APPROXIMATING SEQUENCES AND BIDUAL PROJECTIONS

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[Received 31 July 95; in revised form 28 May 96]

I. Introduction

SINCE the appearance of Enflo's negative solution to the approximation problem **[3]**, only a few positive general results on the approximation properties have been obtained. However, it is shown in **[2]** that any separable Banach space with the metric approximation property (M.A.P.) has the commuting metric approximation property. More precisely, if X has (MAP) there exists a sequence $\{R_n\}$ of finite rank operators such that

$$\lim_{n \to +\infty} \|x - R_n x\| = 0 \tag{1}$$

for all $x \in X$, with $\lim_{n \to +\infty} \|R_n\| = 1$ and

$$R_n R_k = R_k R_n = R_{\inf(k,n)} \tag{2}$$

for all $k \neq n$. Sequences of finite rank operators which satisfy (1) are called approximating sequences in this paper. The result was known much before in the case of shrinking approximating sequences **[11]**, hence in particular in the reflexive case.

In the present work we exhibit tight connections between the existence of a projection with a w^*-closed kernel in the w^*-closure of an approximating sequence and the construction of commuting approximating sequences. This permits us to improve control of commuting approximating sequences when X does not contain $l_1(\mathbb{N})$, and provides an alternative proof of the results of **[2]** in that case. Combined with techniques from **[5]**, these methods allow us to show that (UMAP) implies commuting (UMAP) for arbitrary separable Banach spaces. We recall that a separable Banach space X has the unconditional metric approximation property (UMAP) if there exists an approximating sequence $\{R_n\}$ on X such that

$$\lim_{n \to +\infty} \|I - 2R_n\| = 1. \tag{3}$$

We show that any separable Banach space with (UMAP) has an approximating sequence satisfying (2) and (3).

We now turn to a detailed discussion of our results. In Section 2 we prove three lemmas which provide commuting approximating sequences. In the case when $\{R_n\}$ is w^*-convergent to a projection with w^*-closed kernel (Lemma II.1) or in the case when the norm of the commutators tend to zero (Lemma II.2), slight perturbations of appropriate convex combinations satisfy (2). If we simply know that the w^*-closure of $\{R_n\}$ contains a projection with w^*-closed kernel, then we need perturbations of operators from the convex semi-group generated by $\{R_n\}$ (Lemma II.3). In Section 3 we use the ball topology (see [**6**]) to show that the assumptions of Lemma II.3 are satisfied by any approximating sequence of contractions on a Banach space X not containing $l_1(\mathbb{N})$. An improvement of ([**2**], Theorem 2.4) in the case $X \not\supset l_1(\mathbb{N})$ follows. Note that the commuting approximating sequence obtained by the approach of [**2**] does not necessarily consist of operators which are uniformly close to the convex semi-group generated by a given approximating sequence.

In Section 4 we use Lemma II.1 and techniques from [**5**] to show that any separable space with (UMAP) has commuting (UMAP) (Theorem IV.1). The crucial point is to show that the kernel of the limit projection is w^*-closed (see Claim IV.3). Note that this point is simpler to show when one assumes that X does not contain $c_0(\mathbb{N})$ ([**7**], Prop. 2.8). A corollary is a satisfactory structure theorem for separable spaces with (UMAP) (Corollary IV.4). An Appendix, which concludes Section IV, contains a simpler proof of Theorem IV.1 in the case of the complex (UMAP) on a complex Banach space. This alternative approach relies on the use of Hermitian operators and on a theorem of Sinclair [**21**]. In Section V we exhibit a subspace J of the dual of a Banach space X not containing $l_1(\mathbb{N})$ which plays an important role in dualization of approximating sequences.

We use classical notation, as can be found e.g. in [**17**]. The closed unit ball of a Banach space X is denoted B_X. We refer to [**2**] and references therein for recent progress on the approximation properties. Our reference for classical notions of Banach space theory is [**17**].

Acknowledgement

This work was completed while the first-named author was visiting the University of Missouri-Columbia during the year 1993/94. It is his pleasure to give his warmest thanks to all those who made this stay possible, and to UMC for its hospitality and support.

II. Construction of commuting approximating sequences

We start with

LEMMA II.1. *Let X be a separable Banach space, with an approximating sequence $\{R_n\}$ such that*

$$Px^{**} = w^* - \lim_{n \to +\infty} R_n^{**} x^{**}$$

exists for all $x^{**} \in X^{**}$, with $P \in L(X^{**})$ a projection with w^*-closed kernel. Then there exists a sequence $\{C_k\}$ of successive convex combinations of $\{R_n\}$, and a sequence $\{B_k\}$ of finite rank operators such that

$$\lim \|C_k - B_k\| = 0$$

and for all $n \neq k$,

$$B_n B_k = B_k B_n = B_{\inf(k,n)}.$$

Proof. We denote $M = (Ker(P))_\perp \subseteq X^*$. Let $Q = X^* \to X^*/M$ be the canonical quotient map, and

$$L_n = QR_n^* = X^* \to X^*/M.$$

For any $y^* \in (X^*/M)^*$, we have

$$w^* - \lim_{n \to \infty} L_n^*(y^*) = w^* - \lim_{n \to +\infty} R_n^{**} Q^*(y^*) = 0 \tag{1}$$

and, since $R_n^{**}(X^{**}) \subseteq X$, (1) means that for all $y^* \in (X^*/M)^*$

$$w - \lim_{n \to \infty} L_n^*(y^*) = 0. \tag{2}$$

It follows from (2) and Lebesgue's dominated convergence theorem (see [**12**]) that

$$w - \lim_{n \to +\infty} (L_n) = 0$$

in the space $K(X^*, X^*/M)$. Therefore there exist successive convex combinations $\{D_k\}$ of $\{L_n\}$ such that $\lim \|D_k\| = 0$. In other words, there exist successive convex combinations $\{U_k\}$ of $\{R_n\}$ such that

$$\lim_{n \to +\infty} \|QU_k^*\| = 0. \tag{3}$$

We still have $P = w^* - \lim (U_k^{**})$, hence for all $x^* \in M$ and $x^{**} \in X^{**}$

$$\lim_{k \to +\infty} \langle x^{**}, U_k^* x^* \rangle = \langle Px^{**}, x^* \rangle = \langle x^{**}, x^* \rangle$$

and hence for all $x^* \in M$,

$$w - \lim_{k \to +\infty} U_k^*(x^*) = x^*. \tag{4}$$

By (4), M is separable and there exists a sequence $\{C_n\}$ of successive convex combinations of $\{U_k\}$ such that for all $x^* \in M$

$$\lim_{n \to +\infty} \|C_n^* x^* - x^*\| = 0 \tag{5}$$

and by (3), we still have

$$\lim_{n \to +\infty} \|QC_n^*\| = 0. \tag{6}$$

Since $M^\perp = Ker(P)$ is the kernel of a bounded projection, M is locally complemented in X^* ([13]), that is, there exists $\lambda \in \mathbb{R}$ such that for every finite-dimensional subspace F of X^*, there exists A: $F \to M$ with $\|A\| \leqslant \lambda$ and $L(x^*) = x^*$ for all $x^* \in F \cap M$. If follows from (6) and a proper choice of a finite-dimensional space F_n containing $C_n^*(X^*)$ that there exist operators V_n, with $V_n^* = A_n C_n^*$, such that

$$\lim_{n \to +\infty} \|C_n - V_n\| = 0 \tag{7}$$

and

$$V_n^*(X^*) \subseteq M \tag{8}$$

for all $n \geqslant 1$. Clearly, $\{V_n\}$ is an approximating sequence and $\{V_n^*\}$ satisfies (5). It then follows from (8) that for all $n \geqslant 1$

$$\lim_{k \to +\infty} \|V_k V_n - V_n\| = 0 = \lim_{k \to +\infty} \|V_k^* V_n^* - V_n^*\|.$$

