COMPACT $p$-CONVEX SETS

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Let $A$ be a subset of a topological vector space $X$, and let $Y$ be another topological vector space. We shall say that $A$ can be linearly embedded in $Y$ if there is a linear map $T: \text{lin}(A) \to Y$ (not necessarily continuous) whose restriction to $A$ is a homeomorphism. Until recently it was unknown whether every compact convex subset of a topological vector space could be linearly embedded in a locally convex space. However, J. Roberts [6] has now constructed a non-empty compact convex subset of $L_p = L_p(0, 1)$ $(0 < p < 1)$ which has no extreme points and hence cannot be linearly embedded in a locally convex space (see [4] for some results in the other direction).

In this note we consider a similar problem for $p$-convex sets where $0 < p < 1$ (see the definition below). In view of the example of Roberts it is perhaps somewhat surprising that we are able to show that a compact $p$-convex set can be linearly embedded in a locally $p$-convex topological vector space, and always has $p$-extreme points. We are also able to prove an appropriate version of Choquet’s theorem which for $p < 1$ takes a rather trivial form.

Throughout the paper we shall assume that all vector spaces are over the real field and that all topologies are Hausdorff. A subset $C$ of a vector space is $p$-convex if whenever $x, y \in C$ and $a, b \in R$ with $0 \leq a, b \leq 1$ and $a^p + b^p = 1$ then $ax + by \in C$. $C$ is absolutely $p$-convex if it is $p$-convex and $x \in C$ implies $-x \in C$. A $p$-extreme point of a set $C$ is any point $x \in C$ such that whenever $x = ay_1 + by_2$ with $y_1, y_2 \in C$, $0 \leq a, b \leq 1$ and $a^p + b^p = 1$ then $x = y_1$ or $x = y_2$; the set of $p$-extreme points of $C$ is denoted by $\partial_p C$. If $C$ is any set we denote by $\Gamma_p(C)$ and $\Delta_p(C)$ the smallest $p$-convex and absolutely $p$-convex sets containing $C$. Note that in a topological vector space, if $p < 1$, a closed $p$-convex set always contains 0.

Let $K$ be a compact Hausdorff topological space, and let $C(K)$ be the Banach space of all real-valued continuous functions on $K$. Let $\mathcal{M}(K) = C(K)^*$ be the dual of $C(K)$, i.e. the space of all regular Borel measures on $K$, with the usual dual norm denoted by $\|\cdot\|_1$; we shall denote by $w^*$ the weak*-topology on $\mathcal{M}(K)$ induced by $C(K)$. For $x \in K$ we denote by $\delta_x$ or $\delta(x) \in M(K)$ the unit mass concentrated at $x$; let $\delta(K) = \{\delta(x) : x \in K\}$.

Now suppose \(0 < p < 1\), and let \(\mathcal{M}_p(K)\) be the subspace of \(\mathcal{M}(K)\) of all \(\mu\) of the form

\[
\mu = \sum_{n=1}^{\infty} a_n \delta(x_n)
\]

where \((x_n : n \in \mathbb{N})\) is a sequence of distinct points \(K\) and

\[
\|\mu\|_p = \sum_{n=1}^{\infty} |a_n|^p < \infty.
\]

Let \(U_p = \{\mu : \|\mu\|_p \leq 1\}\) and \(U^+_p = \{\mu \geq 0 : \|\mu\|_p \leq 1\}\). Observe that \(\|\ |_p\) is a \(p\)-norm on \(\mathcal{M}_p(K)\) (see [7] p. 3).

We shall define the topology \(\theta_p\) on \(\mathcal{M}_p(K)\) to be the finest vector topology on \(\mathcal{M}_p(K)\) which agrees with the \(w^*\)-topology on each set \(nU_p, n \in \mathbb{N}\). We can give an explicit basic set of neighbourhoods for \(\theta_p\) (see Wiweger [8]), namely sets of the form

\[
\bigcup_{n=1}^{\infty} \sum_{k=1}^{n} kU_p \cap W_k
\]

where \(\{W_k : k \in \mathbb{N}\}\) is a sequence of \(w^*\)-neighbourhoods of 0. Since each \(U_p\) is \(p\)-convex and the \(w^*\)-topology is locally convex, we conclude:

**Lemma 1.** \(\theta_p\) is a locally \(p\)-convex topology on \(\mathcal{M}_p(K)\).

In the next lemma we combine two results which have essentially the same proof.

