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ASYMPTOTIC UNCONDITIONALITY

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Abstract

We show that a separable real Banach space embeds almost isometrically in a space Y with a shrinking 1-unconditional basis if and only if $\lim_{n\to\infty} \|x^* + x_n^*\| = \lim_{n\to\infty} \|x^* - x_n^*\|$ whenever $x^* \in X^*$, $(x_n^*)_{n=1}^{\infty}$ is a weak*-null sequence and both limits exist. If X is reflexive then Y can be assumed reflexive. These results provide the isometric counterparts of recent work of Johnson and Zheng.

1. Introduction

In this paper we consider only real Banach spaces. Recently, Johnson and Zheng [10] gave an intrinsic characterization of separable Banach spaces which embed isomorphically into a reflexive Banach space with unconditional basis. Precisely a separable reflexive Banach space X embeds into a (reflexive) Banach space with unconditional basis if and only if X has the unconditional tree property (UTP), that is, for some C, every weakly null tree has a C-unconditional branch. The use of tree properties to describe subspaces of certain Banach spaces is a recent development in Banach space theory which originates in [13] and was later developed in [20].

The results of [13, 20] are both, in a certain sense, isomorphic versions of earlier isometric results from [15]. In the latter paper, for $1 < p < \infty$, it is shown that if X is a separable Banach space containing no copy of ℓ_1, then X $(1+\delta)$-embeds in an ℓ_p-sum of finite-dimensional spaces for every $\delta > 0$ if and only if

$$\lim_{n\to\infty} (\|x + x_n\|^p - \|x\|^p - \|x_n\|^p) = 0$$

whenever $x \in X$ and $(x_n)_{n=1}^{\infty}$ is a weakly null sequence. Similarly, again assuming X is separable and contains no copy of ℓ_1, X $(1+\delta)$-embeds into c_0 for every $\delta > 0$ if and only if

$$\lim_{n\to\infty} (\|x + x_n\| - \max(\|x\|, \|x_n\|)) = 0$$

whenever $x \in X$ and $(x_n)_{n=1}^{\infty}$ is weakly null.

In [13] it was shown that a separable Banach space X, containing no copy of ℓ_1, embeds isomorphically into c_0 if and only if every weakly null tree has a c_0-branch; the corresponding result for $1 < p < \infty$ was given in [20] where it was shown that a reflexive Banach space X embeds isomorphically into an ℓ_p-sum of finite-dimensional spaces if and only if every weakly null tree has an ℓ_p-branch. We remark that in [13] the proof of the isomorphic result was given by renorming and reducing to a situation very similar to the isometric result.

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The aim of this paper is to prove an isometric analog of the Johnson–Zheng theorem. We say that a separable Banach space X has *property* (au) if given any $x \in X$ and $\delta > 0$ there is a closed subspace F of finite codimension such that

$$\|x - y\| \le (1 + \delta)\|x + y\|, \quad y \in F.$$

This could be restated as

$$\lim_{d \in D}(\|x + x_d\| - \|x - x_d\|) = 0$$

whenever $x \in X$ and $(x_d)_{d \in D}$ is a bounded weakly null net. If X has separable dual we may replace nets by sequences in this definition. There is also a natural dual notion; a separable Banach space X has *property* (au^*) if given any $x^* \in X^*$ and $\delta > 0$ there is a weak* closed subspace F of finite codimension in X^* such that

$$\|x^* - y^*\| \le (1 + \delta)\|x^* + y^*\|, \quad y^* \in F.$$

This is equivalent to

$$\lim_{n \to \infty}(\|x^* + x_n^*\| - \|x^* - x_n^*\|) = 0$$

whenever $x^* \in X^*$ and $(x_n^*)_{n=1}^{\infty}$ is a weak*-null sequence in X^*. Both these concepts already exist in the literature under different names (see [**16**, **24**]). It is easy to show that (au^*) implies (au) (Proposition 2.3 below) but the converse is false (take $X = \ell_1$).

Our main result (Theorem 4.2) is that a separable Banach space X has property (au^*) if and only if for every $\delta > 0$ there is a Banach space Y with a shrinking 1-unconditional basis and a subspace X_δ of Y with $d(X, X_\delta) < 1 + \delta$; Y may be assumed reflexive when X is reflexive. A special case of this theorem was already implicit in the literature. Recall that a separable Banach space X has the *unconditional metric approximation property* (*UMAP*) [**4**] if there is a sequence of finite rank operators such that $\lim_{n \to \infty} T_n x = x$ for $x \in X$ and $\lim_{n \to \infty} \|I - 2T_n\| = 1$; if additionally $\lim_{n \to \infty} T_n^* x^* = x^*$ for $x^* \in X^*$ we say that X has shrinking (UMAP). Lima [**16**] showed that if X is a separable Banach space with property (au^*) and such that X^* has the approximation property then X has (shrinking) (UMAP). In [**6**, Corollary IV.4] it is shown that if X has shrinking (UMAP) then X can be $(1 + \delta)$-embedded in a space with a shrinking 1-unconditional basis; unfortunately the proof of this result is inaccurate (as Haskell Rosenthal has pointed out to us) and we give a corrected proof below (contained in Proposition 3.3). Thus the novelty in Theorem 4.2 is the removal of the approximation property hypothesis. Let us also remark at this point that Johnson and Zheng [**11**] have informed us that they have extended the methods of [**10**] to show that a separable Banach space X with separable dual embeds isomorphically into a space with a shrinking unconditional basis if and only if X^* has the weak*-(UTP). This provides a complete isomorphic analog of Theorem 4.2.

If X is reflexive (au) is equivalent to (au^*) and so Theorem 4.2 could be restated using property (au). We conjecture that if X contains no copy of ℓ_1 then (au) and (au^*) are equivalent. We are not quite able to prove this, but we do prove a result very close to it. We say that a separable Banach space has *weak alternating Banach–Saks* (*WABS*) property if given any bounded sequence $(x_n)_{n=1}^{\infty}$

we can find a sequence of convex blocks $(y_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \sup_{r_1<r_2<\cdots<r_n} \left\| \frac{1}{n} \sum_{j=1}^{n} (-1)^j y_{r_j} \right\| = 0.$$

This condition is implied by reflexivity or the alternating Banach–Saks (ABS) property. Then X has property (au^*) if and only if X has property (au) and (WABS). The example of the James space [9] shows then there is a space with separable dual and (UTP) which has no equivalent renorming to have property (au).

2. Asymptotic unconditionality

Let X be a separable Banach space. We will say that X is *asymptotically unconditional* (au) if given any $x \in X$ and $\delta > 0$ there is a closed finite-codimensional subspace W of X such that

$$\|x - w\| \leq (1+\delta)\|x + w\|, \quad w \in W.$$

An alternative formulation of this condition is that

$$\lim_{d\in D}(\|x + u_d\| - \|x - u_d\|) = 0$$

whenever $x \in X$ and $(u_d)_{d\in D}$ is a bounded weakly null net.

We shall say that X is *sequentially asymptotically unconditional* $(\omega - au)$ if

$$\lim_{n\to\infty}(\|x + u_n\| - \|x - u_n\|) = 0$$

whenever $x \in X$ and $(u_n)_{n=1}^{\infty}$ is a weakly null sequence. This condition has already been considered by Sims [24] under the acronym WORTH. Note that if X^* is separable then the weak topology is metrizable on bounded sets and so X is $(\omega - au)$ if and only if X is (au).

We shall say that X is **-asymptotically unconditional* (au^*) if

$$\lim_{n\to\infty}(\|x^* + x_n^*\| - \|x^* - x_n^*\|) = 0$$

whenever $x^* \in X^*$ and $(x_n^*)_{n=1}^{\infty}$ is a weak*-null sequence in X^*. This condition has been considered under the name (wM^*) by Lima [16]; later Oja [21] considered a family of more general conditions. Since X is assumed separable, the weak*-topology on bounded sets is metrizable, and so X^* is *-asymptotically unconditional if and only if either given any $x^* \in X^*$ and $\epsilon > 0$ there is a weak*-closed finite-codimensional subspace W of X^* such that

$$\|x^* - w^*\| \leq (1+\epsilon)\|x^* + w^*\|, \quad w^* \in W,$$

or, alternatively,

$$\lim_{d\in D}(\|x^* + u_d^*\| - \|x^* - u_d^*\|) = 0$$

whenever $x^* \in X^*$ and $(u_d^*)_{d\in D}$ is a bounded weak*-null net.

We first state a very simple principle based on compactness that will be used frequently.

LEMMA 2.1 (i) *Let X be a separable Banach space with property (au). Then given any finite-dimensional subspace E of X and $\delta > 0$ there is a closed finite-codimensional subspace F of X such that*

$$\|e - f\| \leq (1+\delta)\|e+f\|, \quad e \in E, f \in F.$$

(ii) *Let X be a separable Banach space with property (au^*). Then given any finite-dimensional subspace E of X^* and $\delta > 0$ there is a weak*-closed finite-codimensional subspace F of X^* such that*

$$\|e^* - f^*\| \leq (1+\delta)\|e^*+f^*\|, \quad e^* \in E, f^* \in F.$$

The following result is a consequence of [16, Proposition 4.1], but we give an independent proof.

