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# CHARACTERISTIC DETERMINANTS AND TRACES

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ABSTRACT. We show how to construct a determinant associated to a *regular* trace on an ideal of compact operators on a Hilbert space and prove a result characterizing those operators in the ideal for which the associated characteristic determinant is slowly growing in terms of uniqueness of trace.

## 1. INTRODUCTION

Suppose $\mathcal{I}$ is a (two-sided) ideal in $\mathcal{K}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space. The subspace Com $\mathcal{I}$ of $\mathcal{I}$ is defined to be the linear span of the commutators $[A,T] = AT - TA$ for $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{I}$. The problem of identifying Com $\mathcal{I}$ at least for the case of the Schatten ideals $\mathcal{I} = \mathcal{C}_p = \{T : \text{tr } |T|^p < \infty\}$ goes back to [10] where it is shown that Com $\mathcal{C}_p = \mathcal{C}_p$ when $1 < p < \infty$. For $0 < p < 1$, it was proved in [1] that Com $\mathcal{C}_p = \{T \in \mathcal{C}_p : \text{ tr } T = 0\}$. Results of [12] and [13] showed that the corresponding result is false when $p = 1$.

In the case of the trace-class $\mathcal{C}_1$ the subspace Com $\mathcal{C}_1$ was identified in [8]. To state the result of [8] let us introduce some notation. If $T \in \mathcal{K}(\mathcal{H})$ we let $(s_n(T))_{n=1}^\infty$ be the sequence of *singular values* of $T$ and $(\lambda_n(T))_{n=1}^\infty$ the *eigenvalue sequence* of $T$. By the latter we mean that the sequence $(\lambda_n(T))$ is the sequence of eigenvalues of $T$ repeated according to algebraic multiplicity and arranged in (some) order so that $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots$ ; we also specify that if $T$ has only finitely many eigenvalues then the sequence is completed by adjoining zeros. In [8] it is shown that if $T \in \mathcal{C}_1$, then $T \in$ Com $\mathcal{C}_1$ if and only if the diagonal operator diag $\{\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\} \in \mathcal{C}_1$, where $\lambda_k = \lambda_k(T)$.

Recently, in [3] it is shown that for an *arbitrary* ideal $\mathcal{I}$, and an *hermitian* operator $H \in \mathcal{I}$ we have $H \in$ Com $\mathcal{I}$ if and only if diag $\{\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\} \in \mathcal{I}$, where $\lambda_n = \lambda_n(H)$. This characterizes Com $\mathcal{I}$ since it is a self-adjoint subspace. Shortly after this in [9] it was shown that this characterization extends to all operators $T \in \mathcal{I}$ under the additional hypothesis that the ideal $\mathcal{I}$ is *geometrically stable* i.e. that

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diag $\{(s_1 \cdots s_n)^{1/n})\} \in \mathcal{I}$ whenever $T \in \mathcal{I}$ and $s_n = s_n(T)$. This condition is satisfied for quasi-Banach ideals (i.e. any ideal with an associated unitarily invariant quasi-norm for which it is complete). However, in general it is not true that we can characterize the condition $T \in \text{Com}\,\mathcal{I}$ in terms of the spectrum of $T$; see [4].

The motivation for this note is a result in the paper [6] where for the case of $\mathcal{C}_1$ it is shown that if $T \in \mathcal{C}_1$ then $T \in \text{Com}\,\mathcal{C}_1$ if and only if its characteristic determinant $f(z) = \det\,(I - zT)$ (see [7]) satisfies

$$\int_0^\infty \int_0^{2\pi} \frac{\log_+ |f(re^{i\theta}|}{r^2} \frac{d\theta}{2\pi} dr < \infty. \tag{1.1}$$

To explain our generalization of this result we recall that a *trace* on an ideal $\mathcal{I}$ is a linear functional satisfying $\tau(AT) = \tau(TA)$ for $T \in \mathcal{I}$ and $A \in \mathcal{B}(\mathcal{H})$ or equivalently $\tau(S) = 0$ for $S \in \text{Com}\,\mathcal{I}$. We call a trace *regular* if $\tau(T) = \text{tr}\,T$ whenever $T \in \mathcal{F}$, the ideal of finite-rank operators. If $\mathcal{I}$ is an ideal with a regular trace $\tau$ which is contained in $\cup_{p<\infty}\mathcal{C}_p$ we show in Theorem 3.1 that it is possible to construct a determinant $\Delta_\tau$ associated to $\tau$.

