UNCONDITIONALITY IN SPACES OF m-HOMOGENEOUS POLYNOMIALS

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Abstract

Let E be a Banach space with an unconditional basis. We prove that for $m \geqslant 2$ the Banach space $\mathcal{P}(^m E)$ of all m-homogeneous polynomials on E has an unconditional basis if and only if E is finite dimensional. This answers a problem of S. Dineen.

1. Introduction

As usual we denote by $\mathcal{P}(^m E)$, E a Banach space and m a natural number, the space of all m-homogeneous (scalar-valued and continuous) polynomials p on E which together with the norm $\|p\| := \sup_{\|x\|\leqslant 1} |p(x)|$ forms a Banach space. Recall that a scalar-valued mapping p on E is said to be an m-homogeneous polynomial whenever there is some $\varphi \in \mathcal{L}_m(E)$ which on its diagonal coincides with p; as usual $\mathcal{L}_m(E)$ stands for the Banach space of all continuous m-linear forms on E^m.

A problem of S. Dineen asks whether there exists an infinite-dimensional Banach space E with an unconditional shrinking basis for which $\mathcal{P}(^m E)$ for $m \geqslant 2$ has an unconditional basis. Dineen [**13**, p. 303] conjectures that this situation *is going to happen rarely and perhaps never*. The following theorem is our main result.

THEOREM 1.1 *Let E be a Banach space with an unconditional basis and $m \geqslant 2$. Then the Banach space $\mathcal{P}(^m E)$ of all m-homogeneous polynomials on E has an unconditional basis if and only if E is finite dimensional.*

Let us also introduce the space $\mathcal{P}_{\text{app}}(^m E)$ of all m-homogeneous polynomials which are *approximable*; this is defined as the closed linear span in $\mathcal{P}(^m E)$ of all polynomials of the type $p(x) = \prod_{k=1}^m x_k^*(x)$, where $x_1^*, \cdots, x_m^* \in E^*$.

Suppose E is a Banach space with a Schauder basis $(e_j)_{j=1}^\infty$ and biorthogonal functionals $(e_j^*)_{j=1}^\infty$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with order $|\alpha| = m$ we call

$$e_\alpha^*(x) := e_1^*(x)^{\alpha_1} \ldots e_n^*(x)^{\alpha_n}, \quad x \in E,$$

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an (m-homogeneous) monomial on E. If the e_j are shrinking, then by a result of [**19**] (see also [**12**, **13**]) the monomials with the so-called square order form a basis of $\mathcal{P}_{\rm app}(^m E)$. For a reflexive space E Alencar proved in [**2**] that the monomials (square order) form a basis of $\mathcal{P}(^m E)$ if and only if $\mathcal{P}(^m E) = \mathcal{P}_{\rm app}(^m E))$ if and only if $\mathcal{P}(^m E)$ is reflexive. See [**13**] for a collection of results on the reflexivity of spaces of m-homogeneous polynomials on Banach spaces; for example, a result of Pełczyński [**18**] from 1957 states that $\mathcal{P}(^m \ell_p)$ is reflexive if and only if $m < p$. As a consequence, the monomials (square order) form a basis of $\mathcal{P}(^m \ell_p)$ if and only if $m < p$.

In [**9**] the authors undertake a systematic study of Dineen's problem following a program originally initiated by Gordon and Lewis in [**15**]. Among other things, it is proved that for each Banach space E which has a dual with an unconditional basis $(e_j^*)_{j=1}^\infty$, the space $\mathcal{P}_{\rm app}(^m E)$ has an unconditional basis if and only if its monomials e_α^* form an unconditional basis; see [**9**, Corollary 2]. As a consequence asymptotically correct estimates for the unconditional basis constant of all m-homogeneous polynomials on ℓ_p^n are determined. These results are used to narrow down considerably the list of natural test candidates E for Dineen's conjecture (in particular, $\mathcal{P}(^m E)$ has no unconditional basis when E is a super-reflexive space or the original Tsirelson space T^*). Our proof of the preceding theorem is based on these results.

We also study when $\mathcal{P}(^m E)$ is isomorphic to a Banach lattice. For spaces E with an unconditional basis $(e_j)_{j=1}^\infty$ it turns out that this happens if and only if the monomials e_α^* form an unconditional basic sequence. It can be seen easily that $\mathcal{P}(^m \ell_1)$ is isomorphic to the Banach lattice ℓ_∞. In contrast we here construct an example of a Banach space E with a symmetric basis which is not isomorphic to ℓ_1 but such that $\mathcal{P}(^m E)$ is isomorphic to a Banach lattice for every $m \geqslant 1$. We conclude with some open problems.