PROPOSITION 2.2 *If X is a separable Banach space with (au^*) then X^* has no proper norming subspace and hence is separable.*

Proof. Let M be a norming subspace of X^*. If $M \neq X^*$ then there exists $x^* \in X^*$ with $\|x^*\| = 1$ and $d(x^*, M) = d > 1/2$. Let (x_n^*) be a sequence in $B_{X^*} \cap M$ which weak*-converges to x^*. Then

$$1 \geq \lim_{n\to\infty} \|x_n^*\| = \lim_{n\to\infty} \|x^* + (x_n^* - x^*)\| = \lim_{n\to\infty} \|2x^* - x_n^*\| \geq 2d > 1.$$

This contradiction gives the result.

PROPOSITION 2.3 *Suppose X is a separable Banach space. Then*

(a) *if X has (au^*) then X has (au);*
(b) *if X is reflexive then X has (au^*) if and only if X has (au).*

Proof. (a) It is enough to show that if $x \in X$ and $(u_d)_{d\in D}$ is a bounded weakly null net with

$$\lim_{d\in D} \|x + u_d\| = 1, \quad \lim_{d\in D} \|x - u_d\| = \theta$$

then $\theta \geq 1$. To do this we may by the Hahn–Banach theorem pick $(x_d^*)_{d\in D}$ with $x_d^*(x+u_d) = \|x+u_d\|$ and $\|x_d^*\| = 1$. We may then pass to a subnet and assume that $(x_d^*)_{d\in D}$ is weak*-convergent to some x^*. Let $x_d^* = x^* + u_d^*$. Then

$$\begin{aligned}1 &= \lim_{d\in D}(x^*(x) + x^*(u_d) + u_d^*(x) + u_d^*(u_d))\\ &= \lim_{d\in D}(x^*(x) - x^*(u_d) - u_d^*(x) + u_d^*(u_d))\\ &\leq \limsup_{d\in D} \|x^* - u_d^*\|\|x - u_d\|\\ &= \theta.\end{aligned}$$

This proves (a).

(b) It is a trivial deduction from (a).

PROPOSITION 2.4 (i) *If X is a Banach space with a shrinking* 1*-unconditional finite dimensional decomposition (1-UFDD) then X has* (*au**).
(ii) *If Y is a separable Banach space with* (*au**) *then any subspace or quotient X of Y also has* (*au**).

Proof. Condition (i) is clear, as is (ii) for quotients. Consider the case when X is a subspace of Y. Suppose $x^* \in X^*$ and (u_n^*) is a weak*-null sequence in X^* such that $\lim_{n\to\infty} \|x^* + u_n^*\| = 1$ but $\lim_{n\to\infty} \|x^* - u_n^*\| = 1 + \delta > 1$. Let $y_n^* \in Y^*$ be extensions to Y with $\|y_n^*\| = \|x^* + u_n^*\|$. Passing to a subsequence we can suppose (y_n^*) converges weak* to y^*. Then $\lim_{n\to\infty} \|2y^* - y_n^*\| = 1$. However $(2y^* - y_n^*)|_X = x^* - u_n^*$ and we have a contradiction.

REMARK Note that property (au) does not pass to quotients since every separable Banach space is a quotient of ℓ_1.

We close this section with a simple Lemma, which will be useful later.

LEMMA 2.5 (i) *Let X be a separable Banach space with property (au), and suppose that $(x_n)_{n=1}^\infty$ is a weakly null sequence which is not norm convergent to* 0. *Then, given $\delta > 0$, there is a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that the sequence $(y_n)_{n=1}^\infty$ is $(1+\delta)$-unconditional.*
(ii) *Let X be a separable Banach space with property (au^*), and suppose that $(x_n^*)_{n=1}^\infty$ is a weak*-null sequence in X^* which is not norm convergent to* 0. *Then, given $\delta > 0$, there is a subsequence $(y_n^*)_{n=1}^\infty$ of $(x_n^*)_{n=1}^\infty$ such that the sequence $(y_n^*)_{n=1}^\infty$ is $(1+\delta)$-unconditional.*

Proof. The proofs of these statements are essentially identical so we prove only (i).

We may suppose, by passing to a subsequence, that $(x_n)_{n=1}^\infty$ is basic (see for example, [**1**, Theorem 1.5.2]). Let K be the basis constant for the sequence $(x_n)_{n=1}^\infty$ and assume that $0 < c \le \|x_k\| \le C < \infty$ for all k.

Choose $(\delta_n)_{n=1}^\infty$ to be a decreasing sequence of positive numbers so that $\prod_{j=1}^\infty (1+\delta_j) < 1+\delta$. We will construct a subsequence $(y_n)_{n=1}^\infty$ and a sequence $(F_n)_{n=1}^\infty$ of closed finite-codimensional subspaces inductively.

Let $y_1 = x_1$ and $F_1 = X$. If $y_1, \ldots, y_{n-1}$ and $F_1, \ldots, F_{n-1}$ have been chosen then we may choose a closed subspace F_n of finite codimension so that if $w \in [y_j]_{j=1}^{n-1}$ and $z \in F_n$ then

$$\|w - z\| \le (1 + \tfrac{1}{4}\delta_n)\|w + z\|.$$

Let $Q_j : X \to X/F_j$ denote the quotient map for $1 \le j \le n$. If $y_{n-1} = x_{m_n}$ we may pick $y_n = x_{m_{n+1}}$ with $m_{n+1} > m_n$ so that

$$\|Q_j y_n\| \le \frac{2^{j-n-1} c\delta_j}{10K}, \quad 1 \le j \le n.$$

Now suppose $w = \sum_{j=1}^{n-1} a_j y_j$ and $z = \sum_{j=n}^{N} a_j y_j$ where $\|w + z\| = 1$. Then we have

$$\|Q_n z\| = \left\| \sum_{j=n}^{N} a_j Q_n y_j \right\| \le 2Kc^{-1} \sum_{j=n}^{\infty} \|Q_n y_j\| \le \frac{\delta_n}{5}.$$

Hence there exists $z' \in F_n$ such that $\|z - z'\| \leq \delta_n/4$ and thus

$$\|w - z\| \leq \|w - z'\| + \tfrac{1}{4}\delta_n \leq (1 + \tfrac{1}{4}\delta_n)\|w + z'\| + \tfrac{1}{4}\delta_n \leq 1 + \delta.$$

Thus we have the inequality

$$\left\| \sum_{j=1}^{n-1} a_j y_j - \sum_{j=n}^{N} a_j y_j \right\| \leq (1 + \delta_n) \left\| \sum_{j=1}^{N} a_j y_j \right\|. \tag{1}$$

Then we claim that if $\epsilon_j = \pm 1$ with $\epsilon_j = 1$ for $j < k$ we have

$$\left\| \sum_{j=1}^{n} \epsilon_j a_j y_j \right\| \leq \prod_{j=k}^{\infty}(1 + \delta_j) \left\| \sum_{j=1}^{n} a_j y_j \right\|. \tag{2}$$

This is proved for fixed n by backwards induction on k. Indeed for $k = n$ it follows from (1). If it is proved for $k + 1$ we simply note that when $\epsilon_k = -1$ but $\epsilon_j = 1$ for $j < k$,

$$\left\| \sum_{j=1}^{n} \epsilon_j a_j y_j \right\| \leq (1 + \delta_k) \left\| \sum_{j=1}^{k} a_j y_j - \sum_{j=k+1}^{n} \epsilon_j a_j y_j \right\| \leq \prod_{j=k}^{\infty}(1 + \delta_j) \left\| \sum_{j=1}^{n} a_j y_j \right\|.$$

3. Embedding in a space with unconditional basis

Let Y be a space with a finite dimensional decomposition (FDD) $(Q_j)_{j=1}^{\infty}$ and let X be a subspace of Y. Then we will say that X satisfies the *density condition* with respect to $(Q_j)_{j=1}^{\infty}$ if there is a dense subset D of X such that if $x \in D$ we have

$$x = \sum_{j=1}^{n} Q_j x$$

for some $n = n(x) \in \mathbb{N}$. The following lemma is similar to [7, Lemma 2.1].

LEMMA 3.1 *Let Y be a space with an (FDD) $(Q_j)_{j=1}^{\infty}$ and let X be a subspace of Y. Then given $\delta > 0$ there exists an automorphism $T : Y \to Y$ so that $\|T - I\| < \delta$ and X satisfies the density condition with respect to the FDD $(TQ_jT^{-1})_{j=1}^{\infty}$.*

Proof. We first prove the following claim.

Claim 1 Suppose $(Q'_j)_{j=1}^{\infty}$ is any (FDD) of Y and $x \in Y$. Then given $n \in \mathbb{N}$ and $\nu > 0$ there exists an automorphism $S : Y \to Y$, with $\|S - I\| < \nu$, such that $SQ'_jS^{-1}(Y) = Q'_j(Y)$ for $1 \leq j \leq n$ and for some $m \geq n$ we have $x \in \sum_{j=1}^{m} SQ'_jS^{-1}(Y)$.

