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COARSE AND UNIFORM EMBEDDINGS INTO REFLEXIVE SPACES

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Abstract

Answering an old problem in nonlinear theory, we show that c_0 cannot be coarsely or uniformly embedded into a reflexive Banach space, but that any stable metric space can be coarsely and uniformly embedded into a reflexive space. We also show that certain quasi-reflexive spaces (such as the James space) also cannot be coarsely embedded into a reflexive space and that the unit ball of these spaces cannot be uniformly embedded into a reflexive space. We give a necessary condition for a metric space to be coarsely or uniformly embeddable in a uniformly convex space.

1. Introduction

Let M_1, M_2 be metric spaces. Suppose that $f : M_1 \to M_2$ is any map and let

$$\varphi_f(t) = \inf\{d(f(x), f(y)) : d(x, y) \geq t\}, \quad t > 0,$$

and

$$\omega_f(t) = \sup\{d(f(x), f(y)) : d(x, y) \leq t\}, \quad t > 0,$$

so that

$$\varphi_f(d(x, y)) \leq d(f(x), f(y)) \leq \omega_f(d(x, y)), \quad x, y \in M_1. \tag{1}$$

Then we say that f is a *coarse embedding* and M_1 *coarsely embeds* into M_2 if $\omega_f(t) < \infty$ for all t and $\lim_{t\to\infty} \varphi_f(t) = \infty$. On the other hand, f is a *uniform embedding* and M_1 *uniformly embeds* into M_2 if $\varphi_f(t) > 0$ for all $t > 0$ and $\lim_{t\to 0} \omega_f(t) = 0$. We shall refer to f as a *strong uniform embedding* if it is both a coarse embedding and a uniform embedding. It is worth mentioning that in the literature this terminology is not yet standardized; coarse embeddings are often called uniform embeddings following the terminology of Gromov [**12**, p. 211] (for example, [**7**]). However, we prefer the term coarse embedding since uniform embedding already has a well-established meaning as above.

A metric space M is called *uniformly discrete* if $\inf_{x\neq y} d(x, y) > 0$. Let us define a *skeleton* M' of a metric space M as a subset such that for suitable constants $0 < a, b < \infty$ we have

$$d(x, y) \geq a, \quad x \neq y, \quad x, y \in M',$$

and

$$d(x, M') \leq b, \quad x \in M.$$

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Then M always coarsely embeds into any of its skeletons and so coarse embedding problems reduce to the consideration of uniformly discrete spaces.

A metric space M is called *locally finite* if any set of finite diameter is finite and has *bounded geometry* if there is a function $\psi : (0, \infty) \to \mathbb{N}$ so that every set of diameter r contains at most $\psi(r)$ points. Recently, in connection with the Novikov conjecture, there has been some considerable interest in the problem whether every metric space with bounded geometry can be coarsely embedded into a uniformly convex Banach space. This results from the work of Yu [**28**] and Kasparov and Yu [**18**] who show that such metric spaces with bounded geometry which coarsely embed into a uniformly convex space satisfy the coarse Novikov conjecture. It is known from results of Johnson and Randrianarivony [**16**] and Randrianarivony [**25**] that there are uniformly convex Banach spaces which do not coarsely embed into a Hilbert space.

Recently Brown and Guentner [**7**] proved that a metric space with bounded geometry coarsely embeds into a reflexive Banach space. However, it was apparently unknown whether there was *any* metric space which did not coarsely embed into a reflexive space. Since every separable metric space Lipschitz embeds into c_0 [**1**] it is natural to ask whether c_0 (or one of its skeletons) coarsely embeds into a reflexive space. In fact a similar problem was open for uniform embeddings. Does c_0 uniformly embed into a reflexive Banach space? This question has been a problem in the area for at least 30 years. It is explicitly raised in the recent book of Benyamini and Lindenstrauss [**6**, p. 184] (in the form of the question whether every separable metric space can be uniformly embedded into a reflexive space). We note that the recent paper of Mendel and Naor [**20**] shows that c_0 cannot be coarsely embedded into a super-reflexive Banach space. However, it is known that c_0 can be uniformly embedded into a Banach space with the Schur property [**17**] and in fact this embedding is also a strong uniform embedding, and hence a coarse embedding. Thus c_0 can be coarsely and uniformly embedded into a Banach space that does not contain c_0 (as a closed linear subspace). One of our main results in this paper shows that c_0 cannot be uniformly or coarsely embedded into a reflexive Banach space.

Let us recall that a metric space M is called *stable* if for any pair of sequences $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ such that both iterated limits exist we have

$$\lim_{m\to\infty}\lim_{n\to\infty} d(x_m, y_n) = \lim_{n\to\infty}\lim_{m\to\infty} d(x_m, y_n).$$

It is well known that c_0 cannot be uniformly embedded into a stable metric space [**6**, pp. 213, 214, **26**] and the same argument works for coarse embeddings. An example of a non-reflexive stable Banach space is L_1 which coarsely and uniformly embeds into a Hilbert space [**2**, **25**].

We now describe our main results. First we prove that a stable metric space can always be coarsely embedded into a reflexive Banach space. We then show that c_0 cannot be coarsely or uniformly embedded into a reflexive space, or more generally into any Banach space all of whose duals are separable (for example, a quasi-reflexive space). We next develop a criterion which enables us to identify a property inherited by metric spaces which can be either uniformly or coarsely embedded into a reflexive space. This reduces to a single property for Banach spaces, the $\mathcal{Q}$-property, which is inherited by all spaces X such that either X coarsely embeds into a reflexive space or B_X uniformly embeds into a reflexive space. Examples of spaces failing the $\mathcal{Q}$-property include the James quasi-reflexive space and any non-reflexive space with non-trivial type.

In some cases analogous results for embeddings into stable metric spaces are already known from work of Raynaud [26], and our methods are based on some of his ideas.

2. Coarse embeddings of stable metric spaces

Let M be a metric space. We distinguish a point 0 in M and define the Banach space $\mathrm{Lip}_0(M,d)$ of all Lipschitz $f : M \to \mathbb{R}$ such that $f(0) = 0$ with the norm

$$\|f\|_{\mathrm{Lip}_0(M,d)} = \sup_{x\neq y}\left\{\frac{|f(x)-f(y)|}{d(x,y)}\right\}.$$

We write $d(x) = d(x, 0)$.

THEOREM 2.1 *Let M be stable metric space. Then M strongly uniformly embeds into a reflexive Banach space.*

Proof. Fix $0 < a < 1$. We will consider a second metric on M given by

$$d_a(x,y) = \begin{cases} d(x,y)^a & \text{if } d(x,y) \le 1, \\ d(x,y) & \text{if } d(x,y) > 1. \end{cases}$$

Let $\tilde{M} = \{(p,q) : \ p,q \in M, \ p \ne q\}$. For $(p,q) \in \tilde{M}$ we define a function

$$g_{p,q}(x) = \max(d(p,q) - d(q,x), 0) - \max(d(p,q) - d(q), 0).$$

Let us observe that

$$|g_{p,q}(x) - g_{p,q}(y)| \le \min(d(x,y), d(p,q)), \quad x,y \in M.$$

Now let

$$f_{p,q} = \min(1, d(p,q)^{a-1})g_{p,q}, \quad (p,q) \in \tilde{M}.$$

We clearly have $f_{p,q} \in \mathrm{Lip}_0(M,d) \subset \mathrm{Lip}_0(M,d_a)$.

Note that

$$\frac{|f_{p,q}(x) - f_{p,q}(y)|}{d_a(x,y)} \le \begin{cases} d(p,q)^{1-a}, & d(p,q) \le 1, \\ d(p,q)^{a-1}, & d(p,q) > 1, \end{cases} \tag{2}$$

and

$$\frac{|f_{p,q}(x) - f_{p,q}(y)|}{d_a(x,y)} \le \begin{cases} d(x,y)^{1-a}, & d(x,y) \le 1, \\ d(p,q)d(x,y)^{-1}, & d(x,y) > 1. \end{cases} \tag{3}$$

Let $W \subset \mathrm{Lip}_0(M,d_a)$ be the set $W = \{f_{p,q} : (p,q) \in \tilde{M}\}$. We will show that W is relatively weakly compact. Note first that our assumptions give that $\|f_{p,q}\|_{\mathrm{Lip}_0(M,d_a)} \le 1$ for all $(p,q) \in \tilde{M}$.

