ON SUBSPACES OF c_0 AND EXTENSION OF OPERATORS INTO $C(K)$-SPACES

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Abstract

Johnson and Zippin recently showed that if X is a weak*-closed subspace of ℓ_1 and $T : X \to C(K)$ is any bounded operator then T can be extended to a bounded operator $\tilde{T} : \ell_1 \to C(K)$. We give a converse result: if X is a subspace of ℓ_1 such that ℓ_1/X has an unconditional finite-dimensional decomposition (UFDD) and every operator $T : X \to C(K)$ can be extended to ℓ_1 then there is an automorphism τ of ℓ_1 such that $\tau(X)$ is weak*-closed. This result is proved by studying subspaces of c_0 and several different characterizations of such subspaces are given.

1. Introduction

In [**15**], Johnson and Zippin proved an extension theorem for operators into $C(K)$-spaces:

THEOREM 1.1 *Let X be a weak*-closed subspace of ℓ_1 (considered as the dual of c_0) and let $T : X \to C(K)$ be a bounded operator. Then T has an extension $\tilde{T} : \ell_1 \to C(K)$.*

Note that this implies the same conclusion for any subspace X so that ℓ_1/X is isomorphic to the dual of a subspace of c_0 (using results of [**17**]). The aim of this paper is to prove a partial converse result to the Johnson–Zippin theorem. We show that if X is a subspace of ℓ_1 such that every bounded operator $T : X \to C[0, 1]$ can be extended and if, additionally, ℓ_1/X has an unconditional finite-dimensional decomposition (UFDD) then ℓ_1/X is isomorphic to the dual of a subspace of c_0, and hence there is an automorphism τ of ℓ_1 such that $\tau(X)$ is weak*-closed. The hypothesis on X can be weakened a little: it suffices that ℓ_1/X be the dual of space which embeds in a space with a UFDD.

The technique of proof depends heavily on ideas developed in [**8**], where subspaces of c_0 are characterized in terms of properties of norms. We also use ideas from [**9**] where trees are used to obtain renormings, to obtain a characterization of subspaces of c_0 in terms of properties of trees in the dual.

If X is a subspace of ℓ_1 which satisfies the conclusion of Theorem 1.1 we show that ℓ_1/X has a property we call the very strong Schur property (the strong Schur property was considered first for subspaces of L_1 by Rosenthal [**22**]; see also [**2**]). In the presence of some unconditionality assumption, for example, if ℓ_1/X has a UFDD this can then be used to show that ℓ_1/X is the dual of a subspace of c_0.

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2. Preliminary results

In this section we gather together some basic definitions and preliminary results.

We start by recalling that a projection P on a Banach space X is an *L-projection* if $\|x\| = \|Px\| + \|x - Px\|$ for any $x \in X$. We shall say that P is a *θ-L-projection* where $0 < \theta \leqslant 1$ if we have $\|x\| \geqslant \|Px\| + \theta\|x - Px\|$. We shall say that X is an *L-summand* (respectively a *θ-L-summand*) if there is an L-projection (respectively a θ-L-projection) of X^{**} onto X; we shall say that X is a crude-L-summand if it can be equivalently renormed to be a θ-L-summand for some $0 < \theta \leqslant 1$. We also recall that X is called an *M-ideal* if the canonical projection π of X^{***} onto X^* is an L-projection. Similarly that a Banach space X is a *θ-M-ideal* if π is a θ-L-projection and a crude M-ideal if it has an equivalent norm so that it is a θ-M-ideal for some $0 < \theta \leqslant 1$. For background on the theory of M-ideals we refer to [**12**]. The notion of a crude M-ideal has also been considered in the literature, originating with the work of Ando [**1**] and more recently as a special case of the so-called $M(r, s)$-inequalities [**3**, **4**, **11**].

Let us recall that X has the *strong Schur property* [**22**] if there is a constant $c > 0$ such that if (x_n) is any normalized sequence with $\|x_m - x_n\| \geqslant \delta > 0$ for any $m \neq n$ then there is a subsequence $(x_n)_{n\in\mathcal{M}}$ such that

$$\left\| \sum_{k\in\mathcal{M}} \alpha_k x_k \right\| \geqslant c\delta \sum_{k\in\mathcal{M}} |\alpha_k|$$

for any finitely non-zero sequence $(\alpha_k)_{k\in\mathcal{M}}$. This notion was first introduced implicitly by Johnson and Odell [**13**], and then explicitly by Rosenthal [**22**] and later studied by Bourgain and Rosenthal [**2**].

We will need some equivalent formulations of the strong Schur property:

PROPOSITION 2.1 *Let X be a Banach space. The following are equivalent.*

(i) *X has the strong Schur property.*

(ii) *There is a constant $c_1 > 0$ such that if (x_n) is any normalized sequence with* $\inf_{m>n} \|x_m - x_n\| = \delta$ *then there exists $x^* \in B_{X^*}$ with*

$$\limsup_{n\to\infty} x^*(x_n) - \liminf_{n\to\infty} x^*(x_n) \geqslant c_1\delta.$$

(iii) *For some fixed $\epsilon > 0$ there exists a constant $c_2 > 0$ such that if (x_n) is a normalized sequence with* $\inf_{m>n} \|x_m - x_n\| \geqslant 1 - \epsilon$ *for any $m \neq n$ then there exists $x^* \in B_{X^*}$ with* $\limsup_{n\to\infty} x^*(x_n) \geqslant c_2$.

(iv) *There is a constant $c_3 > 0$ such that for any sequence (x_n) in X there exists $x^* \in B_{X^*}$ with* $\limsup_{n\to\infty} x^*(x_n) \geqslant c_3 \limsup_{n\to\infty} \|x_n\|$.

Proof. The equivalence of (i) and (ii) is essentially contained in the usual proof of Rosenthal's ℓ_1-theorem (cf. [**6**, pp. 209–211]). (ii) trivially implies (iii).

We now prove that (iii) implies (iv). By the Uniform Boundedness Principle we may suppose (x_n) bounded and by passing to subsequences and renormalizing we may suppose that $\|x_n\| = 1$ for all n. Let $\delta = \inf_{m>n} \|x_m - x_n\|$, and suppose that x^{**} is any weak*-cluster point of (x_n). If $\delta < 1-\epsilon$ then $\|x^{**} - x_n\| \leqslant 1-\epsilon$ for all n and so $\|x^{**}\| \geqslant \epsilon$. In this case there exists $x^* \in B_{X^*}$ with $\limsup_{n\to\infty} x^*(x_n) \geqslant \frac{1}{2}\epsilon$. If $\delta \geqslant 1 - \epsilon$, then we apply (ii). Thus (iii) holds with $c_3 = \min(c_2, \frac{1}{2}\epsilon)$.