Proof of Claim. Let K be the FDD-constant of $(Q'_j)_{j=1}^{\infty}$. If $\sum_{j=1}^{n} Q'_j x = x$ we take $m = n$ and $S = I$. If not we may choose $m > n$ so that

$$\left\| x - \sum_{j=1}^{m} Q'_j x \right\| < \nu(2K)^{-1} \left\| \sum_{j=n+1}^{m} Q'_j x \right\|.$$

Pick $y^* \in Y^*$ with $\|y^*\| = 1$ and $y^*(\sum_{j=n+1}^{m} Q'_j x) = \| \sum_{j=n+1}^{m} Q'_j x\|$. Then let

$$Sy = y + \left\| \sum_{j=n+1}^{m} Q'_j x \right\|^{-1} y^* \left(\sum_{j=n+1}^{m} Q'_j y \right) \left(x - \sum_{j=1}^{m} Q'_j x \right).$$

Then $\|S - I\| < \nu$. Also $SQ'_j = Q'_j$ if $j = 1, 2 \ldots, n$ so that $SQ'_j S^{-1}(Y) = Q'_j(Y)$. We also have $S \sum_{j=1}^{m} Q'_j x = x$. Hence $S^{-1}x = \sum_{j=1}^{m} Q'_j x$ and so $x = \sum_{j=1}^{m} SQ'_j S^{-1} x$. This concludes the proof of the claim.

We now turn to the Lemma. Now suppose $\nu_n > 0$ are such that $\prod_{j=1}^{\infty}(1 + \nu_j) < 1 + \delta$. Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in X. We inductively define automorphisms $S_n : Y \to Y$ with $\|S_n - I\| < \nu_n$ and a non-decreasing sequence of integers $(m_n)_{n=0}^{\infty}$ such that if $T_0 = I$ and then $T_n = \prod_{j=1}^{n} S_j$ we have

$$T_n Q_j T_n^{-1}(Y) = T_{n-1} Q_j T_{n-1}^{-1}(Y)$$

for $1 \leq j \leq m_{n-1}$, and

$$x_n = \sum_{j=1}^{m_n} T_n Q_j T_n^{-1} x_n.$$

To do this pick $m_0 = 1$, say and then proceed inductively using the previous claim. If $m_0, \ldots, m_{n-1}$ and $S_1, \ldots, S_{n-1}$ have been chosen, we pick S_n by the claim so that $\|S_n - I\| < \nu_n$, $S_n T_{n-1} Q_j T_{n-1}^{-1} S_n^{-1}(Y) = T_{n-1} Q_j T_{n-1}^{-1}(Y)$ for $1 \leq j \leq m_{n-1}$ and for suitable $m_n \geq m_{n-1}$ we have

$$x_n = \sum_{j=1}^{m_n} T_n Q_j T_n^{-1} x_n.$$

The sequence (T_n) converges in operator norm to an operator T where $\|T - I\| \leq \prod_{n=1}^{\infty}(1 + \nu_n) - 1 < \delta$. Clearly

$$T Q_j T^{-1} = T_n Q_j T_n^{-1}, \quad 1 \leq j \leq m_n$$

so that for each n,

$$x_n = \sum_{j=1}^{m_n} T Q_j T^{-1} x_n.$$

PROPOSITION 3.2 *Let X be a separable Banach space containing no copy of ℓ_1 (respectively, a separable reflexive Banach space) which is isometrically embedded in a Banach space Y with a 1-UFDD $(Q_j)_{j=1}^\infty$. Suppose X satisfies the density condition with respect to $(Q_j)_{j=1}^\infty$. Then X can be isometrically embedded into a Banach space Z (respectively, a reflexive Banach space) with a shrinking 1-UFDD $(Q'_j)_{j=1}^\infty$ with rank $Q'_j \le$ rank Q_j.*

If further X is λ-complemented in Y then X is λ-complemented in Z.

Proof. We assume, without loss of generality, that $Q_j(Y) = Q_j(X)$ for each j. Let $J : X \to Y$ be an isometric embedding. Define on Y^* the norm

$$|||y^*||| = \sup_n \sup_{\epsilon_j=\pm1} \left\| \sum_{j=1}^n \epsilon_j J^* Q_j^* y^* \right\|.$$

Then $|||\cdot|||$ is weak*-lower semi-continuous and we can define a Banach space $(\tilde{Z}, \|\cdot\|_Z)$ continuously embedded in Y by

$$\|z\|_Z = \sup\{|y^*(z)| : \ |||y^*||| \le 1\}.$$

By assumption $Q_j(Y) = Q_j(X) \subset \tilde{Z}$. If we let Z be the closed linear span of $\bigcup_{j=1}^\infty Q_j(Y)$ in $\tilde{Z}$ then Z^* can be identified with the completion of $(Y^*, |||\cdot|||)$.

Clearly $(Q_j)_{j=1}^\infty$ is a 1-UFDD for Z. We must check that $(Q_j)_{j=1}^\infty$ is shrinking for Z. Indeed if not we can find a blocked sequence $z_j^* \in \sum_{i=N_{j-1}+1}^{N_j} Q_i^*(Y^*)$ where $N_0 = 0 < N_1 < N_2 < \cdots$ which is equivalent to the canonical basis $(e_j)_{j=1}^\infty$ of c_0. Choose $\epsilon_i = \pm1$ so that

$$\left\| J^* \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^* \right\| = |||z_j^*|||.$$

If we let

$$x_j^* = J^* \left(\sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^* \right)$$

then there is a bounded linear operator $T : c_0 \to X^*$ with $Te_j = x_j^*$. Since X^* contains no copy of c_0 this implies that $\|x_j^*\| = |||z_j^*|||$ converges to zero, contrary to assumption.

Also if $x \in X$ then $\|x\|_Y = \|x\|_Z$ so that X is isometrically embedded in $\tilde{Z}$. Since a dense subset of X lies in the linear span of $Q_j(Y)$ it follows that $X \subset Z$. Further since $\|z\|_Z \ge \|z\|_Y$ in general, if there is a projection $P : Y \to X$ with $\|P\| = \lambda$ then $\|P\|_{Z\to X} \le \lambda$.

Finally, if X is reflexive we show that Z is reflexive. To do this it is necessary to show that the UFDD of Z^* given by $(Q_j^*(Z^*))_{j=1}^\infty$ is also shrinking. Suppose not. Then we can find a blocked sequence $z_j^* \in \sum_{i=N_{j-1}+1}^{N_j} Q_i^*(Y^*)$ where $N_0 = 0 < N_1 < N_2 < \cdots$ which is equivalent to the canonical basis $(e_j)_{j=1}^\infty$ of ℓ_1. Let Δ denote the Cantor set $\{-1,+1\}^{\mathbb{N}}$ of all sequences $\epsilon = (\epsilon_i)_{i=1}^\infty$ and consider the compact Hausdorff space $\Omega = \Delta \times B_X$ where B_X has the weak topology. Let $f_j \in \mathcal{C}(\Omega)$ be defined by

$$f_j(\epsilon, x) = \left\langle x, J^* \left(\sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^* \right) \right\rangle.$$

Then $(f_j)_{j=1}^{\infty}$ is equivalent to the ℓ_1-basis and so there exists a probability measure μ on Ω and a Borel function $\varphi \in L_1(\mu)$ so that

$$\int_{\Omega} f_j \varphi \, \mathrm{d}\mu \geq 1, \quad j = 1, 2, \dots.$$

We argue that $\lim_{j\to\infty} f_j(\epsilon, x) = 0$ for every $(\epsilon, x) \in \Omega$ and this contradicts the Dominated Convergence Theorem. Indeed

$$|f_j(\epsilon, x)| = \left| \left\langle \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i J x, z_j^* \right\rangle \right|$$
$$\leq \left\| \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i J x \right\|_Z \|z_j^*\|_{Z^*} \to 0$$

since $\sum_{i=1}^{\infty} \epsilon_i Q_i J x$ converges.

The proof of the next Proposition is standard.

PROPOSITION 3.3 *Let X be a separable Banach space. Then the following conditions on X are equivalent*:

(i) *Given $\delta > 0$ there exists a Banach space Y with a* 1*-UFDD and a subspace X_δ of Y with $d(X, X_\delta) < 1 + \delta$.*
(ii) *Given $\delta > 0$ there exists a Banach space Y with a* 1*-unconditional basis and a subspace X_δ of Y with $d(X, X_\delta) < 1 + \delta$.*
(iii) *Given $\delta > 0$ there exists a Banach space Y containing X* (*isometrically*) *and a sequence of finite-rank operators $A_n : X \to Y$ such that*

$$\left\| \sum_{j=1}^{n} \epsilon_j A_j \right\| < 1 + \delta, \quad \epsilon_j = \pm 1, \ n = 1, 2, \dots$$

and

$$x = \sum_{j=1}^{\infty} A_j x, \quad x \in X.$$

Proof. (i) $\implies$ (ii) It is essentially contained in [17, Theorem 1.g.5, p. 51] that every Banach space with a 1-UFDD is $(1+\delta)$-isomorphic to a subspace of a space with a 1-unconditional basis; in [17] the constants are not tracked, but clearly the same argument would prove this more precise statement.

(ii) $\implies$ (iii) This is clear.