To prove weak compactness it suffices, by the Eberlein–Smulian theorem, to show that any sequence $(f_n = f_{p_n,q_n})_{n=1}^\infty$ has a weakly convergent subsequence. Let E be the subspace of $\mathrm{Lip}_0(M)$ generated by the functions $\{f_n : n = 1, 2, \dots\}$; then E is separable and so we can find a countable subset M_0 of M containing 0 and all $\{p_n, q_n : n = 1, 2, \dots\}$ so that

$$\|f\|_{\mathrm{Lip}_0(M)} = \|f\|_{\mathrm{Lip}_0(M_0)}, \quad f \in E.$$

Without loss of generality we may assume that $(f_n)_{n=1}^\infty$ converges pointwise on M_0 to some function f. We may also assume that $\lim_{n\to\infty} d(p_n, q_n) = r$ where $0 \le r \le \infty$ and that all the limits $\lim_{n\to\infty} d(x, q_n)$ exist in $[0, \infty]$ for $x \in M_0$. Let $s = \lim_{n\to\infty} d(q_n)$ (so that $0 \le s \le \infty$).

Let $\tilde{M}_0 = \tilde{M} \cap (M_0 \times M_0)$. We consider a map $V : \mathrm{Lip}_0(M, d_a) \to \ell_\infty(\tilde{M}_0)$ defined by

$$Vh(x, y) = \frac{h(x) - h(y)}{d_a(x, y)}.$$

Then $\|V\| \le 1$ but $V|_E$ is an isometry. We shall show that $(Vf_n)_{n=1}^\infty$ is weakly convergent to Vf.

Let $\mathcal{A}$ be the real subalgebra of $\ell_\infty(\tilde{M}_0)$ generated by the constants and the functions

$$\begin{aligned}
&(x, y) \to \frac{f(x) - f(y)}{d_a(x, y)},\\
&(x, y) \to \arctan(d(x, u)), \quad u \in M_0,\\
&(x, y) \to \arctan(d(y, u)), \quad u \in M_0,\\
&(x, y) \to \arctan(d(x, y)).
\end{aligned}$$

Let K be the associated compactification of $\tilde{M}_0$ so each $h \in \mathcal{A}$ continuously extends to K; K is then a compact metric space. Note that Vf_n, Vf belong to $\mathcal{C}(K)$ and so it will suffice to show that $\lim_{n\to\infty} Vf_n(\xi) = Vf(\xi)$ for every $\xi \in K$.

First observe that by (2) $\|Vf_n\| \le \min(d(p, q)^{a-1}, d(p, q)^{1-a})$ so that if $r = 0$ or $r = \infty$ the sequence $(Vf_n)_{n=1}^\infty$ is norm convergent to zero. Therefore we can suppose that $0 < r < \infty$.

If $\xi \in K$ we may pick a sequence $(x_m, y_m) \in \tilde{M}_0$ with $(x_m, y_m) \to \xi$. Note that $\lim_{m\to\infty} d(x_m, y_m) = t$ for some t with $0 \le t \le \infty$. Observe that if $t = 0$ or $t = \infty$, we have $Vf(\xi) = 0 = Vf_n(\xi)$ for all n by (3). We therefore reduce to the case $0 < t < \infty$.

We need the remark that if $(u_n)_{n=1}^\infty$, $(v_n)_{n=1}^\infty$ are two sequences in M such that

$$\lim_{n\to\infty} d(u_n) = \infty$$

and $\lim_{m\to\infty} d(u_n, v_m)$ exists in $[0, \infty]$ for all m then

$$\lim_{n\to\infty} \lim_{m\to\infty} d(u_n, v_m) = \lim_{m\to\infty} \lim_{n\to\infty} d(u_n, v_m) = \infty.$$

Indeed, $\lim_{n\to\infty} d(u_n, v_m) = \infty$ for all m. If for some k we have $\lim_{m\to\infty} d(u_k, v_m) < \infty$ we note that

$$d(u_n, v_m) \ge d(u_n, u_k) - d(u_k, v_m)$$

and so that

$$\lim_{m\to\infty} d(u_n, v_m) \geq d(u_n, u_k) - \lim_{m\to\infty} d(u_k, v_m),$$

which converges to ∞ as $n \to \infty$. Thus the stability of (M, d) applies even to extended valued limits. Now

$$\lim_{n\to\infty} f_n(x_m) = f(x_m) = \min(1, r^{a-1}) \Big(\max\Big(r - \lim_{n\to\infty} d(q_n, x_m), 0\Big) - \max(r - s, 0)\Big)$$

while

$$\begin{aligned}\lim_{m\to\infty} f_n(x_m) = \min(1, d(p_n, q_n)^{a-1})(&\max(d(p_n, q_n) - \lim_{m\to\infty} d(q_n, x_m), 0)\\ &- \max(d(p_n, q_n) - d(q_n), 0)\end{aligned}$$

and so

$$\lim_{m\to\infty} f(x_m) = \lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m).$$

Similarly,

$$\lim_{m\to\infty} f(y_m) = \lim_{n\to\infty} \lim_{m\to\infty} f_n(y_m).$$

Thus

$$\begin{aligned}Vf(\xi) &= \lim_{m\to\infty} Vf(x_m, y_m)\\ &= \min(t^{-a}, t^{-1}) \lim_{m\to\infty} (f(x_m) - f(y_m))\\ &= \lim_{n\to\infty} \lim_{m\to\infty} \min(d(x_m, y_m)^{-a}, d(x_m, y_m)^{-1})(f_n(x_m) - f_n(y_m))\\ &= \lim_{n\to\infty} Vf_n(\xi).\end{aligned}$$

Thus $(Vf_n)_{n=1}^{\infty}$ is weakly convergent and therefore so is $(f_n)_{n=1}^{\infty}$.

We conclude that W is weakly compact. This implies that the map $S : \ell_1(\tilde{M}) \to \mathrm{Lip}_0(M)$ defined by

$$S(\xi) = \sum_{p,q\in\tilde{M}} \xi_{p,q} f_{p,q}$$

is weakly compact. If we let $e_{x,y}$ be the canonical basis vectors we have $Se_{x,y} = f_{x,y}$ for $(x, y) \in \tilde{M}$. Clearly $\|S\| \leq 1$. Consider the adjoint $S^* : \mathrm{Lip}_0(M)^* \to \ell_\infty(\tilde{M})$ which is also weakly compact by Gantmacher's theorem. By the factorization theorem of Davis *et al.* [10] there is a reflexive space Z and maps $T : \mathrm{Lip}_0(M)^* \to Z$ and $U : Z \to \ell_\infty(\tilde{M})$ so that $\|U\|, \|T\| \leq 1$ and $S^* = UT$.

Let us define a map $\delta : (M, d_a) \to \mathrm{Lip}_0(M, d_a)^*$ by

$$\delta(x)(f) = f(x).$$

Then δ is an isometric embedding. Now, if $x \neq y \in M$

$$\|T\delta(x) - T\delta(y)\| \le \|\delta(x) - \delta(y)\| \le d_a(x, y)$$

while

$$\begin{aligned} \|T\delta(x) - T\delta(y)\| &\ge \|S^*\delta(x) - S^*(\delta(y))\| \\ &\ge |(\delta(x) - \delta(y))(Se_{x,y})| \\ &= |f_{x,y}(x) - f_{x,y}(y)| \\ &= \min(d(x, y), d(x, y)^a). \end{aligned}$$

This shows that $T \circ \delta$ is a strong uniform embedding of M into Z with

$$\min(d(x, y), d(x, y)^a) \le \|T \circ \delta(x) - T \circ \delta(y)\| \le \max(d(x, y), d(x, y)^a), \quad x, y \in M.$$

Brown and Guentner [7] showed that every metric space with bounded geometry coarsely embeds into a reflexive space; our original motivation for the above result was that any locally compact metric space is automatically stable, so that we have a corollary.