We say $T \in \mathcal{I}$ is uniquely traceable if all regular traces agree on $T$ (this is equivalent to $T \in \mathcal{F} + \text{Com}\,\mathcal{I}$). In our main theorem 4.3 we assume two conditions on $\mathcal{I}$, namely that it is exponentially stable and satisfies the $m$-Boyd condition for some $m \in \mathbb{N}$. These conditions which are precisely defined in Section 4 apply if $\mathcal{I}$ is a quasi-Banach ideal with upper Boyd index $q_\mathcal{I} < m$. We refer to [2] for a discussion of the Boyd indices. Then we show that if $T \in \mathcal{I}$ and $f(z) = \Delta_\tau(I - zT)$ that the growth condition that there exists a positive operator $P \in \mathcal{I}$ and a constant $C$ so that $|f(z)| \le Ce^{N(r)}$ where $N(r) = |\{n : s_n(P) \ge r\}|$ is equivalent to the statement that $T^k$ is uniquely traceable in the ideal $\mathcal{I}^k$ for $1 \le k \le m-1$. Here $\mathcal{I}^k$ is the ideal generated by all $T_1 \cdots T_k$ for $T_1, \cdots, T_k \in \mathcal{I}$. This reduces to the result of [6] when $\mathcal{I} = \mathcal{C}_1$ when we can take $m = 2$ and the condition characterizes uniquely traceable operators; with the additional constraint that $f'(z) = 0$ one would have $T \in \text{Com}\,\mathcal{C}_1$.

We refer the reader to [7] for background materials on compact operators and determinants; also see [11] for determinants in a Banach space setting. A very nice recent result on traces for ideals of operators on Banach spaces is given in [14].

## 2. Regular and unique traces

Let $\mathcal{I}$ be a (two-sided) ideal of compact operators on a separable Hilbert space $\mathcal{H}$. It was shown in [3] that $\mathcal{I}$ admits a regular trace if and only if diag $\{1/n\} \notin \mathcal{I}$. If $\mathcal{I}$ admits a regular trace, we will say that $T$ is *uniquely traceable* with unique trace $t$ if for every regular trace $\tau$ on $\mathcal{I}$ we have $\tau(T) = t$. We will say that a subspace $\mathcal{V}$ of $\mathcal{I}$ is uniquely traceable if each $T \in \mathcal{V}$ is uniquely traceable.

**Lemma 2.1.** *Suppose $\mathcal{I}$ is an ideal of compact operators admitting a regular trace. Then $T \in \mathcal{I}$ is uniquely traceable if and only if $T \in \text{Com}\,\mathcal{I} + \mathcal{F}$.*

**Proof.** If $T \in \mathcal{I}$ and $\tau$ is a regular trace it is easily seen that $T$ is uniquely traceable if and only if $T - F \in \text{Com}\,\mathcal{I}$ where $F$ is any finite-rank operator with tr $F = \tau(T)$. □

**Lemma 2.2.** *Suppose $\mathcal{I}$ is an ideal of compact operators admitting a regular trace. Suppose $H$ is an hermitian operator in $\mathcal{I} \cap \mathcal{C}_1$. Then the following are equivalent:*



*(i) diag* $\{\frac{1}{n}\sum_{k=n+1}^\infty \lambda_k(H)\} \in \mathcal{I}$.
*(ii) $H$ is uniquely traceable and the unique trace of $H$ is tr $H$.*