(iii) $\implies$ (i) We first note that by blocking the series $\sum A_j$ suitably, that is, replacing $(A_j)_{j=1}^{\infty}$ by $A'_j = \sum_{N_{j-1}+1}^{N_j} A_i$ for suitable $N_0 = 0 < N_1 < \cdots$, we can assume that for a dense set of $x \in X$ we have $\sum_{j=1}^{\infty} \|A_j x\| < \infty$. We define Z to be the space of all sequences $(y_j)_{j=1}^{\infty}$ with $y_j \in A_j(X)$ such that $\sum_{j=1}^{\infty} y_j$ converges unconditionally in Y, under the norm

$$\|(y_j)_{j=1}^{\infty}\| = \sup_n \sup_{\epsilon_j=\pm 1} \left\| \sum_{j=1}^{n} \epsilon_j y_j \right\|.$$

This space has a 1-UFDD. X can be $(1+\delta)$-embedded into Z via the map $x \to (A_j x)_{j=1}^{\infty}$ (it suffices to note that Z is closed in the larger space of weakly unconditionally Cauchy series with the same norm, and a dense subset of X is mapped into Z by our assumptions).

Combining Proposition 3.2, Proposition 3.3 and Lemma 3.1 gives the following result. Part (ii) is contained in [6, Corollary IV.4] (where the proof is inaccurate); as remarked in [17, p. 51] one cannot hope for (ii) to hold with Y having an unconditional basis.

PROPOSITION 3.4 *Suppose X is a separable Banach space containing no complemented copy of ℓ_1. Suppose, given $\delta > 0$ there exists a Banach space Y containing X (isometrically) and a sequence of finite-rank operators $A_n : X \to Y$ such that*

$$\left\| \sum_{j=1}^{n} \epsilon_j A_j \right\| < 1 + \delta, \quad \epsilon_j = \pm 1, \; n = 1, 2, \ldots$$

and

$$x = \sum_{j=1}^{\infty} A_j x, \quad x \in X.$$

Then

(i) *For any $\delta > 0$, there is a Banach space Z with a shrinking 1-unconditional basis and a subspace X_δ of Z such that $d(X, X_\delta) < 1 + \delta$.*
(ii) *If X is reflexive then we may take Z reflexive in* (i).
(iii) *If for every $\delta > 0$ we can take $Y = X$ (that is, X has (UMAP)) then for any $\delta > 0$, there is a Banach space Z with a shrinking 1-UFDD and a $(1+\delta)$-complemented subspace X_δ of Z such that $d(X, X_\delta) < 1 + \delta$.*
(iv) *If X is reflexive then we may take Z reflexive in* (iii).

4. The main result

LEMMA 4.1 *Let Y be a Banach space and suppose X is a closed subspace of Y. Denote by Q the quotient map $Q : Y \to Y/X$. Suppose $(B_n)_{n=1}^{\infty}$ is a uniformly bounded sequence of operators on Y, such that*

$$\lim_{n\to\infty} \|QB_n\|_{X\to Y/X} = 0 \tag{3}$$

and

$$\limsup_{n\to\infty} \|B_n\|_{X\to Y} \leq 1. \tag{4}$$

Then given $\delta > 0$ there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that if $n_1 < n_2 < \cdots < n_k$ with $n_j \in \mathbb{M}$ for $1 \leq j \leq k$ then

$$\|B_{n_1} B_{n_2} \ldots B_{n_k}\|_{X\to Y} < 1 + \delta.$$

Proof. We suppose $\|B_n\| \leq M$ for all n. We assume $\delta < 1/2$. It suffices to prove this for $\mathbb{M} = \mathbb{N}$ when $\|QB_n\|_{X\to Y/X} < \nu_n/3$ and $\|B_n\|_{X\to Y} \leq 1 + \nu_n/3$ where $(\nu_n)_{n=1}^{\infty}$ is the decreasing positive sequence given by $\nu_n = (3M+6)^{-n+1}\delta$. We will prove by induction on k that

$$\|QB_{n_1} \ldots B_{n_k}\|_{X\to Y/X} < \nu_{n_1} \tag{5}$$

and

$$\|B_{n_1} \ldots B_{n_k}\|_{X\to Y} < 1 + \nu_{n_1}. \tag{6}$$

Under these hypotheses the conclusion is obviously true for $k = 1$. We next assume it is true for k and prove it for products of length $k+1$. Consider $m < m_1 < \cdots < m_k$. Then if $S = B_{m_1} \ldots B_{m_k}$ we have

$$\|QS\|_{X\to Y/X} < \nu_{m+1}, \quad \|S\|_{X\to Y} < 1 + \nu_{m+1}.$$

Now if $x \in X$ with $\|x\| \leq 1$ there exists $x' \in X$ so that

$$\|x' - Sx\| < \nu_{m+1}$$

and then

$$\|x'\| < 1 + 2\nu_{m+1}.$$

Now

$$B_m Sx = B_m x' + B_m(Sx - x')$$

and so we have

$$\begin{aligned} \|QB_m Sx\| &< \tfrac{1}{3}\nu_m(1 + 2\nu_{m+1}) + M\nu_{m+1} \\ &< \tfrac{2}{3}\nu_m + M\nu_{m+1} < \nu_m \end{aligned}$$

and

$$\|B_m Sx\| \le (1 + \tfrac{1}{3}\nu_m)(1 + 2\nu_{m+1}) + M\nu_{m+1}$$
$$< 1 + \tfrac{2}{3}\nu_m + (M+2)\nu_{m+1} = 1 + \nu_m$$

establishing both inductive hypotheses (5) and (6).

THEOREM 4.2 *Let X be a separable Banach space. Then the following conditions are equivalent*:

(i) *X has (au*).*
(ii) *For any $\delta > 0$ there is a Banach space Y with a shrinking 1-unconditional basis and a subspace X_δ of Y such that $d(X, X_\delta) < 1 + \delta$.*

Proof. That (ii) $\implies$ (i) follows from Proposition 2.4. We turn to the proof of (i) $\implies$ (ii).

By Proposition 2.2. X^* is separable. We start by using the result of Zippin [27] that X can be embedded in a space Y with a shrinking basis (we can assume the embedding is isometric). Let S_n denote the partial sum operators with respect to this basis, and let $Q : Y \to Y/X$ be the quotient map. We also denote by J the inclusion $J : X \to Y$.

We will prove the following lemma.

LEMMA 4.3 *Given $\nu > 0$ and $n \in \mathbb{N}$ there exists T in the convex hull of $\{S_k : k > n\}$ such that $\|QT\|_{X\to Y/X} < \nu$ and $\|I - 2T\|_{X\to Y} < 1 + \nu$.*

Proof. First we will argue that for every $n \in \mathbb{N}$ there exists $m > n$ such that

$$\|J^*(S_n^* y^* + S_m^* y^* - y^*)\| \le \|J^*(S_n^* y^* - S_m^* y^* + y^*)\| + \tfrac{1}{2}\nu\|y^*\|, \quad y^* \in Y^*. \tag{7}$$

If (7) fails we may find a sequence $(y_m^*)_{m>n}$ such that $\|y_m^*\| = 1$ and

$$\|J^*(S_n^* y_m^* + S_m^* y_m^* - y_m^*)\| > \|J^*(S_n^* y_m^* - S_m^* y_m^* + y_m^*)\| + \tfrac{1}{2}\nu, \quad m > n.$$

We may pass to a subsequence $\mathbb{M}$ of $\{n+1, n+2, \ldots\}$ so that $\lim_{m\in\mathbb{M}} y_m^* = y^*$ weak* for some $y^* \in Y^*$. Since S_n is finite rank $\lim_{m\in\mathbb{M}} \|S_n^*(y^* - y_m^*)\| = 0$. Hence

$$\liminf_{m\in\mathbb{M}} (\|J^* S_n^* y^* + J^*(S_m^* y_m^* - y_m^*)\| - \|J^* S_n^* y^* - J^*(S_m^* y_m^* - y_m^*)\|) \ge \tfrac{1}{2}\nu.$$

Now $(S_m^* y_m^* - y_m^*)_{m=1}^\infty$ is weak*-null in Y^* since for $y \in Y$,

$$|\langle y, S_m^* y_m^* - y_m^*\rangle| = |\langle S_m y - y, y_m^*\rangle| \le \|S_m y - y\|.$$

Hence the sequence $(J^*(S_m^* y_m^* - y_m^*))_{m=1}^\infty$ is weak*-null in X^*. Thus we have a contradiction to (au^*) for X. This shows that (7) holds for some $m = m(n) > n$.