COROLLARY 2.2 *If M is a locally compact metric space then M strongly uniformly embeds into a reflexive space.*

However, the author has learned of a more recent result of Baudier and Lancien [3] that every locally finite metric space Lipschitz embeds into any Banach space failing cotype (and in particular into a reflexive space). This is, of course, a much stronger conclusion for this class of spaces.

3. Embeddings of infinite graphs

Let G be a connected graph. Then G has an associated metric space structure given by setting $d(u, v)$ to be the length of the shortest path from u to v.

Let $\mathbb{M}$ be any subset of $\mathbb{N}$. For $r \in \mathbb{N}$ we define $\mathcal{P}_r(\mathbb{M})$ to be the set of all subsets of $\mathbb{M}$ of size r. For $r = 0$ we define $\mathcal{P}_0(\mathbb{M})$ to be the singleton $\{\emptyset\}$. We make $\mathcal{P}_r(\mathbb{M})$ into a graph by declaring two distinct subsets $\sigma = \{m_1, m_2, \ldots, m_r\}$ and $\tau = \{n_1, n_2, \ldots, n_r\}$ (written in increasing order) if they *interlace* so that either

$$m_1 \le n_1 \le m_2 \le \cdots \le m_r \le n_r \quad \text{or} \quad n_1 \le m_1 \le n_2 \le \cdots \le n_r \le m_r.$$

We write $\sigma < \tau$ if $m_r < n_1$. It is clear that as long as $|\mathbb{M}| \ge 2r$, the diameter of this graph (and of its associated metric space) is r. It is also easy to verify that

$$d(\sigma, \tau) = r$$

if and only if either $\sigma < \tau$ or $\tau < \sigma$.

For $r = 0$ we define $\mathcal{P}_0(\mathbb{M})$ to be the singleton $\{\emptyset\}$, and thus a function on $\mathcal{P}_0(\mathbb{M})$ is a constant. The following proposition is elementary.

PROPOSITION 3.1 *Suppose M is a stable metric space and $f : \mathcal{P}_r(\mathbb{N}) \to M$ is a bounded map, where $r \geq 1$. Then, for any $\epsilon > 0$, there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that for any $\sigma, \tau \in \mathcal{P}_r(\mathbb{M})$ with $\sigma < \tau$ we have $d(f(\sigma), f(\tau)) < \omega_f(1) + \epsilon$.*

Proof. By a standard application of Ramsey theory we can pass to an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that for some constant a and every $\sigma, \tau \in \mathcal{P}_r(\mathbb{M})$ with $\sigma < \tau$ we have $|d(f(\sigma), f(\tau)) - a| < \epsilon/2$.

Now let $\mathcal{U}$ be any non-principal ultrafilter on $\mathbb{N}$ containing $\mathbb{M}$. Then

$$\lim_{m_1 \in \mathcal{U}} \lim_{n_1 \in \mathcal{U}} \cdots \lim_{m_r \in \mathcal{U}} \lim_{n_r \in \mathcal{U}} d(f(m_1, \dots, m_r), f(n_1, \dots, n_r)) \leq \omega_f(1).$$

Thus using stability [6, p. 213] we also have

$$\lim_{m_1 \in \mathcal{U}} \cdots \lim_{m_r \in \mathcal{U}} \lim_{n_1 \in \mathcal{U}} \cdots \lim_{n_r \in \mathcal{U}} d(f(m_1, \dots, m_r), f(n_1, \dots, n_r)) \leq \omega_f(1).$$

Hence for some $\sigma, \tau \in \mathcal{P}_r(\mathbb{A})$ with $\sigma < \tau$ we have

$$d(f(\sigma), f(\tau)) < \omega_f(1) + \frac{\epsilon}{2}$$

and so

$$a < \omega_f(1) + \epsilon,$$

and the result follows.

Now suppose X is a Banach space and $f : \mathcal{P}_r(\mathbb{N}) \to X$ is a bounded map, where $r \geq 1$. Fix some non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then we can define a bounded map $\partial_{\mathcal{U}} f : \mathcal{P}_{r-1}(\mathbb{N}) \to X^{**}$ by setting

$$\partial_{\mathcal{U}} f(m_1, \dots, m_{r-1}) = w^* - \lim_{m_r \in \mathcal{U}} f(m_1, \dots, m_r).$$

In the case $r = 1$ we can regard $\partial_{\mathcal{U}} f$ as an element of X^{**}. We can then iterate this procedure and define bounded maps $\partial^k_{\mathcal{U}} f : \mathcal{P}_{r-k}(\mathbb{N}) \to X^{(2k)}$ for $1 \leq k \leq r$, where $X^{(k)}$ denotes the kth dual of X. We consider $\partial^r_{\mathcal{U}} f$ as an element of $X^{(2r)}$.

Of course if X is reflexive each $\partial^k_{\mathcal{U}} f$ maps $\mathcal{P}_{r-k}(\mathbb{N})$ into X. We shall also use this notation when $X = \mathbb{R}$.

LEMMA 3.2 *Let $h : \mathcal{P}_r(\mathbb{N}) \to \mathbb{R}$ be a bounded map. Then, given $\epsilon > 0$, there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that*

$$|h(\sigma) - \partial^r_{\mathcal{U}} h| < \epsilon, \quad \sigma \in \mathcal{P}_r(\mathbb{M}).$$

Proof. We select $\mathbb{M} = \{m_1, m_2, \dots\}$ inductively so that for each k if $\sigma \subset \{m_1, \dots, m_k\}$ and $1 \leq |\sigma| = s \leq \min(k, r)$ then $|\partial^{r-s}_{\mathcal{U}} h(\sigma) - \partial h^r_{\mathcal{U}}| < \epsilon$. To start the induction select m_1 so that $|\partial^{r-1}_{\mathcal{U}} h(m_1) - \partial^r_{\mathcal{U}} h| < \epsilon$.

Assume $m_1 < m_2 < \cdots < m_k$ have been chosen. For each subset σ of $\{m_1, \ldots, m_k\}$ with $s = |\sigma| \le r - 1$ there is a set $\mathbb{A}_\sigma \in \mathcal{U}$ so that if $m \in \mathbb{A}_\sigma$ we have $m > m_k$ and $|\partial_{\mathcal{U}}^{r-s-1} h(\sigma \cup \{m\}) - \partial_{\mathcal{U}}^r h| < \epsilon$. The intersection $\mathbb{A}$ of all such $\mathbb{A}_\sigma$ also belongs to $\mathcal{U}$. Hence we may pick $m_{k+1} \in \mathbb{A}$ and complete the inductive step.

Suppose X is a Banach space and that $f : \mathcal{P}_r(\mathbb{N}) \to X$ and $g : \mathcal{P}_r(\mathbb{N}) \to X^*$ are bounded maps. Define $f \otimes g : \mathcal{F}_{2r}(\mathbb{N}) \to \mathbb{R}$ by

$$f \otimes g(m_1, m_2, \ldots, m_{2r}) = \langle f(m_2, m_4, \ldots, m_{2r}), g(m_1, m_3, \ldots, m_{2r-1})\rangle.$$

LEMMA 3.3 *Under the above hypotheses,*

$$\partial_{\mathcal{U}}^2 (f \otimes g) = \partial_{\mathcal{U}} f \otimes \partial_{\mathcal{U}} g.$$

Proof. We have

$$\begin{aligned}\partial_{\mathcal{U}}(f \otimes g)(m_1, \ldots, m_{2r-1}) &= \lim_{m_{2r} \in \mathcal{U}} \langle f(m_2, \ldots, m_{2r}), g(m_1, m_3, \ldots, m_{2r-1})\rangle \\ &= \langle \partial_{\mathcal{U}} f(m_2, \ldots, m_{2r-2}), g(m_1, m_3, \ldots, m_{2r-1})\rangle.\end{aligned}$$

Now letting $m_{2r-1} \to \infty$ through $\mathcal{U}$ gives the conclusion.

