UNCONDITIONALITY IN SPACES OF $m$-HOMOGENEOUS POLYNOMIALS

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[Received 22 January 2004]

Abstract

Let $E$ be a Banach space with an unconditional basis. We prove that for $m \geq 2$ the Banach space $P^m(E)$ of all $m$-homogeneous polynomials on $E$ has an unconditional basis if and only if $E$ is finite dimensional. This answers a problem of S. Dineen.

1. Introduction

As usual we denote by $P^m(E)$, $E$ a Banach space and $m$ a natural number, the space of all $m$-homogeneous (scalar-valued and continuous) polynomials $p$ on $E$ which together with the norm $\|p\| := \sup \{|p(x)| : \|x\| \leq 1\}$ forms a Banach space. Recall that a scalar-valued mapping $p$ on $E$ is said to be an $m$-homogeneous polynomial whenever there is some $\varphi \in L^m(E)$ which on its diagonal coincides with $p$; as usual $L^m(E)$ stands for the Banach space of all continuous $m$-linear forms on $E^m$.

A problem of S. Dineen asks whether there exists an infinite-dimensional Banach space $E$ with an unconditional shrinking basis for which $P^m(E)$ for $m \geq 2$ has an unconditional basis. Dineen [13, p. 303] conjectures that this situation is going to happen rarely and perhaps never. The following theorem is our main result.

THEOREM 1.1 Let $E$ be a Banach space with an unconditional basis and $m \geq 2$. Then the Banach space $P^m(E)$ of all $m$-homogeneous polynomials on $E$ has an unconditional basis if and only if $E$ is finite dimensional.

Let us also introduce the space $P_{app}^m(E)$ of all $m$-homogeneous polynomials which are approximable; this is defined as the closed linear span in $P^m(E)$ of all polynomials of the type $p(x) = \prod_{k=1}^m x_k^\alpha(x)$, where $x_1, \ldots, x_m \in E^*$.

Suppose $E$ is a Banach space with a Schauder basis $(e_j)_{j=1}^\infty$ and biorthogonal functionals $(e_j^*)_{j=1}^\infty$.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with order $|\alpha| = m$ we call

$$e_\alpha^*(x) := e_1^\alpha(x)^{\alpha_1} \cdots e_n^\alpha(x)^{\alpha_n}, \quad x \in E,$$

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Quart. J. Math. 56 (2005), 53–64; doi: 10.1093/qmath/hah022

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an \((m\text{-homogeneous})\) monomial on \(E\). If the \(e_j\) are shrinking, then by a result of [19] (see also [12, 13]) the monomials with the so-called square order form a basis of \(P_{\text{app}}\(mE\)). For a reflexive space \(E\) Alencar proved in [2] that the monomials (square order) form a basis of \(P(mE)\) if and only if \(P(mE) = \mathcal{P}_{\text{app}}(mE)\) if and only if \(P(mE)\) is reflexive. See [13] for a collection of results on the reflexivity of spaces of \(m\)-homogeneous polynomials on Banach spaces; for example, a result of Pelczyński [18] from 1957 states that \(P(m\ell_p)\) is reflexive if and only if \(m < p\). As a consequence, the monomials (square order) form a basis of \(P(m\ell_p)\) if and only if \(m < p\).

In [9] the authors undertake a systematic study of Dineen’s problem following a program originally initiated by Gordon and Lewis in [15]. Among other things, it is proved that for each Banach space \(E\) which has a dual with an unconditional basis \((e_j^*)_{j=1}^\infty\), the space \(P_{\text{app}}\(mE\)) has an unconditional basis if and only if its monomials \(e_n^*\) form an unconditional basis; see [9, Corollary 2]. As a consequence asymptotically correct estimates for the unconditional basis constant of all \(m\)-homogeneous polynomials on \(\ell_p^n\) are determined. These results are used to narrow down considerably the list of natural test candidates \(E\) for Dineen’s conjecture (in particular, \(P(mE)\) has no unconditional basis when \(E\) is a super-reflexive space or the original Tsirelson space \(T^*\)). Our proof of the preceding theorem is based on these results.

We also study when \(P(mE)\) is isomorphic to a Banach lattice. For spaces \(E\) with an unconditional basis \((e_j)_{j=1}^\infty\) it turns out that this happens if and only if the monomials \(e_n^*\) form an unconditional basic sequence. It can be seen easily that \(P(m\ell_1)\) is isomorphic to the Banach lattice \(\ell_\infty\). In contrast we here construct an example of a Banach space \(E\) with a symmetric basis which is not isomorphic to \(\ell_1\) but such that \(P(mE)\) is isomorphic to a Banach lattice for every \(m \geq 1\). We conclude with some open problems.