Now a perturbation lemma (see [22], Proof of Lemma III.9.2, p. 315-316) provides a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ and a sequence $\{B_k\}$ of finite rank operators such that

$$\lim_{k \to +\infty} \|V_{n_k} - B_k\| = 0, \tag{9}$$

and for all $n > k$, $B_n B_k = B_k$ and $B_n^* B_k^* = B_k^*$, thus $B_k B_n = B_k$. The lemma now follows from (7) and (9).

We observe now that ([2], Cor. 2.2) can be obtained through convex combinations.

LEMMA II.2. *Let X be a separable Banach space with an approximating sequence $\{R_n\}$ such that*

$$\lim_{n \to +\infty} \left\{ \sup_{k \geqslant n} \|[R_k, R_n]\| \right\} = 0. \tag{10}$$

Then there exists a sequence $\{C_k\}$ of successive convex combinations of $\{R_n\}$, and a sequence $\{B_k\}$ of finite rank operators with

$$\lim \|C_k - B_k\| = 0$$

and for all $n \neq k$

$$B_n B_k = B_k B_n = B_{\inf(k,n)}.$$

Proof. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We let

$$Px^{**} = w^* - \lim_{n \to \mathcal{U}} (R_n^{**} x^{**}).$$

We have

$$\begin{aligned} P^2 x^{**} &= w^* - \lim_{n \to \mathcal{U}} R_n^{**}(Px^{**}) \\ &= w^* - \lim_{n \to \mathcal{U}} R_n^{**}\left(w^* - \lim_{k \to \mathcal{U}} R_k^{**} x^{**}\right) \\ &= w^* - \lim_{n \to \mathcal{U}} \left(w^* - \lim_{k \to \mathcal{U}} R_n^{**} R_k^{**} x^{**}\right). \end{aligned}$$

It follows from (10) that

$$P^2 x^{**} = w^* - \lim_{n \to \mathcal{U}} \left(w^* - \lim_{k \to \mathcal{U}} R_k^{**} R_n^{**} x^{**}\right).$$

Since $\{R_k\}$ is an approximating sequence, we have

$$w^* - \lim_{k \to \mathcal{U}} R_k^{**} R_n^{**} x^{**} = R_n^{**} x^{**} \tag{11}$$

and it follows that $P^2 x^{**} = Px^{**}$. We claim that $Ker(P)$ is w^*-closed. Indeed denote

$$\varepsilon_n = \sup\{\|[R_k, R_n]\|; k \geq n\}.$$

It follows from (11) that for any $x^{**} \in X^{**}$

$$\lim_{k \to \mathcal{U}} \|R_n^{**} R_k^{**} x^{**} - R_n^{**} x^{**}\| \leq \varepsilon_n \|x^{**}\|.$$

If $x^{**} \in Ker(P)$ we have, since R_n^{**} is w^*-to-norm continuous, that

$$\lim_{k \to \mathcal{U}} \|R_n^{**} R_k^{**} x^{**}\| = 0$$

and thus

$$\|R_n^{**} x^{**}\| \leq \varepsilon_n \|x^{**}\|. \tag{12}$$

Hence

$$Ker(P) \cap B_{X^{**}} = \{x^{**} \in B_{X^{**}}; \|R_n^{**} x^{**}\| \leq \varepsilon_n \text{ for all } n \geq 1\} \tag{13}$$

since the reverse inclusion is clear, and (13) shows that $Ker(P) \cap B_{X^{**}}$ is

w^*-closed. Hence $Ker(P)$ is w^*-closed by the Banach-Dieudonné theorem. We also observe that (13) shows that $Ker(P)$ does not depend upon the choice of $\mathcal{U}$. Let $M \subseteq X^*$ be such that $Ker(P) = M^{\perp}$. For any $x^* \in M$ and any free ultrafilter $\mathcal{U}$, we have

$$\lim_{n \to \mathcal{U}} \langle x^{**}, R_n^* x^* \rangle = \langle x^{**}, x^* \rangle;$$

hence for all $x^* \in M$,

$$w - \lim_{n \to +\infty} R_n^* x^* = x^*. \tag{14}$$

Now we let again Q: $X^* \to X^*/M$ be the canonical quotient map, and $L_n = QR_n^*$. Since $Q^*((X^*/M)^*) = M^{\perp}$, it follows from (12) that for any $y^* \in (X^*/M)^*$,

$$\lim_{n \to +\infty} \|L_n^* y^*\| = 0. \tag{15}$$

Note that by (14), we have for any sequence $\{U_k\}$ of successive convex combinations of $\{R_n\}$ and any $x^* \in M$ that

$$w - \lim_{k \to +\infty} U_k^* x^* = x^*. \tag{16}$$

We may now finish the proof along the lines of the proof of Lemma II.1, substituting (15) and (16) to (2) and (4).

Our next lemma addresses the slightly more complicated situation when there is a projection P with w^*-closed kernel in the w^*-closure of an approximating sequence.

LEMMA II.3. *Let X be a separable Banach space with an approximating sequence $\{R_n\}$ such that for some ultrafilter $\mathcal{U}$,*

$$Px^{**} = w^* - \lim_{n \to \mathcal{U}} R_n^{**} x^{**}$$

defines a projection $P \in L(X^{**})$ with ω^*-closed kernel. Then there exists an approximating sequence $\{C_k\}$ of convex combinations of the products $\{R_i R_j;\ j > i \geqslant 1\}$, and a sequence $\{B_k\}$ of finite rank operators such that

$$\lim \|C_k - B_k\| = 0$$

and for all $n \neq k$

$$B_n B_k = B_k B_n = B_{\inf(k,n)}.$$

Proof. Letting as before $M = (KerP)_{\perp}$, we have for all $x^* \in M$

$$w - \lim_{n \to \mathcal{U}} R_n^* x^* = x^*. \tag{17}$$

Since the R_n's are finite rank operators, the space

$$Z = \overline{\text{span}}\{R_n^* x^*; x^* \in X^*, n \geq 1\}$$

is separable, and so is M which is by (17) a subspace of Z. Note that if $Q = X^* \to X^*/M$ is the canonical quotient map, we have for any $x^* \in X^*$ and any $x^{**} \in M^\perp$ that

$$\lim_{n \to \mathcal{U}} \langle x^{**}, QR_n^* x^* \rangle = \langle PQ^* x^{**}, x^* \rangle = 0;$$

hence for any $x^* \in X^*$

$$w - \lim_{n \to \mathcal{U}} QR_n^* x^* = 0. \tag{18}$$

It now follows from (17), (18) and the separability of Z that there exists a sequence $\{D_n\}$ of successive convex combinations of $\{R_k\}$ such that for all $x^* \in Z$

$$\lim_{n \to +\infty} dist(D_n^* x^*, M) = 0, \tag{19}$$

and for all $x^* \in M$,

$$\lim_{n \to +\infty} \|x^* - D_n^* x^*\| = 0. \tag{20}$$

Clearly $\{D_n\}$ is still an approximating sequence. We now observe that if S and T are operators such that $\|Sx - x\| < \varepsilon$ and $\|Tx - x\| < \varepsilon$ for some $x \in X$ and $\varepsilon > 0$ then

$$\begin{aligned} \|STx - x\| &= \|S(Tx - x) + (Sx - x)\| \\ &\leq (\|S\| + 1)\varepsilon. \end{aligned}$$