2. Some preliminaries

We shall use standard notation and notions from Banach space theory, as presented, for example, in [**6**] or [**17**].

A Banach space E has cotype q for $2 \leqslant q < \infty$ if there is a constant C such that

$$\left(\sum_{k=1}^n \|x_k\|^q\right)^{1/q} \leqslant C\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^q\right)^{1/q}, \qquad x_1, \cdots, x_n \in X,$$

where $(\epsilon_1, \cdots, \epsilon_n)$ denotes a sequence of mutually independent Rademachers on some probability space.

We say that E contains uniformly complemented ℓ_p^ns if there exists C such that for every $n \in \mathbb{N}$ there are operators $S_n : \ell_p^n \to E$ and $T_n : E \to \ell_p^n$ with $T_n S_n = \mathrm{Id}_{\ell_p^n}$ (the identity on ℓ_p^n) and $\|S_n\|\|T_n\| \leqslant C$. It is well known that E has some non-trivial cotype $q < \infty$ if and only if E does not contain uniformly complemented ℓ_∞^ns [**17**].

A normalized basic sequence $(e_j)_{j=1}^\infty$ in a Banach space E is called *democratic* if there is a constant C such that if A, B are finite subsets of $\mathbb{N}$ with $|A| \leqslant |B|$ then

$$\left\|\sum_{j\in A} e_j\right\| \leqslant C\left\|\sum_{j\in B} e_j\right\|.$$

A basic sequence which is both unconditional and democratic is called *greedy*. In fact, greedy bases were originally defined in terms of approximation rates, and it is a theorem of Konyagin and

Temlyakov [**16**] that this is equivalent to our definition. We refer to [**10**, **11**] for more information on greedy bases.

If $(e_j)_{j=1}^{\infty}$ is a greedy basic sequence then we define its fundamental function to be

$$\phi(n) = \sup\left\{ \left\| \sum_{j\in A} e_j \right\| : \ |A| \leqslant n \right\}.$$

Thus ϕ is increasing and there is a constant Δ (the democratic constant) such that for any finite set A

$$\Delta^{-1}\phi(|A|) \leqslant \left\| \sum_{j\in A} e_j \right\| \leqslant \phi(|A|).$$

An important principle we shall need is the following special case of [**10**, Proposition 5.3].

PROPOSITION 2.1 *Suppose E is a Banach space with non-trivial cotype and $(e_j)_{j=1}^{\infty}$ is an unconditional basis of E. Then $(e_j)_{j=1}^{\infty}$ has a subsequence $(e_{j_n})_{n=1}^{\infty}$ which is greedy.*

3. Remarks on a theorem of Tzafriri

A well-known result of Tzafriri [**20**] states that each infinite-dimensional Banach space X with an unconditional basis contains uniformly complemented ℓ_p^ns for some $p \in \{1, 2, \infty\}$. We shall here modify the proof a little to obtain some additional information on greedy bases.

THEOREM 3.1 *Suppose E has a greedy basis $(e_j)_{j=1}^{\infty}$ with fundamental function ϕ. Suppose E has non-trivial cotype $q < \infty$ and that for some $p > 1$ we have*

$$\liminf_{n\to\infty} n^{-1/p}\phi(n) = 0.$$

Then E contains uniformly complemented ℓ_2^ns.

Proof. For convenience we suppose the basis is 1-unconditional. Let C be the cotype q constant of E and let Δ be the democratic constant; clearly, we may assume that $q \geqslant p$. We first remark that if $|A| = mn$ then by splitting it into m subsets of size n we have

$$\phi(n) \leqslant C\Delta m^{-1/q}\phi(mn), \qquad m, n \in \mathbb{N}. \tag{3.1}$$

On the other hand the set $\mathbb{A}$ of all n such that if $0 \leqslant k \leqslant n$ we have

$$2^{-k/p}\phi(2^k) \geqslant 2^{-n/p}\phi(2^n)$$

is infinite. If $n \in \mathbb{A}$ let $N = 2^n$. It follows that if $1 \leqslant m < N$ then if we choose k with $2^k \leqslant m < 2^{k+1}$ we have

$$\phi(m) \geqslant \phi(2^k) \geqslant 2^{(k-n)/p}\phi(2^n).$$

Thus we have

$$\phi(m) \geqslant \frac{1}{2}\left(\frac{m}{N}\right)^{1/p}\phi(N), \qquad 1 \leqslant m \leqslant N. \tag{3.2}$$

On the other hand similar reasoning shows that (3.1) implies that

$$\phi(m) \leqslant 2C\Delta\left(\frac{m}{N}\right)^{1/q}\phi(N), \qquad 1 \leqslant m \leqslant N. \tag{3.3}$$