2. Some preliminaries

We shall use standard notation and notions from Banach space theory, as presented, for example, in [6] or [17].

A Banach space \(E\) has cotype \(q\) for \(2 \leq q < \infty\) if there is a constant \(C\) such that

\[
\left( \sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq C \left( \sum_{k=1}^{n} \| e_k x_k \|^q \right)^{1/q}, \quad x_1, \ldots, x_n \in X,
\]

where \((e_1, \ldots, e_n)\) denotes a sequence of mutually independent Rademachers on some probability space.

We say that \(E\) contains uniformly complemented \(\ell_p^n\)s if there exists \(C\) such that for every \(n \in \mathbb{N}\) there are operators \(S_n : \ell_p^n \rightarrow E\) and \(T_n : E \rightarrow \ell_q^n\) with \(T_n S_n = \text{Id}_{\ell_p^n}\) (the identity on \(\ell_p^n\)) and \(\|S_n\|\|T_n\| \leq C\). It is well known that \(E\) has some non-trivial cotype \(q < \infty\) if and only if \(E\) does not contain uniformly complemented \(\ell_p^n\)s [17].

A normalized basic sequence \((e_j)_{j=1}^\infty\) in a Banach space \(E\) is called democratic if there is a constant \(C\) such that if \(A, B\) are finite subsets of \(\mathbb{N}\) with \(|A| \leq |B|\) then

\[
\left\| \sum_{j \in A} e_j \right\| \leq C \left\| \sum_{j \in B} e_j \right\|.
\]

A basic sequence which is both unconditional and democratic is called greedy. In fact, greedy bases were originally defined in terms of approximation rates, and it is a theorem of Konyagin and
Temlyakov [16] that this is equivalent to our definition. We refer to [10, 11] for more information on greedy bases.

If \((e_j)_{j=1}^\infty\) is a greedy basic sequence then we define its fundamental function to be

\[
\phi(n) = \sup \left\{ \left\| \sum_{j \in A} e_j \right\| : |A| \leq n \right\}.
\]

Thus \(\phi\) is increasing and there is a constant \(\Delta\) (the democratic constant) such that for any finite set \(A\)

\[
\Delta^{-1}\phi(|A|) \leq \left\| \sum_{j \in A} e_j \right\| \leq \phi(|A|).
\]

An important principle we shall need is the following special case of [10, Proposition 5.3].

**Proposition 2.1** Suppose \(E\) is a Banach space with non-trivial cotype and \((e_j)_{j=1}^\infty\) is an unconditional basis of \(E\). Then \((e_j)_{j=1}^\infty\) has a subsequence \((e_{jn})_{n=1}^\infty\) which is greedy.

**3. Remarks on a theorem of Tzafriri**

A well-known result of Tzafriri [20] states that each infinite-dimensional Banach space \(X\) with an unconditional basis contains uniformly complemented \(\ell_p^n\)s for some \(p \in \{1, 2, \infty\}\). We shall here modify the proof a little to obtain some additional information on greedy bases.

**Theorem 3.1** Suppose \(E\) has a greedy basis \((e_j)_{j=1}^\infty\) with fundamental function \(\phi\). Suppose \(E\) has non-trivial cotype \(q < \infty\) and that for some \(p > 1\) we have

\[
\liminf_{n \to \infty} n^{-1/p} \phi(n) = 0.
\]

Then \(E\) contains uniformly complemented \(\ell_2^n\)s.

**Proof.** For convenience we suppose the basis is 1-unconditional. Let \(C\) be the cotype \(q\) constant of \(E\) and let \(\Delta\) be the democratic constant; clearly, we may assume that \(q \geq p\). We first remark that if \(|A| = mn\) then by splitting it into \(m\) subsets of size \(n\) we have

\[
\phi(n) \leq C \Delta m^{-1/q} \phi(mn), \quad m, n \in \mathbb{N}.
\]

On the other hand the set \(A\) of all \(n\) such that if \(0 \leq k \leq n\) we have

\[
2^{-k/p} \phi(2^k) \geq 2^{-n/p} \phi(2^n)
\]

is infinite. If \(n \in A\) let \(N = 2^n\). It follows that if \(1 \leq m < N\) then if we choose \(k\) with \(2^k \leq m < 2^{k+1}\) we have

\[
\phi(m) \geq \phi(2^k) \geq 2^{(k-n)/p} \phi(2^n).
\]

Thus we have

\[
\phi(m) \geq \frac{1}{2} \left( \frac{m}{N} \right)^{1/p} \phi(N), \quad 1 \leq m \leq N.
\]
On the other hand similar reasoning shows that (3.1) implies that

$$
\phi(m) \leq 2C \Delta \left( \frac{m}{N} \right) \frac{1}{q} \phi(N), \quad 1 \leq m \leq N.
$$