LEMMA 3.4 *Suppose $f : \mathcal{P}_r(\mathbb{N}) \to X$ is a bounded map. Then for any $\epsilon > 0$ there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that*

$$\|f(\sigma)\| < \|\partial_{\mathcal{U}}^r f\| + \omega_f(1) + \epsilon, \quad \sigma \in \mathcal{P}_r(\mathbb{M}).$$

Proof. Using the Hahn–Banach theorem, we define $g : \mathcal{F}_r(\mathbb{N}) \to X^*$ so that $\|g(\sigma)\| = 1$ and

$$\langle f(\sigma), g(\sigma)\rangle = \|f(\sigma)\|, \quad \sigma \in \mathcal{P}_r(\mathbb{N}).$$

Now by Lemma 3.3 and induction we have

$$\partial_{\mathcal{U}}^{2r}(f \otimes g) = \partial_{\mathcal{U}}^r f \otimes \partial_{\mathcal{U}}^r g = \langle \partial_{\mathcal{U}}^r f, \partial_{\mathcal{U}}^r g\rangle \le \|\partial_{\mathcal{U}}^r f\|.$$

By Lemma 3.2, we can find an infinite set $\mathbb{A} \subset \mathbb{N}$ so that

$$|f \otimes g(\sigma)| < \|\partial_{\mathcal{U}}^r f\| + \epsilon, \quad \sigma \in \mathcal{P}_{2r}(\mathbb{A}).$$

Let $\mathbb{A} = \{m_1, n_1, m_2, n_2, \ldots\}$, where $m_1 < n_1 < m_2 < \cdots$. Let $\mathbb{M} = \{m_1, m_2, \ldots\}$.

Suppose $\sigma = \{m_{u_1}, m_{u_2}, \ldots, m_{u_r}\}$, where $u_1 < u_2 < \cdots < u_r$. Then

$$\begin{aligned}\|f(\sigma)\| &= \langle f(\sigma), g(\sigma)\rangle \\ &\le \omega_f(1) + \langle f(n_{u_1}, n_{u_2}, \ldots, n_{u_r}), g(m_{u_1}, m_{u_2}, \ldots, m_{u_r})\rangle \\ &\le \|\partial_{\mathcal{U}}^r f\| + \omega_f(1) + \epsilon.\end{aligned}$$

THEOREM 3.5 *Suppose* $r \in \mathbb{N}$. *Let* X *be a Banach space such that* $X^{(2k)}$ *is separable. Then given any uncountable family of bounded maps* $f_i : \mathcal{P}_r(\mathbb{N}) \to X$ *for* $i \in I$ *and* $\epsilon > 0$, *there exist* $i \neq j$ *and an infinite subset* $\mathbb{M}$ *of* $\mathbb{N}$ *so that*

$$\| f_i(\sigma) - f_j(\sigma)\| < \omega_{f_i}(1) + \omega_{f_j}(1) + \epsilon, \quad \sigma \in \mathbb{M}.$$

Proof. Since $X^{(2r)}$ is separable we may find $i \neq j$ so that $\|\partial^r_{\mathcal{U}} f_i - \partial^r_{\mathcal{U}} f_j\| < \epsilon/2$. Now $\|\partial^r_{\mathcal{U}}(f_i - f_j)\| < \epsilon/2$ and so using Lemma 3.4 gives the conclusion.

THEOREM 3.6 c_0 *cannot be uniformly or coarsely embedded into a Banach space* X *such that every dual space* $X^{(r)}$ *is separable. In particular,* c_0 *cannot be uniformly or coarsely embedded into a reflexive Banach space.*

Proof. Suppose $h : c_0 \to X$ is any map. Let $(e_k)_{k=1}^{\infty}$ be the canonical basis of c_0. For any $r > 0$ and $0 < \theta < \infty$ and any infinite subset A of $\mathbb{N}$ we define

$$s_n(A) = \sum_{\substack{k \le n \\ k \in A}} e_k$$

and then

$$f_{A,\theta,r}(n_1, \ldots, n_r) = \theta \sum_{j=1}^{r} s_{n_j}(A).$$

We then consider $h \circ f : \mathcal{P}_r(\mathbb{N}) \to X$. Fix θ, r. By Theorem 3.5 we can find $A \neq B$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that

$$\|hf_{A,\theta,r}(\sigma) - hf_{B,\theta,r}(\sigma)\| < \omega_{hf}(1) + \epsilon, \quad \sigma \in \mathbb{M}.$$

Since $A \neq B$ there exists σ so that

$$\| f_{A,\theta,r}(\sigma) - f_{A,\theta,r}(\sigma)\| = r\theta.$$

Thus

$$\varphi_h(r\theta) \le \omega_{hf}(1) + \epsilon \le \omega_h(\theta) + \epsilon.$$

It follows that

$$\varphi_h(r\theta) \le \omega_h(\theta), \quad 0 < \theta < \infty.$$

If $\omega_h(\theta) < \infty$ for some θ this implies $\lim_{\theta \to \infty} \varphi_h(\theta) < \infty$. If $\lim_{\theta \to 0} \omega_h(\theta) = 0$ this implies that $\varphi_h(\theta) = 0$ for all $\theta > 0$.

COROLLARY 3.7 *There is a separable Schur space* X *such that* B_X *cannot be uniformly embedded into a Banach space* Z *such that every dual of* Z *is separable.*

Proof. By [17, Proposition 5.2] there is a Schur space X which is uniformly homeomorphic to $Y \oplus c_0$ for some Banach space Y. In particular B_{c_0} is uniformly homeomorphic to a subset of B_X. Since c_0 is uniformly into B_{c_0} by [1], the result follows from Theorem 3.6.

Odell and Schlumprecht [21] showed that if X has an unconditional basis, then B_X is uniformly homeomorphic to B_{ℓ_2} if and only if X has non-trivial cotype; the corresponding result for Banach lattices is due to Chaatit [8]; see also [9]. We now present a simple generalization of this result.

THEOREM 3.8 *Let X be a separable Banach lattice. Then the following are equivalent*:

(i) *B_X is uniformly homeomorphic to a subset of a reflexive Banach space;*
(ii) *B_X is uniformly homeomorphic to B_Y for some reflexive Banach lattice;*
(iii) *X contains no subspace isomorphic to c_0.*

Proof. (iii) $\Rightarrow$ (ii) X can be represented as an order-continuous Banach function space on some probability space (Ω, μ) (which may contain atoms). For $1 < p < \infty$ we define Y to be the p-concavification of X [19, pp. 53, 54]. Thus Y is the space of measurable functions $y : \Omega \to \mathbb{R}$ such that $|y|^p \in X$ with the norm $\|y\|_Y = \||y|^p\|_X^{1/p}$. This space is p-convex and hence contains no copy of ℓ_1. On the other hand, if Y contains a copy of c_0 then it contains a disjoint sequence $(y_n)_{n=1}^\infty$ equivalent to the c_0-basis [19, p. 35]; but then $(|y_n|^p)_{n=1}^\infty$ would be equivalent to the c_0-basis in X, which yields a contradiction. Thus Y is reflexive [19, Theorem 1.c.5, p. 35]. Define $f : B_X \to B_Y$ by $f(x) = |x|^{1/p}\operatorname{sgn} x$. Then f is onto.

If $x_1, x_2 \in B_X$ with $\|x_1 - x_2\|_X = \theta$. Let $u = |x_1| + |x_2|$ and $v = |x_1 - x_2|$. Define a measurable function a with $0 \le a \le 1$ so that $v = au$.