Finally we show that (iv) implies (ii). Indeed there exists $x^* \in B_{X^*}$ with $\limsup x^*(x_{2n} - x_{2n-1}) \geqslant c_3\delta$ and so

$$\limsup_{n\to\infty} x^*(x_n) - \liminf_{n\to\infty} x^*(x_n) \geqslant c_3\delta.$$

We will be interested in conditions which guarantee that a Banach space X embeds into c_0. We next state a criterion from [**16**] (the almost isometric case) and [**8**].

THEOREM 2.2 *Let X be a separable Banach space. Suppose there is a constant $c > 0$ such that if $x^* \in X^*$ and (x_n^*) is any weak*-null sequence then*

$$\liminf_{n\to\infty} \|x^* + x_n^*\| \geqslant \|x^*\| + c \liminf_{n\to\infty} \|x_n^*\|.$$

Then X is isomorphic to a subspace of c_0.

Note here that we can replace lim inf by lim sup or consider only the case when both limits exist without changing the criterion. A norm with this property is called *Lipschitz-UKK^**. We now give a simple application in the spirit of later results. We refer also to [**10**] for connections between embeddability into c_0 and the strong Schur property.

THEOREM 2.3 *Suppose that X is a separable Banach space such that X^* has the strong Schur property and suppose that X is a crude M-ideal. Then X is isomorphic to a subspace of c_0.*

Proof. We may suppose X is a θ-M-ideal for some $0 < \theta \leqslant 1$. Suppose $x^* \in X^*$ and that (x_n^*) is a weak*-null sequence. Then there exists $x^{**} \in B_{X^{**}}$ such that $\limsup x^{**}(x_n^*) \geqslant c_3 \limsup \|x_n^*\|$. Hence $(x_n^*)_{n=1}^\infty$ has a weak*-cluster point $x^{***} \in X^{***}$ with $\|x^{***}\| \geqslant c_3 \limsup \|x_n^*\|$. Clearly $x^{***} \in X^\perp$ and so

$$\begin{aligned}\limsup_{n\to\infty} \|x^* + x_n^*\| &\geqslant \|x^* + x^{***}\| \geqslant \|x^*\| + \theta\|x^{***}\| \\ &\geqslant \|x^*\| + c_3\theta \limsup_{n\to\infty} \|x_n^*\|.\end{aligned}$$

We can now apply the result of [**8**] to deduce that X embeds into c_0.

Another important concept we use concerns unconditionality. We shall say that a Banach space X is of *unconditional type* if whenever $x \in X$ and (x_n) is a weakly null sequence in X we have

$$\lim_{n\to\infty} (\|x + x_n\| - \|x - x_n\|) = 0.$$

We shall say that X is of *shrinking unconditional type* if whenever $x^* \in X^*$ and (x_n^*) is weak*-null in X^* then

$$\lim_{n\to\infty} (\|x^* + x_n^*\| - \|x^* - x_n^*\|) = 0.$$

These notions were introduced and studied (with different terminology) by Neuwirth [**19**]. We first note the following.

LEMMA 2.4 *If X is a separable Banach space which has shrinking unconditional type then X has unconditional type.*

Proof. Suppose $x \in X$ and (x_n) is weakly null and that $\|x + x_n\| > \|x - x_n\| + \epsilon$ for all n, where $\epsilon > 0$. Choose $y_n^* \in B_{X^*}$ such that $y_n^*(x + x_n) = \|x + x_n\|$. By passing to a subsequence we can suppose y_n^* converges to some $x^* \in X^*$. Then $\lim_{n\to\infty} \|2x^* - y_n^*\| = \lim_{n\to\infty} \|y_n^*\| = 1$. Now $\lim_{n\to\infty}(\|x + x_n\| - y_n^*(x_n)) = x^*(x)$ and so

$$\lim_{n\to\infty}(\langle x - x_n, 2x^* - y_n^*\rangle - \|x + x_n\|) = 0.$$

This implies that $\liminf(\|x - x_n\| - \|x + x_n\|) \geqslant 0$ and gives the lemma.

Let us recall that a separable Banach space X has the *unconditional metric approximation property* (UMAP) if there is a sequence of finite-rank operators (T_n) such that $\lim_{n\to\infty} T_n x = x$ for $x \in X$ and $\lim_{n\to\infty} \|I - 2T_n\| = 1$ (see [**5**, **7**]); we say X has *shrinking* (UMAP) if, in addition, $\lim_{n\to\infty} T_n^* x^* = x^*$ for $x^* \in X^*$. It is shown in [**7**] that X has (UMAP) if and only if for every $\epsilon > 0$ X is isometric to a one-complemented subspace of a space V_ϵ with a $(1 + \epsilon)$-(UFDD).

LEMMA 2.5 *Let X be a Banach space with (UMAP); then X is of unconditional type. If X has shrinking (UMAP) X is of shrinking unconditional type.*

Proof. Suppose $x \in X$ and (x_n) is weakly null. It is enough to show that $\lim_{n\to\infty} \|x + x_n\| \leqslant \lim_{n\to\infty} \|x - x_n\|$ under the assumption that both limits exist;

$$\begin{aligned}\lim_{n\to\infty} \|x + x_n\| &= \lim_{k\to\infty} \lim_{n\to\infty} \|(2T_k - 1)x + x_n\| \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \|(2T_k - I)x + (I - 2T_k)x_n\| \\ &\leqslant \lim_{n\to\infty} \|x - x_n\|.\end{aligned}$$

The shrinking case is similar.

LEMMA 2.6 *Let X be a separable Banach space of shrinking unconditional type. Then any subspace or quotient of X has shrinking unconditional type.*

Proof. If Y is a subspace of X then $(X/Y)^*$ can be identified with $Y^\perp$ and trivially X/Y has shrinking unconditional type. Now Y^* can be identified with $X^*/Y^\perp$. Let $Q : X^* \to Y^*$ be the canonical quotient map. Suppose $y^* \in Y^*$ and that (y_n^*) is weak*-null in Y^*. Suppose that $\|y^* + y_n^*\| < \|y^* - y_n^*\| - \epsilon$, where $\epsilon > 0$. We may pick (by the Hahn–Banach theorem) $x_n^* \in X^*$ such that $\|x_n^*\| = \|y^* + y_n^*\|$ and $Qx_n^* = y^* + y_n^*$. Passing to a subsequence we can suppose that x_n^* converges weak* to x^*. Now

$$\lim_{n\to\infty}(\|2x^* - x_n^*\| - \|x_n^*\|) = 0$$

and $Q(2x^* - x_n^*) = 2Qx^* - y^* - y_n^* = y^* - y_n^*$ by the weak*-continuity of Q. Hence

$$\limsup_{n\to\infty}(\|y^* - y_n^*\| - \|y^* + y_n^*\|) \leqslant 0,$$

which yields a contradiction; thus Y is of shrinking unconditional type.