Using this observation, we find that for any subsequence $\{U_k\}$ of $\{D_n\}$, if we let

$$V_k = U_k U_{k+1}$$

then $\{V_k\}$ is still an approximating sequence and $\{V_k^*\}$ still satisfies (20). Since the D_n's are finite rank operators, (19) shows that the subsequence can be chosen in such a way that

$$\lim_{k \to +\infty} \|QV_k^*\| = \lim_{k \to +\infty} \|QU_{k+1}^* U_k^*\| = 0. \tag{21}$$

It now follows from (20) (with the V_k^*'s) that for any $n \geq 1$

$$\limsup_{k \to +\infty} \|V_k^* V_n^* - V_n^*\| \leq (1 + M)\, \|QV_n^*\| \tag{22}$$

with $M = \sup\{\|V_k^*\|: k \geq 1\}$, while on the other hand

$$\lim_{k\to+\infty} \|V_k V_n - V_n\| = 0. \tag{23}$$

It is easy to deduce from (21), (22) and (23) that there exists a subsequence $\{W_k\}$ of $\{V_k\}$ such that

$$\lim_{n\to+\infty} (\sup\{\|[W_k, W_n]\|; k \geq n\}) = 0.$$

We can now apply Lemma II.2 to the sequence $\{W_k\}$, and this provides $\{C_k\}$ which satisfies the conclusion of Lemma II.3.

Remarks II.4. **1)** The proof of Lemma II.2 shows that when an approximating sequence $\{R_n\}$ satisfies

$$\lim_{n\to+\infty} \left\{\sup_{k\geq n} (\|[R_k, R_n]\|)\right\} = 0$$

then any of its w^*-cluster points in $L(X^{**})$ is a projection with w^*-closed kernel. This shows that finding such a projection is essentially a necessary step in our constructions. For instance, if an approximating sequence $\{R_n\}$ is w^*-convergent (for the Frechet filter) in $L(X^{**})$, then it satisfies the conclusion of Lemma II.1 if and only if its w^*-limit is a projection with w^*-closed kernel.

2) The projection P from the proof of Lemma II.2 depends in general upon the ultrafilter $\mathcal{U}$. For instance, if $\{R_n\}$ are the partial sums associated to the summing basis of $c_0(\mathbb{N})$, and (e_k^*) is the canonical basis of $l_1(\mathbb{N})$, we have for $n \geq k$

$$R_n^*(e_k^*) = e_k^* - e_{n+1}^*.$$

Hence if $P = w^* - \lim_{n\to\mathcal{U}} (R_n^{**})$ and $u = (u(k)) \in l_\infty(\mathbb{N})$, we have for all $k \geq 1$

$$P(u)(k) = u(k) - \lim_{u\to\mathcal{U}} u(n).$$

III. Commuting approximation in spaces which do not contain $l_1(\mathbb{N})$

The main result of this section is an improvement of ([2], Theorem 2.4) in the special case when the space on which approximation is performed fails to contain $l_1(\mathbb{N})$.

THEOREM III.1. *Let X be a separable space not containing $l_1(\mathbb{N})$, with the metric approximation property. For any approximating sequence $\{T_n\}$ of contractions, there exists an approximating sequence $\{C_k\}$ in the convex semi-group $\mathcal{S}$ generated by the sequence $\{T_n\}$ and a sequence $\{B_k\}$ of finite rank operators such that:*

(i) $\lim_{k\to+\infty} \|C_k - B_k\| = 0$,

(ii) $B_k B_n = B_n B_k = B_{\inf(n,k)}$ *for all* $n \neq k$.

Proof. We denote by $\mathcal{S}^*$ the closure of $\{T^{**}; T \in \mathcal{S}\}$ in $L(X^{**})$ equipped with the w^*-operator topology, and

$$S_0 = \{T \in \mathcal{S}^*;\ T_{\restriction X} = Id_X\}.$$

Note that $\mathcal{S}_0 \neq \varnothing$ since $\{R_n\}$ is an approximating sequence. We equip $\mathcal{S}_0$ with the order relation $\leqslant$ defined by: $S \leqslant T$ if $\|Sx^{**}\| \leqslant Tx^{**}\|$ for all $x^{**} \in X^{**}$.

It follows from w^*-compactness that the set $(\mathcal{S}_0, \leqslant)$ is (downwards) inductive. We denote by P a minimal element.

The set $\mathcal{S}_0$ is a convex semi-group. Indeed convexity is clear, and to check that $(UV) \in \mathcal{S}_0$ when $U \in \mathcal{S}_0$ and $V \in \mathcal{S}_0$, we write

$$U = w^* - \lim U_\alpha^{**},$$
$$V = w^* - \lim V_\beta^{**}$$

and then

$$UV = w^* - \lim_\alpha \lim_\beta (U_\alpha V_\beta)^{**}$$

and $(U_\alpha V_\beta) \in \mathcal{S}$ provided that $U_\alpha \in \mathcal{S}$ and $V_\beta \in \mathcal{S}$.

We now claim that P is a projection. Indeed since P is minimal and $\|S\| \leqslant 1$ for all $S \in \mathcal{S}_0$, we have $\|SPx^{**}\| = \|Px^{**}\|$ for all $S \in \mathcal{S}_0$ and all $x^{**} \in X^{**}$.

Applying this observation to

$$S_n = \frac{1}{n}\left(\sum_{i=1}^{n} P^i\right)$$

provides

$$\begin{aligned}\|(S_nP^2 - S_nP)x^{**}\| &= \|S_nP(Px^{**} - x^{**})\| \\ &= \|P(Px^{**} - x^{**})\| \\ &= \|P^2x^{**} - Px^{**}\|.\end{aligned}$$

But since we have

$$S_nP^2 - S_nP = \frac{1}{n}(P^{n+2} - P^2),$$

we have $\|P^2x^{**} - Px^{**}\| \leqslant 2n^{-1}$ for all $n \geqslant 1$, hence $P^2 = P$. Clearly, we have $\|P\| = 1$. We need the following crucial claim.