(3.3)

Now if \( n \in \mathbb{A} \) and \( N = 2^n \) we let \( \Omega \) be the set \( \{1, 2, \ldots, N\} \) equipped with normalized counting measure \( \mu(A) = |A|/N \). Fix \( 1 < r < p < q < s < \infty \). We define a map \( U : L_{rs}(\Omega, \mu) \to E \) by

$$
Uf = \frac{1}{\phi(N)} \sum_{j=1}^{N} f(j) e_j
$$

and a map \( V : X \to L_{r}(\Omega, \mu) \) by

$$
Vx(j) = \phi(N)e_j^*(x), \quad 1 \leq j \leq N.
$$

Let us estimate \( \|U\| \). If \( \|f\|_s \leq 1 \) let \( A_k = \{j : 2k \leq |f(j)| < 2^{k+1}\} \) for \( k \in \mathbb{Z} \). Then by (3.3)

\[
\|Uf\| \leq \sum_{k \in \mathbb{Z}} \|Uf \chi_{A_k}\|
\leq \sum_{k \in \mathbb{Z}} 2^{k+1} \phi(N)^{-1} \|f\| \sum_{j \in A_k} e_j
\leq \sum_{k \in \mathbb{Z}} 2^{k+1} \phi(N)^{-1} \phi(|A_k|)
\leq 4C \Delta \sum_{k \in \mathbb{Z}} 2^{k} \mu(A_k)^{1/q}
\leq 4C \Delta \left( 1 + \sum_{k \geq 0} 2^{k} \mu(A_k)^{1/q} \right).
\]

However,

$$
\sum_{k \geq 0} 2^{k} \mu(A_k)^{1/q} \leq \left( \sum_{k \geq 0} 2^{k} \mu(A_k) \right)^{1/q} \left( \sum_{k \geq 0} 2^{-q'k(s/q-1)} \right)^{1/q'},
$$

where \( q' \) is conjugate to \( q \). Thus \( \|U\| \leq C' \) where

$$
C' = 4C \Delta \left( 1 + \left( \sum_{k \geq 0} 2^{-q'k(s/q-1)} \right)^{1/q'} \right).
$$

The estimate for \( V \) is similar. Suppose \( \|x\| = 1 \) and \( A_k = \{j : 2k \leq \phi(N)|e_j^*(x)| < 2^{k+1}\} \) for \( k \in \mathbb{Z} \). Then

$$
\left| \sum_{j \in A_k} e_j \right| \leq \left| \sum_{j \in A_k} \phi(N) \frac{1}{2^k} |e_j^*(x)| e_j \right| \leq 2^{-k} \phi(N)
$$
and hence $\phi(|A_k|) \leq \Delta 2^{-k} \phi(N)$. Then together with (3.2) this yields

$$\|Vx\|_r \leq 2 \left( \sum_{k \in \mathbb{Z}} 2^{kr} \mu(A_k) \right)^{1/r} \leq 2 + 2 \left( \sum_{k \geq 0} 2^{kr} \mu(A_k) \right)^{1/r} \leq 2 + 2^{1+p/r} \left( \sum_{k \geq 0} 2^{kr} (\phi(|A_k|))^{p/(p-1)} \right)^{1/r} \leq 2 + 2^{1+p/r} \Delta^{p/r} \left( \sum_{k \geq 0} 2^{k(r-p)} \right)^{1/r} = C'$$

say.

Now since $N = 2^n$ we can identify $\Omega$ with $\{-1, +1\}^n$ and thus find $n$ Rademacher functions $\epsilon_1, \ldots, \epsilon_n$ on $\Omega$. Define $L : \ell_2^n \rightarrow L_s(\Omega, \mu)$ by $L(\xi) = \sum_{k=1}^n \xi_k \epsilon_k$ and $R : L_s(\Omega, \mu) \rightarrow \ell_2^n$ by $Rf = (\int f, d\mu)^{\otimes n}_{\otimes 1}$ and both $\|L\|$, $\|R\|$ are uniformly bounded independent of $n$. If we define $S = UL$ and $T = RV$ then $\|T\| \|S\|$ is uniformly bounded independent of $n$ and $TS = 1d_{\ell_2^n}$.

**Proposition 3.2** Suppose $E$ has an unconditional basis and $m \geq 2$. If $\mathcal{P}(mE)$ is separable then either $E$ contains uniformly complemented $\ell_2^n$s or $E$ contains uniformly complemented $\ell_\infty^n$s.