Let us put $R_n = S_{m(n)} - S_n$. Thus we have

$$\begin{aligned}\|J^*(I-2S_n)^*y^*\| &\le \|J^*(S_n^*y^* + S_m^*y^* - y^*)\| + \|R_n^*y^*\| \\ &\le \|J^*(y^* - S_m^*y^* + S_n^*y^*)\| + \|R_n^*y^*\| + \tfrac{1}{2}\nu\|y^*\| \\ &\le \|J^*y^*\| + 2\|R_n^*y^*\| + \tfrac{1}{2}\nu\|y^*\|.\end{aligned}$$

Thus we have

$$\|J^*(I-2S_n)^*y^*\| \le (1+\tfrac{1}{2}\nu)\|y^*\| + 2\|R_n^*y^*\|, \quad y^* \in Y^*. \tag{8}$$

We next consider two sequences of finite-rank operators. First we consider the sequence $(QS_nJ)_{n=1}^\infty$ in $\mathcal{K}(X, Y/X)$. Note that if $z^* \in (Y/X)^*$ then $J^*Q^*z^* = 0$. Since $\lim_{n\to\infty}\|S_n^*Q^*z^* - Q^*z^*\| = 0$ (as the basis is shrinking) we conclude that $\lim_{n\to\infty}\|J^*S_n^*Q^*z^*\| = 0$. This implies [12] that $(QS_nJ)_{n=1}^\infty$ is a weakly null sequence in $\mathcal{K}(X, Y/X)$.

Next consider $\tilde{R}_n : c_0(Y) \to Y$ defined by $\tilde{R}_n(y_k)_{k=1}^\infty = R_ny_n$. Then $\tilde{R}_n^* : Y^* \to \ell_1(Y^*)$ is given by $\tilde{R}_n^*y^* = (0, \dots, 0, R_n^*y^*, 0, \dots)$ with the non-zero entry in the nth slot. Since the basis of Y is shrinking we have $\lim_{n\to\infty}\|R_n^*y^*\| = 0$ for $y^* \in Y^*$ and so also $\lim_{n\to\infty}\|\tilde{R}_n^*y^*\| = 0$. This implies that $(\tilde{R}_n)_{n=1}^\infty$ is weakly null in $\mathcal{K}(c_0(Y), Y)$ again using [12].

Combining these statements with Mazur's theorem for any n we can find $r > n$ and $(\alpha_j)_{j=n+1}^r$ with $\alpha_j \ge 0$ and $\sum_{j=n+1}^r \alpha_j = 1$ such that

$$\left\|\sum_{j=n+1}^r \alpha_j QS_jJ\right\| < \nu, \quad \left\|\sum_{j=n+1}^r \alpha_j \tilde{R}_j\right\| < \frac{1}{4}\nu.$$

Let $T = \sum_{j=n+1}^r \alpha_j S_j$. Then $\|QT\|_{X\to Y/X} = \|QTJ\| < \nu$.

Also if $y^* \in Y^*$, using (8),

$$\begin{aligned}\|J^*(I-2T)^*y^*\| &\le \sum_{j=n+1}^r \alpha_j\|J^*(I-2S_j)^*y^*\| \\ &\le \left(1+\frac{1}{2}\nu\right)\|y^*\| + 2\sum_{j=n+1}^r \alpha_j\|R_j^*y^*\| \\ &= \left(1+\frac{1}{2}\nu\right)\|y^*\| + 2\left\|\sum_{j=n+1}^r \alpha_j\tilde{R}_j^*y^*\right\|.\end{aligned}$$

By the selection of α_j this implies that $\|(I-2T)J\| = \|I-2T\|_{X\to Y} < 1+\nu$. This completes the proof of Lemma 4.3.

We now turn to the proof of Theorem 4.2. Using Lemmas 4.1 and 4.3 we can find a sequence of convex combinations

$$T_j = \sum_{i=N_{j-1}+1}^{N_j} \alpha_i S_i,$$

where $N_0 = 0 < N_1 < N_2 < \cdots$ and $\alpha_i \geq 0$ are such that $\sum_{i=N_{j-1}+1}^{N_j} \alpha_i = 1$ for all j with the property that

$$\|(I - 2T_{n_1})(I - 2T_{n_2}) \cdots (I - 2T_{n_k})\|_{X \to Y} < 1 + \delta$$

whenever $n_1 < n_2 < \cdots < n_k$. Note that the $(T_j)_{j=1}^{\infty}$ are a commuting approximating sequence in Y and that $T_j T_k = T_k$ if $j > k$.

Let $A_j = T_j - T_{j-1}$ where $T_0 = 0$. We now repeat a calculation in [4, Theorem 3.8] with a correction to a small misprint. Note that if $\epsilon_j = \pm 1$ we have

$$\epsilon_n T_n \prod_{j=1}^{n-1} (I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j} T_{n-j}) = \sum_{j=1}^{n} \epsilon_j A_j.$$

(Here the index $n - j + 1$ replaces $n - j - 1$.) Since $T_n = (1/2)(I - (I - 2T_n))$, it follows that

$$\left\| \sum_{j=1}^{n} \epsilon_j A_j \right\|_{X \to Y} < 1 + \delta.$$

The result now follows by Proposition 3.4.

COROLLARY 4.4 *Let X be a reflexive Banach space. Then X has property (au) if and only if for any $\delta > 0$ there is a reflexive Banach space Y with a* 1*-unconditional basis and a subspace X_δ of Y such that $d(X, X_\delta) < 1 + \delta$.*

Proof. This follows from Theorem 4.2 and Propositions 2.3 and 3.4.

The next corollary is due to Johnson and Zheng [11] by a quite different proof.

COROLLARY 4.5 *Any quotient of a Banach space X with a shrinking unconditional basis is isomorphic to a subspace of a Banach space with a shrinking unconditional basis.*

Proof. X can be renormed to have (au^*) and so this follows from Proposition 2.4.

5. Skipped unconditional bases

Let us say that a basic sequence $(e_k)_{k=1}^{N}$ (where $1 \leq N \leq \infty$) in a (finite or infinite-dimensional) Banach space X is *skipped λ-unconditional* if whenever $0 = m_0 < m_1 < \cdots < m_n < \infty$ with $m_j - m_{j-1} \geq 2$ for $1 \leq j \leq n$ and $y_j \in [e_i]_{m_{j-1}+1}^{m_j - 1}$ then for any choice of signs $(\epsilon_j)_{j=1}^{n}$,

$$\left\| \sum_{j=1}^{n} \epsilon_j y_j \right\| \leq \lambda \left\| \sum_{j=1}^{n} y_j \right\|.$$

We shall say that $(e_k)_{k=1}^{\infty}$ is *asymptotically skipped* 1*-unconditional* if for every $\lambda > 1$ there exists n so that if $x \in [e_k]_{k=1}^{n} \setminus \{0\}$ then the basic sequence $\{x, e_{n+1}, e_{n+2}, \ldots\}$ is skipped λ-unconditional.

We will define a basis $(f_k)_{k=1}^N$ of a finite-dimensional Banach space to be *dual skipped* λ-*unconditional* when the dual basis $(f_k^*)_{k=1}^N$ is skipped λ-unconditional. We will need the following simple lemma.

LEMMA 5.1 *Let X be a Banach space with a basis $(e_k)_{k=1}^N$ where $1 \le N \le \infty$. Suppose $1 \le m_1 < m_2 < \cdots < m_n < N$, and that for every $x \in [e_j]_{j=1}^{m_1}$ the basic sequence $\{x, (e_k)_{k=m_1+1}^N\}$ is skipped λ-unconditional. Suppose $x^*, y^* \in X^* \setminus \{0\}$ are such that $x^* \in [e_k^*]_{k=1}^{m_1}$, $y^*(e_j) = 0$ for $1 \le j \le m_n$. Then $\{x^*, e_{m_2}^*, \ldots, e_{m_{n-1}}^*, y^*\}$ is a dual skipped λ-unconditional basis of its linear span.*

In particular if $(e_k)_{k=1}^N$ is skipped λ-unconditional then the finite sequence $\{x^*, e_{m_2}^*, \ldots, e_{m_{n-1}}^*, y^*\}$ is a dual skipped λ-unconditional basis of its linear span.

Proof. Define a map $T : X \to \mathbb{R}^n$ by

$$Tx = (x^*(x), e_{m_2}^*(x), \ldots, e_{m_{n-1}}^*(x), y^*(x))$$

and consider the quotient norm $\|\xi\| = \inf\{\|x\| : Tx = \xi\}$ on $\mathbb{R}^n$. Then it is easy to check that the canonical basis of $\mathbb{R}^n$ is skipped λ-unconditional and its biorthogonal functionals are isometric to $\{x^*, e_{m_2}^*, \ldots, e_{m_{n-1}}^*, y^*\}$.

Our next result concerns the unconditionality of the biorthogonal sequence $(e_k^*)_{k=1}^\infty$ in X^*. If $\mathbb{A}$ is a finite subset of $\mathbb{N}$ we denote by $\text{ubc}(e_j^*)_{j\in\mathbb{A}}$ the unconditional basis constant of $(e_j^*)_{j\in\mathbb{A}}$.