Claim III.2. If $X \not\supset l_1(\mathbb{N})$ *and* $P = X^{**} \to X^{**}$ *is a projection with* $\|P\| = 1$ *and* $P(X^{**}) \subseteq X$, *the space* $Ker(P)$ *is* w^**-closed.*

Proof of Claim III.2.

Recall that the ball topology b_Y on a Banach space Y equipped with a given norm, is the coarsest topology for which the closed balls are closed (see [6]). If $X \not\supset l_1(\mathbb{N})$ and $X \subseteq Y \subseteq X^{**}$ then the restriction of b_Y to the bounded subsets of Y is Hausdorff ([6], Th. 9.3) when Y is equipped with the norm induced by the bidual norm.

We let $Y = P(X^{**})$. Let $(x_\alpha) \subseteq B_Y$ be a w^*-convergent net in X^{**}, and put $x^{**} = w^* - \lim (x_\alpha)$. For any $y \in Y$, we have

$$\begin{aligned} \liminf \|x_\alpha - y\| &\geqslant \|x^{**} - y\| \\ &\geqslant \|Px^{**} - y\| \end{aligned}$$

and thus

$$P(x^{**}) = b_Y - \lim (x_\alpha).$$

Pick $x^{**} \in Ker(P)$ with $\|x^{**}\| \leqslant 1$. There is a net (x_α) in B_X, hence in B_Y, such that $x^{**} = w^* - \lim (x_\alpha)$ and by the above $b_Y - \lim (x_\alpha) = 0$.

Conversely, if there is a net (x_α) in B_Y such that $x^{**} = w^* - \lim (x_\alpha)$ and $0 = b_Y - \lim (x_\alpha)$, then we have

$$P(x^{**}) = b_Y - \lim (x_\alpha)$$

and since b_Y is Hausdorff it follows that $P(x^{**}) = 0$. Thus we have

$$Ker(P) \cap B_X^{**} = \bigcap \{\bar{V}^*;\ V \text{ is a } b_Y \text{ neighbourhood of } 0 \text{ in } B_Y\}.$$

Thus $Ker(P) \cap B_{X^{**}}$ is w^*-closed, and Claim III.2 follows by the Banach-Dieudonné theorem.

To conclude the proof of Theorem III.1 we observe that since the semi-group $\mathscr{S}$ is uniformly separable, we can find a sequence $\{R_n\}$ in $\mathscr{S}$ such that

$$Px^{**} = w^* - \lim_{n \to \mathscr{U}} R_n^{**} x^{**}$$

for some ultrafilter $\mathscr{U}$ and for all $x^{**} \in X^{**}$. By Claim III.2 we may apply Lemma II.3 which concludes the proof.

The following corollary is a restatement of Theorem III.1 in isomorphic terms.

Corollary III.3. *Let X be a separable Banach space which does not contain $l_1(\mathbb{N})$. Let S be a uniformly bounded convex semi-group of finite rank operators such that Id_X belongs to the closure of $\mathscr{S}$ in the weak operator topology. Then there exist a sequence $\{S_n\}$ in $\mathscr{S}$, and a sequence $\{B_n\}$ of finite rank operators such that*

(i) $\lim_{n \to +\infty} \|S_n - B_n\| = 0$;

(ii) $\lim_{n\to+\infty} \|x - B_n x\| = 0$ *for all* $x \in X$;

(iii) $B_n B_k = B_k B_n = B_{\inf(n,k)}$ *for all* $n \neq k$.

Proof. If $|.|$ denotes the original norm we define

$$\|x\| = \sup\{|Sx|;\ S \in \mathscr{S}\}.$$

It follows from our assumptions that $\|.\|$ is an equivalent norm on X. Clearly $\|S\| \leq 1$ for all $S \in \mathscr{S}$. Using the separability of X, we easily construct an approximating sequence $\{T_n\}$ in $\mathscr{S}$. Now the corollary is an immediate consequence of Theorem III.1.

We do not know whether Corollary III.3 (or equivalently, Theorem III.1) holds true for an arbitrary separable Banach space.

IV. The unconditional metric approximation property

We recall that a separable Banach space X has the unconditional metric approximation property (in short, (UMAP)) if there exists an approximating sequence $\{R_n\}$ such that $\lim_{n\to+\infty} \|I - 2R_n\| = 1$. This notion is defined and studied in [2]. The main result of this section, which answers positively ([5], Questions 6 and 8), asserts in particular that as soon as the (UMAP) holds, it can be achieved by commuting operators.

THEOREM IV.1. *Let X be a separable Banach space, with an approximating sequence $\{R_n\}$ such that*

$$\lim_{n\to+\infty} \|I - 2R_n\| = 1.$$

Then there exists a sequence $\{C_k\}$ of successive convex combinations of $\{R_n\}$, and a sequence $\{B_k\}$ of finite rank operators such that:

(i) $\lim_{k\to+\infty} \|C_k - B_k\| = 0$,

(ii) $B_n B_k = B_k B_n = B_{\inf(k,n)}$ *for all* $n \neq k$.

In particular, every separable Banach space with (UMAP) has the commuting (UMAP).

Proof. Let $\{R_n\}$ be as above. Then, by ([5], Th. 6.5 and Th. 7.5), for any $x^{**} \in X^{**}$,

$$Tx^{**} = w^* - \lim_{n\to+\infty} R_n^{**} x^{**}$$

exists, and $T \in L(X^{**})$ is a projection from X^{**} onto the w^*-sequential closure $Ba(X)$ of X in X^{**}, such that $\|I - 2T\| = 1$. To prove the theorem, it suffices by Lemma II.1 to check that $Ker(T)$ is w^*-closed.

Claim IV.2. If X has (UMAP), and $\{x_\alpha\}$ is a weakly null net in B_X, then for any $x \in X$

$$\lim_\alpha (\|x + x_\alpha\| - \|x - x_\alpha\|) = 0.$$

Proof of Claim IV.2.

With the above notation, for any $x \in X$ and $\varepsilon > 0$ there is $k_0 \in \mathbf{N}$ such that

$$\|x - R_{k_0}x\| < \varepsilon/4$$

and

$$\|I - 2R_{k_0}\| < 1 + \varepsilon.$$

Since $\{x_\alpha\}$ is weakly null we have for $\alpha \geqslant \alpha_0$

$$\|R_{k_0}(x_\alpha)\| < \varepsilon/4.$$

Hence for $\alpha \geqslant \alpha_0$

$$\|x - R_{k_0}(x - x_\alpha)\| < \varepsilon/2$$

and

$$\|x_\alpha - (I - R_{k_0})(x_\alpha - x)\| < \varepsilon/2.$$

By addition, it follows that for $\alpha \geqslant \alpha_0$

$$\|(x + x_\alpha) - (I - 2R_{k_0})(x_\alpha - x)\| < \varepsilon$$

and thus if $\alpha \geqslant \alpha_0$

$$\|x + x_\alpha\| \leqslant (I + \varepsilon)\ \|x - x_\alpha\| + \varepsilon.$$

Claim IV.2 follows since $\varepsilon > 0$ is arbitrary.

We now prove another crucial result.