**Proof.** Assume that $E$ neither contains uniformly complemented $\ell_2^n$s nor contains uniformly complemented $\ell_\infty^n$s. Then $E$ has cotype and by Proposition 2.1 $E$ has a complemented subspace $F$ with a greedy basis $(e_j)_{j=1}^\infty$ and biorthogonal functionals $(e_j^*)_F = (e_j)^{\otimes n}_{\otimes 1}$. We may assume $(e_j)_{j=1}^\infty$ is 1-unconditional. Then $\mathcal{P}(mF)$ is also separable. Pick $1 < p < m$. Then by Theorem 3.1 the fundamental function $\phi$ satisfies $\phi(n) \geq cn^{1/p}$ for some $c > 0$. Now if $x \in F$ with $\|x\| = 1$, let $A_k = \{ j : 2^k \leq |e_j^*(x)| < 2^{k+1} \}$. Then $\phi(A_k) \leq \Delta 2^{-k}$ where $\Delta$ is the democratic constant. We have

$$\sum_{j=1}^\infty |e_j^*(x)|^m \leq 2^m \sum_{k \leq 0} 2^{mk} |A_k| \leq 2^m c^{-1} \sum_{k \leq 0} 2^{mk} \phi(|A_k|)^p \leq 2^m c^{-1} \Delta \sum_{k \leq 0} 2^{(m-p)k}.$$ 

Thus the series $\sum_{j=1}^\infty \delta_j |e_j^*(x)|^m$ converges pointwise in $\mathcal{P}(mF)$ for any choice of signs $\delta_j = \pm 1$ and it is easily seen that this then defines an uncountable 1-separated set, contradicting separability.

**Proof of Theorem 1.1.** Suppose $E$ is infinite-dimensional. If $\mathcal{P}(mE)$ has an unconditional basis (where $m \geq 2$) then by Proposition 3.2 it follows that either $E$ contains uniformly complemented $\ell_2^n$s or $E$ contains uniformly complemented $\ell_\infty^n$s. Now by [9, Corollary 4] we are done.
4. \( \mathcal{P}(mE) \) as a Banach lattice

If \( E = \ell_1 \) then the space \( \mathcal{L}_m(E) \) of bounded \( m \)-linear forms is isometric to \( \ell_\infty \) and it follows that \( \mathcal{P}(mE) \) (which is isomorphic to a complemented subspace of \( \mathcal{L}_m(E) \)) is then also isomorphic to \( \ell_\infty \) and is thus isomorphic to a Banach lattice.

**Proposition 4.1** Let \( E \) be a Banach space with an unconditional basis \( (e_j)_{j=1}^\infty \) and biorthogonal functionals \( (e_j^*)_{j=1}^\infty \). Then for each \( m \) the following are equivalent.

1. The monomials \( (e_j^*) \) form an unconditional basic sequence in \( \mathcal{P}(mE) \).

2. \( \mathcal{P}(mE) \) is isomorphic to a Banach lattice.

**Proof.** Suppose we have (1). We may suppose that \( (e_j)_{j=1}^\infty \) is a 1-unconditional basis. Let \( S_n \) denote the partial sum projections \( S_n x = \sum_{k=1}^n e_k^*(x) e_k \). Then for \( p \in \mathcal{P}(mE) \) we have \( p \circ S_n \in \mathcal{P}_{\text{app}}(mE) \) and for each multi-index \( \alpha \) with \( |\alpha| = m \) we can define \( \hat{p}(\alpha) \) so that

\[
p \circ S_n = \sum_{\alpha \leq n} \hat{p}(\alpha) e_\alpha^*,
\]

where \( \alpha \leq n \) means that \( \alpha(k) = 0 \) for \( k > n \). It is clear that

\[
\| p \| = \sup_n \| p \circ S_n \|.
\]

Conversely, if \( (\hat{p}(\alpha))_{|\alpha|=m} \) are scalars such that

\[
\sup_n \left\| \sum_{\alpha \leq n} \hat{p}(\alpha) e_\alpha^* \right\|_{\mathcal{P}(mE)} < \infty
\]

then we can define \( p \in \mathcal{P}(mE) \) by

\[
p(x) = \lim_{n \to \infty} \sum_{\alpha \leq n} \hat{p}(\alpha) e_\alpha^*(x), \quad x \in E.
\]

Thus the map \( p \to (p(\alpha))_{|\alpha|=m} \) gives \( \mathcal{P}(mE) \) the structure of a Banach lattice.