LEMMA 5.2 *Let X be a finite-dimensional Banach space with a skipped 1-unconditional basis $(e_k)_{k=1}^{2N+1}$. Suppose $\text{ubc}(e_{2j-1}^*)_{j=1}^{N+1} = \mu > 1$. Then*

$$\text{ubc}(e_j^*)_{j=1}^{2N+1} \ge 1 + 2(\mu - 1).$$

Proof. By assumption there exist real numbers $(\alpha_j)_{j=1}^{N+1}$ and signs $(\epsilon_j)_{j=1}^{N+1}$ so that

$$\left\| \sum_{j=1}^{N+1} \alpha_j e_{2j-1}^* \right\| = 1$$

and

$$\left\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* \right\| = \mu.$$

Let $E = [e_{2j-1}]_{j=1}^{N+1}$. Then we have

$$\left\| \sum_{j=1}^{N+1} \alpha_j e_{2j-1}^* |_E \right\| \le 1$$

and so by the skipped unconditionality condition,

$$\left\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e^*_{2j-1} \big|_E \right\| \le 1.$$

By the Hahn–Banach theorem there exists $(\beta_j)_{j=1}^N$ such that

$$\left\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e^*_{2j-1} + \sum_{j=1}^{N} \beta_j e^*_{2j} \right\| \le 1.$$

Thus

$$2\mu \le 1 + \left\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e^*_{2j-1} - \sum_{j=1}^{N} \beta_j e^*_{2j} \right\|$$

so that

$$\mathrm{ubc}(e^*_j)_{j=1}^{2N+1} \ge 2\mu - 1.$$

LEMMA 5.3 *Suppose* $N \in \mathbb{N}$. *Let* X *be a Banach space of dimension* $2^N + 1$ *with a dual skipped* 1*-unconditional basis* $(f_k)_{k=1}^{2^N+1}$. *Suppose* $\mathrm{ubc}(f_1, f_{2^N+1}) = \mu > 1$. *Then*

$$\mathrm{ubc}(f_j)_{j=1}^{2^N+1} \ge 1 + 2^N(\mu - 1).$$

Proof. This is proved by induction on N. If $N = 1$ it is immediate from Lemma 5.2. Suppose now that the lemma is proved for $N - 1$. Then $\{f_1, f_3, \dots, f_{2^N+1}\}$ is a dual skipped 1-unconditional basis of its linear span by Lemma 5.1. By the inductive hypothesis

$$\mathrm{ubc}(f_{2j-1})_{j=1}^{2^{N-1}+1} \ge 1 + 2^{N-1}(\mu - 1).$$

Now applying Lemma 5.2 we have

$$\mathrm{ubc}(f_j)_{j=1}^{2^N+1} \ge 1 + 2^N(\mu - 1).$$

PROPOSITION 5.4 *Let* X *be a Banach space containing no copy of* ℓ_1 *and with a skipped unconditional basis* $(e_k)_{k=1}^\infty$. *Then*

(i) $(e_k)_{k=1}^\infty$ *is shrinking, and*
(ii) *if* X *contains no copy of* c_0 *then either* X *is reflexive or* X *is quasi-reflexive of order one.*

Proof. (i) Let $(u_k)_{k=1}^\infty$ be any normalized block basic sequence with respect to $(e_k)_{k=1}^\infty$. Then $(u_{2k})_{k=1}^\infty$ (respectively, $(u_{2k-1})_{k=1}^\infty$) is an unconditional basic sequence and hence weakly null; thus $(u_k)_{k=1}^\infty$ is weakly null. Hence $(e_k)_{k=1}^\infty$ is shrinking.

(ii) We may assume $\|e_k\| = 1$ for all k. Suppose $x^{**} \in X^{**}$ is such that $\lim_{k\to\infty} x^{**}(e_k^*) = 0$. Select a strictly increasing sequence $(m_k)_{k=0}^\infty$ (with $m_0 = 0$) such that $|x^{**}(e_{m_k}^*)| < 2^{-k}$ for $k \geq 1$. Then the series

$$\sum_{k=1}^\infty \left(\sum_{i=m_{k-1}+1}^{m_k-1} x^{**}(e_i^*)e_i \right)$$

is a WUC series and hence convergent in X. On the other hand the series $\sum_{k=1}^\infty x^{**}(e_{m_k}^*)e_k$ is absolutely convergent and so $x^{**} \in X$.

Now suppose X is non-reflexive and $x_0^{**} \in X^{**}\backslash X$. Then $\liminf_k |x_0^{**}(e_k^*)| > 0$. For any $x^{**} \in X^{**}$ we may find $\lambda \in \mathbb{R}$ so that $\liminf_k |(x^{**} - \lambda x_0^{**})(e_k^*)| = 0$ and hence $x^{**} - \lambda x_0^{**} \in X$. Thus $\dim X^{**}/X = 1$.

PROPOSITION 5.5 *Let X be a Banach space containing no copy of ℓ_1 and with a normalized asymptotically skipped* 1*-unconditional basis $(e_k)_{k=1}^\infty$. Suppose X fails to have property (au^*). Then*

(i) *no subsequence of $(e_k^*)_{k=1}^\infty$ is unconditional, and*
(ii) *every spreading model of $(e_k)_{k=1}^\infty$ is equivalent to the standard ℓ_1-basis.*

Proof. Since $(e_k)_{k=1}^\infty$ is shrinking by Proposition 5.4 we can assume the existence of $\mu > 1$, $r \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $x^* \in [e_k^*]_{k=1}^r$ and a sequence $(y_n^*)_{n>r}$ with $y_n^*(e_j) = 0$ for $j < n$ such that $\|x^*\| = \|y_n^*\| = 1$ and $\|\alpha x^* - \beta y_n^*\| = 1$ for all n but $\|\alpha x^* + \beta y_n^*\| \geq \mu$. Let K be the basis constant of $(e_k)_{k=1}^\infty$. We first argue that for $n \in \mathbb{N}$ with $n > 80K/(\mu - 1)$ there exists $k = k(n)$ so that if $k < m_1 < m_2 < \cdots < m_n$ then

$$\mathrm{ubc}(e_{m_1}^*, \ldots, e_{m_n}^*) \geq \frac{(\mu-1)n}{10K^2}. \tag{9}$$

Assume not. Then for each $k > r$ we may select $k < m_{k,1} < \cdots < m_{k,n}$ so that

$$\mathrm{ubc}(e_{m_{k,1}}^*, \ldots, e_{m_{k,n}}^*) \leq \frac{(\mu-1)n}{10K^2}.$$

Hence since the basis constant of $(x^*, e_{m_{k,1}}^*, \ldots, e_{m_{k,n}}^*, y_{m_{k,n}+1}^*)$ is at most K we have that if $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+2} = \pm 1$ and $\xi_1, \ldots, \xi_{n+2}$ are real numbers,

$$\begin{aligned}
&\left\| \epsilon_1 \xi_1 x^* + \sum_{j=1}^n \epsilon_{j+1}\xi_{j+1} e_{m_{k,j}}^* + \epsilon_{n+2}\xi_{n+2} y_{m_{k,n}+1}^* \right\| \\
&\quad \leq |\xi_1| + |\xi_{n+2}| + \frac{(\mu-1)n}{10K^2} \left\| \sum_{j=1}^n \xi_{j+1} e_{m_{k,j}}^* \right\| \\
&\quad \leq \left(4K + \frac{(\mu-1)n}{5}\right) \left\| \xi_1 x^* + \sum_{j=1}^n \xi_{j+1} e_{m_{k,j}}^* + \xi_{n+2} y_{m_{k,n}+1}^* \right\| \\
&\quad \leq \frac{(\mu-1)n}{4} \left\| \xi_1 x^* + \sum_{j=1}^n \xi_{j+1} e_{m_{k,j}}^* + \xi_{n+2} y_{m_{k,n}+1}^* \right\|.
\end{aligned}$$

Thus

$$\mathrm{ubc}(x^*, e^*_{m_{k,1}}, \dots, e^*_{m_{k,n}}, y^*_{m_{k,n}+1}) \le \frac{\mu - 1}{4} n.$$

Note the basis $(x^*, e^*_{m_{k,1}}, \dots, e^*_{m_{k,n}}, y^*_{m_{k,n}+1})$ is dual λ_k-skipped where $\lim_k \lambda_k = 1$.

Let us define a norm on $\mathbb{R}^{n+2}$ by

$$\|(\xi_1, \dots, \xi_{n+2})\| = \lim_{\mathcal{U}} \left\| \xi_1 x^* + \sum_{j=1}^{n} \xi_{j+1} e^*_{m_{k,j}} + \xi_{n+2} y^*_{m_{k,n}+1} \right\|,$$

where $\mathcal{U}$ is some non-principal ultrafilter. The canonical basis $(f_1, \dots, f_{n+2})$ is then dual skipped 1-unconditional and

$$\mathrm{ubc}(f_1, \dots, f_{n+2}) \le \frac{1}{4}(\mu - 1)n.$$

Also $\mathrm{ubc}(f_1, f_{n+2}) \ge \mu$. Hence by Lemma 5.3 (and utilizing Lemma 5.1 since $n + 1$ need not be a power of 2)

$$\mathrm{ubc}(f_1, \dots, f_{n+2}) \ge \frac{1}{2}(\mu - 1)(n + 1).$$

This gives a contradiction and (9) is established.

Condition (i) is now immediate.