Claim IV.3. If X has (UMAP), then

$$S = \{x^{**} \in B_{X^{**}};\ \|x^{**} + x\| = \|x^{**} - x\| \text{ for all } x \in X\}$$

is a w^*-closed subset of X^{**}.

Proof. Pick $u^{**} \in \bar{S}^{w^*}$. There exists a net $\{u_\alpha\}$ in B_X such that

$$u^{**} = w^* - \lim_\alpha (u_\alpha) \tag{1}$$

and

$$\tau(x) = \lim_\alpha \|x + u_\alpha\| \tag{2}$$

exists for all $x \in X$ and satisfies $\tau(x) = \tau(-x)$. The local reflexivity principle implies (see e.g. [9]) the existence of a net $\{v_\beta\}$ in B_X such that

$$u^{**} = w^* - \lim (v_\beta), \tag{3}$$

and for all $x \in X$

$$\|x + u^{**}\| = \lim_{\beta} \|x + v_\beta\|. \tag{4}$$

By Claim IV.2 it suffices to show that for any $x \in X$, $\delta > 0$ and W a weak neighbourhood of 0 in X, there exists $t \in W$ such that

$$|\|x + u^{**}\| - \|x + t\|| < \delta. \tag{5}$$

Pick such x, δ and W and let $\eta > 0$ and $N \geqslant 1$ to be chosen later. By (3) and (4), there is β_0 such that

$$|\|x + w\| - \|x + u^{**}\|| < \eta \tag{6}$$

for all $w \in conv\{v_\beta; \beta \geqslant \beta_0\}$. Let $M = 4N + 1$, and pick $\beta_1, \beta_2, \ldots, \beta_M \geqslant \beta_0$. Since $\tau(y) = \tau(-y)$ for all $y \in X$, there exists

$$\alpha_0 = \alpha_0(\beta_1, \beta_2, \ldots, \beta_M)$$

such that if $\alpha \geqslant \alpha_0$ then for all $\varepsilon_i \in \{-1, 1\}$ $(1 \leqslant i \leqslant M)$ one has

$$\left|\left\|x + \frac{1}{M}\sum_{i=1}^{M} \varepsilon_i v_{\beta_i} + \frac{v_\alpha}{M}\right\| - \left\|x + \frac{1}{M}\sum_{i=1}^{M} \varepsilon_i v_{\beta_i} - \frac{v_\alpha}{M}\right\|\right| < \eta.$$

It follows from (1), (3) and Claim IV.2 that there exist

$$B_M = B_M^0(\beta_1, \beta_2, \ldots, \beta_{M-1})$$

and

$$\alpha_1 = \alpha_1(\beta_1, \beta_2, \ldots, \beta_{M-1})$$

such that if $\beta_M \geqslant \beta_M^0$ and $\alpha \geqslant \alpha_1$, then

$$\left|\left\|x + \frac{1}{M}\sum_{i=1}^{M-1} v_{\beta_i} + \frac{1}{M}(v_{\beta_M} - u_\alpha)\right\| - \left\|x + \frac{1}{M}\sum_{i=1}^{M-1} v_{\beta_i} - \frac{1}{M}(v_{\beta_M} - u_\alpha)\right\|\right| < \eta.$$

Hence if $\beta_M \geqslant B_M^0$ and $\alpha \geqslant \sup\{\alpha_0, \alpha_1\}$, we have

$$\left|\left\|x + \frac{1}{M}\sum_{i=1}^{M-1} v_{\beta_i} + \frac{1}{M}(v_{\beta_M} + u_\alpha)\right\| - \left\|x + \frac{1}{M}\sum_{i=1}^{M-1} v_{\beta_i} - \frac{1}{M}(v_{\beta_M} + u_\alpha)\right\|\right| < 3\eta.$$

We now proceed to the iterative part of the proof.

By (1) and (3), the net

$$\{v_{\beta_{M-2}} + v_{\beta_{M-1}} - v_{\beta_M} - u_\alpha\}$$

is weakly null. Hence by Claim IV.2 there exist for $j \in \{M-2, M-1, M\}$

$$B_j^1 = B_j^1(\beta_1, \beta_2, \ldots, \beta_{M-3})$$

and

$$\alpha_2 = \alpha_2(\beta_1, \beta_2, \ldots, \beta_{M-3})$$

such that if $\beta_j \geq B_j$ and $\alpha \geq \alpha_2$ then

$$\left| \left\| x + \frac{1}{M} \sum_{i=1}^{M-1} v_{\beta_i} - \frac{1}{M}(v_{\beta_M} + u_\alpha) \right\| - \left\| x + \frac{1}{M} \sum_{i=1}^{M-3} v_{\beta_i} - \frac{1}{M}(v_{\beta_{M-2}} + v_{\beta_{M-1}}) + \frac{1}{M}(v_{\beta_M} + u_\alpha) \right\| \right| < \eta.$$

Hence if $\beta_{M-2} \geq B^1_{M-2}$, $\beta_{M-1} \geq B^1_{M-1}$, $\beta_M \geq \sup\{B^0_M, B^1_M\}$, $\alpha \geq \sup\{\alpha_0, \alpha_1, \alpha_2\}$, we have

$$\left| \left\| x + \frac{1}{M} \sum_{i=1}^{M} v_{\beta_i} + \frac{u_\alpha}{M} \right\| - \left\| x + \frac{1}{M} \sum_{i=1}^{M-3} v_{\beta_i} - \frac{1}{M}(v_{\beta_{M-2}} + v_{\beta_{M-1}}) + v_{\beta_M} - \frac{1}{M} u_\alpha \right\| \right| < 5\eta.$$

We continue in this manner, this time using the fact that the net

$$\{v_{\beta_{M-4}} + v_{\beta_{M-3}} - v_{\beta_{M-2}} - v_{\beta_{M-1}} + v_{\beta_M} - u_\alpha\}$$

is weakly null.

Using again Claim IV.2 we find that under suitable conditions on the β_i's, $(M-4 \leq i \leq M)$ and α,

$$\left| \left\| x + \frac{1}{M} \sum_{i=1}^{M} v_{\beta_i} + \frac{u_\alpha}{M} \right\| - \left\| x + \frac{1}{M} \sum_{i=1}^{M-5} v_{\beta_i} - \frac{1}{M}(v_{\beta_{M-4}} + v_{\beta_{M-3}}) + \frac{1}{M}(v_{\beta_{M-2}} + v_{\beta_{M-1}}) - \frac{1}{M} v_{\beta_M} - \frac{1}{M} u_\alpha \right\| \right| < 7\eta.$$

After $(2N-1)$ iterations of that procedure, we obtain D_1, $D_j(\beta_{j-1}, \beta_{j-2}, \ldots, \beta_1)(2 \leq j \leq M)$, all greater than β_0, and $A(\beta_1, \beta_2, \ldots, \beta_M)$ such that if $\beta_j \geq D_j (1 \leq j \leq M)$ and $\alpha \geq A$, then

$$\left| \left\| x + \frac{1}{M} \sum_{i=1}^{M} v_{\beta_i} + \frac{u_\alpha}{M} \right\| - \left\| x + z - \frac{u_\alpha}{M} \right\| \right| < (4N+3)\eta,$$

where

$$z = \frac{1}{M} \left[\sum_{j=0}^{N-1} (-v_{\beta_{4j+1}} - v_{\beta_{4j+2}} + v_{\beta_{4j+3}} + v_{\beta_{4j+4}}) \right] - \frac{v_{\beta_M}}{M}.$$