Now if $n \in \mathbb{A}$ and $N = 2^n$ we let Ω be the set $\{1, 2, \cdots, N\}$ equipped with normalized counting measure $\mu(A) = |A|/N$. Fix $1 < r < p \leqslant q < s < \infty$. We define a map $U : L_s(\Omega, \mu) \to E$ by

$$Uf = \frac{1}{\phi(N)}\sum_{j=1}^{N} f(j)e_j$$

and a map $V : X \to L_r(\Omega, \mu)$ by

$$Vx(j) = \phi(N)e_j^*(x), \qquad 1 \leqslant j \leqslant N.$$

Let us estimate $\|U\|$. If $\|f\|_s \leqslant 1$ let $A_k = \{j : 2^k \leqslant |f(j)| < 2^{k+1}\}$ for $k \in \mathbb{Z}$. Then by (3.3)

$$\begin{aligned}\|Uf\| &\leqslant \sum_{k\in\mathbb{Z}} \|Uf\chi_{A_k}\| \\ &\leqslant \sum_{k\in\mathbb{Z}} 2^{k+1}\phi(N)^{-1}\|\sum_{j\in A_k} e_j\| \\ &\leqslant \sum_{k\in\mathbb{Z}} 2^{k+1}\phi(N)^{-1}\phi(|A_k|) \\ &\leqslant 4C\Delta\sum_{k\in\mathbb{Z}} 2^k\mu(A_k)^{1/q} \\ &\leqslant 4C\Delta\Big(1 + \sum_{k\geqslant 0} 2^k\mu(A_k)^{1/q}\Big).\end{aligned}$$

However,

$$\sum_{k\geqslant 0} 2^k\mu(A_k)^{1/q} \leqslant \Big(\sum_{k\geqslant 0} 2^{ks}\mu(A_k)\Big)^{1/q}\Big(\sum_{k\geqslant 0} 2^{-q'k(s/q-1)}\Big)^{1/q'},$$

where q' is conjugate to q. Thus $\|U\| \leqslant C'$ where

$$C' = 4C\Delta\bigg(1 + \Big(\sum_{k\geqslant 0} 2^{-q'k(s/q-1)}\Big)^{1/q'}\bigg).$$

The estimate for V is similar. Suppose $\|x\| = 1$ and $A_k = \{j : 2^k \leqslant \phi(N)|e_j^*(x)| < 2^{k+1}\}$ for $k \in \mathbb{Z}$. Then

$$\Big\|\sum_{j\in A_k} e_j\Big\| \leqslant \Big\|\sum_{j\in A_k}\frac{\phi(N)}{2^k}|e_j^*(x)|e_j\Big\| \leqslant 2^{-k}\phi(N)$$

and hence $\phi(|A_k|) \leqslant \Delta 2^{-k}\phi(N)$. Then together with (3.2) this yields

$$\begin{aligned}\|Vx\|_r &\leqslant 2\Big(\sum_{k\in\mathbb{Z}} 2^{kr}\mu(A_k)\Big)^{1/r}\\ &\leqslant 2+2\Big(\sum_{k\geqslant 0} 2^{kr}\mu(A_k)\Big)^{1/r}\\ &\leqslant 2+2^{1+p/r}\Big(\sum_{k\geqslant 0} 2^{kr}(\phi(|A_k|))^p(\phi(N))^{-p}\Big)^{1/r}\\ &\leqslant 2+2^{1+p/r}\Delta^{p/r}\Big(\sum_{k\geqslant 0} 2^{k(r-p)}\Big)^{1/r}\\ &= C'',\end{aligned}$$

say.

Now since $N = 2^n$ we can identify Ω with $\{-1,+1\}^n$ and thus find n Rademacher functions $\epsilon_1, \cdots, \epsilon_n$ on Ω. Define $L : \ell_2^n \to L_s(\Omega,\mu)$ by $L(\xi) = \sum_{k=1}^n \xi_k\epsilon_k$ and $R : L_r(\Omega,\mu) \to \ell_2^n$ by $Rf = (\int f\epsilon_k\, d\mu)_{k=1}^n$ and both $\|L\|, \|R\|$ are uniformly boundedly independent of n. If we define $S = UL$ and $T = RV$ then $\|T\|\|S\|$ is uniformly bounded independent of n and $TS = \mathrm{Id}_{\ell_2^n}$.