Conversely, assume (2). Then we show that for each \( n \) the finite sequence \( (e_\alpha^*)_{\alpha \leq n} \) has a bounded unconditional basis constant that is uniformly bounded in \( n \). Indeed, if \( E_n = [e_j]_{j=1}^n \) the spaces \( \mathcal{P}(mE_n) \) are 1-complemented in \( \mathcal{P}(mE) \) by the projections \( p \to p \circ S_n \). We may then use [9, Theorem 2].

We next construct a Banach space with a symmetric basis which is not isomorphic to \( \ell_1 \) but such that the equivalent conditions of Proposition 4.1 hold for every \( m \in \mathbb{N} \).

Let us choose an increasing sequence of natural numbers \( (a_r)_{r=0}^\infty \) with \( a_0 = 1 \) and for \( r = 1, 2, \ldots \), \( a_r > 3^{r+1} a_{r-1} \). Then we define \( w_1 = 1 \) and then \( w_k = 2^{-r} \) if \( a_{r-1} < k \leq a_r \). Consider the Lorentz sequence space \( d(w, 1) \) consisting of all sequences \( (\xi_k)_{k=1}^\infty \) such that

\[
\| \xi \| = \sup \sum_{k=1}^\infty w_k |\xi_k| < \infty.
\]
where \( \pi \) runs through all permutations of \( \mathbb{N} \). See [17, pp. 175ff] for background on such Lorentz sequence spaces; note that by [17, Theorem 4.e.2] this space is also an Orlicz sequence space.

Let us denote the canonical basis of \( d(w, 1) \) by \( (e_n)_{n=1}^\infty \). The fundamental function for \( d(w, 1) \) is given by \( \phi(n) = \sum_{k=1}^n w_k \).

For \( A \subset \mathbb{N} \) define \( \Sigma_r(A) \) to be the collection of all elements \( \xi \in d(w, 1) \) of the form
\[
\xi = 2^r a_r^{-1} \sum_{k \in B} e_k e_k, \quad e_k = \pm 1, \quad |B| = a_r, \quad B \subset A.
\]

Observe that \( \Sigma_r(A) = \emptyset \) if \( |A| < a_r \) and that if \( |A| = N \geq a_r \) then \( |\Sigma_r(A)| = \left( \frac{N}{a_r} \right) 2^a_r \). Let \( \Sigma(A) = \bigcup_{r \geq 0} \Sigma_r(A) \). Then if \( |A| = N \) we have
\[
|\Sigma(A)| \leq \sum_{k=0}^N \left( \frac{N}{k} \right) 2^k = 3^N. \tag{4.1}
\]

**Lemma 4.2** (1) For \( r \geq 0 \) we have \( 2^{-r} a_r \leq \phi(a_r) \leq 2 \times 2^{-r} a_r \).

(2) Suppose \( \xi^* \in d(w, 1)^* \). Then
\[
\frac{1}{2} \|\xi^*\| \leq \sup_{\xi \in \Sigma(N)} \xi^*(\xi) \leq 2\|\xi^*\|.
\]

(3) For each \( \varphi \in \mathcal{L}_m(d(w, 1)) \) we have
\[
\frac{1}{2^m} \sup_{a_j \in \Sigma(N)} |\varphi(u_1, \ldots, u_m)| \leq \|\varphi\| \leq \left( \frac{1}{2^m} \right) \sup_{a_j \in \Sigma(N)} |\varphi(u_1, \ldots, u_m)|.
\]

**Proof.** We first observe that \( 2^{-r} a_r \leq \phi(a_r) \). Next by induction we see that \( \phi(a_r) \leq 2 \times 2^{-r} a_r \). Indeed this is trivially true when \( r = 0 \) and then if we assume it is true for \( r-1 \) we have
\[
\phi(a_r) = \phi(a_{r-1}) + 2^{-r} (a_r - a_{r-1})
\]
so that, since \( a_{r-1}/a_r < \frac{1}{2} \),
\[
\phi(a_r) = a_{r-1} a_r^{-1} \phi(a_{r-1}) + \left( \frac{1}{a_r} - \frac{1}{a_{r-1}} \right) 2^{-r}
\]
\[
\leq \frac{4 \times 2^{-r}}{3} + \frac{2 \times 2^{-r}}{3} = 2 \times 2^{-r}.
\]

Now suppose \( \xi^* \in d(w, 1)^* \) is such that
\[
\sup_{\xi \in \Sigma(N)} \xi^*(\xi) = 1.
\]
Without loss of generality we may suppose that if \( b_j = \xi^*(e_j) \) then \( (b_j)_{j=1}^\infty \) is a decreasing non-negative sequence so that
\[
\sup_r 2^{r} a_r^{-1} \sum_{j=1}^{a_r} b_j = 1.
\]
Then if \( a_{r-1} < n \leq a_r \) we have
\[
\frac{1}{n} \sum_{j=1}^{n} b_j \leq 2 \times 2^{-r} \leq 2 \frac{\phi(a_r)}{a_r} \leq 2 \frac{\phi(n)}{n}.
\]