By (6), we have

$$\left| \left\| x + \frac{1}{M} \sum_{i=1}^{M} v_{\beta_i} + \frac{u_\alpha}{M} \right\| - \|x + u^{**}\| \right| < \eta + \frac{1}{M}.$$

Hence if the β_is and α satisfy the above conditions, we have

$$\left| \|x + u^{**}\| - \left\|x + z - \frac{u_\alpha}{M}\right\| \right| < (4N + 4)\eta + \frac{1}{M}. \tag{7}$$

We now choose N and η. There exists $\lambda > 0$ and W' a weak neighbourhood of 0 in X such that

$$(W' + \lambda B_X) \subseteq W.$$

We choose N such that $M = 4N + 1$ satisfies

$$M \geq \sup\{2/\delta, 2/\lambda\}$$

and then $\eta > 0$ such that

$$(4N + 4)\eta + 1/M < \delta. \tag{8}$$

Since β_1, then β_2, then $\beta_3, \ldots$, then β_M can be chosen arbitrarily large such that there exists α for which (7) holds, (3) shows that we can ensure that

$$\frac{1}{M}\left[\sum_{j=0}^{N-1} (-v_{\beta_{4j+1}} - v_{\beta_{4j+2}} + v_{\beta_{4j+3}} + v_{\beta_{4j+4}})\right] \in W'$$

and then

$$t = \left(z - \frac{u_\alpha}{M}\right) \in W.$$

By (7) and (8), t satisfies (5), and this concludes the proof of Claim IV.3.

To conclude the proof of Theorem IV.1 it suffices to check that $Ker(T)$ is w^*-closed, or by the Banach-Dieudonné theorem, that

$$K = Ker(T) \cap B_{X^{**}}$$

if w^*-closed. The set $\bar{K}^{w^*}$ is convex and balanced and by Claim IV.3, it is contained in S. If K is not w^*-closed, we pick $x^{**} \in \bar{K}^{w^*} \setminus K$, and we write $x^{**} = b + s$, with $b \in Ba(X)\setminus\{0\}$ and $s \in K$. Since $b = x^{**} - s$, we have $(b/2) \in \bar{K}^{w^*}$ and hence $b \in S$. But we have $Ba(X) \cap S = \{0\}$ by a result from [**18**]. We can also observe that $K_u(b) = 1$ ([**5**], Lemma 8.1) and thus $Ker(b)$ is not a norming subspace of X^* ([**4**], Lemma 6.3), while $Ker(y^{**})$ is norming for all $y^{**} \in S$.

Let us mention that if we assume that $X \not\supset l_1(\mathbb{N})$ in Theorem IV.1 then $Ker(T) = \{0\}$, while if we assume that $X \not\supset c_0(\mathbb{N})$, the proof that $Ker(T)$ is w^*-closed can be simplified (see [**7**], Prop. 2.8).

Our next result provides a complete description of spaces with (UMAP).

COROLLARY IV.4. *Let X be a separable Banach space. Then:*

1) *X has (UMAP) if and only if for any $\varepsilon > 0$, X is isometric to a 1-complemented subspace of a space V_ε with a $(1+\varepsilon)$-unconditional F.D.D.*

2) *The unconditional F.D.D. in* 1) *can be made shrinking if and only if X does not contain $l_1(\mathbb{N})$.*

3) *The unconditional F.D.D. in* 1) *can be made boundedly complete if and only if X does not contain $c_0(\mathbb{N})$.*

Proof. It is clear that 1-complemented subspaces of spaces with (UMAP) have (UMAP). The "if" implications of Corollary IV.4 follow from this observation. We now show the reverse implications.

Pick $\varepsilon > 0$. if X has (UMAP), there exists ([2], Th. 3.8) a sequence $\{A_n\}$ of finite rank operators such that

$$S_k = \sum_{i=1}^{k} A_i$$

is an approximating sequence and

$$\sup \left\{ \left\| \sum_{i=1}^{N} \varepsilon_i A_i \right\|;\ N \geqslant 1,\ \varepsilon_i = \pm 1 \right\} < 1 + \varepsilon.$$

To prove 1), it suffices to apply ([19]): the space V_ε is defined to be the completion of $(\sum \oplus A_n(X))$ equipped with the norm

$$|||(a_n)||| = \sup \{ \|\sum \varepsilon_n a_n\|_X;\ \varepsilon_n = \pm 1 \}.$$

The map $Q((a_n)) = \sum a_n$ is a quotient map from V_ε onto X, whose right inverse is given by

$$\sigma(x) = (A_n(x)) \in V_\varepsilon.$$

The assertion 2) is in ([5], Th. 9.3). We recall a simple proof, based on a well-known interpolation argument (see [16]). We denote by $\mathscr{G}$ the group of isomorphisms J of $V_\varepsilon = (\sum \oplus E_n)$ defined by $J(\sum e_n) = (\sum \varepsilon_n e_n)$, where $\varepsilon_n \in \{-1, 1\}^{\mathbb{N}}$ is a given choice of signs. We call

$$Q = V_\varepsilon^* \to X^*$$

the canonical quotient map, and we define a new norm of V_ε^* by

$$|||v^*||| = \sup \{ \|QJv^*\|_{X^*};\ J \in \mathscr{G} \}. \tag{9}$$

We denote $\hat{V}_\varepsilon$ the completion of $(\sum' \oplus E_n)$ with respect to the predual norm $|||.|||_*$. It is easily seen that

$$\|x\|_X \geq |||x|||_* \tag{10}$$

for all $x \in S$. Since the F.D.D. $\{E_n\}$ is $(1+\varepsilon)$-unconditional on V_ε, we also have

$$|||v|||_* \geq (1+\varepsilon)^{-1} \|v\| \tag{11}$$

for all $v \in (\sum' \oplus E_n)$. It follows from (10) and (11) that X is isomorphic to a complemented subspace of $\tilde{V}_\varepsilon$. It follows from (9) that $\tilde{V}_\varepsilon$ has a 1-unconditional F.D.D. Since $X \not\supset l_1(\mathbb{N})$, X^* does not contain $c_0(\mathbb{N})$. It follows that $\tilde{V}_\varepsilon^*$ does not contain $c_0(\mathbb{N})$. Indeed if not, there exists a sequence of blocks $(w_n^*) \in \tilde{V}_\varepsilon^*$ with

$$|||w_n^*||| = 1 \tag{12}$$

and

$$M = \sup\left\{ \left|\left|\left| \sum_{l=1}^{N} \varepsilon_l w_l^* \right|\right|\right| ;\ N \geq 1,\ \varepsilon_l = \pm 1 \right\} < \infty. \tag{13}$$

By (12), there exists $J_n \in \mathscr{C}$ such that

$$\|QJ_n w_n^*\|_{X^*} \geq \tfrac{1}{2}. \tag{14}$$

If $x_n^* = QJ_n w_n^*$, for any choice of signs $\eta_l = \pm 1$, there exists since (w_n^*) is a sequence of blocks $\varepsilon_k = \pm 1$ such that

$$\sum{}' \eta_l x_l^* = \sum{}' \varepsilon_k w_k^*$$

and then it follows from (13) that

$$\sup\left\{ \left\| \sum_{l=1}^{N} \eta_l x_l^* \right\|_{X^*} ;\ N \geq 1,\ \eta_l = \pm 1 \right\} \leq M. \tag{15}$$

But (14) and (15) would imply that $X^* \supset c_0(\mathbb{N})$, contradicting $X \not\supset l_1(\mathbb{N})$. This shows 2).