PROPOSITION 3.2 *Suppose E has an unconditional basis and $m \geqslant 2$. If $\mathcal{P}(^mE)$ is separable then either E contains uniformly complemented ℓ_2^ns or E contains uniformly complemented ℓ_∞^ns.*

Proof. Assume that E neither contains uniformly complemented ℓ_2^ns nor contains uniformly complemented ℓ_∞^ns. Then E has cotype and by Proposition 2.1 E has a complemented subspace F with a greedy basis $(e_j)_{j=1}^\infty$ and biorthogonal functionals $(e_j^*)_{j=1}^\infty$. We may assume $(e_j)_{j=1}^\infty$ is 1-unconditional. Then $\mathcal{P}(^mF)$ is also separable. Pick $1 < p < m$. Then by Theorem 3.1 the fundamental function ϕ satisfies $\phi(n) \geqslant cn^{1/p}$ for some $c > 0$. Now if $x \in F$ with $\|x\| = 1$, let $A_k = \{j : 2^k \leqslant |e_j^*(x)| < 2^{k+1}\}$. Then $\phi(A_k) \leqslant \Delta 2^{-k}$ where Δ is the democratic constant. We have

$$\begin{aligned}\sum_{j=1}^\infty |e_j^*(x)|^m &\leqslant 2^m \sum_{k\leqslant 0} 2^{mk}|A_k|\\ &\leqslant 2^m c^{-1}\sum_{k\leqslant 0} 2^{mk}\phi(|A_k|)^p\\ &\leqslant 2^m c^{-1}\Delta\sum_{k\leqslant 0} 2^{(m-p)k}.\end{aligned}$$

Thus the series $\sum_{j=1}^\infty \delta_j(e_j^*(x))^m$ converges pointwise in $\mathcal{P}(^mF)$ for any choice of signs $\delta_j = \pm 1$ and it is easily seen that this then defines an uncountable 1-separated set, contradicting separability.

Proof of Theorem 1.1. Suppose E is infinite-dimensional. If $\mathcal{P}(^mE)$ has an unconditional basis (where $m \geqslant 2$) then by Proposition 3.2 it follows that either E contains uniformly complemented ℓ_2^ns or E contains uniformly complemented ℓ_∞^ns. Now by [**9**, Corollary 4] we are done.

4. $\mathcal{P}(^m E)$ as a Banach lattice

If $E = \ell_1$ then the space $\mathcal{L}_m(E)$ of bounded m-linear forms is isometric to ℓ_∞ and it follows that $\mathcal{P}(^m E)$ (which is isomorphic to a complemented subspace of $\mathcal{L}_m(E)$) is then also isomorphic to ℓ_∞ and is thus isomorphic to a Banach lattice.

PROPOSITION 4.1 *Let E be a Banach space with an unconditional basis $(e_j)_{j=1}^\infty$ and biorthogonal functionals $(e_j^*)_{j=1}^\infty$. Then for each m the following are equivalent.*

(1) *The monomials (e_α^*) form an unconditional basic sequence in $\mathcal{P}(^m E)$.*

(2) *$\mathcal{P}(^m E)$ is isomorphic to a Banach lattice.*

Proof. Suppose we have (1). We may suppose that $(e_j)_{j=1}^\infty$ is a 1-unconditional basis. Let S_n denote the partial sum projections $S_n x = \sum_{k=1}^n e_k^*(x)e_k$. Then for $p \in \mathcal{P}(^m E)$ we have $p \circ S_n \in \mathcal{P}_{\text{app}}(^m E)$ and for each multi-index α with $|\alpha| = m$ we can define $\hat{p}(\alpha)$ so that

$$p \circ S_n = \sum_{\alpha \leqslant n} \hat{p}(\alpha) e_\alpha^*,$$

where $\alpha \leqslant n$ means that $\alpha(k) = 0$ for $k > n$. It is clear that

$$\|p\| = \sup_n \|p \circ S_n\|.$$

Conversely, if $(\hat{p}(\alpha))_{|\alpha|=m}$ are scalars such that

$$\sup_n \left\| \sum_{\alpha \leqslant n} \hat{p}(\alpha) e_\alpha^* \right\|_{\mathcal{P}(^m E)} < \infty$$

then we can define $p \in \mathcal{P}(^m E)$ by

$$p(x) = \lim_{n \to \infty} \sum_{\alpha \leqslant n} \hat{p}(\alpha) e_\alpha^*(x), \qquad x \in E.$$

Thus the map $p \to (p(\alpha))_{|\alpha|=m}$ gives $\mathcal{P}(^m E)$ the structure of a Banach lattice.

Conversely, assume (2). Then we show that for each n the finite sequence $(e_\alpha^*)_{\alpha \leqslant n}$ has a bounded unconditional basis constant that is uniformly bounded in n. Indeed, if $E_n = [e_j]_{j=1}^n$ the spaces $\mathcal{P}(^m E_n)$ are 1-complemented in $\mathcal{P}(^m E)$ by the projections $p \to p \circ S_n$. We may then use [**9**, Theorem 2].