**Proof.** Assume (i). Let $\tau$ be a regular trace on $\mathcal{I}$. We show that $\tau(H) = \text{tr}\,H$. then follow by applying the same argument to $K$. We may suppose $H = \text{diag}\,\{\lambda_n\}$ where $\lambda_n = \lambda_n(H)$. Let $Q = \text{diag}\,\{a - \text{tr}\,H, -a, \lambda_1, \lambda_2, \cdots\}$ where $a$ is chosen so that this sequence is decreasing in absolute value. We show that $Q \in \text{Com}\,\mathcal{I}$ by using the criterion of [3]. In fact if $n \ge 3$ we have (letting $Q = \text{diag}\,\{q_n\}$ and using Lidskii's theorem),

$$|\frac{1}{n}\sum_{k=1}^n q_k| = \frac{1}{n}\sum_{k=n-2}^\infty \lambda_k$$

and it follows from our condition on $H$ that $Q \in \text{Com}\,\mathcal{I}$. Thus $\tau(Q) = 0$ and hence

$$\tau(H) = \tau(Q) - \tau(\text{diag}\,\{a - \text{tr}\,H, -a, 0, \cdots\}) = \text{tr}\,H.$$

Now assume (ii). Defining $Q$ as above we must have $Q \in \text{Com}\,\mathcal{I}$ and we obtain (i) by the same argument. □

**Lemma 2.3.** *Suppose $\mathcal{I}$ is an ideal of compact operators admitting a regular trace. Suppose $T$ is a positive operator which is uniquely traceable. Then $T \in \mathcal{C}_1$ and the unique trace of $T$ is tr $T$.*

**Proof.** Without loss of generality we write $T = \text{diag}\,\{s_n\}$ where $s_n = s_n(T)$. If $\tau$ is any regular trace on $\mathcal{I}$ let $Q = \text{diag}\,\{a - \tau(T), -a, s_1, s_2, \cdots\}$ where again $a$ is chosen so this sequence decreases in absolute value. Then $Q \in \text{Com}\,\mathcal{I}$ and hence there exists a positive operator $P = \text{diag}\,\{d_n\} \in \mathcal{I}$ with $d_n \downarrow 0$ such that if $n \ge 3$

$$|\sum_{k=1}^{n-2} s_k - \tau(T)| \le nd_n.$$

Now since $\mathcal{I}$ admits a regular trace we have by the result of [3] that

$$\liminf_{n\to\infty} nd_n = 0.$$

This implies that $\tau(T) = \sum_{k=1}^\infty s_k$ and proves the lemma. □

**Theorem 2.4.** *Let $\mathcal{I}$ be an ideal in $\mathcal{K}(\mathcal{H})$. In order that $\mathcal{I}$ admits a unique regular trace it is necessary and sufficient that $\mathcal{I} \subset \mathcal{C}_1$ and that if diag $\{s_n\} \in \mathcal{I}$ with $s_n \downarrow 0$ then diag $\{\frac{1}{n}\sum_{k=n}^\infty s_k\} \in \mathcal{I}$.*

**Proof.** This is immediate from the previous Lemmas 2.2 and 2.3. □

**Proposition 2.5.** *Let $\mathcal{I}$ be an ideal contained in $\mathcal{C}_p$ for some $p < \infty$. Suppose $m > p+1$. Then $\mathcal{I}^m$ is uniquely traceable in $\mathcal{I}$ with unique trace tr .*

**Proof.** It is enough to consider the case of a positive $T = \text{diag}\,\{d_n\} \in \mathcal{I}^m$, where $d_n \downarrow 0$. Then there exists a positive $P = \text{diag}\,\{s_n\} \in \mathcal{I}$ with $s_n \downarrow 0$ and $d_n \le s_n^m$. Now

$$\sum_{k=n+1}^\infty d_k \le s_n \sum_{k=n+1}^\infty s_n^{m-1}$$

so that we can apply Lemma 2.2 to obtain the result. □

## 3. Determinants

Our next theorem shows how to construct a determinant from a regular trace at least for operators in $\cup_{p<\infty}\mathcal{C}_p$. Let us define a sequence $(\varphi_n)$ of polynomials in $n$ variables, for $n = 1, 2, \cdots$ by the formal expansion

$$\exp(-\sum_{k=1}^{\infty}\frac{a_k}{k}z^k) = 1 + \sum_{k=1}^{\infty}\varphi_k(a_1,\cdots,a_k)z^k.$$