**Lemma 2.**

(i) \(U_p\) and \(U^+_p\) are \(\theta_p\)-compact.

(ii) Suppose \(T : \mathcal{M}_p(K) \to X\) is a linear map into a topological vector space satisfying (a) \(T\) is continuous for the \(w^*\)-topology on \(\delta(K)\) and (b) whenever \(\mu_n \in U_p\) and \(\|\mu_n\|_1 \to 0\) then \(T\mu_n \to 0\); then \(T\) is continuous for the topology \(\theta_p\).

**Proof.** The operator \(T\) in (ii) will be continuous if its restriction to \(U_p\) is continuous for the \(w^*\)-topology. Using this and the observation that the unit ball of \(\mathcal{M}(K)\) is \(w^*\)-compact and contains \(U_p\), we see that if either (i) or (ii) is false we can construct a net \(\{\mu_\alpha\}\) in \(U_p\) such that \(\mu_\alpha \to \mu \ w^*\) and either (1) \(\mu \notin U_p\) or (2) \(\mu \in U_p\) and there is a neighbourhood \(V\) of 0 in \(X\) such that \(T(\mu - \mu_\alpha) \notin V\) for all \(\alpha\).

In either case, by replacing \(\mu_\alpha\) by a subnet, we may assume that when each \(\mu_\alpha\) is written in the form

\[
\mu_\alpha = \sum_{n=1}^{\infty} a_{\alpha,n} \delta(x_{\alpha,n})
\]

where \((x_{\alpha,n}, n \in \mathbb{N})\) is a sequence of distinct points of \(K\) and \(|a_{\alpha,n}| \geq \)
$|a_{n+1}| (n \in \mathbb{N})$ then the limits $\lim_{n} a_{\alpha,n} (= a_{n}, \text{say})$ and $\lim_{n} x_{\alpha,n} (= x_{n})$ exist. To see this consider the net $(a_{\alpha,n}; x_{\alpha,n})$ in the compact space $[-1, 1]^{\mathbb{N}} \times K^{\mathbb{N}}$.

Now

$$n |a_{\alpha,n}|^p \leq \sum_{k=1}^{n} |a_{\alpha,k}|^p \leq 1$$

so that $|a_{\alpha,n}| \leq n^{-1/p}$ and hence

$$\|\mu - \sum_{k=1}^{n} a_{\alpha,k} \delta(x_{\alpha,k})\|_1 = \sum_{n+1}^{\infty} |a_{\alpha,k}| \leq (n + 1)^{1-1/p} \sum_{n+1}^{\infty} |a_{\alpha,k}|^p \leq (n + 1)^{1-1/p}.$$ 

By the lower-semi-continuity of $\|\cdot\|_1$ with respect to $w^*$ we have

$$\|\mu - \sum_{k=1}^{n} a_{k} \delta(x_{k})\|_1 \leq (n + 1)^{1-1/p}$$

and hence $\Sigma a_{k} \delta(x_{k}) = \mu$ in $\|\cdot\|_1$. However we clearly have $\Sigma |a_{k}|^p \leq 1$ and hence (after combining terms where $x_{k} = x_{i}$) it is clear that $\mu \in U_p$ contradicting (1).

For (2) pick a symmetric neighbourhood $W$ of 0 in $X$ such that $W + W + W \subseteq V$. Then there exists $n \in \mathbb{N}$ such that $\|\mu\|_p \leq 1$ and $\|\mu\| \leq (n + 1)^{1-1/p}$ implies $T\mu \in W$. Since $T$ is continuous for the $w^*$-topology on $\delta(K)$ we may choose $\alpha$ such that

$$T\left(\sum_{k=1}^{n} a_{\alpha,k} \delta(x_{\alpha,k}) - \sum_{k=1}^{n} a_{k} \delta(x_{k})\right) \in W.$$ 

Then

$$T(\mu_{\alpha} - \mu) = T\left(\sum_{n+1}^{\infty} a_{\alpha,k} \delta(x_{\alpha,k})\right) + T\left(\sum_{k=1}^{n} a_{\alpha,k} \delta(x_{\alpha,k}) - \sum_{k=1}^{n} a_{k} \delta(x_{k})\right) - T\left(\sum_{n+1}^{\infty} a_{k} \delta(x_{k})\right) \in W + W + W \subseteq V$$

contradicting (2). This completes the proof.