We next construct a Banach space with a symmetric basis which is not isomorphic to ℓ_1 but such that the equivalent conditions of Proposition 4.1 hold for every $m \in \mathbb{N}$.

Let us choose an increasing sequence of natural numbers $(a_r)_{r=0}^\infty$ with $a_0 = 1$ and for $r = 1, 2, \cdots$ $a_r > 3^{r a_{r-1}} a_{r-1}$. We then define $w_1 = 1$ and then $w_k = 2^{-r}$ if $a_{r-1} < k \leqslant a_r$. Consider the Lorentz sequence space $d(w, 1)$ consisting of all sequences $(\xi_k)_{k=1}^\infty$ such that

$$\|\xi\| = \sup_\pi \sum_{k=1}^\infty w_k |\xi_{\pi(k)}| < \infty,$$

where π runs through all permutations of $\mathbb{N}$. See [**17**, pp. 175ff] for background on such Lorentz sequence spaces; note that by [**17**, Theorem 4.e.2] this space is also an Orlicz sequence space.

Let us denote the canonical basis of $d(w,1)$ by $(e_n)_{n=1}^{\infty}$. The fundamental function for $d(w,1)$ is given by $\phi(n) = \sum_{k=1}^{n} w_k$.

For $A \subset \mathbb{N}$ define $\Sigma_r(A)$ to be the collection of all elements $\xi \in d(w,1)$ of the form

$$\xi = 2^r a_r^{-1} \sum_{k \in B} \epsilon_k e_k, \qquad \epsilon_k = \pm 1,\ |B| = a_r,\ B \subset A.$$

Observe that $\Sigma_r(A) = \emptyset$ if $|A| < a_r$ and that if $|A| = N \geqslant a_r$ then $|\Sigma_r(A)| = \binom{N}{a_r} 2^{a_r}$. Let $\Sigma(A) = \cup_{r \geqslant 0} \Sigma_r(A)$. Then if $|A| = N$ we have

$$|\Sigma(A)| \leqslant \sum_{k=0}^{N} \binom{N}{k} 2^k = 3^N. \tag{4.1}$$

LEMMA 4.2 (1) *For* $r \geqslant 0$ *we have* $2^{-r} a_r \leqslant \phi(a_r) \leqslant 2 \times 2^{-r} a_r$.

(2) *Suppose* $\xi^* \in d(w,1)^*$. *Then*

$$\frac{1}{2}\|\xi^*\| \leqslant \sup_{\xi \in \Sigma(\mathbb{N})} \xi^*(\xi) \leqslant 2\|\xi^*\|.$$

(3) *For each* $\varphi \in \mathcal{L}_m(d(w,1))$ *we have*

$$\frac{1}{2^m} \sup_{u_j \in \Sigma(\mathbb{N})} |\varphi(u_1, \ldots, u_m)| \leqslant \|\varphi\| \leqslant 2^m \sup_{u_j \in \Sigma(\mathbb{N})} |\varphi(u_1, \ldots, u_m)|.$$

Proof. We first observe that $2^{-r} a_r \leqslant \phi(a_r)$. Next by induction we see that $\phi(a_r) \leqslant 2 \times 2^{-r} a_r$. Indeed this is trivially true when $r = 0$ and then if we assume it is true for $r - 1$ we have

$$\phi(a_r) = \phi(a_{r-1}) + 2^{-r}(a_r - a_{r-1})$$

so that, since $a_{r-1}/a_r < \frac{1}{3}$,

$$\begin{aligned} \frac{\phi(a_r)}{a_r} &= \frac{a_{r-1}}{a_r} \frac{\phi(a_{r-1})}{a_{r-1}} + \left(1 - \frac{a_{r-1}}{a_r}\right) 2^{-r} \\ &\leqslant \frac{4 \times 2^{-r}}{3} + \frac{2 \times 2^{-r}}{3} = 2 \times 2^{-r}. \end{aligned}$$

Now suppose $\xi^* \in d(w,1)^*$ is such that

$$\sup_r \sup_{\xi \in \Sigma_r(\mathbb{N})} \xi^*(\xi) = 1.$$

Without loss of generality we may suppose that if $b_j = \xi^*(e_j)$ then $(b_j)_{j=1}^{\infty}$ is a decreasing non-negative sequence so that

$$\sup_r 2^r a_r^{-1} \sum_{j=1}^{a_r} b_j = 1.$$

Then if $a_{r-1} < n \leqslant a_r$ we have

$$\frac{1}{n}\sum_{j=1}^{n} b_j \leqslant 2 \times 2^{-r} \leqslant 2\frac{\phi(a_r)}{a_r} \leqslant 2\frac{\phi(n)}{n}.$$