Thus if \( \xi = \sum_{j=1}^{\infty} \xi_j e_j \) with \( (\xi_j) \) non-negative and decreasing,
\[
\xi^*(\xi) = \sum_{j=1}^{\infty} b_j \xi_j = \sum_{j=1}^{\infty} (b_1 + \cdots + b_j)(\xi_j - \xi_{j+1}) \leq 2 \sum_{j=1}^{\infty} \phi(j)(\xi_j - \xi_{j+1}) = 2 \sum_{j=1}^{\infty} w_j \xi_j.
\]

Thus \( \|\xi^*\| \leq 2 \).

On the other hand if \( \xi \in \Sigma_r(\mathbb{N}) \) then \( \|\xi\| \leq \phi(a_r)2^r a_r^{-1} \leq 2 \) so that \( \|\xi^*\| \geq \frac{1}{2} \). Finally, (3) is a straightforward consequence of (2).

**Theorem 4.3** For every \( m \in \mathbb{N} \) the monomials \( (e^*_i)_a \) form an unconditional basic sequence in \( \mathcal{P}_{\text{app}}(m d(w, 1)) \), and hence \( \mathcal{P}(m d(w, 1)) \) is isomorphic to a Banach lattice.

**Proof.** It will suffice to show that the elements \( e^*_1 \otimes \cdots \otimes e^*_m \) form an unconditional basic sequence in \( \mathcal{L}_m(d(w, 1)) \) for every choice of \( m \). Indeed the monomials in \( \mathcal{P}(m d(w, 1)) \) are equivalent to an unconditional block basic sequence of this basis.

More precisely we show by induction that there is a constant \( C_m \) such that if \( \psi \) is an \( m \)-linear form given by
\[
\psi(x_1, \ldots, x_m) = \sum_{i_1, \ldots, i_m} b_{i_1, \ldots, i_m} e^*_i(x_1) \cdots e^*_m(x_m),
\]
where the array \( (b_{i_1, \ldots, i_m}) \) is finitely non-zero and if
\[
||\psi||_{(x_1, \ldots, x_m)} = \sum_{i_1, \ldots, i_m} |b_{i_1, \ldots, i_m}| |e^*_i(x_1) \cdots e^*_m(x_m)|
\]
then \( ||\psi|| \leq C_m ||\psi|| \).

The case \( m = 1 \) is trivial and indeed \( C_1 = 1 \). Let us now suppose the theorem is proved for \( k < m \). We shall assume \( ||\psi|| = 1 \) and let \( ||\psi|| = M \). Then by Lemma 4.2 we can find \( u_j \in \Sigma_{r_j}(\mathbb{N}) \) for \( 1 \leq j \leq m \) such that
\[
||\psi(u_1, \ldots, u_m) \geq 2^{-m} M.
\]

In fact each \( u_j \) can be taken of the form
\[
u_j = \frac{2r_j}{a_{r_j}} \sum_{k \in B_j} e_k,
\]
where \( |B_j| = a_{r_j} \).
By reordering if necessary we shall assume that \( r_m = \max_{1 \leq j \leq m} r_j \). Let us consider the case when \( r_m < m - 1 \). In this case

\[
M \leq 2^{m + m r_m} \max |\varphi(e_{i_1}, \cdots, e_{i_m})| \leq 2^{m^2}.
\]

We continue with the assumption that \( r_m \geq m - 1 \). If \( 1 \leq j < m \) and \( r_j = r_m \) we can write

\[
u_j = \frac{2^{r_m}}{a_{r_m}} \sum_{k \in B_j} e_k = 2 \left( \frac{a_{r_m}}{a_{r_m-1}} \right)^{-1} \sum_{D \subset B_j, |D| = a_{r_m-1}} 2^{r_m-1} a_{r_m-1}^{-1} \sum_{k \in D} e_k.
\]

Expanding each such \( u_j \) out in this way we see that we can find \( B_j \) with \( |B_j| = a_{r_j} \), where \( r_j < r_m \) if \( j < m \) and such that if \( v_j = \frac{2^{r_j}}{a_{r_j}} \sum_{k \in B_j} e_k \) then

\[
|\varphi|(v_1, \cdots, v_m) \geq 2 - (m-1) \sum_{k \in D_j} e_k.
\]