LEMMA 2.7 *Let X be a subspace of a space with (UMAP); if X does not contain ℓ_1 then X has shrinking unconditional type.*

Proof. Suppose $\epsilon > 0$ and that $x^* \in X^*$ and that (x_n^*) is any weakly null sequence; assume that $\sup_n \|x^* + x_n^*\| \leqslant 1$ and $\|x^* - x_n^*\| > \|x^* + x_n^*\| + \epsilon$ for all $n \in \mathbb{N}$, where $\epsilon > 0$. By results of [**7**] we may suppose X is isometric to a subspace of a space $V = V_\epsilon$ with a $(1+\epsilon/2)$-(UFDD). Indeed suppose (Q_n) are finite rank projections defining a $(1+\epsilon/2)$-(UFDD). Let $T_n = \sum_{k=1}^n Q_i$. Let $j : X \to V$ be the isometric embedding.

If $v^* \in V^*$ we have $j^*v^* = \sum_{k=1}^\infty j^* Q_k^* v^*$ unconditionally in the weak*-topology. Since X^* does not contain c_0 (or equivalently ℓ_∞) this series converges in norm so that $\lim_{k\to\infty} \|j^*v^* - j^*T_k^*v^*\| = 0$.

Now by the Hahn–Banach theorem we can find $v_n^* \in V^*$ such that $\|v_n^*\| = \|x^* + x_n^*\|$ and $j^*v_n^* = x^* + x_n^*$. By passing to a subsequence we can suppose that v_n^* converges weak* to some v^*. Clearly $j^*v^* = x^*$. Then

$$\begin{aligned}\limsup_{n\to\infty} \|x^* - x_n^*\| &= \limsup_{n\to\infty} \|2j^*v^* - j^*v_n^*\| \\ &= \limsup_{k\to\infty} \limsup_{n\to\infty} \|2j^*T_k^*v^* - j^*v_n^* - 2j^*(T_k^*v^* - T_k v_n^*)\| \\ &\leqslant \limsup_{k\to\infty} \limsup_{n\to\infty} \|(2T_k - I)v_n^*\| \\ &\leqslant (1 + \tfrac{1}{2}\epsilon) \limsup_{n\to\infty} \|x^* + x_n^*\|.\end{aligned}$$

This contradiction establishes the lemma.

3. Subspaces of c_0 and trees

Consider the set $\mathcal{F}\mathbb{N}$ of all finite subsets of $\mathbb{N}$ with the following partial order. If $a = \{n_1, n_2, \dots, n_k\}$ where $n_1 < n_2 < \cdots < n_k$ and $b = \{m_1, m_2, \dots, m_l\}$ where $m_1 < m_2 < \cdots < m_l$, then $a \leqslant b$ if and only if $k \leqslant l$ and $m_i = n_i$ where $1 \leqslant i \leqslant k$ (that is, a is an initial segment of b). We say that b is a *successor* of a if $|b| = |a| + 1$ and $a \leqslant b$; the collection of successors of a is denoted by $a+$. If $a \neq \emptyset$ then $a-$ denotes the unique predecessor of a; that is, a is a successor of $a-$. Let S be a subset of $\mathcal{F}\mathbb{N}$. We say that S is a *full tree* whenever

1. $\emptyset \in S$;
2. each $a \in S$ has infinitely many successors in S;
3. if $a \in S$ and $\emptyset \neq a \in S$ then $a- \in S$.

It is easy to see that any full tree is isomorphic as an ordered set to $\mathcal{F}\mathbb{N}$. If S is any full tree we will say that a sequence $\beta = \{a_n\}_{n=0}^\infty$ is a *branch* of S if $a_n \in S$ for all n, $a_0 = \emptyset$ and a_{n+1} is a successor of a_n for all $n \geqslant 0$.

Now let V be a vector space. We define a *tree-assignment* to be a map $a \to x_a$ defined on a full tree S. We define a *tree-map* to be a tree-assignment $a \to x_a$ with the properties that $x_\emptyset = 0$ and for every branch β the set $\{a : x_a \neq 0 : a \in \beta\}$ is finite. Given any tree-map we define a *height function* h which assigns to each a a countable ordinal; to do this we define $h(a) = 0$ if $x_b = 0$ for $b \geqslant a$ and then inductively $h(a)$ is defined by $h(a) \leqslant \eta$ if and only if $h(b) < \eta$ for every $b > a$. The *height* of the tree-map is defined to be $h(\emptyset)$. Note that the tree-map $a \to x_a$ has finite height $m \leqslant n$ if and only if $x_a = 0$ whenever $|a| > n$.

The following easy lemma, proved in [**9**], is a restatement of the fact that certain types of games (which are not used in this paper) are determined.

LEMMA 3.1 *Suppose $(x_a)_{a\in S}$ is a tree-map and that A is any subset of V. Then either there is a full tree $T \subset S$ such that $\sum_{a\in\beta} x_a \in A$ for every branch $\beta \subset T$ or there is a full tree $T \subset S$ such that $\sum_{a\in\beta} x_a \notin A$ for every branch $\beta \subset T$.*

Now suppose $V = X$ is a Banach space. If τ is a topology on X (for example, the weak topology or for dual spaces the weak*-topology) we say that a tree-map $(x_a)_{a\in S}$ is *τ-null* if for every $a \in S$ the set $\{x_b\}_{b\in a+}$ is a τ-null sequence.

We now introduce a definition which will characterize subspaces of c_0. We say that a Banach space X has the *bounded tree property* with constant $\sigma > 0$ if every weakly null tree-map $(x_a)_{a\in S}$ has a full subtree T such that $\left\| \sum_{a\in\beta} x_a \right\| \leqslant 1$ for every branch β.