Thus if $\xi = \sum_{j=1}^{\infty} \xi_j e_j$ with (ξ_j) non-negative and decreasing,

$$\begin{aligned}\xi^*(\xi) &= \sum_{j=1}^{\infty} b_j \xi_j \\ &= \sum_{j=1}^{\infty} (b_1 + \cdots + b_j)(\xi_j - \xi_{j+1}) \\ &\leqslant 2\sum_{j=1}^{\infty} \phi(j)(\xi_j - \xi_{j+1}) \\ &= 2\sum_{j=1}^{\infty} w_j \xi_j.\end{aligned}$$

Thus $\|\xi^*\| \leqslant 2$.

On the other hand if $\xi \in \Sigma_r(\mathbb{N})$ then $\|\xi\| \leqslant \phi(a_r)2^r a_r^{-1} \leqslant 2$ so that $\|\xi^*\| \geqslant \frac{1}{2}$. Finally, (3) is a straightforward consequence of (2).

THEOREM 4.3 *For every $m \in \mathbb{N}$ the monomials $(e_\alpha^*)_\alpha$ form an unconditional basic sequence in $\mathcal{P}_{\mathrm{app}}({}^m d(w,1))$, and hence $\mathcal{P}({}^m d(w,1))$ is isomorphic to a Banach lattice.*

Proof. It will suffice to show that the elements $e_{i_1}^* \otimes \cdots \otimes e_{i_m}^*$ form an unconditional basic sequence in $\mathcal{L}_m(d(w,1))$ for every choice of m. Indeed the monomials in $\mathcal{P}({}^m d(w,1))$ are equivalent to an unconditional block basic sequence of this basis.

More precisely we show by induction that there is a constant C_m such that if φ is an m-linear form given by

$$\varphi(x_1, \cdots, x_m) = \sum_{i_1,\cdots,i_m} b_{i_1,\cdots,i_m} e_{i_1}^*(x_1) \cdots e_{i_m}^*(x_m),$$

where the array $(b_{i_1,\cdots,i_m})$ is finitely non-zero and if

$$|\varphi|(x_1, \cdots, x_m) = \sum_{i_1,\cdots,i_m} |b_{i_1,\cdots,i_m}| e_{i_1}^*(x_1) \cdots e_{i_m}^*(x_m)$$

then $\||\varphi|\| \leqslant C_m \|\varphi\|$.

The case $m = 1$ is trivial and indeed $C_1 = 1$. Let us now suppose the theorem is proved for $k < m$. We shall assume $\|\varphi\| = 1$ and let $\||\varphi|\| = M$. Then by Lemma 4.2 we can find $u_j \in \Sigma_{r_j}(\mathbb{N})$ for $1 \leqslant j \leqslant m$ such that

$$|\varphi|(u_1, \cdots, u_m) \geqslant 2^{-m} M.$$

In fact each u_j can be taken of the form

$$u_j = \frac{2^{r_j}}{a_{r_j}} \sum_{k \in B_j} e_k,$$

where $|B_j| = a_{r_j}$.

By reordering if necessary we shall assume that $r_m = \max_{1\leqslant j\leqslant m} r_j$. Let us consider the case when $r_m < m-1$. In this case

$$M \leqslant 2^{m+mr_m} \max |\varphi(e_{i_1}, \cdots, e_{i_n})| \leqslant 2^{m^2}. \tag{4.2}$$

We continue with the assumption that $r_m \geqslant m-1$. If $1 \leqslant j < m$ and $r_j = r_m$ we can write

$$\begin{aligned} u_j &= \frac{2^{r_m}}{a_{r_m}} \sum_{k\in B_j} e_k \\ &= 2\binom{a_{r_m}}{a_{r_m-1}}^{-1} \sum_{\substack{D\subset B_j \\ |D|=a_{r_m-1}}} 2^{r_m-1} a_{r_m-1}^{-1} \sum_{k\in D} e_k. \end{aligned}$$

Expanding each such u_j out in this way we see that we can find B_j with $|B_j| = a_{r_j}$, where $r_j < r_m$ if $j < m$ and such that if

$$v_j = \frac{2^{r_j}}{a_{r_j}} \sum_{k\in B_j} e_k$$

then

$$|\varphi|(v_1, \cdots, v_m) \geqslant 2^{-2m} M. \tag{4.3}$$

Now, for each $k \in B_m$ we define $\psi_k \in \mathcal{L}_{m-1}(d(w,1))$ by

$$\psi_k(x_1, \cdots, x_{m-1}) = \sum_{i_1\in B_1} \cdots \sum_{i_{m-1}\in B_{m-1}} b_{i_1,\cdots,i_{m-1},k} e^*_{i_1}(x_1) \cdots e^*_{i_{m-1}}(x_{m-1}).$$