Also, if $m \in \mathbb{N}$ we define the entire function $E_m(z)$ by

$$E_m(z) = (1-z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^{m-1}}{m-1}).$$

(When $m = 1$, $E_1(z) = (1-z)$.) We note the estimate $|E_m(z) - 1| \le C|z|^m$ for $|z| < \frac{1}{2}$.

**Theorem 3.1.** *Suppose $\mathcal{I}$ is an ideal of compact operators and that $\tau$ is a regular trace on $\mathcal{I}$. Assume that $T \in \mathcal{I}$ is such that $T \in \mathcal{C}_p$ for some $p < \infty$. Then the series $\sum_{k=1}^{\infty}\varphi_k(\tau(T),\cdots,\tau(T^k))$ converges.*

*Furthermore if we set*

$$\Delta_\tau(I-T) = 1 + \sum_{k=1}^{\infty}\varphi_k(\tau(T),\cdots,\tau(T^k)) \tag{3.1}$$

*then we have:*

*(i) If $m \in \mathbb{N}$ with $m > p+1$ then*

$$\Delta_\tau(I-zT) = \exp(-\sum_{k=1}^{m-1}\tau(T^k)\frac{z^k}{k})\prod_{j=1}^{\infty}E_m(\lambda_j z) \tag{3.2}$$

*where $\lambda_j = \lambda_j(T)$.*

*(ii) If $A, B \in \mathcal{B}(\mathcal{H})$ with $I - A, I - B \in \mathcal{I}\cap\mathcal{C}_p$ for some $p < \infty$ then $\Delta_\tau(AB) = \Delta_\tau(A)\Delta_\tau(B)$.*

**Proof.** We begin by fixing $m \in \mathbb{N}$ so that $m \ge p+1$. It follows from Proposition 2.5 (applied to $\mathcal{I}\cap\mathcal{C}_p$) that, if $\lambda_j = \lambda_j(T)$,

$$\tau(T^k) = \operatorname{tr}(T^k) = \sum_{j=1}^{\infty}\lambda_j^k$$

whenever $k \ge m$.

Notice that we also have $\sum_{j=1}^{\infty}|\lambda_j|^m < \infty$ and so the product $\prod_{j=1}^{\infty}E_m(\lambda_j z)$ converges uniformly on compact sets to an entire function $F$. If $|z| < |\lambda_1|^{-1}$ then $F$ is nonvanishing and indeed $F(z) = e^{G(z)}$ where

$$G(z) = -\sum_{k=m}^{\infty}\frac{\tau(T^k)}{k}z^k.$$

It follows that the function

$$H(z) = \exp(-\sum_{k=1}^{m-1}\frac{\tau(T^k)}{k}z^k)F(z)$$



is entire and clearly this implies equation (3.1) defines $\Delta_\tau(I-T)$ properly. We also have (i) immediately.

In order to prove (ii), let $S = I - A$ and $T = I - B$. Let $B(w) = I - wT$ for $w \in \mathbb{C}$. Note that it follows easily from the definition that the function $\phi(z,w) = \Delta_\tau(I-zS)(I-wT)$ is entire on $\mathbb{C}^2$. For small enough $z, w$ we can write

$$\phi(z,w) = \exp(-\sum_{k=1}^{\infty}\frac{1}{k}\tau(zSB(w)+wT)^k)$$

Note that the properties of the trace $\tau$ yield that

$$\frac{\partial}{\partial z}\tau(zSB(w)+wT)^k = k\tau(SB(w)(zSB(w)+wT)^{k-1}).$$