THEOREM 3.2 *Let X be a separable Banach space containing no copy of ℓ_1 with the bounded tree property. Then X is isomorphic to a subspace of c_0.*

Proof. cf. [**9**, Theorem 4.1]. Define for $x \in X$, $f(x)$ to be the infimum of all $\lambda > 0$ such that for every weakly null tree-map $(x_a)_{a\in S}$ with $\|x_a\| \leqslant \sigma$ there is a full tree $T \subset S$ with $\left\|x + \sum_{a\in\beta} x_a\right\| \leqslant \lambda$ for every branch β. Note that $\|x\| \leqslant f(x)$, $f(0) \leqslant 1$, $f(-x) = f(x)$ and that $|f(x) - f(y)| \leqslant \|x - y\|$. In particular we have $\|x\| \leqslant f(x) \leqslant \|x\| + 1$. We now argue exactly as in [**9**] that f is convex. For the convenience of the reader we repeat the argument. Let $u = tx + (1-t)y$, where $0 < t < 1$. Suppose $\lambda > f(x)$ and $\mu > f(y)$. Let $(x_a)_{a\in S}$ be any weakly null tree-map of height k with $\|x_a\| \leqslant \sigma$ for all $a \in S$. Then we can find a full subtree $T_1 \subset S$ such that for every branch β we have

$$\left\| x + \sum_{a\in\beta} x_a \right\| \leqslant \lambda$$

and then a full subtree $T_2 \subset T_1$ such that for every branch $\beta \subset T_2$

$$\left\| y + \sum_{a\in\beta} x_a \right\| \leqslant \mu.$$

Obviously for every branch $\beta \subset T_2$

$$\left\| u + \sum_{a\in\beta} x_a \right\| \leqslant t\lambda + (1-t)\mu$$

so that $f_k(u) \leqslant t\lambda + (1-t)\mu$.

Next we note that if $\|x_n\| \leqslant \sigma$ and $\lim_{n\to\infty} x_n = 0$ weakly then $\limsup f(x + x_n) \leqslant f(x)$. Assume that $\lambda < \limsup_{n\to\infty} f(x + x_n)$. By passing to a subsequence we can suppose that $\lambda < f(x + x_n)$ for every n. Then for each n there is a weakly null tree-map $(y_a^{(n)})_{a\in S_n}$ of height k such that $\|y_a^{(n)}\| \leqslant \sigma$ for all $a \in S_n$ and

$$\left\| x + x_n + \sum_{a\in\beta} y_a^{(n)} \right\| > \lambda$$

for every branch $\beta \subset S_n$. Now let T be the tree consisting of all sets $\{m_1, \ldots, m_l\}$, where $m_1 < m_2 < \cdots < m_l$ such that if $l > 1$ then $\{m_2, \ldots, m_l\} \in S_{m_1}$. We define a weakly null tree-map by

$$z_{m_1,\ldots,m_l} = \begin{cases} x_{m_1} & \text{if } l = 1, \\ y^{(m_1)}_{m_2,\ldots,m_l} & \text{if } l > 1. \end{cases}$$

Then for every branch $\beta \subset T$ we have

$$\Big\| x + \sum_{a\in\beta} z_a \Big\| > \lambda$$

so that $f(x) \geqslant \lambda$. This implies our claim.

Let $|\cdot|$ be the Minkowski functional of the set $\{x : f(x) \leqslant 2\}$. Then $|\cdot|$ is a norm on X satisfying $\frac{1}{2}\|x\| \leqslant |x| \leqslant \|x\|$. Suppose $|x| = 1$ and (x_n) is a weakly null sequence with $|x_n| \leqslant \frac{1}{2}\sigma$. Then $\|x_n\| \leqslant \sigma$ and so $\limsup f(x + x_n) \leqslant 2$. Hence

$$\limsup |x + x_n| \leqslant 1.$$

Now by [**9**, Proposition 2.7] we have that X^* is separable. We can then apply [**9**, Proposition 2.6] to deduce that if $|x^*| = 1$ and (x_n^*) is a weak*-null sequence in X^* with $\|x_n^*\| = \tau$ then

$$\liminf_{n\to\infty} |x^* + x_n^*| \geqslant 1 + \frac{\sigma}{12}\tau.$$

Thus X^* has a Lipschitz-UKK* norm and by the results of [**8**] (see also [**16**]) this implies that X embeds into c_0.

We next introduce a dual notion. We say that X^* has the *weak* summable tree property* with constant $c > 0$ if for every weak*-null tree-map $(x_a^*)_{a\in S}$ on X^* satisfying the boundedness property

$$\sup_{a\in S} \Big\| \sum_{b\leqslant a} x_b \Big\| < \infty, \tag{3.1}$$

and for every $\epsilon > 0$ there is a full subtree T such that

$$\Big\| \sum_{a\in\beta} x_a^* \Big\| > c \sum_{a\in\beta} \|x_a^*\| - \epsilon$$

for every branch β. Notice that if X is a subspace of c_0 then [**16**]

$$\liminf_{n\to\infty} \|x^* + x_n^*\| \geqslant \|x^*\| + \liminf_{n\to\infty} \|x_n^*\|$$

and this implies directly that X^* has the weak* summable tree property with constant one.

THEOREM 3.3 *Suppose X is a separable Banach space such that X^* has the weak*-summable tree property. Then X is isomorphic to a subspace of c_0.*

Proof. We show that X contains no subspace isomorphic to ℓ_1 and that X has the bounded tree property. To show X contains no copy of ℓ_1 it suffices to show that ℓ_2 does not embed in X^* by [**21**]. Suppose then u_n^* is a weak*-null sequence in X^* so that $\|u_n^*\| = 1$ and $\| \sum_{i\in A} u_i \| \leqslant C|A|^{1/2}$ for any finite subset of A of $\mathbb{N}$, where C is an absolute constant. Then define, for any N, the tree-map on $\mathcal{F}\mathbb{N}$ by

$$x_a^* = u_{m_n}^* \qquad \text{if } a = \{m_1, \ldots, m_n\} \text{ and } 1 \leqslant n \leqslant N$$

and x_a^* otherwise. It is clear that any full subtree T has a branch β with $\| \sum_{a\in\beta} u_a^* \| \leqslant CN^{1/2}$

while $\sum_{a\in\beta}\|u_a^*\| = N$, and therefore if N is large enough we obtain a contradiction to the weak* summable tree property.

Next we need a duality argument. We assume that X^* has the weak* summable tree property with constant $c > 0$. We show that X has the bounded tree property with constant σ for any $0 < \sigma < c/2$. Indeed if not there is by Lemma 3.1 a weakly null tree-map $(x_a)_{a\in S}$ with the properties that $\|x_a\| \leqslant \sigma$ for all a and $\|\sum_{a\in\beta} x_a\| > 1$ for every branch β. For each branch β pick $u_\beta^* \in X^*$ with $\|u_\beta^*\| = 1$ and $\langle\sum_{a\in\beta} x_a, u_\beta^*\rangle > 1$. Let h be the height function of the given tree-map. For each $a \in S$ we define y_a^* by transfinite induction on $h(a)$. If $h(a) = 0$ let $y_a^* = u_\beta^*$ where β is any branch to which a belongs. Then if (y_a^*) has been defined for $h(a) < \eta$ and if $h(b) = \eta$ we define (y_b^*) to be any weak*-cluster point of $(y_a^*)_{a\in b+}$. (Note that according to our definition (y_a^*) is a tree-assignment but not necessarily a tree-map because it is not supported on a well-founded tree and we might have $y_\emptyset^* \neq 0$.)