For each $k \in B_m$ there exists at least one $(\xi_1, \cdots, \xi_{m-1}) \in \Sigma(B_1) \times \cdots \times \Sigma(B_{m-1})$ so that

$$\psi_k(\xi_1, \cdots, \xi_{m-1}) \geqslant 2^{-(m-1)} \|\psi_k\|.$$

It follows that we can partition B_m into subsets $D_1, \cdots, D_N$, where by (4.1) and since all $r_j \leqslant r_m - 1$

$$N \leqslant 3^{a_{r_1}+\cdots+a_{r_{m-1}}} \leqslant 3^{(m-1)a_{r_m-1}} \tag{4.4}$$

so that for each j there exists a choice $(\xi_1, \cdots, \xi_{m-1}) \in \Sigma(B_1) \times \cdots \times \Sigma(B_{m-1})$ with

$$\psi_k(\xi_1, \cdots, \xi_{m-1}) \geqslant 2^{-(m-1)} \|\psi_k\|, \qquad k \in D_j.$$

Let $|D_j| = s_j$. By Lemma 4.2 we have $\|\xi_j\| \leqslant 2$ for $1 \leqslant j \leqslant m$, hence

$$2^{-(m-1)} \sum_{k\in D_j} \|\psi_k\| \leqslant \varphi\left(\xi_1, \cdots, \xi_{m-1}, \sum_{k\in D_j} e_k\right) \leqslant 2^{m-1}\phi(s_j).$$

By the inductive hypothesis we have $\||\psi_k|\| \leqslant C_{m-1}\|\psi_k\|$. Returning to (4.3) we have (again noting that each $\|v_j\| \leqslant 2$)

$$\begin{aligned} M &\leqslant 2^{2m}|\varphi|(v_1,\cdots,v_m) \\ &\leqslant 2^{2m+r_m}a_{r_m}^{-1}\sum_{k\in B_m}|\psi_k|(v_1,\cdots,v_{m-1}) \\ &\leqslant C_{m-1}2^{3m-1+r_m}a_{r_m}^{-1}\sum_{k\in B_m}\|\psi_k\| \\ &= C_{m-1}2^{3m-1+r_m}a_{r_m}^{-1}\sum_{j=1}^{N}\sum_{k\in D_j}\|\psi_k\| \\ &\leqslant C_{m-1}2^{5m-3+r_m}a_{r_m}^{-1}\sum_{j=1}^{N}\phi(s_j). \end{aligned}$$

Finally, we estimate $\sum_{j=1}^{N}\phi(s_j)$. If $r \geqslant 1$ and $a_{r-1} < s_j \leqslant a_r$ then $\phi(s_j) \leqslant s_j a_{r-1}^{-1}\phi(a_{r-1}) \leqslant 4\times 2^{-r}s_j$ (by Lemma 4.2 and the fact that $\phi(n)/n$ is decreasing in n). Thus

$$\sum_{a_{r-1}<s_j\leqslant a_r}\phi(s_j) \leqslant 4\times 2^{-r}\sum_{a_{r-1}<s_j\leqslant a_r}s_j. \tag{4.5}$$

Define $\sigma_r := |\{j : a_{r-1} < s_j \leqslant a_r\}|$ and notice that $\sum_{s_j=1}\phi(s_j) = \sigma_0$, where $\sigma_0 = |\{j : s_j = 1\}|$. Then by (4.5) we have

$$\sum_{a_{r-1}<s_j\leqslant a_r}\phi(s_j) \leqslant 4\times 2^{-r}\sigma_r a_r.$$

Now if $r \leqslant r_m - 1$ we have $2^{-r}a_r \leqslant 2^{1-r_m}a_{r_m-1}$ (use again $a_r/a_{r+1} < \frac{1}{3} \leqslant \frac{1}{2}$), and as a consequence from (4.4)

$$\sum_{s_j\leqslant a_{r_m-1}}\phi(s_j) \leqslant 2^{3-r_m}a_{r_m-1}\sum_{r=0}^{r_m-1}\sigma_r \leqslant 3^{(m-1)(a_{r_m-1})}2^{3-r_m}a_{r_m-1}.$$