Hence for $(z,w)$ in a neigborhood of the origin we have:

$$\frac{\partial}{\partial z}\phi(z,w) = -\phi(z,w)\sum_{k=1}^{\infty}\tau(SB(w)(zSB(w)+wT)^{k-1}).$$

Let $m$ be any integer greater than $p$. Now, if $k \ge m+1$ we obtain from Proposition 2.5 that

$$\tau(SB(w)(zSB(w)+wT)^{k-1}) = \operatorname{tr}(SB(w)(zSB(w)+wT)^{k-1}).$$

In a suitable neighborhood of the origin $(I - zS)B(w)$ is invertible and

$$\begin{aligned}\sum_{k=m+1}^{\infty}\operatorname{tr}(SB(w)(zSB(w)+wT)^{k-1}) &= \operatorname{tr}(S(I-zS)^{-1}(zSB(w)+wT)^m)\\ &= \tau(S(I-zS)^{-1}(zSB(w)+wT)^m).\end{aligned}$$

By simple linearity

$$\sum_{k=1}^{m}\tau(SB(w)(zSB(w)+wT)^{k-1}) = \tau(S(I-zS)^{-1}(I-(zSB(w)+wT)^m)).$$

We finally obtain

$$\frac{\partial}{\partial z}\phi(z,w) = -\tau(S(I-zS)^{-1})\phi(z,w)$$

for $(z,w)$ in a suitable neighborhood of the origin. This leads immediately to the deduction that $\phi(z,w) = \phi(z,0)\phi(0,w)$ for $(z,w)$ small enough. From the fact that $\phi$ is entire we obtain the same equation globally and hence deduce (ii). □

## 4. Growth of the characteristic determinant

We now introduce the functionals defined in [9]. Suppose $T$ is any compact operator on $\mathcal{H}$ and that $\lambda_n = \lambda_n(T)$. We define

$$\nu(T) = \sum_{|\lambda_n|\ge 1}1, \quad \mu(T) = \sum_{n=1}^{\infty}\log_+|\lambda_n|, \quad \text{and } \chi(T) = \sum_{|\lambda_n|\ge 1}\lambda_n.$$

The next theorem characterizes operators in $\mathcal{I}$ with slowly growing determinants.

**Theorem 4.1.** *Let $\mathcal{I}$ be an ideal of compact operators admitting a regular trace $\tau$ and contained in $\cup_{p<\infty}\mathcal{C}_p$. Suppose $T \in \mathcal{I}$, and let $\lambda_n = \lambda_n(T)$. Then the following conditions on $T$ are equivalent:*

*(i) There exists a positive operator $P \in \mathcal{I}$ and a constant $C$ so that for all $z \in \mathbb{C}$ we have*

$$|\Delta_\tau(I - zT)| \le Ce^{\nu(|z|P)} \tag{4.1}$$

*(ii) There exists a positive operator $P = diag\,\{d_n\} \in \mathcal{I}$ where $d_n \downarrow 0$ such that $\mu(rT) \le \nu(rP)$ for all $r > 0$ and for every $k, n \in \mathbb{N}$*

$$\frac{1}{n}|\tau(T^k) - \sum_{j=1}^{n} \lambda_j^k| \le d_n^k \tag{4.2}$$

*(iii) There exists a positive operator $P \in \mathcal{I}$ so that for every $r > 0, k \in \mathbb{N}$,*

$$\mu(rT) + |\tau((rT)^k) - \chi((rT)^k)| \le \nu(rP) \tag{4.3}$$

**Proof.** We first prove the equivalence of (i) and (iii). We will denote $\Delta_\tau(1 - zT)$ by $f(z)$. Let $M(r) = \max_{|z|=r} \log|f(z)|$ and $V(r) = \sup_k |\tau((rT)^k - \chi((rT)^k)|$. Let us assume that $T \in \mathcal{C}_p$ and that $m > p+1$. We have by Lemma 2.5 that $\tau(T^k) = \sum_{j=1}^\infty \lambda_j^k$ for $k \ge m$.