Let us now make a tree-map by defining $x_\emptyset^* = 0$ and then if $h(a-) \geqslant 1$ we define $x_a^* = y_a^* - y_{a-}^*$. If $h(a-) = 0$ we define $x_a^* = 0$. This is clearly a tree-map which also satisfies (3.1) and we have that for each $a \in S$, zero is a weak*-cluster point of $(x_b^*)_{b\in a+}$. It is then easy to see that we can pass to a full subtree T so that $(x_a^*)_{a\in T}$ is weak*-null. Let $x^* = y_\emptyset^*$.

Now pick $\epsilon > 0$ such that $3\epsilon + 2c^{-1}\sigma < 1$. We can use the definition of the weak*-summable tree property and also [**9**, Lemma 3.3] to pass to a further full subtree (still labelled T) so that we have $|\langle x_a, x^*\rangle| < \epsilon/2^{|a|}$ when $|a| > 0$ and for any branch $\beta \subset T$

$$\left|\left\langle\sum_{a\in\beta} x_a, \sum_{a\in\beta} x_a^*\right\rangle - \sum_{a\in\beta}\langle x_a, x_a^*\rangle\right| \leqslant \epsilon$$

$$c\left(\sum_{a\in\beta}\|x_a^*\| - \epsilon\right) \leqslant \left\|\sum_{a\in\beta} x_a^*\right\|.$$

For any branch β let b be the first point for which $h(b) = 0$. Then

$$\left\|x^* + \sum_{a\in\beta} x_a^*\right\| = \|y_b^*\| = 1.$$

It follows that

$$\left\|\sum_{a\in\beta} x_a^*\right\| \leqslant 2.$$

Now we have

$$\begin{aligned} 1 < \left\langle\sum_{a\in\beta} x_a, y_b^*\right\rangle &\leqslant \sum_{a\in\beta}|\langle x_a, x^*\rangle| + \left|\left\langle\sum_{a\in\beta} x_a, \sum_{a\in\beta} x_a^*\right\rangle\right| \\ &\leqslant 2\epsilon + \sum_{a\in\beta}|\langle x_a, x_a^*\rangle| \leqslant 2\epsilon + \sigma\sum_{a\in\beta}\|x_a^*\| \\ &\leqslant 3\epsilon + c^{-1}\sigma\left\|\sum_{a\in\beta} x_a^*\right\| \leqslant 3\epsilon + 2c^{-1}\sigma. \end{aligned}$$

This gives a contradiction and so we deduce that X has the bounded tree property and we can apply Theorem 3.2 to obtain the result.

4. The very strong Schur property

We shall say that a tree-assignment $(x_a)_{a\in S}$ in X is δ-*separated* if $\|x_b - x_{b'}\| \geqslant \delta$ whenever $b, b' \in S$ are such that $b, b' \in a+$ for some $a \in S$. Let us say that a Banach space X has the *very strong Schur property* if there is a constant $c > 0$ such that whenever $(x_a)_{a\in S}$ is a δ-separated bounded tree-assignment then there is a branch β and $x^* \in B_{X^*}$ with $|x^*(x_a)| \geqslant c\delta$ whenever $\emptyset \neq a \in \beta$.

We first justify this terminology.

PROPOSITION 4.1 *Suppose X is a Banach space with the very strong Schur property. Then X has the strong Schur property.*

Proof. We verify condition (iii) of Proposition 2.1 with $\epsilon = \frac{1}{2}$. Let (x_n) be a normalized sequence with $\inf_{m>n} \|x_m - x_n\| \geqslant \frac{1}{2}$. Form a tree-assignment $(y_a)_{a\in\mathcal{F}\mathbb{N}}$ by putting $y_\emptyset = 0$ and then $y_a = x_n$ if $n = \max a$. Then (y_a) is a bounded $\frac{1}{2}$-separated tree-assignment and so there is a branch β and $x^* \in B_{X^*}$ with $|x^*(y_a)| \geqslant \frac{1}{2}c$. This leads to a subsequence $(x_{n_k})_{k=1}^\infty$, where $|x^*(x_{n_k})| \geqslant \frac{1}{2}c$ and Proposition 2.1 (iii) holds with either x^* or $-x^*$.

There is an important situation when the converse is true.

THEOREM 4.2 *Suppose that Y is a crude L-ideal. If X is a closed subspace of Y with the strong Schur property then X has the very strong Schur property.*

Proof. We may suppose that Y is a θ-L-ideal where $0 < \theta \leqslant 1$. Suppose P is the associated L-projection. We also use Proposition 2.1 (iv) to deduce that there is a constant $c > 0$ such that if $(x_n)_{n\in\mathbb{N}}$ is any bounded sequence in X with $\inf_{m\neq n} \|x_m - x_n\| \geqslant \delta > 0$ then (x_n) has a subsequence (w_n) such that we have an estimate

$$\left\| \sum_{k=1}^{\infty} \alpha_k w_k \right\| \geqslant c \sum_{k=1} |\alpha_k| \tag{4.1}$$

for all finitely non-zero sequences (α_k).

Now suppose $(x_a)_{a\in S}$ is a δ-separated tree-assignment. Let $\sigma = \frac{1}{4}c\theta$. We shall show by an inductive construction that there is a branch β and for each $a \in \beta$, $x_a^* \in X^*$ with $\|x_a^*\| < 1$ so that $|x_a^*(x_b)| \geqslant \sigma\delta$ if $\emptyset \neq b \leqslant a$. This will complete the proof since then we can take x^* as any weak* cluster point of $\{x_a^* : a \in \beta\}$.

We start the branch with $\emptyset$. Now suppose $a \in \beta$; we must choose a successor $b \in a+$ and a corresponding x_b^*. First let y_a^* be any norm-preserving extension of x_a^* to Y. Next we pick a subsequence $w_n = x_{b_n}$ of $\{x_b : b \in a+\}$ satisfying (4.1). Let x^{**} be any weak*-cluster point of $(w_n)_{n=1}^\infty$.