Hence as $r_m \geqslant m-1$ we deduce from the defining property of the a_rs that

$$\sum_{s_j\leqslant a_{r_m-1}}\phi(s_j) \leqslant 2^{3-r_m}a_{r_m}.$$

On the other hand, by (4.5),

$$\sum_{a_{r_m-1}<s_j\leqslant a_{r_m}}\phi(s_j) \leqslant 4\times 2^{-r_m}\sum_{a_{r_m-1}<s_j\leqslant a_{r_m}}s_j \leqslant 2^{2-r_m}a_{r_m}$$

(recall that the sum over all s_j equals a_{r_m}). Combining we have

$$\sum_{j=1}^{N}\phi(s_j) \leqslant 2^{4-r_m}a_{r_m}$$

and hence

$$M \leqslant C_{m-1}2^{5m+1}. \tag{4.6}$$

Combining (4.2) and (4.6) we have $C_m \leqslant \max(2^{5m+1}C_{m-1}, 2^{m^2})$ and this completes the proof.

5. Some related open problems

The space created in Theorem 4.3 is not reflexive. We therefore ask the following.

Let E be a reflexive Banach space with an unconditional basis and suppose $m \geqslant 2$. Can $\mathcal{P}({}^m E)$ be a Banach lattice?

Notice in this situation Proposition 4.1 implies that $\mathcal{P}({}^m E)$ is isomorphic to a Banach lattice if and only if $\mathcal{P}_{\mathrm{app}}({}^m E)$ has an unconditional basis.

Finally, we relate our study of unconditionality in spaces of m-homogeneous polynomials with complex analysis. Let $E = (\mathbb{C}^n, \|.\|)$ be a finite-dimensional Banach space such that its canonical basis vectors e_k form a normalized 1-unconditional basis. The Bohr radius of its open unit ball B_E is defined to be

$$K(B_E) := \sup r,$$

where the supremum is taken over all $0 \leqslant r \leqslant 1$ such that whenever the power series $\sum_\alpha a_\alpha z^\alpha$ satisfies $|\sum_\alpha a_\alpha z^\alpha| \leqslant 1$ for all $z \in B_E$, it follows that $\sum_\alpha |a_\alpha z^\alpha| \leqslant 1$ for all $z \in r B_E$.

In this notation Bohr's power series theorem from [**5**] states that the Bohr radius of the open unit disc in $\mathbb{C}$ equals $\frac{1}{3}$, $K(B_{\mathbb{C}}) = \frac{1}{3}$.

Upper and lower estimates for Bohr radii in higher dimensions show two in a sense extreme cases. The sequence $(K(B_{\ell_\infty^n})$ of the Bohr radii of the n-dimensional polydiscs tends to zero essentially like $\sqrt{\log n/n}$, whereas the sequence $(K(B_{\ell_1^n})$ of the Bohr radii of the n-dimensional hypercones is uniformly bounded from below by some strictly positive constant. More precisely, there is a constant $c > 0$ such that for each $n \geqslant 2$

$$\frac{1}{c\sqrt{\log\log n}}\sqrt{\frac{\log n}{n}} \leqslant K(B_{\ell_\infty^n}) \leqslant c\sqrt{\frac{\log n}{n}}$$

(see [**4, 14**] for the upper estimate and [**7**] for the lower one) and

$$\frac{1}{c} \leqslant K(B_{\ell_1^n}) \leqslant c$$

(a result of [**1**]). See [**3, 7**] for the asymptotic behaviour of the whole scale of sequences $(K(B_{\ell_p^n})$, $1 < p < \infty$, and [**8**] for an extension of these estimates within the framework of local Banach space theory.

There is a basic link to unconditional basis constants of spaces of m-homogeneous polynomials [**8**, Theorem 2.2]. Define

$$r(E) := \sup_m \chi_{\mathrm{mon}}(\mathcal{P}({}^m E))^{1/m},$$

where $\chi_{\mathrm{mon}}(\mathcal{P}({}^m E))$ stands for the unconditional basis constant of the monomials in $\mathcal{P}({}^m E)$. Then

$$\frac{1}{3}\frac{1}{r(E)} \leqslant K(B_E) \leqslant \min\left(\frac{1}{3}, \frac{1}{r(E)}\right)$$

(for $E = \mathbb{C}$ this is obviously Bohr's result).

In view of this link the following problem seems to be a sort of uniform analogue of Dineen's problem. Let E be a Banach sequence space (that is, $\ell_1 \subset E \subset c_0$ and the e_ks form a 1-unconditional basis of E), and let $E_n = [e_k]_{k=1}^n$.

Does E necessarily equal ℓ_1 whenever $\inf_n K(B_{E_n}) > 0$ or, equivalently, is $E = \ell_1$ whenever there is some constant $C > 0$ such that $\chi_{\mathrm{mon}}(\mathcal{P}({}^m E_n)) \leqslant C^m$ for all n and m?

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