Now, for each \( k \in B_m \) we define \( \psi_k \in \mathcal{L}_{m-1}(d(w, 1)) \) by

\[
\psi_k(x_1, \cdots, x_{m-1}) = \sum_{i_1 \in B_1} \cdots \sum_{i_{m-1} \in B_{m-1}} b_{i_1, \cdots, i_{m-1}, k} e_{i_1}^*(x_1) \cdots e_{i_{m-1}}^*(x_{m-1}).
\]

For each \( k \in B_m \) there exists at least one \( (\xi_1, \cdots, \xi_{m-1}) \in \Sigma(B_1) \times \cdots \times \Sigma(B_{m-1}) \) so that

\[
\psi_k(\xi_1, \cdots, \xi_{m-1}) \geq 2^{-(m-1)} \|\psi_k\|.
\]

It follows that we can partition \( B_m \) into subsets \( D_1, \cdots, D_N \), where by (4.1) and since all \( r_j \leq r_m - 1 \)

\[
N \leq 3^{a_1 + \cdots + a_{m-1}} \leq 3^{(m-1)a_{r_m-1}}
\]

so that for each \( j \) there exists a choice \( (\xi_1, \cdots, \xi_{m-1}) \in \Sigma(B_1) \times \cdots \times \Sigma(B_{m-1}) \) with

\[
\psi_k(\xi_1, \cdots, \xi_{m-1}) \geq 2^{-(m-1)} \|\psi_k\|, \quad k \in D_j.
\]

Let \( |D_j| = s_j \). By Lemma 4.2 we have \( \|\xi_j\| \leq 2 \) for \( 1 \leq j \leq m \), hence

\[
2^{-(m-1)} \sum_{k \in D_j} \|\psi_k\| \leq \varphi \left( \xi_1, \cdots, \xi_{m-1} \sum_{k \in D_j} e_k \right) \leq 2^{m-1} \phi(s_j).
\]

By the inductive hypothesis we have \( \|\psi_k\| \leq C_{m-1} \|\psi_k\| \). Returning to (4.3) we have (again
noting that each \( \|v_j\| \leq 2 \)

\[
M \leq 2^{2m} |\psi|(v_1, \ldots, v_m)
\]
\[
\leq 2^{2m + r_m a_{r_m}^{-1}} \sum_{k \in B_m} |\psi_k|(v_1, \ldots, v_{m-1})
\]
\[
\leq C_{m-1} 2^{3m-1 + r_m a_{r_m}^{-1}} \sum_{k \in B_m} \|\psi_k\|
\]
\[
= C_{m-1} 2^{3m-1 + r_m a_{r_m}^{-1}} \sum_{j=1}^{N} \sum_{k \in D_j} \|\psi_k\|
\]
\[
\leq C_{m-1} 2^{5m-3 + r_m a_{r_m}^{-1}} \sum_{j=1}^{N} \phi(s_j).
\]

Finally, we estimate \( \sum_{j=1}^{N} \phi(s_j) \). If \( r \geq 1 \) and \( a_{r-1} < s_j \leq a_r \) then \( \phi(s_j) \leq s_j a_{r-1}^{-1} \phi(a_{r-1}) \leq 4 \times 2^{-r} s_j \) (by Lemma 4.2 and the fact that \( \phi(n)/n \) is decreasing in \( n \)). Thus

\[
\sum_{a_{r-1} < s_j \leq a_r} \phi(s_j) \leq 4 \times 2^{-r} \sum_{a_{r-1} < s_j \leq a_r} s_j. \tag{4.5}
\]

Define \( \sigma_r := |\{j : a_{r-1} < s_j \leq a_r\}| \) and notice that \( \sum_{j=1}^{N} \phi(s_j) = \sigma_0 \), where \( \sigma_0 = |\{j : s_j = 1\}| \). Then by (4.5) we have

\[
\sum_{a_{r-1} < s_j \leq a_r} \phi(s_j) \leq 4 \times 2^{-r} \sigma_r a_r.
\]

Now if \( r \leq r_m - 1 \) we have \( 2^{-r} a_r \leq 2^{1-r_m} a_{r_m-1} \) (use again \( a_r/a_{r+1} < \frac{1}{2} \), and as a consequence from (4.4)

\[
\sum_{s_j \leq a_{r_m-1}} \phi(s_j) \leq 2^{3-r_m} a_{r_m-1} \sum_{r=0}^{r_m-1} \sigma_r \leq 3^{(m-1)(a_{r_m-1})} 2^{3-r_m} a_{r_m-1}.
\]

Hence as \( r_m \geq m - 1 \) we deduce from the defining property of the \( a_r \)s that

\[
\sum_{s_j \leq a_{r_m-1}} \phi(s_j) \leq 2^{3-r_m} a_{r_m}.
\]