To prove 3), we observe that by the proof of Theorem IV.1 when X has (UMAP) and $X \not\supset c_0(\mathbb{N})$, we have

$$X^{**} = X \oplus_u X_s$$

with $X_s = M^\perp$ a w^*-closed subspace. Moreover if $\{R_n\}$ is an approximating sequence on X with $\lim \|I - 2R_n\| = 1$, we have for all $x^{**} \in X^{**}$

$$w^* - \lim_{n \to +\infty} R_n^{**} x^{**} = Tx^{**} \tag{16}$$

with $T = X^{**} \to X$ the projection of kernel X_s. It follows from (16) that for all $x^* \in M$

$$w - \lim_{n \to +\infty} R_n^* x^* = x^*.$$

We may now follow the lines of the proof of Lemma II.1 to obtain finite rank operators V_n with $V_n^*(X^*) \subseteq M$ and convex combinations $\{C_n\}$ of $\{R_n\}$ such that

$$\lim_{n \to +\infty} \|V_n - C_n\| = 0 \tag{17}$$

and for all $x^* \in M$

$$\lim_{n \to +\infty} \|x^* - V_n^* x^*\| = 0. \tag{18}$$

Clearly, (17) and (18) show that predual M of X has (UMAP). Obviously $M \not\supset l_1(\mathbb{N})$ since $M^* = X$ is separable. Hence 2) applies to M, and then 3) follows by dualization.

Remarks IV.5. It is instructive to compare Corollary IV.4.3) with some negative results. There exist: (a) a separable Banach lattice U with the Radon-Nikodym property such that if $U^{**} = U \oplus S$ is the decomposition of U^{**} in orthogonal bands, S is not w^*-closed [23], although U is the dual of a Banach lattice [24]. (b) A translation invariant subspace X of $L^1(\mathbf{T})$ which is isometric to a dual space, and such that $X^{**} = X \oplus_1 X_s$ but X_s is not w^*-closed [8]. (c) A subspace V of a space with an unconditional basis, such that V has (PCP) but not (RNP). In particular, $V \not\supset c_0(\mathbb{N})$ but V does not embed into a space with boundedly complete unconditional F.D.D. [10].

All these spaces are failing (UMAP). We refer to [7] for (UMAP) in certain subspaces of L^1, and more relevant examples.

We recall that a 1-complemented subspace of a space with a 1-unconditional basis has, in the complex case, a 1-unconditional basis ([14]; see [4], [20]). It is not so in the real case [15]. This yields to the idea that Theorem IV.1 should have a simpler proof in the complex case. It is indeed so, as shown in the Appendix below.

Appendix. An alternative proof of Theorem IV.1 in the complex case.

We recall that a complex Banach space X has complex (UMAP) if there exists an approximating sequence $\{R_n\}$ on X such that $\lim \|I - (1 + \lambda)R_n\| = 1$ for any $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ (see [5], § 8). Using Hermitian operators, we can give a simpler proof of "(UMAP) implies (UCMAP)" in the complex case.

If X has $\mathbb{C}$-UMAP, by ([5], Lemma 8.1) there exists a sequence $\{A_n\}$ of finite rank operators such that

$$x = \sum_{n=1}^{+\infty} A_n x$$

for all $x \in X$ and

$$\sup\left\{\left\|\sum_{j=1}^{n} \lambda_j A_j\right\|;\ \lambda_j \in \mathbb{C},\ |\lambda_j| = 1,\ n \geq 1\right\} < 1 + \varepsilon. \tag{1}$$

As in the proof of Theorem IV.1 we have $w^* - \sum A_n^{**} = P$ pointwise on X^{**}, with P the Hermitian projection from X^{**} onto $Ba(X)$. If $\|x\| = \|x^*\| = x^*(x) = 1$, we have

$$1 = \sum_{n=1}^{+\infty} x^*(A_n x) \tag{2}$$

and by (1)

$$\sum_{n=1}^{+\infty} |x^*(A_n x)| < 1 + \varepsilon. \tag{3}$$

Given $\delta > 0$, we can find $\varepsilon > 0$ such that (2) and (3) imply

$$\sum_{n=1}^{+\infty} |Im(x^*(A_n x))| < \delta.$$

Hence if $S_n = \sum_{k=1}^{n} A_k$, $|Im(x^*(S_n x))| < \delta$.

It follows that there exists an approximating sequence $\{R_n\}$ such that $R_k R_n = R_n$ if $k > n$, $\lim \|I - 2R_n\| = 1$, and $\lim (v_n) = 0$, with

$$v_n = \sup\{|Im(x^*(R_n x))|;\ \|x^*\| = \|x\| = x^*(x) = 1\}.$$

For all $t \in \mathbf{R}$, we have (see [1])

$$\|\exp(itR_n)\| \leq \exp(v_n |t|). \tag{4}$$

If $k > n$, we have

$$[R_k, R_n]^2 = R_n(I - R_k)R_n(I - R_k) = 0 \tag{5}$$

since $(I - R_k)R_n = 0$. We now use an ultraproduct argument. Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$, let $\tilde{S}$ and $\tilde{R}$ be the elements provided by $\{R_n\}$ and a subsequence in the ultrapower algebra

$$\tilde{L}(X) = (L(X))_{\mathcal{U}}.$$

By (4), $\tilde{R}$ and $\tilde{S}$, and thus $i[\tilde{R}, \tilde{S}]$ are Hermitian. By (5), we have

$$[\tilde{R}, \tilde{S}]^2 = 0.$$

Hence we have

$$(i[\tilde{R}, \tilde{S}])^2 = 0$$

and then [21] allows us to conclude that $[\tilde{R}, \tilde{S}] = 0$; which implies since $\mathscr{U}$ is arbitrary that

$$\lim_{n \to +\infty} \{\sup\{\|[R_k, R_n]\|;\ k \geqslant n\}\} = 0,$$

and then Lemma II.2 concludes the proof.

V. Minimal projections and a distinguished subspace of certain dual spaces

For a given Banach space X, we set

$$\mathscr{P}_X = \{Y \subseteq X^{**};\ Y = Ker(P), \text{ with } P^2 = P,\ \|P\| = 1,\ P(X^{**}) \supseteq X\}.$$

The following geometrical statement is related to our results.