For (ii) observe that any spreading model of $(e_k)_{k=1}^\infty$ is 1-unconditional. For any n there exists $k(n)$ so that (9) holds. Suppose $k(n) < m_1 < \cdots < m_n$. Then there exist $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $(\epsilon_1, \dots, \epsilon_n) \in \{-1, 1\}^n$ so that

$$\left\| \sum_{j=1}^{n} \alpha_j e^*_{m_j} \right\| = 1, \qquad \left\| \sum_{j=1}^{n} \epsilon_j \alpha_j e^*_{m_j} \right\| \ge \frac{\mu - 1}{10K^2} n.$$

Since $\|e^*_k\| \le 2K$ we thus have

$$\sum_{j=1}^{n} |\alpha_j| \ge \frac{\mu - 1}{20K^3} n$$

and so for a suitable choice of signs η_j we have

$$\left\| \sum_{j=1}^{n} \eta_j e_{m_j} \right\| \ge \frac{\mu - 1}{20K^3} n.$$

Thus in any spreading model with basis $(f_j)_{j=1}^\infty$ we have $\|f_1 + \cdots + f_n\| \ge cn$ for suitable $c > 0$. This implies that $(f_j)_{j=1}^\infty$ is equivalent to the canonical basis of ℓ_1 (since it is a 1-unconditional spreading model).

6. The WABS property

We recall that a Banach space X is said to have the (ABS) property if every bounded sequence $(x_n)_{n=1}^{\infty}$ in X has a subsequence $(y_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \sup_{r_1<r_2<\cdots<r_n} \left\| \frac{1}{n} \sum_{j=1}^{n} (-1)^j y_{r_j} \right\| = 0. \tag{10}$$

This is equivalent to the requirement that some spreading model of $(x_n)_{n=1}^{\infty}$ is not equivalent to the ℓ_1-basis (see [2]).

We shall say that X has the $(WABS)$ property if every bounded sequence $(x_n)_{n=1}^{\infty}$ in X has a convex block sequence $(y_n)_{n=1}^{\infty}$ such that (10) holds. Here $(y_n)_{n=1}^{\infty}$ is a convex block sequence if

$$y_n = \sum_{j=p_{n-1}+1}^{p_n} \lambda_j x_j,$$

where $p_0 = 0 < p_1 < p_2 < \cdots, \lambda_j \geq 0$, and $\sum_{p_{n-1}+1}^{p_n} \lambda_j = 1$ for every n. Note that if $(y_n)_{n=1}^{\infty}$ satisfies (10) then so does every further sequence of convex blocks.

Let us recall at this point that a Banach space X has Pełczyński's property (u) if for every weakly Cauchy sequence $(x_n)_{n=1}^{\infty}$ there is a weakly null sequence $(z_n)_{n=1}^{\infty}$ so that if $u_n = x_n - z_n$ then the series $\sum_{n=1}^{\infty}(u_n - u_{n-1})$ (where $u_0 = 0$) is weakly unconditionally Cauchy (WUC). Any Banach space with an unconditional basis has property (u) [18, p. 31; 22] Let us note the following, which shows the connection with the (WABS) property.

Proposition 6.1 *Let X be a separable Banach space. Then X contains no copy of ℓ_1 and has property (u) if and only if every bounded sequence $(x_n)_{n=1}^{\infty}$ has a convex block sequence $(y_n)_{n=1}^{\infty}$ such that*

$$\sup_{n} \sup_{r_1<r_2<\cdots<r_n} \left\| \sum_{j=1}^{n} (-1)^j y_{r_j} \right\| < \infty. \tag{11}$$

Proof. If X contains no copy of ℓ_1, we can assume $(x_n)_{n=1}^{\infty}$ is weakly Cauchy [23]. If X has property (u) we can write $x_n = u_n + z_n$ where (z_n) is weakly null and $\sum(u_n - u_{n-1})$ is a WUC series. We may then pass to convex blocks $(\hat{x}_n)_{n=1}^{\infty}$ so that the corresponding convex blocks $(\hat{z}_n)_{n=1}^{\infty}$ and $(\hat{u}_n)_{n=1}^{\infty}$ satisfy $\|\hat{z}_n\| < 2^{-n}$. Then $(\hat{x}_n)_{n=1}^{\infty}$ satisfies our requirements.

Conversely, it is clear X cannot contain ℓ_1. If $(x_n)_{n=1}^{\infty}$ is weakly Cauchy we may pass to convex blocks $(y_n)_{n=1}^{\infty}$ verifying (11). But then $\sum(y_n - y_{n-1})$ is a WUC series and $x_n - y_n$ is weakly null.

In [8] Haydon *et al.* introduced the class of Baire-1/2 functions: if Ω is a compact metric space then a bounded function f on Ω is Baire-1/2 if for every $\epsilon > 0$ there exist bounded lower-semi-continuous functions φ, ψ such that $|f(s) - (\varphi(s) - \psi(s))| < \epsilon$ for $s \in \Omega$.

Suppose X is a separable Banach space and $x^{**} \in X^{**} \setminus X$. We can generate a sequence $\chi_n = \chi_n(x^{**}) \in X^{(2n)}$ by $\chi_1 = x^{**}$ and then $\chi_n = j_{n-1}^{**} x^{**}$ where j_{n-1} is the canonical embedding $X \subset X^{**} \subset \cdots \subset X^{2(n-1)}$. The sequence $(\chi_n)_{n=1}^{\infty}$ is considered in the transfinite dual X^{ω} defined as the completion of $\bigcup_{n\geq 1} X^{(2n)}$.

The following theorem follows easily from [5, 8].

THEOREM 6.2 *If X is a separable Banach space then the following are equivalent*:

(i) *X has the (WABS) property.*
(ii) *Every $x^{**} \in X^{**}$ is Baire-1/2 as a function on B_{X^*} with the weak*-topology.*
(iii) *There is no $x^{**} \in X^{**} \backslash X$ so that $(\chi_n(x^{**}))_{n=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_1.*

Proof. (i) $\implies$ (ii). Since X contains no copy of ℓ_1, every $x^{**} \in X^{**} \backslash X$ is the weak*-limit of a sequence $(x_n)_{n=1}^{\infty}$ [19]. We pass to a sequence of convex blocks $(y_n)_{n=1}^{\infty}$ so that (10) holds. Now apply [8, Theorem B] to deduce that x^{**} is Baire-1/2.

(ii) $\iff$ (iii). This is Farmaki [5, Theorem 11] (since (iii) also implies that X contains no copy of ℓ_1 by [5, Proposition 6]).

(ii) $\implies$ (i). Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in X. If $(x_n)_{n=1}^{\infty}$ has a weakly convergent subsequence then Mazur's theorem quickly yields a sequence of convex blocks satisfying (10). By Rosenthal's theorem [23] we may therefore pass to the case when $(x_n)_{n=1}^{\infty}$ is weakly Cauchy and converging weak* to some $x^{**} \in X^{**} \backslash X$. By [8, Theorem 3.7 and Lemma 3.8] there is a bounded sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{C}(B_{X^*})$ converging pointwise to x^{**} so that $(f_n)_{n=1}^{\infty}$ satisfies (10). By Mazur's theorem, we may find a sequence of convex blocks $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and a sequence of convex blocks $(g_n)_{n=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that $\|y_n - g_n\| < 2^{-n}$ (considering X as a subspace of $\mathcal{C}(B_{X^*})$). Then $(y_n)_{n=1}^{\infty}$ satisfies (10).

We next give a very similar argument to Lemma 2.5 for the case when $(x_n)_{n=1}^{\infty}$ converges weak* to some $x^{**} \in X^{**} \backslash X$.

LEMMA 6.3 *Let X be a separable Banach space with property (au), and suppose that $(x_n)_{n=1}^{\infty}$ is a weakly Cauchy sequence in X converging weak* to some $x^{**} \in X^{**} \backslash X$. Then there is a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that the sequence $(y_n - y_{n-1})_{n=1}^{\infty}$ (where $y_0 = 0$) is an asymptotically skipped 1-unconditional basic sequence.*

Proof. We may suppose, by passing to a subsequence, that $(x_n)_{n=1}^{\infty}$ is basic (see for example, [1, Theorem 1.5.6]), and that if $x^* \in X^*$ is such that $x^{**}(x^*) = 1$ then $|x^*(x_n) - 1| < 2^{-n}$. This implies the existence of $y^* \in X^*$ with $y^*(x_n) = 1$ for all n and so $(x_n - x_{n-1})_{n=1}^{\infty}$ (with $x_0 = 0$) is also a basic sequence (see [25, pp. 308–311]); note this remark applies to all subsequences of $(x_n)_{n=1}^{\infty}$. Let K be the basis constant for the sequence $(x_n)_{n=1}^{\infty}$ and assume that $0 < c \leq \|x_k\| \leq C < \infty$ for all k.

Let $(\delta_n)_{n=1}^{\infty}$ be a decreasing sequence of positive numbers with the property that $\sum_{n=1}^{\infty} \delta_n < \infty$. We will construct a subsequence $(y_n)_{n=1}^{\infty}$ and a sequence $(F_n)_{n=1}^{\infty}$ of closed finite-codimensional subspaces inductively.