Suppose $y \in Y$. Let $x^{**} - y$ belong to the weak*-closed convex hull W_k of $\{w_n - y\}_{n=k}^\infty$; then 0 is in the norm-closure of the set $W_k - \|x^{**} - y\| B_Y$. We deduce that for any $\epsilon > 0$ we can find convex combinations $\sum_{j=1}^k \alpha_j(w_n - y)$ and $\sum_{j=k+1}^l \alpha_j(w_n - y)$ of norm at most $\|x^{**} - y\| + \epsilon/2$. Hence

$$2c\delta \leqslant \left\| \sum_{j=1}^{k} \alpha_j w_j - \sum_{j=k+1}^{l} \alpha_j w_j \right\| \leqslant 2\|x^{**} - y\| + \epsilon.$$

Thus $d(x^{**}, Y) \geqslant c\delta$.

In particular, $\|x^{**} - Px^{**}\| \geqslant c\delta$. Let E be the linear span of $\{x_d : d \leqslant a\} \cup \{Px^{**}, x^{**} - Px^{**}\}$.

We define a linear functional φ on E by $\varphi(e) = y_a^*(e)$ if $e \in E \cap Y$ and $\varphi(x^{**} - Px^{**}) = 2\sigma$ if $y_a^*(Px^{**}) \geqslant 0$ and $\varphi(x^{**} - Px^{**}) = -2\sigma$ if $y_a^*(Px^{**}) < 0$. For any $e \in E$ we have $e = e_0 + \lambda(x^{**} - Px^{**})$, where $e_0 \in E \cap Y$ and $\lambda \in \mathbb{R}$. Then

$$|\varphi(e)| \leqslant |y_a^*(e_0)| + 2|\lambda||\sigma| \leqslant \|x_a^*\|\|e_0\| + \tfrac{1}{2}\theta\|e - Pe\|.$$

Hence $\|\varphi\| < 1$. It follows that φ has a weak*-continuous extension $y^* \in Y^*$ with $\|y^*\| < 1$. Now $|\langle y^*, x^{**}\rangle| \geqslant 2\sigma$ and hence we can pick n such that $|y^*(w_n)| = |y^*(x_{b_n})| \geqslant \sigma$. We thus select $b = b_n$ and set $x_b^* = y^*|_X$. This inductive process establishes our result.

Observe that a closed subspace of L_1 has the very strong Schur property if and only if it has the strong Schur property since L_1 is an L-ideal in its bidual. It follows therefore that the examples constructed by Bourgain and Rosenthal [**2**] show that for subspaces of L_1, the very strong Schur property does not imply embeddability into ℓ_1 or even the Radon–Nikodym property. However, Johnson and Odell [**13**] showed that a subspace of L_1 with a UFDD and the strong Schur property is isomorphic to a subspace of ℓ_1; see also [**20**]. Thus the presence of some unconditionality is crucial here. This motivates our next theorem.

THEOREM 4.3 *Let X be a separable Banach space with the property that X^* has the very strong Schur property. Assume that X is linearly isomorphic to a subspace of a Banach space with UFDD. Then X is linearly isomorphic to a subspace of c_0.*

REMARK. The assumption that X embeds into a space with UFDD is equivalent to the assumption that X embeds in a space with unconditional basis [**18**, p. 51]. As will be seen in the proof, the theorem holds if X is assumed to have shrinking unconditional type.

Proof. Note first that X cannot contain ℓ_1 by results of [**21**] since X^* has the Schur property. Therefore we can apply Lemma 2.7 to deduce that X can be given an equivalent norm so that it has shrinking unconditional type. We complete the proof by showing that X has the weak*-summable tree property and applying Theorem 3.3.

Assume that X^* has the very strong Schur property with constant c. We will show that X has the weak*-summable tree property with constant $c/2$. Suppose $(x_a^*)_{a\in S}$ is a weak*-null tree such that

$$\sup_{a\in S}\Big\|\sum_{b\leqslant a} x_b^*\Big\| = M < \infty.$$

Assume that (x_a^*) fails to have a full subtree such that

$$\Big\|\sum_{a\in\beta} x_a^*\Big\| > \frac{c}{2}\sum_{a\in\beta}\|x_a^*\| - \epsilon.$$

Then by considering the tree-map $(x_a^*, \|x_a^*\|)$ in $X^* \times \mathbb{R}$ and using Lemma 3.1 we can find a full subtree $(x_a^*)_{a\in S_1}$ such that for every branch we have

$$\Big\|\sum_{a\in\beta} x_a^*\Big\| \leqslant \frac{c}{2}\sum_{a\in\beta}\|x_a^*\| - \epsilon.$$

Next we can pass to a full subtree S_2 such that for each $a \in S_2$ either $\inf_{b\in a+}\|x_a^*\| > 0$ or

$\sup_{b\in a+}\|x_b^*\| \leqslant 2^{-|a|-3}\epsilon$. We then define a tree assignment (u_b^*) as follows. Put $u_\emptyset^* = 0$. If $a \in S_2$ is such that $\inf_{b\in a+}\|x_b^*\| = 0$ then let $\{u_b^* : b \in a+\}$ be assigned to be any fixed weak*-null normalized sequence. If $a \in S_2$ and $\inf_{b\in a+}\|x_b^*\| > 0$ we let $u_b^* = x_b^*/\|x_b^*\|$ if $b \in a+$. Then $(u_a^*)_{a\in S_2}$ is weak*-null and, using the weak*-lower-semicontinuity of the norm we may pass to a full subtree S_3 so that for any $a \in S_3$ we have

$$\inf_{b,b'\in a+}\|u_b^* - u_{b'}^*\| \geqslant \tfrac{1}{2}.$$

Next we use the fact that X has shrinking unconditional type. For each $a \in S_3$ there is a closed absolutely convex weak*-neighbourhood of the origin W_a such that if $w^* \in W_a$ and $\|w^*\| \leqslant 2M$ then

$$\left| \left\| \sum_{b\leqslant a}\eta_b x_b^* + w^*\right\| - \left\|\sum_{b\leqslant a}\eta_b x_b^* - w^*\right\|\right| \leqslant 2^{-|a|-2}\epsilon$$

for every choice of signs $\eta_b = \pm 1$ for $b \leqslant a$. Let $T = \{a \in S_3 :\ x_a^* \in 2^{|b|-|a|}W_b,\ \text{if } b < a\}$. Then T is a full subtree of S_3.