If $a(s) = 1$ then $x_1(s), x_2(s)$ have opposite signs and so

$$|f(x_1(s)) - f(x_2(s))|^p = (|x_1(s)|^{1/p} + |x_2(s)|^{1/p})^p.$$

Thus

$$u(s) \le |f(x_1(s)) - f(x_2(s))|^p \le 2^{p-1}u(s).$$

If $a(s) < 1$ then $x_1(s), x_2(s)$ have the same signs and

$$|f(x_1(s)) - f(x_2(s))|^p = \left(\left(\frac{1}{2}(1+a)\right)^{1/p} - \left(\frac{1}{2}(1-a)\right)^{1/p}\right)^p u(s).$$

Note that $\varphi(t) = ((1/2)(1+t))^{1/p} - ((1/2)(1-t))^{1/p}$ is a convex function of t on $[0, 1]$ and so $\varphi(t) \ge \varphi'(0)t$. However, $\varphi'(0) = 2^{1-1/p}p^{-1}$.

Thus

$$2^{p-1}p^{-p}a^p u(s) \le |f(x_1(s)) - f(x_2(s))|^p \le au(s).$$

Thus in general we conclude

$$2^{p-1}p^{-p}a^p u \le |f(x_1) - f(x_2)|^p \le 2^{p-1}au.$$

The right-hand inequality gives

$$\|f(x_1) - f(x_2)\|_Y \le 2^{1-1/p}\|au\|^{1/p} = 2^{1-1/p}\theta^{1/p}.$$

We also note that $\|u\| \leq 2$ so that if $A = \{s : a < (\theta/4)\}$ then $\|au\chi_A\|_X \leq \theta/2$. Hence if B is the complement of A we have $\|au\chi_B\|_X \geq \theta/2$. Now

$$\|f(x_1) - f(x_2)\|_Y \geq 2^{1-1/p} p^{-1} \|a^p u \chi_B\|_X^{1/p} \geq 2^{1-1/p} p^{-1} \left(\frac{\theta}{4}\right)^{1-1/p} \left(\frac{\theta}{2}\right)^{1/p} \geq \frac{\theta}{2p}.$$

Now (ii) $\Rightarrow$ (i) is trivial and (i) $\Rightarrow$ (iii) follows from Theorem 3.5.

4. Embeddings of metric spaces into reflexive spaces

We now consider the problem of embedding a metric space coarsely or uniformly into a reflexive Banach space. We will use the following theorem to develop necessary conditions for the existence of such embeddings.

THEOREM 4.1 *Suppose $r \in \mathbb{N}$. Let X be a reflexive Banach space and suppose $f : \mathcal{P}_r(\mathbb{N}) \to X$ is a bounded map. Then, given $\epsilon > 0$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $x \in X$ so that*

$$\|f(\sigma) - x\| \leq \omega_f(1) + \epsilon, \quad \sigma \in \mathcal{P}_r(\mathbb{M}).$$

Proof. This follows directly from Lemma 3.4. Since X is reflexive we have $\partial^r_{\mathcal{U}} f = x \in X$. Let $f'(\sigma) = f(\sigma) - x$ so that $\partial^r_{\mathcal{U}} f' = 0$ and apply Lemma 3.4.

If M is a metric space and $\epsilon > 0$ and $\delta \geq 0$ we shall say that M has *property* $\mathcal{Q}(\epsilon, \delta)$ if for every $r \in \mathbb{N}$ if $f : \mathcal{P}_r(\mathbb{N}) \to M$ is a map with $\omega_f(1) \leq \delta$ there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ with

$$d(f(\sigma), f(\tau)) \leq \epsilon, \quad \sigma < \tau, \ \sigma, \tau \in \mathcal{P}_r(\mathbb{M}).$$

Let us define $\Delta_M(\epsilon)$ to be the supremum of all $\delta \geq 0$ so that M has property $\mathcal{Q}(\epsilon, \delta)$.

Note first that if M is a stable metric space then $\Delta_M(\epsilon) \geq \epsilon$ for every $\epsilon > 0$ by an application of Proposition 3.1.

THEOREM 4.2 *Suppose M is a metric space.*

(i) *If M uniformly embeds into a reflexive Banach space then $\Delta_M(\epsilon) > 0$ for every $\epsilon > 0$.*
(ii) *If M coarsely embeds into a reflexive space then* $\lim_{\epsilon\to\infty} \Delta_M(\epsilon) = \infty$.

Proof. Let $h : M \to X$ be a map of M into a reflexive Banach space X. Let $f : \mathcal{P}_r(\mathbb{N}) \to M$ be a map with $\omega_f(1) \leq \delta$. Then $\omega_{hf}(1) \leq \omega_h(\delta)$. Hence there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $x \in X$ so that

$$\|hf(\sigma) - x\| \leq 2\omega_h(\delta), \quad \sigma \in \mathcal{P}_r(\mathbb{M}).$$

Thus

$$\|hf(\sigma) - hf(\tau)\| \leq 4\omega_h(\delta), \quad \sigma, \tau \in \mathcal{P}_r(\mathbb{M}).$$

From this we have

$$\varphi_h(d(f(\sigma), f(\tau))) \leq 4\omega_h(\delta), \quad \sigma < \tau, \ \sigma, \tau \in \mathcal{P}_r(\mathbb{M}).$$

If h is a uniform embedding then

$$\lim_{\delta \to 0} \omega_h(\delta) = 0$$

and $\varphi_h(\epsilon) > 0$. Thus $\Delta_M(\epsilon) > 0$ for all $\epsilon > 0$. If h is a coarse embedding $\lim_{\epsilon\to\infty} \varphi_h(\epsilon) = \infty$ and $\omega_h(\delta) < \infty$ for each $\delta > 0$ so that $\lim_{\epsilon\to\infty} \Delta_M(\epsilon) = \infty$.

If X is a Banach space it is clear that $\Delta_X(\epsilon) = a\epsilon$ for some constant a and that $\Delta_{B_X}(\epsilon) = \Delta_X(\epsilon)$ if $0 < \epsilon \le 1$. We denote by $\mathcal{Q}_X$ the constant such that $\Delta_X(\epsilon) = \mathcal{Q}_X\epsilon$. We will say that X has the *$\mathcal{Q}$-property* if $\mathcal{Q}_X > 0$.

We remark that if X is a stable Banach space then $\mathcal{Q}_X = 1$ while if X is reflexive we have $\mathcal{Q}_X \ge 1/2$ by Theorem 4.1. Thus all stable or reflexive Banach spaces have the $\mathcal{Q}$-property. On the other hand, c_0 fails the $\mathcal{Q}$–property; this can be proved by the methods of Theorem 3.5 but will also follow from the more delicate examples given below, since every separable Banach space Lipschitz embeds into c_0 by the result of Aharoni [1].

COROLLARY 4.3 *If X is a Banach space which fails the $\mathcal{Q}$-property then*

(i) *B_X cannot be uniformly embedded into a reflexive Banach space,*
(ii) *X cannot be coarsely embedded into a reflexive Banach space.*

THEOREM 4.4 *Let X be a Banach space with the $\mathcal{Q}$-property. Then given $\epsilon > 0$ and a bounded sequence $(x_n)_{n=1}^\infty$ with a weak*-cluster point $x^{**} \in X^{**}$ there is a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that*

$$\left\| \sum_{j=1}^{2r} (-1)^j y_{n_j} \right\| \ge (1-\epsilon)\mathcal{Q}_X r d(x^{**}, X), \quad n_1 < n_2 < \cdots < n_{2r}, \;\; r = 1, 2, \ldots .$$

Proof. Let $\theta = d(x^{**}, X)$ and let $B = \sup_n \|x_n\|$. We can assume $\theta > 0$. We will pick $\lambda > 1$ and $\alpha > 0$ small enough so that

$$\lambda^{-2}\mathcal{Q}_X - 2\alpha B - \alpha > (1-\epsilon)\mathcal{Q}_X.$$

We can extract a subsequence $(v_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that

$$\left\| \sum_{j=1}^{m} a_j v_j - \sum_{k=m+1}^{n} a_j v_j \right\| > \lambda^{-1}\theta$$

whenever $1 \le m < n$, $a_j \ge 0$ and

$$\sum_{j=1}^{m} a_j = \sum_{j=m+1}^{n} a_j = 1.$$

Using Ramsey's theorem and a diagonalization argument we can extract a further subsequence $(y_n)_{n=1}^\infty$ so that for each $r \in \mathbb{N}$ there is a constant b_r such that if $\alpha r \le n_1 < n_2 < \cdots < n_{2r}$ then

$$b_r - \alpha\theta \le \left\| \sum_{j=1}^{2r} (-1)^j y_{n_j} \right\| \le b_r.$$