For any $r > 0$ we rewrite $f$ in the form

$$f(z) = g_r(z) \prod_{r|\lambda_j|\ge 1} (1 - \lambda_j z) \tag{4.4}$$

Then $g_r$ is non-vanishing in a neighborhood of $\{z : |z| \le r\}$. In fact from Theorem 3.1 (3.2) we have for $|z| \le r$,

$$g_r(z) = \exp(\sum_{k=1}^\infty \frac{z^k}{k}(r^{-k}\chi(r^kT) - \tau(T^k))) \tag{4.5}$$

We deduce, using the fact that $\log|g_r(0)| = 0$ that

$$\begin{aligned} V(r) &\le 2\int_0^{2\pi} |\log|g_r(re^{i\theta})||\frac{d\theta}{2\pi} \\ &= 4\int_0^{2\pi} \log_+ |g_r(re^{i\theta})|\frac{d\theta}{2\pi} \end{aligned}$$

Combining with an obvious lower estimate we have:

$$\int_0^{2\pi} \log_+ |g_r(\frac{1}{2}re^{i\theta})|\frac{d\theta}{2\pi} \le V(r) \le 4\int_0^{2\pi} \log_+ |g_r(re^{i\theta})|\frac{d\theta}{2\pi} \tag{4.6}$$

Now for any $s > 0$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \int_0^{2\pi} |\log|1 - s\lambda e^{i\theta}||\frac{d\theta}{2\pi} &= \int_0^{2\pi} 2\log_+ |1 - s\lambda e^{i\theta}| - \log|1 - s\lambda e^{i\theta}|\frac{d\theta}{2\pi} \\ &\le 2(\log 2 + \log_+ s|\lambda|) - \log_+ s|\lambda| \\ &\le \log 4 + \log_+ s|\lambda|. \end{aligned}$$

It follows then from (4.4) that if $0 < s < r$

$$\left|\int_0^{2\pi} \log_+ |f(se^{i\theta})| - \log_+ |g_r(se^{i\theta})|\frac{d\theta}{2\pi}\right| \le \nu(rT)\log 4 + \mu(sT) \le \mu(4rT) \tag{4.7}$$



Also we observe the standard

$$\frac{1}{3}M(\frac{1}{2}r) \le \int_0^{2\pi} \log_+ |f(re^{i\theta}|\frac{d\theta}{2\pi} \le M(r). \tag{4.8}$$

Combining equations (4.6), (4.7) and (4.8) gives us that

$$\frac{1}{3}M(\frac{r}{4}) - \mu(4rT) \le V(r) \le 4M(r) + 4\mu(4rT) \tag{4.9}$$

We note also Jensen's inequality gives that $\mu(rT) \le M(r)$. Combining this with (4.9) easily yields the equivalence of (i) and (iii).

The equivalence of (ii) and (iii) is essentially formal. We have only to show that (4.2) and (4.3) are equivalent conditions. In fact assume (ii); without loss of generality we may suppose that $\nu(rT) \le \nu(rP)$ for all $r$ (just replace $P$ by $P \oplus P \oplus P$). Thus $|\lambda_j| \le d_j$ for all $j$. Then suppose $r > 0$; let $n$ be chosen so that $d_n r < 1 \le d_{n-1}r$.

$$|\tau(r^kT^k) - \chi(r^kT^k)| \le r^k|\tau(T^k) - \sum_{j=1}^n \lambda_j^k| + n(rd_n)^k$$

so that

$$|\tau(r^kT^k) - \chi(r^kT^k)| \le 2n(rd_n)^k \le 4\nu(rP).$$

Conversely assume (iii). Again we may assume $\nu(rT) \le \nu(rP)$ for all $r$ or equivalently $|\lambda_n| \le d_n$ for all $n$. We may assume $d_n \ne 0$ for all $n$ (the other case is trivial). Then setting $r = d_n^{-1}$ we have

$$|\tau(T^k) - r^{-k}\chi(r^kT^k)| \le nd_n^k$$

and (ii) follows at once for $P$ replaced by $2P$, since

$$|r^{-k}\chi(r^kT^k) - \sum_{j=1}^n \lambda_j^k| \le nd_n^k \quad \square$$

Now we will say that $\mathcal{I}$ is *exponentially stable* if for every $T \in \mathcal{I}$ then $\oplus_{j=0}^\infty 2^{-j}T \in \mathcal{I}$. This condition is trivially satisfied by quasi-Banach ideals.