Let β be any branch in T. We write $\beta = \{a_0, a_1, a_2, \dots\}$, where $a_0 < a_1 < a_2 < \cdots$. Let

$$\sigma_n := \max_{\eta_k=\pm1}\left\|\sum_{k=0}^{n}\eta_k x_{a_k}^* + \sum_{k=n+1}^{\infty}x_{a_k}^*\right\|.$$

Then $\sigma_0 = \left\|\sum_{k=1}^{\infty}x_{a_k}^*\right\|$. Notice that if $\eta_n = -1$ where $n \geqslant 1$ then since $\sum_{k=n+1}^{\infty}x_{a_k}^* \in W_{a_n}$ and $\left\|\sum_{k=n+1}^{\infty}x_{a_k}^*\right\| \leqslant 2M$,

$$\left\|\sum_{k=0}^{n}\eta_k x_{a_k}^* + \sum_{k=n+1}^{\infty}x_{a_k}^*\right\| \leqslant 2^{-n-2}\epsilon + \left\|\sum_{k=0}^{n-1}\eta_k x_{a_k}^* - \sum_{k=n}x_{a_k}^*\right\|.$$

Thus

$$\sigma_n \leqslant \sigma_{n-1} + 2^{-n-2}\epsilon.$$

It follows that

$$\sigma_n \leqslant \sum_{k=1}^{n}2^{-k-2}\epsilon + \left\|\sum_{k=0}^{\infty}x_{a_k}^*\right\|.$$

Thus we conclude that for any branch and any choice of signs η_a we have

$$\left\|\sum_{a\in\beta}\eta_a x_a^*\right\| \leqslant \left\|\sum_{a\in\beta}x_a^*\right\| + \tfrac{1}{4}\epsilon.$$

Next we can use the very strong Schur property and the fact that $(u_a^*)_{a\in T}$ is $\frac{1}{2}$-separated to find a branch β and $u^{**} \in B_{X^{**}}$ with $|u^{**}(u_a^*)| \geqslant \frac{1}{2}c$ for $a \in \beta$. By the construction of (u_a^*) we have

$$\|x_a^* - \|x_a^*\|u_a^*\| \leqslant 2^{-|a|-2}\epsilon$$

so that

$$|u^{**}(x_a^*)| \geqslant \frac{c}{2}\|x_a^*\| - 2^{-|a|-2}\epsilon.$$

Choose $\eta_a = \pm 1$ such that $\eta_a u^{**}(x_a^*) \geqslant 0$. Then

$$u^{**}\Big(\sum_{a\in\beta} \eta_a x_a^*\Big) \geqslant \frac{c}{2}\sum_{a\in\beta} \|x_a^*\| - \tfrac{1}{2}\epsilon.$$

Hence

$$\Big\|\sum_{a\in\beta} x_a^*\Big\| \geqslant \frac{c}{2}\sum_{a\in\beta} \|x_a^*\| - \tfrac{3}{4}\epsilon.$$

This is a contradiction and shows that X has the weak* summable tree property. The proof is complete.

COROLLARY 4.4 *Suppose X and Y are Banach spaces such that X^* and Y^* are isomorphic. Suppose X is isomorphic to a subspace of c_0. Then Y is isomorphic to a subspace of c_0 if and only if Y embeds in a space with UFDD.*

REMARK. We do not know if one can conclude that X and Y are isomorphic if both embed into c_0.

5. The extension property

Let us recall that if X is a Banach space X and E is a closed subspace of X then the pair (E, X) is said to have the λ-*extension property* (λ-(EP)) if, for any compact Hausdorff space K, every bounded operator $T : E \to C(K)$ has a bounded extension $\tilde{T} : X \to C(K)$ with $\|\tilde{T}\| \leqslant \lambda\|T\|$ (Johnson and Zippin [**15**]). We say (E, X) has the EP if it has λ-(EP) for some $\lambda \geqslant 1$. Johnson and Zippin [**15**] showed that if X is a weak*-closed subspace of $\ell_1 = c_0^*$ then (X, ℓ_1) has the EP, although curiously it is unknown whether it has $(1+\epsilon)$-(EP) for any $\epsilon > 0$. See [**23, 24**] for recent progress on extension properties.

As observed in [**15**, Corollary 1.1], using the results of [**17**], the extension property of (X, ℓ_1) depends only on the quotient space ℓ_1/X; hence it follows that if ℓ_1/X is isomorphic to Y^* where Y is a closed subspace of c_0 then (X, ℓ_1) has the extension property (because there is an automorphism τ of ℓ_1 such that $\tau(X)$ is weak*-closed). The aim of this section is to show how the results of the paper can give a partial converse to this theorem.

THEOREM 5.1 *Suppose X is a closed subspace of ℓ_1 so that (X, ℓ_1) has the EP. Then ℓ_1/X has the very strong Schur property.*

REMARK. The result that ℓ_1/X has the Schur property was obtained earlier by the author and A. Pełczyński by somewhat similar arguments. This answered a question of Zippin concerning the case $\ell_1/X \approx L_1$.

Proof. We suppose that (X, ℓ_1) has λ-(EP). Let $Y = \ell_1/X$ and denote by Q_Y the quotient map of ℓ_1 onto Y.

We start by supposing that $(y_a)_{a\in S}$ is a bounded δ-separated tree assignment in $Y = \ell_1/X$. Let E_n be an increasing sequence of finite-dimensional subspaces of Y whose union is dense. We start by observing that for each $a \in S$ and each $n \in \mathbb{N}$ there is an infinite number of $b \in a+$ such that $d(y_b, E_n) > \delta/4$. Indeed, if not there are infinitely many $b \in a+$ such that $d(y_b, E_n) \leqslant \delta/4$ and for each such b we can find $e_b \in E_n$ with $\|y_b - e_b\| \leqslant \delta/4$. The set of such e_b is bounded and so by compactness arguments we obtain $b \neq b'$ with $\|y_b - y_{b'}\| \leqslant 3\delta/4$.

Now we may pass to a full subtree $(y_a)_{a\in T}$ such that there exists a map $\psi : T \setminus \{\emptyset\} \to \mathbb{N}$ with the properties that $d(y_a, E_{\psi(a)}) \geqslant \delta/4$ and if $a = \{n_1, \dots, n_k\}$ where $n_1 < n_2 < \cdots < n_k$ then we have

$$|\{b \in a+ : \psi(b) = m\}| = \begin{cases} 0 & \text{if } m \leqslant n_1 + \cdots + n_k, \\ 1 & \text{if } m > n_1 + \cdots + n_k. \end{cases}$$

Now for each $a \in T \setminus \emptyset$ we can choose $y_a^* \in B_{Y^*} \cap E_{\psi(a)}^{\perp}$ such that $|y^*(y_a)| \geqslant \delta/4$. Note that the set $\{y_a^* : a \in T\}$ forms a weak*-null sequence. For convenience let $y_\emptyset^* = 0$.