For fixed r let $\mathbb{M} = \{n \in \mathbb{N} : n \geq \alpha r\}$. Define $f : \mathcal{P}_r(\mathbb{M}) \to X$ by

$$f(\sigma) = \sum_{k \in \sigma} y_k.$$

Then $\omega_f(1) \leq b_r$ and so there exist $\sigma < \tau$ in $\mathcal{P}_r(\mathbb{M})$ with

$$\|f(\sigma) - f(\tau)\| \leq \lambda \mathcal{Q}_X^{-1} b_r.$$

Hence

$$\lambda^{-1} r\theta \leq \lambda \mathcal{Q}_X^{-1} b_r$$

that is,

$$b_r \geq \lambda^{-2} \mathcal{Q}_X r\theta.$$

Now if $\{n_1, \ldots, n_{2r}\} \in \mathcal{P}_{2r}(\mathbb{N})$ there exist $\{m_1, \ldots, m_{2r}\}$ so that $\alpha r \leq m_1$ and

$$\left\| \sum_{j=1}^{2r} (-1)^j y_{n_j} + \beta \sum_{j=1}^{2r} (-1)^j y_{m_j} \right\| \leq 2\alpha r B$$

for either $\beta = 1$ or $\beta = -1$. Thus

$$\left\| \sum_{j=1}^{2r} (-1)^j y_{n_j} \right\| \geq \lambda^{-2} \mathcal{Q}_X r\theta - 2\alpha B r\theta - \alpha\theta > (1 - \epsilon) \mathcal{Q}_X r\theta.$$

We now recall [**4**] that a Banach space X has the *alternating Banach–Saks property* if every bounded sequence $(x_n)_{n=1}^{\infty}$ in X has a subsequence (y_n) so that the Cesaro means $(1/n) \sum_{k=1}^{n} (-1)^k y_k$ converge to 0. Beauzamy [**4**] proves that X has the alternating Banach–Saks property if and only if X has no spreading model equivalent to the ℓ_1−basis.

THEOREM 4.5 *Suppose X has the alternating Banach–Saks property. If X has the $\mathcal{Q}$-property then X is reflexive.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in X. Let x^{**} be a weak*-cluster point of $(x_n)_{n=1}^{\infty}$ in X^{**}. According to Theorem 4.4 there is a subsequence $(y_n)_{n=1}^{\infty}$ and $c > 0$ so that

$$\left\| \sum_{j=1}^{2r} (-1)^j y_{n_j} \right\| \geq c r d(x^{**}, X), \quad n_1 < n_2 < \ldots < n_{2r}, \quad r = 1, 2, \ldots.$$

By the alternating Banach–Saks property we have $d(x^{**}, X) = 0$ and so by the Eberlein–Smulian theorem, X is reflexive.

We are now in a position to give some more examples of spaces failing the $\mathcal{Q}$-property. A space with non-trivial type has the alternating Banach–Saks property [**4**].

COROLLARY 4.6 *Let X be a non-reflexive Banach space with non-trivial type. Then X fails property $\mathcal{Q}$.*

We note that examples of such spaces were constructed by James [**15**] and later by Pisier and Xu [**24**]. Pisier and Xu also gave quasi-reflexive examples. This result does not cover the original James quasi-reflexive space [**13**, **14**] but this can be done directly from the definition. We recall that J is defined to be the space of all real sequences $(\xi_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty}\xi_n = 0$ and

$$\|\xi\|_J = \sup_{i_0<i_1<i_2<\cdots<i_n}\left(\sum_{j=1}^{n}(\xi_j-\xi_{j-1})^2\right)^{1/2} < \infty.$$

PROPOSITION 4.7 *The James space J and its dual J^* fail the $\mathcal{Q}$-property.*

Proof. In J consider the sequence $x_n = e_1 + \cdots + e_n$ which converges weak* to an element $\chi \in J^{**}$ with $\|\chi\| = 1$. However,

$$\left\|\sum_{j=1}^{2r}(-1)^j x_{n_j}\right\| J = (2r-1)^{1/2}, \quad n_1 < n_2 < \cdots < n_{2r}.$$

We can then directly apply Theorem 4.4.

Similarly in J^* consider the sequence $(e_n^*)_{n=1}^{\infty}$ (the dual basis). Then x_n converges weak* to an element of J^{***} of norm one but

$$\left\|\sum_{j=1}^{2r}(-1)^j e_{n_j}^*\right\| J^* \le r^{1/2}, \quad n_1 < n_2 < \cdots < n_{2r}.$$

However, we do not know whether it is possible to find a non-reflexive quasi-reflexive space X so that B_X embeds uniformly in a reflexive space, or X embeds coarsely. We will show that there are quasi-reflexive spaces with the $\mathcal{Q}$-property. First let us define the ω-dual of a Banach space X as the space $X^{(\omega)}$ obtained by completing $\bigcup_{k\ge 0} X^{(2k)}$; where X^{2k-2} is considered as a subspace of X^{2k} for $k \ge 1$ via the canonical embedding.

THEOREM 4.8 *Let X be a separable Banach space so that* $\dim X^{**}/X = 1$. *Then the following are equivalent:*

(i) $X^{(\omega)} \approx X \oplus \ell_1$*;*
(ii) *X has the $\mathcal{Q}$-property.*

Proof. Although our proof is essentially self-contained, we use ideas due to Perrott [**22**] and Bellenot [**5**].

We start with some general remarks on $X^{(\omega)}$. Pick some $e_1 \in X^{**}\setminus X$ so that $\|e_1\| = 1$. Let $\theta = d(e_1, X)$. Then there is a weakly Cauchy sequence $(u_n)_{n=1}^{\infty}$ converging weak* to e_1 with $\|u_n\| \le 1$. We then define $e_k \in X^{(2k)}$ by setting e_k to be the weak*-limit of $(u_n)_{n=1}^{\infty}$ considered as a sequence in $X \subset X^{(2k)}$. It is clear that $X^{(2k)} = [X, e_1, \ldots, e_k]$. We consider the sequence $(e_n)_{n=1}^{\infty}$ in $X^{\omega} = \bigcup_{k\geq 0} X^{(2k)}$.

Now by taking convex combinations of the $(u_n)_{n=1}^{\infty}$ and using a diagonalization argument it is possible to find a sequence $(v_n)_{n=1}^{\infty}$ in X with $\|v_n\| \leq 1$ so that

$$\lim_{n\to\infty} \|\xi + v_n\| = \|\xi + e_k\|, \quad \xi \in X^{(2k-2)}, \ \ k = 1, 2, \ldots.$$

Now it follows that if $x \in X$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ we have

$$\left\| x + \sum_{j=1}^{r} \alpha_j e_j \right\| = \lim_{n_1\to\infty} \lim_{n_2\to\infty} \cdots \lim_{n_r\to\infty} \left\| x + \sum_{j=1}^{r} \alpha_{r-j+1} v_{n_j} \right\|.$$

From this it follows easily that $(e_k)_{k=1}^{\infty}$ is spreading over X, that is,

$$\left\| x + \sum_{j=1}^{r} \lambda_j e_j \right\| = \left\| x + \sum_{j=1}^{r} \lambda_j e_{n_j} \right\|, \quad x \in X, \ \ \lambda_1, \ldots, \lambda_r \in \mathbb{R}, \ \ n_1 < n_2 < \cdots < n_r. \tag{4}$$

It also follows that if $1 \leq k < r$ then

$$\left\| x + \sum_{j<k} \lambda_j e_j + (\lambda_k + \lambda_{k+1}) e_k + \sum_{j>k+1} \lambda_j e_j \right\| \leq \left\| x + \sum_{j=1}^{r} \lambda_j e_j \right\|, \quad x \in X, \ \ \lambda_1, \ldots, \lambda_r \in \mathbb{R}. \tag{5}$$

Both (4) and (5) are due to Perrott [**22**]. By (4) and (5) the sequence $(e_j)_{j=1}^{\infty}$ is a *dual neighborly* sequence in the sense of Bellenot [**5**].