**Proposition 4.2.** *If $\mathcal{I}$ is exponentially stable then:*
*(i) If $T \in \mathcal{I}$ then there exists a positive operator $P$ so that $\mu(rT) \le \nu(rP)$ for all $r > 0$.*
*(ii) $\mathcal{I}$ is geometrically stable.*

**Proof.** (i) Let $P = \oplus_{j=0}^\infty 2^{-j}|T|$. Let $\lambda_j(T) = \lambda_j$ and $s_j(T) = s_j$. For $r > 0$ let $n$ be the largest integer so that $r|\lambda_n| \ge 1$. Then

$$\mu(rT) = \log|\lambda_1 \cdots \lambda_n| + n\log r \le \log s_1 \cdots s_n + n\log r.$$

Let $m$ be the largest integer $1 \le m \le n$ so that $rs_m \ge 1$. Then

$$\mu(rT) \le \log(s_1 \cdots s_m) + m\log r.$$

Now

$$\nu(rP) = \sum_{rs_j\ge 1} (1 + \log_2 rs_j) \ge \mu(rT).$$

(ii) Suppose $T = \text{diag}\ \{s_n\} \in \mathcal{I}$ where $s_n \downarrow 0$. Pick $P = \text{diag}\ \{t_n\}$ as in (i). Then

$$\begin{aligned} n &\geq \sum_{s_j \geq t_n} \log(s_j/t_n) \\ &\geq \sum_{j=1}^{n} \log(s_j/t_n) \\ &\geq \log(s_1 \cdots s_n) - n \log t_n. \end{aligned}$$

Hence

$$(s_1 \cdots s_n)^{1/n} \leq e t_n$$

so that diag $\{(s_1 \cdots s_n)^{1/n}\} \in \mathcal{I}$. □

Suppose $m \in \mathbb{N}$. We will say that an ideal $\mathcal{I}$ satisfies the *m-Boyd condition* if whenever diag $\{s_n\} \in \mathcal{I}$ where $s_n \downarrow 0$ then diag $\{t_n\} \in \mathcal{I}$ where $t_n = \sum_{k=0}^{\infty} 2^{k/m} s_{2^k n}$. The reason for the terminology is that if $\mathcal{I}$ is a quasi-Banach ideal then $\mathcal{I}$ satisfies the $m$-Boyd condition if $m > q_{\mathcal{I}}$ where $q_{\mathcal{I}}$ is the upper Boyd index of $\mathcal{I}$. We refer to [2] for a discussion of the Boyd indices. Note that if $\mathcal{I}$ satisfies the $m$-Boyd condition then $\sum_{k=0}^{\infty} 2^{k/m} s_{2^k}(T) < \infty$ for all $T \in \mathcal{I}$ and so $\mathcal{I} \subset \mathcal{C}_m$.

We now state our main theorem:

**Theorem 4.3.** *Suppose $\mathcal{I}$ is an exponentially stable ideal of compact operators, and suppose $\mathcal{I}$ admits a regular trace $\tau$ and satisfies the m-Boyd condition. Then for $T \in \mathcal{I}$ the following conditions are equivalent:*

*(i) There exists a positive operator $P \in \mathcal{I}$ and a constant $C$ so that for all $z \in \mathbb{C}$ we have*

$$|\Delta_\tau(I - zT)| \leq C e^{\nu(|z|P)}$$

*(ii) For $1 \leq k \leq m-1$ the operator $T^k$ is uniquely traceable in the ideal $\mathcal{I}^k$.*