PROPOSITION V.1. *Let X be a Banach space not containing $l_1(\mathbb{N})$. Then $\mathscr{P}_X$ consists of w^*-closed subspaces of X^{**}, and $\mathscr{P}_X$ has a largest element L.*

Proof. Claim III.2 asserts precisely that $\mathscr{P}_X$ consists of w^*-closed subspace. We denote

$$\mathscr{S} = \{T \in L(X^{**});\ \|T\| = 1,\ T_{\restriction X} = Id_X\}.$$

The proof of Theorem III.1 shows that $\mathscr{S}$ is inductive when equipped with the order: $S \leqslant T$ if $\|Sx^{**}\| \leqslant \|Tx^{**}\|$ for all $x^{**} \in X^{**}$, and that the non-empty set $\mathscr{M}$ of minimal elements of $\mathscr{S}$ consists of projections.

We pick P and Q two projections in $\mathscr{M}$. By minimality, we have

$$\|QPx^{**}\| = \|Px^{**}\|,\ \|PQx^{**}\| = \|Qx^{**}\|$$

for all $x^{**} \in X^{**}$. Hence (QP) and (PQ) belong to $\mathscr{M}$ and are projections. Moreover

$$Ker(QP) = Ker(P),$$

hence

$$(I - QP)(X^{**}) = Ker(P).$$

Thus

$$PQP = P$$

and therefore

$$P(X^{**}) = PQ(X^{**}).$$

Hence P and PQ are two projections in $\mathscr{M}$ with the same range. Observe now that by the proof of Claim III.2 we have for any projection R in $\mathcal{S}$ with $R(X^{**}) = Y$ that

$$B_{X^{**}} \cap Ker(R) = \bigcap \{\bar{V}^*;\ V\ b_Y\text{—neighbourhood } 0 \text{ in } B_Y\}$$

and in particular R is determined by its range. Since $P(X^{**}) = PQ(X^{**})$, it follows that $P = PQ$, hence $Ker(Q) \subseteq Ker(P)$. Since P and Q were arbitrary elements of $\mathcal{M}$, we conclude that $L = Ker(P)$ does not depend upon the choice of $P \in \mathcal{M}$.

If we pick now $Y = Ker(Q) \in \mathcal{P}_X$, we find $P \in \mathcal{M}$ with $P \leqslant Q$, and then for every $x^{**} \in X^{**}$

$$\|Px^{**}\| \leqslant \|Qx^{**}\|$$

and thus $Ker(Q) = Y \subseteq L = Ker(P)$.

We denote J the subspace of X^* such that $J^\perp = L$. The space J is closely related to the dualization of approximating sequences. For instance, one has the following proposition.

PROPOSITION V.2. a) *If X is separable with the metric approximation property and X does not contain $l_1(\mathbb{N})$, then for every approximating sequence $\{T_n\}$ of contractions, there exists an approximating sequence $\{C_k\}$ in the convex semi-group $\mathcal{S}$ generated by $\{T_n\}$ such that we have $\lim_{k\to+\infty} \|x^* - C_k^* x^*\| = 0$ for all $x^* \in J$. In particular J is separable.*

b) *If X^* is separable and has the approximation property, there exists an approximating sequence $\{E_k\}$ on X with $\|E_k\| \leqslant 1$ and $E_k^*(X^*) \subseteq J$ for all $k \geqslant 1$, and*

$$\lim_{k\to+\infty} \|x^* - E_k^* x^*\| = 0$$

for all $x^* \in J$.

Note that b) means that, at least when X^* is separable with A.P., the space J is the largest space for which a) holds true.

Proof. a) It follows from Theorem III.1 that there is an approximating sequence $\{C_k\}$ in $\mathcal{S}$ such that

$$\lim_{k\to+\infty} \sup\{\|[C_n, C_k]\|;\ n \geqslant k\} = 0.$$

By the proof of Lemma II.2 we have

$$w - \lim_{n\to+\infty} C_n^* x^* = x^*. \tag{1}$$

For all $x^* \in M$, where M is a subspace of X^* such that $M^\perp$ is the kernel of a contractive projection. By Proposition 1, we have $M^\perp \subseteq L = J^\perp$ hence $J \subseteq M$. Since (1) implies that M is separable, we can conclude the proof of a) by a convex combination argument.

b) If X^* is separable with A.P. then it has M.A.P. and thus X has M.A.P. (see [17], § 1.e). Moreover we have

$$K(X)^* = X^{**} \hat{\otimes} X^*$$

and

$$K(X)^{**} = L(X^{**}). \tag{2}$$

We denote $P \in L(X^{**})$ a projection with $\|P\| = 1$ and $P(X^{**}) \supseteq X$ such that $Ker(P) = L = J^\perp$. It follows from (2) that there exists an approximating sequence $\{R_n\}$ and a ultrafilter $\mathcal{U}$ such that

$$Px^{**} = w^* - \lim_{n \to \mathcal{U}} R_n^{**} x^{**}$$

for all $x^{**} \in X^{**}$. We denote by $Q = X^* \to X^*/J$ the canonical quotient map. Reproducing the proof of Lemma II.3 with $M = J$, we construct an approximating sequence $\{V_n\}$ of contractions such that

$$\lim_{n \to +\infty} \|QV_n^*\| = 0 \tag{3}$$

and for all $x^* \in J$

$$\lim_{n \to +\infty} \|V_n^* x^* - x^*\| = 0.$$

Now since $J^\perp$ is the kernel of a bounded projection, J is locally complemented in X^*. It then follows from (3) that there exists a sequence $\{E_n\}$ of finite rank operators such that

$$\lim_{n \to +\infty} \|E_n - V_n\| = 0$$

and $E_n^*(X^*) \subseteq J$. This shows b).

Remarks V.3. 1) We do not know whether J is always a strict subspace of X^* when X non containing $l_1(\mathbb{N})$ is separable but X^* is not. In fact, we do not know whether J, which clearly is a norming subspace of X^*, always coincide with the minimal norming subspace N_X of X^* for all spaces X not containing $l_1(\mathbb{N})$ (see [6], Th. 5.6 for the existence of the space N_X).

2) In general norm-one projections on X^{**} with kernel $J^\perp$ are not unique. For instance, take $X = Z^{**}$ a non-reflexive bidual not containing $l_1(\mathbb{N})$. Then $L = (Z^*)^\perp \subset Z^{***}$ and $J = Z^*$. But if i_k: $Z^{(k)} \to Z^{(k+2)}$ denotes the canonical injection, the canonical projection $i_2 i_1^*$: $Z^{(4)} \to i_2(Z^{**})$ and the projection $i_0^{**} i_1^*$: $Z^{(4)} \to i_0^{**}(Z^{**}) = i_0(Z)^{\perp\perp}$ are distinct contractive projections with kernel $i_1(Z^*)^\perp$.

3) If X has (UMAP), then by ([5], Lemma 6.3 and Lemma 8.1) $Ker(x^{**})$ is not norming if $x^{**} \in Ba(X) \setminus \{0\}$. If moreover X does not contain $l_1(\mathbb{N})$, it follows that $N_X = J = X^*$. It can be shown along the

same lines that the same conclusion holds when X is an order-continuous Banach lattice not containing $l_1(\mathbb{N})$.

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