We note furthermore that

$$\|x + e_1 - e_2\| = \lim_{n\to\infty} \|x + e_1 - u_n\| \geq \theta, \quad x \in X. \tag{6}$$

(i) $\Rightarrow$ (ii) We show first that $(e_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of ℓ_1. Indeed by Rosenthal's theorem [**27**] if this fails (e_n) is weakly Cauchy. However, $X^{(\omega)}$ has codimension one in the subspace of $(X^{(\omega)})^{**}$ consisting of all sequential limits in the weak*-topology. Hence there exists $\alpha \in \mathbb{R}$ so that $e_n - \alpha v_n$ is weakly convergent to some $z \in X^{(\omega)}$. Given $\epsilon > 0$ we can find by Mazur's theorem $n_1 < n_2 < n_3$ and $a_j \geq 0$ for $n_1 + 1 \leq j \leq n_2$ such that

$$\sum_{j=n_1+1}^{n_2} a_j = \sum_{j=n_2+1}^{n_3} a_j = 1$$

and

$$\left\| z - \sum_{j=n_1+1}^{n_2} a_j (e_j - \alpha v_j) \right\|, \quad \left\| z - \sum_{j=n_2+1}^{n_3} a_j (e_j - \alpha v_j) \right\| < \frac{\theta}{2}.$$

Subtracting shows that there exists $x \in X$ with

$$\left\| x + \sum_{j=n_1+1}^{n_2} a_j e_j - \sum_{j=n_2+1}^{n_3} a_j e_j \right\| < \theta.$$

By (5) this implies

$$\|x + e_1 - e_2\| < \theta,$$

contradicting (6). It follows that there is a constant C so that

$$\sum_{j=1}^{r} |\lambda_j| \le C \left\| \sum_{j=1}^{r} \lambda_j e_j \right\|, \quad \lambda_1, \ldots, \lambda_r \in \mathbb{R}.$$

Now suppose $f : \mathcal{P}_r(\mathbb{N}) \to X$ is a bounded map. Then

$$\partial_{\mathcal{U}} f(n_1, \ldots, n_{r-1}) = g_1(n_1, \ldots, n_{r-1}) + \psi_1(n_1, \ldots, n_{r-1}) e_1$$

for suitable bounded functions $g_1 : \mathcal{P}_{r-1}(\mathbb{N}) \to X$ and $\psi_1 : \mathcal{P}_{r-1}(\mathbb{N}) \to \mathbb{R}$. Thus, again for suitable bounded maps $g_2 : \mathcal{P}_{r-2}(\mathbb{N}) \to X$ and $\psi_2 : \mathcal{P}_{r-2}(\mathbb{N}) \to \mathbb{R}$,

$$\partial_{\mathcal{U}}^2 f(n_1, \ldots, n_{r-2}) = g_2(n_1, \ldots, n_{r-2}) + \partial_{\mathcal{U}} \psi_1(n_1, \ldots, n_{r-2}) e_1 + \psi_2(n_1, \ldots, n_{r-2}) e_2$$

and iterating we have

$$\partial_{\mathcal{U}}^k f(n_1, \ldots, n_{r-k}) = g_k(n_1, \ldots, n_{r-k}) + \sum_{j=1}^{k} \partial_{\mathcal{U}}^{k-j} \psi_j(n_1, \ldots, n_{r-k}) e_j,$$

where $g_k : \mathcal{P}_{r-k}(\mathbb{N}) \to X$ and $\psi_k : \mathcal{P}_{r-k}(\mathbb{N}) \to \mathbb{R}$ are bounded maps. In the case $k = r$ these maps are constants. Let $\lambda_j = \partial_{\mathcal{U}}^{r-j} \psi_j \in \mathbb{R}$. Thus

$$\partial_{\mathcal{U}}^r f = g_r + \sum_{j=1}^{r} \lambda_j e_j.$$

Now define $h : \mathcal{P}_{2r}(M) \to X$ by $h(n_1, \ldots, n_{2r}) = f(n_1, n_3, \ldots, n_r) - f(n_2, n_4, \ldots, n_{2r})$. Then

$$\partial_{\mathcal{U}}^{2r} h = \sum_{j=1}^{r} \partial_{\mathcal{U}}^{r-j} \lambda_j e_{2j} - \sum_{j=1}^{r} \partial_{\mathcal{U}}^{r-j} \lambda_j e_{2j-1}.$$

Hence

$$\sum_{j=1}^{r} |\lambda_j| \le \frac{C}{2} \sup_{\sigma \in \mathcal{P}_{2r}(\mathbb{N})} \|h(\sigma)\| \le C \omega_f(1).$$

Now define

$$f'(n_1, \ldots, n_r) = g_r + \sum_{j=1}^{r} \lambda_{r-j} v_{n_j}.$$

Then

$$\omega_{f'}(1) \leq 2\sum_{j=1}^{r}|\lambda_j| \leq C\omega_f(1).$$

Thus $\partial_{\mathcal{U}}^r(f - f') = 0$ and $\omega_{(f-f')}(1) \leq (C+1)\omega_f(1)$. Hence by Lemma 3.4 we can find an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that

$$\|f(\sigma) - f'(\sigma)\| \leq (C+2)\omega_f(1), \quad \sigma \in \mathcal{P}_r(\mathbb{N}).$$

Hence if $\sigma < \tau$ in $\mathcal{P}_r(\mathbb{M})$ we have

$$\begin{aligned}\|f(\sigma) - f(\tau)\| &\leq 2(C+2)\omega_f(1) + \|f'(\sigma) - f'(\tau)\| \\ &\leq 2(C+2)\omega_f(1) + 2\sum_{j=1}^{r}|\lambda_j| \\ &\leq (3C+2)\omega_f(1).\end{aligned}$$

This shows that X has the $\mathcal{Q}$-property.

(ii) $\Rightarrow$ (i) We first observe that from Theorem 4.4 we can assume that

$$\left\|\sum_{j=1}^{2r}(-1)^j v_{n_j}\right\| \geq cr, \quad r = 1, 2, \ldots$$

for some constant $c > 0$. Thus

$$\left\|\sum_{j=1}^{r} u_j\right\| \geq cr, \quad r = 1, 2, \ldots,$$

where $u_j = e_{2j} - e_{2j-1}$.

We will show that $(e_n)_{n=1}^{\infty}$ is equivalent to the ℓ_1-basis. Suppose not: then as before $(e_n)_{n=1}^{\infty}$ is weakly Cauchy. Thus $(u_n)_{n=1}^{\infty}$ is weakly null. Hence we can find a convex combination $\sum_{j=1}^{r} a_j u_j$ so that

$$\left\|\sum_{j=1}^{r} a_j u_j\right\| < \frac{c}{2}.$$

For any integer N we have

$$\left\|\sum_{k=1}^{N}\sum_{j=1}^{r} a_j u_{j+k}\right\| \leq \frac{cN}{2}.$$

This implies

$$\left\|\sum_{j=r+1}^{N} u_j\right\| \leq \frac{cN}{2} + 4r.$$

Thus

$$c(N-r) \leq \frac{cN}{2} + 4r.$$

For large enough N this is a contradiction.

