Then, if we denote \( \lambda(n) = \|\sum_{j} e_i\| \), the 1-subsymmetry of the norm implies that the distribution of the random variables \( \psi \left( -\frac{l}{n} \sum_{j} \eta_i \right) \) and \( \|\frac{l}{2\lambda(n)} (X - Y)\| \) also coincides.
Since \( E \left( \frac{1}{n} \sum_{i}^{k} \eta_{i} \right) = \frac{k}{2n} \) we will compute the probability of deviation of \( \frac{1}{2\lambda(n)} (X - Y) \) from \( \psi \left( \frac{k}{2n} \right) \).

\[
A = \mathbb{P} \left( \omega \mid || \frac{1}{2\lambda(n)} (X - Y) \| - \psi \left( \frac{k}{2n} \right) \| > \varepsilon \left( \frac{k}{2n} \right) \right) \leq e^{\varepsilon \left( \frac{k}{2n} \right)}
\]

\[
\leq \mathbb{P} \left( \omega \mid \psi \left( \frac{1}{n} \sum_{i}^{k} \eta_{i} \right) - \psi \left( \frac{k}{2n} \right) \| > \varepsilon \left( \frac{k}{2n} \right) \right) \leq \mathbb{P} \left( \omega \mid \psi \left( \frac{1}{n} \sum_{i}^{k} \eta_{i} \right) - \psi \left( \frac{k}{2n} \right) \| > \varepsilon \left( \frac{k}{16} \right) \right)
\]

Note that \( a \leq \frac{1}{2n} \leq \frac{k}{2n} \), and so \( \frac{k}{2n} \in [a - \frac{\delta}{2}, a + \frac{\delta}{2}] \) if

\[ n > \frac{128}{\varepsilon}. \]

Thus, \( \left| \frac{1}{n} \sum_{i}^{k} \eta_{i} - \frac{k}{2n} \right| \leq \frac{\delta}{2} \) implies

\[ \left| \psi \left( \frac{1}{n} \sum_{i}^{k} \eta_{i} \right) - \psi \left( \frac{k}{2n} \right) \right| < \varepsilon \frac{1}{16} \]

by (**).

Since

\[ \frac{1}{n} \sum_{i}^{k} \eta_{i} - \frac{k}{2n} = \frac{1}{2n} \sum_{i}^{k} \varepsilon_{i}, \]

we have

\[ A \leq \mathbb{P} \left( \omega \mid \psi \left( \frac{1}{n} \sum_{i}^{k} \eta_{i} \right) - \psi \left( \frac{k}{2n} \right) \| > \varepsilon \left( \frac{1}{16} \right) \right) \leq \mathbb{P} \left( \omega \mid \frac{1}{n} \sum_{i}^{k} \varepsilon_{i} > \varepsilon \frac{1}{128} \right) \leq 2 \exp \left( - \frac{\varepsilon^{2} n^{3}}{C k} \right) \leq 2 \exp \left( - \frac{\varepsilon^{2} n^{3}}{C} \right) \]

where \( C \) is a numerical constant. In this last step we have used the well known probabilistic deviation inequality,

\[ \mathbb{P} \left( \omega \mid \sum_{i}^{m} \varepsilon_{i} > \lambda \sqrt{m} \right) \leq \exp \left( - \frac{\lambda^{2}}{2} \right) \]

\( \lambda > 0, m \in \mathbb{N} \)


Consider now a natural number \( N \) such that \( n > \frac{2C}{\varepsilon^{2}} \log N \). If \( \left| X_{i} \right|^{\theta} \) is an
i.i.d. sequence of copies of \( X \), then
\[
P\{\omega; \parallel \frac{1}{2\lambda(n)} (X_i - X_j) - \psi(\frac{k}{2n}) \parallel \leq \varepsilon \psi(\frac{k}{2n}), \text{ for all } i \neq j \} \geq 1 - \left( \frac{N}{2} \right) 2\exp\left(-\frac{\varepsilon^2 n}{2C}\right) > 0
\]

Hence, there exists \( \omega \) in the probability space, such that the corresponding points
\[
x_i = \frac{X(\omega)}{2\lambda(n) \psi\left(\frac{k}{2n}\right)} \quad 1 \leq i \leq N
\]
satisfy the conclusion of the theorem.

**Corollary.** The \( l_\infty^* \)-cube is \((1 + \varepsilon)\)-embedded in any finite-dimensional \( 1 \)-subsymmetric space \( E \), provided that \( \dim E > \frac{C}{\varepsilon^2} n \). (\( C \) is an absolute constant)

**Remarks.**

i) Since
\[
\| \sum_{i=1}^{k} e_i \| \leq 2 \left\| \sum_{i=1}^{k/2} e_i \right\| + 1
\]
it is easy to prove that \( \| x_i \| \leq \frac{3}{2} \quad 1 \leq i \leq N \).

ii) The asymptotic estimate \( n > K \log N \) is essentially best possible. Indeed, in a ball of radius \( r \) of \( E \), the number \( N \) of balls of radius \( r/2 \) we can pack into (with disjoint interior) is given by
\[
r^\text{vol} (B_i) \geq N\left( \frac{r}{2} \right)^n \text{vol} (B_i)
\]

\( (\text{vol} (B_i) \) is the \( n \)-dimensional volume of the unit ball).

iii) When \( E = l_p \), \( 1 \leq p < \infty \), we can improve slightly the numerical constant. Indeed, by taking \( a = 1/2 \) and using the mean value theorem we obtain the following:
a) If $\varepsilon < \frac{2}{p^{2/m}}$ then $n > \frac{C}{\varepsilon^{2/p^2}} \log N$

b) If $\varepsilon > \frac{1}{p^{2/m}}$ then $n > C \log N$

($C$ is a numerical constant). These expressions say that $p = \infty$ is the best possible situation, because, an isometric embedding ($\varepsilon = 0$) is possible in this case.

References


[8] Pisier, G., (Lecture given in the University of Murcia, 1985)


Jesús Bastero & Julio Bernués
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Zaragoza
50006 Zaragoza (Spain)

Nigel Kalton
Department of Mathematics
University of Missouri Columbia
Columbia MO 65211 (USA)