On the other hand, by (4.5),

\[
\sum_{a_{r_m-1} < s_j \leq a_{r_m}} \phi(s_j) \leq 4 \times 2^{-r_m} \sum_{a_{r_m-1} < s_j \leq a_{r_m}} s_j \leq 2^{2-r_m} a_{r_m}
\]

(recall that the sum over all \( s_j \) equals \( a_{r_m} \)). Combining we have

\[
\sum_{j=1}^{N} \phi(s_j) \leq 2^{4-r_m} a_{r_m}
\]

and hence

\[
M \leq C_{m-1} 2^{5m+1} \tag{4.6}
\]

Combining (4.2) and (4.6) we have \( C_m \leq \max(2^{5m+1} C_{m-1}, 2^{m^2}) \) and this completes the proof.
5. Some related open problems

The space created in Theorem 4.3 is not reflexive. We therefore ask the following.

Let $E$ be a reflexive Banach space with an unconditional basis and suppose $m \geq 2$. Can $\mathcal{P}(^m E)$ be a Banach lattice?

Notice in this situation Proposition 4.1 implies that $\mathcal{P}(^m E)$ is isomorphic to a Banach lattice if and only if $\mathcal{P}_{\text{app}}(^m E)$ has an unconditional basis.

Finally, we relate our study of unconditionality in spaces of $m$-homogeneous polynomials with complex analysis. Let $E = (\mathbb{C}^n, \| \cdot \|)$ be a finite-dimensional Banach space such that its canonical basis vectors $e_k$ form a normalized 1-unconditional basis. The Bohr radius of its open unit ball $B_E$ is defined to be

$$K(B_E) := \sup r,$$

where the supremum is taken over all $0 \leq r < 1$ such that whenever the power series $\sum a_\alpha z^{\alpha}$ satisfies $|\sum a_\alpha z^{\alpha}| \leq 1$ for all $z \in B_E$, it follows that $|\sum a_\alpha z^{\alpha}| \leq 1$ for all $z \in rB_E$.

In this notation Bohr’s power series theorem from [5] states that the Bohr radius of the open unit disc in $\mathbb{C}$ equals $1/3$, $K(B_\mathbb{C}) = 1/3$.

Upper and lower estimates for Bohr radii in higher dimensions show two in a sense extreme cases. The sequence $(K(B_{\ell^n_\infty}))$ of the Bohr radii of the $n$-dimensional polydiscs tends to zero essentially like $\sqrt{\log n}/n$, whereas the sequence $(K(B_{\ell^n_1}))$ of the Bohr radii of the $n$-dimensional hypercones is uniformly bounded from below by some strictly positive constant. More precisely, there is a constant $c > 0$ such that for each $n \geq 2$

$$\frac{1}{c \sqrt{\log n}} \sqrt{\frac{\log n}{n}} \leq K(B_{\ell^n_\infty}) \leq c \sqrt{\frac{\log n}{n}}$$

(see [4, 14] for the upper estimate and [7] for the lower one) and

$$\frac{1}{c} \leq K(B_{\ell^n_1}) \leq c$$

(a result of [1]). See [3, 7] for the asymptotic behaviour of the whole scale of sequences $(K(B_{\ell^n_p}))$, $1 < p < \infty$, and [8] for an extension of these estimates within the framework of local Banach space theory.

There is a basic link to unconditional basis constants of spaces of $m$-homogeneous polynomials [8, Theorem 2.2]. Define

$$r(E) := \sup_{m} \chi_{\text{mon}}(\mathcal{P}(^m E))^{1/m},$$

where $\chi_{\text{mon}}(\mathcal{P}(^m E))$ stands for the unconditional basis constant of the monomials in $\mathcal{P}(^m E)$. Then

$$\frac{1}{3} \frac{1}{r(E)} \leq K(B_E) \leq \min \left( \frac{1}{3}, \frac{1}{r(E)} \right)$$

(for $E = \mathbb{C}$ this is obviously Bohr’s result).

In view of this link the following problem seems to be a sort of uniform analogue of Dineen’s problem. Let $E$ be a Banach sequence space (that is, $\ell_1 \subset E \subset c_0$ and the $e_k$’s form a 1-unconditional basis of $E$), and let $E_n = [e_k]_{k=1}^n$.

Does $E$ necessarily equal $\ell_1$ whenever $\inf_n K(B_{E_n}) > 0$ or, equivalently, is $E = \ell_1$ whenever there is some constant $C > 0$ such that $\chi_{\text{mon}}(\mathcal{P}(^m E_n)) \leq C^n$ for all $n$ and $m$?
Acknowledgement
The second author acknowledges support from NSF grant DMS-0244515.

References