Thus $(e_n)_{n=1}^{\infty}$ is equivalent to the ℓ_1-basis, that is, there is a constant C so that

$$\sum_{j=1}^{r}|\lambda_j| \leq C\left\|\sum_{j=1}^{r}\lambda_j e_j\right\|, \quad \lambda_1,\dots,\lambda_r \in \mathbb{R}.$$

Next note that if $x \in X$ and $\lambda_1,\dots,\lambda_r \in \mathbb{R}$ then

$$\begin{aligned}
\|x\| &\leq \left\|x+\sum_{j=1}^{r}\lambda_j e_j\right\| + \left\|\sum_{j=1}^{r}\lambda_j e_j\right\| \\
&\leq \left\|x+\sum_{j=1}^{r}\lambda_j e_j\right\| + \frac{1}{2}C\left\|\sum_{j=1}^{r}\lambda_j(e_{2j}-e_{2j-1})\right\| \\
&\leq \left\|x+\sum_{j=1}^{r}\lambda_j e_j\right\| + \frac{1}{2}C\left\|x+\sum_{j=1}^{r}\lambda_j e_{2j}\right\| + \frac{1}{2}C\left\|x+\sum_{j=1}^{r}\lambda_j e_{2j-1}\right\| \\
&\leq (C+1)\left\|x+\sum_{j=1}^{r}\lambda_j e_j\right\|.
\end{aligned}$$

Thus $X^{(\omega)} = X \oplus [e_n] \approx X \oplus \ell_1$.

We now observe that Bellenot [5] has constructed a quasi-reflexive space X so that $X^{(\omega)} \approx X \oplus \ell_1$. Bellenot's space is thus an example of a quasi-reflexive space with the $\mathcal{Q}$-property. However, we do not know if this space can be coarsely embedded in a reflexive space or its unit ball is uniformly homeomorphic to a subset of a reflexive space.

5. Embeddings in uniformly convex spaces

Our results have some applications to uniform and coarse embeddings in uniformly convex spaces (or super-reflexive spaces). We recall the results of Enflo [11] (see Pisier [23] for an improvement) that a Banach space is super-reflexive if and only if it has an equivalent uniformly convex norm.

We recall that Raynaud [26] proved that a non-reflexive Banach space X with non-trivial type has the property that B_X cannot be uniformly embedded into a stable metric space. Our first application is an extension of this result (by Theorem 2.1).

THEOREM 5.1 *Let X be a Banach space with non-trivial type. Assume that either*

(i) *B_X embeds uniformly in a uniformly convex space, or*
(ii) *X coarsely embeds into a uniformly convex space.*

Then X is super-reflexive (*that is, has an equivalent uniformly convex norm*).

Proof. It follows quickly in either case that every ultraproduct of X has the same property and therefore has the $\mathcal{Q}$-property. Hence every ultraproduct is reflexive by Corollary 4.6. Thus X is super-reflexive.

We may also give a criterion for coarse or uniform embeddability of a metric space into a super-reflexive space. We denote by $[N]$ the set $\{1, 2, \ldots, N\}$.

THEOREM 5.2 *Let M be a metric space.*

(i) *In order that M coarsely embeds into a uniformly convex space it is necessary that there exists a function $\phi : (0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \phi(t) = \infty$ and a function $N : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ so that if $r \in \mathbb{N}$ and $f : \mathcal{P}_r([N(r, t)]) \to M$ satisfies $\omega_f(1) \le \phi(t)$ then there exist $\sigma < \tau$ in $\mathcal{P}_r([N(r, t)])$ with*

$$d(f(\sigma), f(\tau)) \le t.$$

(ii) *In order that M uniformly embeds into a uniformly convex space it is necessary that there exists a function $\phi : (0, \infty) \to (0, \infty)$ and a function $N : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ so that if $r \in \mathbb{N}$ and $f : \mathcal{P}_r([N(r, t)]) \to M$ satisfies $\omega_f(1) \le \phi(t)$ then there exist $\sigma < \tau$ in $\mathcal{P}_r([N(r, t)])$ with*

$$d(f(\sigma), f(\tau)) \le t.$$

Proof. Suppose X is a super-reflexive space. Let $h : M \to X$ be a map which is either a coarse embedding or a uniform embedding. Suppose that $t \in \mathbb{R}$ and $\theta > 0$ is such that for some $r \in \mathbb{N}$ and every m there is a map $f_m : \mathcal{P}_r([m]) \to M$ such that $\omega_{f_m}(1) \le \theta$ but

$$d(f_m(\sigma, \tau)) > t, \quad \sigma < \tau, \ \ \sigma, \tau \in \mathcal{P}_r([m]).$$

(If $m < 2r$ this always holds vacuously.) Let $y_m = hf_m(1, 2, \ldots, r)$. Consider the ultraproduct $X_{\mathcal{U}}$, where $\mathcal{U}$ is some non-principal ultrafilter on $\mathbb{N}$. Thus $X_{\mathcal{U}}$ is the space of all bounded sequences $(x_k)_{k=1}^{\infty}$ in X under the seminorm

$$\|(x_k)_{k=1}^{\infty}\|_{X_{\mathcal{U}}} = \lim_{k\in\mathcal{U}} \|x_k\|.$$

We define a map

$$g : \mathcal{P}_r(\mathbb{N}) \to X_{\mathcal{U}}$$

by

$$g(\sigma) = (x_m)_{m=1}^{\infty}$$

where

$$x_m = \begin{cases} 0 & \text{if } m < \max\sigma, \\ hf_m(\sigma) - y_m & \text{if } m \ge \max\sigma. \end{cases}$$

Then $\omega_g(1) \le \omega_h(\theta)$. Hence since $X_{\mathcal{U}}$ is reflexive there exists $\sigma < \tau$ in $\mathbb{N}$ with

$$\|g(\sigma) - g(\tau)\| < 3\omega_h(\theta).$$

It follows that for large enough m

$$\|hf_m(\sigma) - hf_m(\tau)\| < 3\omega_h(\theta).$$

However, for large enough m we have $d(f_m(\sigma), f_m(\tau)) > t$. Thus

$$\varphi_h(t) \leq 3\omega_h(\theta).$$

It follows that if $\omega_h(\theta) < (1/3)\varphi_h(t)$ we can choose $N(r,t)$ so that if $f : \mathcal{P}_r([N(r,t)]) \to M$ has $\omega_f(1) \leq \theta$ then there exist $\sigma < \tau$ with $d(f(\sigma), f(\tau)) \leq t$. It can now be checked easily that we can define the functions ϕ in cases (i) and (ii).

Note that a space with bounded geometry always fulfills (i) of Theorem 5.2.

6. Problems

There are a number of obvious problems that we leave unsolved.

PROBLEM 6.1 Does every (separable) reflexive Banach space coarsely (or uniformly) embed into a stable metric space? Is the converse of Theorem 2.1 valid?

We suspect the answer to Problem 1 is negative. However, we may perhaps have some hope for a positive answer to:

PROBLEM 6.2 If X is a separable reflexive Banach space, does B_X uniformly embed into a stable metric space?

Here we note that the results of [**8**; **9**; **21**; **6**, pp. 199–206] show that for a very wide class of super-reflexive spaces, B_X is uniformly homeomorphic to B_{ℓ_2}; see also [**17**].

PROBLEM 6.3 If X is separable super-reflexive, is B_X uniformly homeomorphic to B_{ℓ_2}?

We also do not know the answers to the following.

PROBLEM 6.4 Suppose X is a separable Banach space which coarsely embeds into a reflexive space; must X be weakly sequentially complete? Similarly if B_X uniformly embeds into a reflexive space, must X be weakly sequentially complete?

PROBLEM 6.5 If X is a separable Banach space is the $\mathcal{Q}$-property for X equivalent to coarse embeddability of X into a reflexive space (or to uniform embeddability of B_X into a reflexive space)?

Finally we note that Raynaud [**26**] proves that if X is a separable Banach space which uniformly embeds into a super-stable Banach (see [**26**] for the definition) then X contains a copy of some ℓ_p where $1 < p < \infty$. The result of Randrianarivony [**25**] shows that if X coarsely embeds into ℓ_2 then the same conclusion holds. Therefore it is natural to ask for an analog of Raynaud's result for coarse embeddings.

PROBLEM 6.6 If X is a separable Banach space which coarsely embeds into a super-stable Banach space, must X contain a copy of some ℓ_p, where $1 < p < \infty$?

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