Consider the closed unit interval $I = [0, 1]$ and let D be the set of dyadic rationals $k/2^n$, where $1 \leqslant k \leqslant 2^n - 1$ and $n \in \mathbb{N}$. Let Z be the space of all real-valued functions f on I which are continuous on $I \setminus D$ and such that on D both left and right limits $f(q-)$ and $f(q+)$ exist:

$$f(q) = \tfrac{1}{2}(f(q-) + f(q+)).$$

It is easy to see that Z equipped with the sup-norm is isometric to $C(\Delta)$, where Δ is the Cantor set. Then $C(I)$ is a closed subspace of Z and $Z/C(I) \approx c_0(D)$ with the quotient map being given by $Qf = (f(q+) - f(q-))_{q \in D}$.

Now we can define a one–one map $\varphi : T \to D$ with the property that $\lim_{b \in a+} \varphi(b) = \varphi(a)$ and $|\{b : \varphi(b) > \varphi(a)\}| = |\{b : \varphi(b) < \varphi(a)\}| = \infty$ for $a \in T$.

Next define an operator $L : Y \to c_0(D)$ by putting $Ly(q) = y_a^*(y)$ if $\varphi(a) = q$ and $Ly(q) = 0$ otherwise; then $\|L\| \leqslant 1$. Then $LQ_Y : \ell_1 \to c_0(D)$ can be lifted to an operator $U : \ell_1 \to Z$ such that $QU = LQ_Y$ and $\|U\| \leqslant 2$. Then U maps X into $C(I)$ and by assumption this restriction $U|_X$ has an extension $V : \ell_1 \to C(I)$ with $\|V\| \leqslant 2\lambda$. Now $U - V$ factors to an operator $U - V = RQ_Y$, where $R : Y \to Z$ satisfies $\|R\| \leqslant 2(\lambda + 1)$ and $QR = L$.

We can then write R in the form

$$Ry(q) = \langle y, h(q) \rangle,$$

where $h : I \to Y^*$ is weak*-continuous except on points of D and has left and right weak*-limits $h(q-)$ and $h(q+)$ on D with

$$h(q) = \tfrac{1}{2}(h(q-) + h(q+)).$$

Note that $\|h(q)\| \leqslant \lambda + 1$:

$$h(q+) - h(q-) = \begin{cases} y_a^* & \text{if } q = \varphi(a), \\ 0 & \text{if } q \notin \varphi(T). \end{cases}$$

Finally we build a branch $\beta = \{a_0, a_1, \dots\}$ such that for each $n \geqslant 1$ there exists $y_n^* = h(\varphi(a_n)+)$ or $y_n^* = h(\varphi(a_n)-)$ such that

$$|\langle y_{a_k}, y_n^* \rangle| > \delta/10$$

for $1 \leqslant k \leqslant n$. This is done by induction. Let $a_0 = \emptyset$ and a_1 be any element of T with $|a_1| = 1$. Then since

$$\frac{\delta}{4} \leqslant \langle y_{a_1}, y_{a_1}^* \rangle = \langle y_{a_1}, h(\varphi(a_1)+) - h(\varphi(a_1)-) \rangle$$

we can choose an appropriate sign so that the inductive hypothesis holds when $n = 1$. Now suppose $a_0, \dots, a_{n-1}$ have been chosen and that

$$|\langle y_{a_k}, y_{n-1}^* \rangle| > \frac{\delta}{10}$$

for $1 \leqslant k \leqslant n-1$. We shall assume that $y^*_{n-1} = h(\varphi(a_{n-1})+)$; the other case is similar. Then there exists $\eta > 0$ such that if $\varphi(a_{n-1}) < q < \varphi(a_{n-1}) + \eta$ we have, for some $\rho > 0$,

$$|\langle y_{a_k}, h(q)\rangle| > \frac{\delta}{10} + \rho$$

for $1 \leqslant k \leqslant n-1$. Then we can choose $a_n \in a_{n-1}+$ such that $\varphi(a_{n-1}) < \varphi(a_n) < \varphi(a_n) + \eta$. Then

$$|\langle y_{a_k}, h(\varphi(a_n)\pm)\rangle| > \frac{\delta}{10}$$

for $1 \leqslant k \leqslant n-1$. Now

$$\langle y_{a_n}, h(\varphi(a_n)+) - h(\varphi(a_n)-)\rangle \geqslant \frac{\delta}{4}$$

so that we can choose $y^*_n = h(\varphi(a_n)\pm)$ to satisfy the inductive hypothesis. This completes the inductive construction of the branch β. Finally we let y^* be any weak*-cluster point of the sequence $((2\lambda+2)^{-1}y^*_n)_{n=1}^{\infty}$ so that $\|y^*\| \leqslant 1$ and

$$|y^*(y_a)| \geqslant \frac{\delta}{20(\lambda+1)}$$

for all $a \in \beta$. This shows that Y has the very strong Schur property with constant $1/20(\lambda+1)$.

Our next theorem is then a partial converse of the Johnson–Zippin theorem of [**15**].

THEOREM 5.2 *Suppose X is a closed subspace of X such that (X, ℓ_1) has the EP and one of the following holds:*

(i) *ℓ_1/X has a UFDD.*

(ii) *ℓ_1/X is isomorphic to the dual of a Banach space Y which embeds into a space with a UFDD.*

Then ℓ_1/X is isomorphic to the dual of a subspace of c_0 and there is an automorphism τ of ℓ_1 such that $\tau(X)$ is weak*-closed.

Proof. First note that (i) implies (ii). In fact by Theorem 5.1 ℓ_1/X is a Schur space and hence any UFDD is boundedly complete so that ℓ_1/X is a dual of a space with UFDD. If we assume (ii) then Theorem 4.3 and Theorem 5.1 together yield the result.

REMARK. We can replace (i) by the assumption that ℓ_1/X has (UMAP) (in some equivalent norm). Indeed if ℓ_1/X has (UMAP) it is shown in [**7**] that it has *commuting* (UMAP) and hence by [**19**, Lemma 5.2], ℓ_1/X is the dual of a space with (UMAP). Hence by Lemma 2.7 and the remarks following Theorem 4.3 we obtain that ℓ_1/X is the dual of a subspace of c_0. As observed above the classical results of [**17**] yield the existence of the desired automorphism.

Let us also remark that, in the case when ℓ_1/X has a UFDD one can easily deduce (from, say, results of [**14**]) that ℓ_1/X is isomorphic to an ℓ_1-sum of finite-dimensional spaces.

Note what we have proved.

THEOREM 5.3 *Let X be a separable Banach space with a UFDD. If X has the very strong Schur property then X is isomorphic to the dual of a subspace of c_0.*

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