Let $y_1 = x_1$ and $F_1 = X$. If $y_1, \ldots, y_{n-1}$ and $F_1, \ldots, F_{n-1}$ have been chosen then we may choose a closed subspace F_n of finite codimension so that if $w \in [y_j]_{j=1}^{n-1}$ and $z \in F_n$ then

$$\|w - z\| \leq (1 + \tfrac{1}{4}\delta_n)\|w + z\|.$$

Let $Q_j : X \to X/F_j$ denote the quotient map for $1 \leq j \leq n$. If $y_{n-1} = x_{m_n}$ we may pick $y_n = x_{m_{n+1}}$ with $m_{n+1} > m_n$ so that

$$\|Q_j y_n - Q_j^{**} x^{**}\| \leq \frac{2^{j-n-1} c\delta_j}{10K}, \quad 1 \leq j \leq n.$$

Now suppose $w = \sum_{j=1}^{n-1} a_j y_j$ and $z = \sum_{j=n}^{N} a_j y_j$ where $\|w + z\| = 1$ and $\sum_{j=n}^{N} a_j = 0$. Then we have

$$\|Q_n z\| = \left\| \sum_{j=n}^{N} a_j (Q_n y_j - Q_n^{**} x^{**}) \right\| \leq 2Kc^{-1} \sum_{j=n}^{\infty} \|Q_n y_j - Q_n^{**} x^{**}\| \leq \delta_n / 5.$$

Hence there exists $z' \in F_n$ such that $\|z - z'\| \leq \delta_n/4$ and thus

$$\|w - z\| \leq \|w - z'\| + \tfrac{1}{4}\delta_n \leq (1 + \tfrac{1}{4}\delta_n)\|w + z'\| + \tfrac{1}{4}\delta_n \leq 1 + \delta.$$

Thus we have the inequality

$$\left\| \sum_{j=1}^{n-1} a_j y_j - \sum_{j=n}^{N} a_j y_j \right\| \leq (1 + \delta_n) \left\| \sum_{j=1}^{N} a_j y_j \right\|, \quad \text{if } \sum_{j=n}^{N} a_j = 0. \tag{12}$$

Now let $z_n = y_n - y_{n-1}$ and suppose $v_j = \sum_{m_{j-1}+1}^{m_j - 1} a_j z_j$ for $1 \leq j \leq n$ where $m_0 = 0 < m_1 < \cdots < m_n$ with $m_j - m_{j-1} \geq 2$ for $j \geq 2$. Then we claim that if $\epsilon_j = \pm 1$ we have

$$\left\| \sum_{j=1}^{n} \epsilon_j v_j \right\| \leq \prod_{j=1}^{n} (1 + \delta_{m_j}) \left\| \sum_{j=1}^{n} v_j \right\|. \tag{13}$$

This is proved by induction on $n \geq 2$. For $n = 2$ it follows from (12). Assume it is proved for $n - 1$. Then

$$\begin{aligned} \left\| \sum_{j=1}^{n} \epsilon_j v_j \right\| &\leq (1 + \delta_{m_1}) \left\| v_1 + v_2 + \sum_{j=3}^{n} \epsilon_2 \epsilon_j v_j \right\| \\ &\leq \prod_{j=1}^{n} (1 + \delta_{m_j}) \left\| \sum_{j=1}^{n} v_j \right\|. \end{aligned}$$

Hence $(y_j - y_{j-1})_{j=1}^{\infty}$ is asymptotically skipped 1-unconditional.

THEOREM 6.4 *Let X be a separable Banach space. Then the following are equivalent*:

(i) *X has properties (au) and (WABS)*,
(ii) *for any $\delta > 0$ there is a Banach space Y with a shrinking 1-unconditional basis and a subspace X_δ of Y such that $d(X, X_\delta) < 1 + \delta$.*

Proof. Of course by Theorem 4.2 (ii) is equivalent to the fact that X has (au^*).

(ii) $\implies$ (i). We observe that (ii) implies X has property (u) and hence (WABS). Property (au) follows trivially from (ii).

(i) $\Longrightarrow$ (ii). Clearly X contains no copy of ℓ_1. Suppose $x^{**} \in X^{**}\backslash X$; by the Odell–Rosenthal theorem [19] and property (WABS) there is a sequence $(x_n)_{n=1}^{\infty}$ converging weak* to x^{**} with the property that

$$\lim_{n\to\infty} \sup_{r_1<r_2<\cdots<r_n} \left\| \frac{1}{n} \left(\sum_{k=1}^{n} (-1)^k x_{r_k} \right) \right\| = 0.$$

According to Lemma 6.3, by passing to a subsequence we can assume that $(x_n - x_{n-1})_{n=1}^{\infty}$ is asymptotically skipped 1-unconditional. But then no spreading model of $(x_n - x_{n-1})_{n=1}^{\infty}$ (with $x_0 = 0$) is equivalent to the ℓ_1-basis. Thus, by Proposition 5.5 we have that the space $E = [x_n - x_{n-1}]_{n=1}^{\infty}$ has property (au^*). In particular by Theorem 4.2 E has property (u). Since x^{**} is in the weak*-closure of E we conclude that X has property (u).

We next show that X has property (au^*). Suppose not. Then there exists $x^* \in X^*$ and a weak*-null sequence $(x_n^*)_{n=1}^{\infty}$ such that $\|x^* + x_n^*\| \le 1$ and $\|x^* - x_n^*\| > 1 + \delta$ for some $\delta > 0$. Pick $x_n \in X$ so that $\|x_n\| = 1$ but $x^*(x_n) - x_n^*(x_n) > 1 + \delta$. If $(x_n)_{n=1}^{\infty}$ is weakly convergent to some x then we obtain a contradiction since

$$\lim_{n\to\infty} x^*(2x - x_n) + x_n^*(2x - x_n) = \lim_{n\to\infty} x^*(x_n) - x_n^*(x_n) > 1 + \delta$$

but

$$\lim_{n\to\infty} \|2x - x_n\| = 1.$$

Thus we can assume, passing to a subsequence, that $(x_n)_{n=1}^{\infty}$ is a basic sequence which converges weak* to some $x^{**} \in X^{**}\backslash X$. Since X has property (u) there is a sequence $(y_n)_{n=1}^{\infty}$ in X so that $(y_n)_{n=1}^{\infty}$ also converges weak* to x^{**} and is equivalent to the summing basis of c_0. Let $G = [y_n]_{n=1}^{\infty}$. By Sobczyk's theorem (see [26] or for example, [1, Theorem 2.5.8]) there is a projection $P : X \to G$. Then $(P^{**}x_n)_{n=1}^{\infty}$ converges weak* to x^{**} and so if $Q = I - P$ the sequence $(Qx_n)_{n=1}^{\infty}$ is weakly null.

Now, by Lemma 2.5, passing to a further subsequence of $(x_n)_{n=1}^{\infty}$ we can suppose that either (a) $\|Qx_n\| < 2^{-n}$ or (b) $(Qx_n)_{n=1}^{\infty}$ is an unconditional basis for its closed linear span Z. We may also suppose that $z_n = x_n - x_{n-1}$ (where $x_0 = 0$) defines an asymptotically skipped 1-unconditional basis of Z. In case (a) the space $E = [x_n]_{n=1}^{\infty}$ is isomorphic to a subspace of c_0. In case (b) E is isomorphic to a subspace of $Z \oplus G$. In either case E embeds (isomorphically, not isometrically) into a space with a shrinking unconditional basis. In particular the biorthogonal sequence $(z_n^*)_{n=1}^{\infty}$ in Z^* (which is weak*-null) has a subsequence which is an unconditional basic sequence (again by Lemma 2.5). By Proposition 5.5 this means that Z has property (au^*). Now $\|(x^* + x_n^*)|_Z\| \le 1$ and so $\limsup_{n\to\infty} \|(x^* - x_n^*)|_Z\| \le 1$. However $(x^* - x_n^*)(x_n) > 1 + \delta$ and we have a contradiction.

REMARK We do not know whether it is possible to replace the (WABS)-condition in (i) by the assumption that X contains no copy of ℓ_1 (or even that X^* is separable). This problem reduces to the question of whether one can find a space Y with an asymptotically skipped 1-unconditional basis, which contains no copy of ℓ_1 but does not have property (au^*). If one further imposes the condition that Y contains no copy of c_0 then Y would be quasi-reflexive of order one by Proposition 5.4. It is certainly possible to find such quasi-reflexive spaces which fail the (WABS) property; this is the requirement that the transfinite dual $Y^{\omega} \approx Y \oplus \ell_1$. Examples have been given by Bellenot [3] and by Haydon *et al.* [**8**]. However, it seems difficult to impose the extra condition that Y has an asymptotically skipped 1-unconditional basis and therefore leads us to speculate that Theorem 6.4 can be improved.

Note that the James space [**9**] (or see [**1**, p. 62]) is quasi-reflexive and does have (WABS). It therefore fails (au^*) (it does not even have property (u)). By Theorem 6.4 the James space cannot have (au) under any equivalent norming. However, it does have the (UTP) of Johnson and Zheng [**10**].

REMARK The (WABS)-condition also appears implicitly in [**14**] where Theorem 4.5 could be rephrased as saying that a separable Banach space with the (WABS) property and the $\mathcal{Q}$-property is reflexive; this implies that if X is a space with the (WABS) property such that X coarsely embeds into a reflexive space or B_X uniformly embeds into a reflexive space then X is reflexive. There is a clear link with the problems considered here. For example, if X is a separable Banach space with an unconditional basis containing no copy of c_0 then B_X uniformly embeds in a reflexive space ([**14**, Theorem 3.8]).

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