*(iii) For every $k \in \mathbb{N}$ the operator $T^k$ is uniquely traceable in the ideal $\mathcal{I}^k$.*

*In particular if $\mathcal{I}$ satisfies the 2-Boyd condition then $T$ is uniquely traceable if and only if (i) holds, and $T \in$ Com $\mathcal{I}$ if and only if $\tau(T) = 0$ and (i) holds.*

**Proof.** We first prove (i) implies (iii). It follows from Proposition 4.2 that $\mathcal{I}$ and hence $\mathcal{I}^k$ for all $k \in \mathbb{N}$ are geometrically stable. Now it follows from Theorem 3.3 of [9] that $T^k$ is uniquely traceable in $\mathcal{I}^k$ if and only if the operator diag $\{-\tau(T^k), \lambda_1^k, \lambda_2^k, \cdots\}$ $\in$ Com $\mathcal{I}^k$. As in Lemma 2.2 this is equivalent to diag $\{a - \tau(T^k), -a, \lambda_1^k, \cdots\} \in$ Com $\mathcal{I}^k$. This follows immediately from Theorem 4.1 (ii) and the criterion of [3] or [9] Theorem 3.1.

Now (iii) implies (ii) trivially. Assume therefore (ii). Again from geometric stability the operator $S = \text{diag}\ \{\lambda_n\} \in \mathcal{I}$.

We start by observing that $\mathcal{I} \subset \mathcal{C}_m$ and that $\mathcal{I}^m$ admits a unique regular trace tr . Indeed in diag $\{s_n\} \in \mathcal{I}$ then

$$\begin{aligned} \frac{1}{n} \sum_{k=n+1}^{\infty} s_k^m &\leq \sum_{k=0}^{\infty} 2^k s_{2^k n}^m \\ &\leq (\sum_{k=0}^{\infty} 2^{k/m} s_{2^k n})^m \end{aligned}$$

so that we can apply Theorem 2.4.



In particular if we let $b_n = \sum_{k=0}^{\infty} 2^{k/m} |\lambda_{2^k n}|$ then $B = \text{diag}\ \{b_n\} \in \mathcal{I}$ and

$$\frac{1}{n} \sum_{j=n+1}^{\infty} |\lambda_j|^m \leq b_n^m.$$

But then if $k > m$ we have

$$\frac{1}{n} \sum_{j=n+1}^{\infty} |\lambda_j|^k \leq b_n^m |\lambda_n|^{k-m} \leq b_n^k.$$

Now since tr $(T^k) = \tau(T^k)$ for $k \geq m$ this implies that (4.2) is satisfied for $k \geq m$ with $(b_n)$ replacing $(d_n)$.

It remains to observe that for $1 \leq k \leq m-1$ we have that $T^k$ is uniquely traceable in $\mathcal{I}^k$ so that, repeating the argument from (i) implies (iii), we can find operators $P_k = \text{diag}\ \{d_{kn}\}_{n=1}^{\infty}$ for $1 \leq k \leq m-1$, in $\mathcal{I}$ with

$$|\tau(T^k) - \sum_{j=1}^{n} \lambda_j^k| \leq d_{kn}^k$$

for $n \in \mathbb{N}$. Finally by Proposition 4.2 there is a operator $P_0 = \text{diag}\ \{d_{0n}\}_{n=1}^{\infty} \in \mathcal{I}$ with $\mu(rT) \leq \nu(rP_0)$ for all $r > 0$. If we finally let $P = P_0 + P_1 + \cdots + P_{m-1} + B = \text{diag}\ \{d_n\}$ then (ii) of Theorem 4.1 holds and so (i) holds.

The last part of the statement of the Theorem follows trivially. □

**Concluding Remarks.** As observed in the introduction, in the case when $\mathcal{I} = \mathcal{C}_1$ this reduces to the result of [6] when we take $\tau$ as the standard trace so that $\Delta_\tau$ is the standard determinant det . We omit the elementary verification that (i) is equivalent to (1.1).

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