We remark that it is now clear that $\theta_p$ is the finest topology agreeing with the $w^*$-topology on $U_p$ (by a result of Waelbroeck [7] p. 48).
LEMMA 3. Let $X$ be a topological vector space and suppose $x(t) \in X$ for $0 \leq t \leq 1$. Suppose that the set $\Delta_p(A)$ is relatively compact where

$$A = \{(t-s)^{-1/p}(x(t) - x(s)) : 0 \leq s < t \leq 1\}.$$ 

Then $x(t) = x(0)$ for $0 \leq t \leq 1$.

**Proof.** Let $E$ be the space of real functions on $[0, 1]$ of the form

$$\varphi = \sum_{i=1}^{n} c_i \chi_i$$

where $\chi_1 \ldots \chi_n$ are characteristic functions of disjoint intervals. We may define a linear map $T : E \rightarrow X$ so that

$$T \chi_{[s,t]} = T \chi_{(s,t)} = T \chi_{[s,t]} = x(t) - x(s).$$

If $\varphi \in E$ is given by (*), and

$$\int_{0}^{1} |\varphi(t)|^p \, dt \leq 1$$

then

$$T \varphi = \sum_{i=1}^{n} c_i (t_i - s_i)^{1/p} [(t_i - s_i)^{-1/p}(x(t_i) - x(s_i))]$$

where $s_i < t_i$ are the endpoints of the interval whose characteristic function is $\chi_i$.

As

$$\int_{0}^{1} |\varphi(t)|^p \, dt = \sum_{i=1}^{n} |c_i|^p (t_i - s_i)$$

we have $T \varphi \in \Delta_p(A)$ and so $T$ extends uniquely to a compact operator $T : L_p \rightarrow X.$ Hence $T = 0$, by the results of [5] and so $x(t) = x(0), 0 \leq t \leq 1$.

LEMMA 4. Let $K$ be a compact subset of a topological vector space $X$ and suppose $\Delta_p(K)$ is relatively compact. Then the map $T : (\mathcal{M}_p(K), \theta_p) \rightarrow X$ defined by

$$T \left( \sum_{n=1}^{\infty} a_n \delta(x_n) \right) = \sum_{n=1}^{\infty} a_n x_n$$

is continuous. (Note that the series necessarily converges since $\Delta_p(K)$ is bounded).

**Proof.** We use Lemma 2(ii). Clearly (a) is satisfied by $T$. To prove (b) suppose the contrary that there is a sequence $\mu_m \in U_p$ such that $\|\mu_m\|_{p} \leq 1$ and $\|\mu_m\|_1 \rightarrow 0$, but that for some neighbourhood $V$ of 0 in $X$ we have $T \mu_m \notin V$. 

Let

\[ \mu_m = \sum_{n=1}^{\infty} a_{m,n} \delta(x_{m,n}) \]

where \((x_{m,n} : n \in \mathbb{N})\) is a sequence of distinct points of \(K\). Define \(y_m(t), 0 \leq t \leq 1\) as follows:

\[ y_m(t) = 0, \quad 0 \leq t < |a_{m,1}|^p \]

\[ = \sum_{n=1}^{k} a_{m,n} x_{m,n}, \quad \sum_{n=1}^{k} |a_{m,n}|^p \leq t < \sum_{n=1}^{k+1} |a_{m,n}|^p \]

\[ = \sum_{n=1}^{\infty} a_{m,n} x_{m,n}, \quad \sum_{n=1}^{\infty} |a_{m,n}|^p \leq t \leq 1. \]

We shall show that if \(1 \geq t > s \geq 0\) and \(t - s \geq 2 \|\mu_m\|_1^p\) then \((t - s)^{-(1/p)}(y_m(t) - y_m(s)) \in \left(\frac{3}{2}\right)^{(1/p)} \Delta_p(K)\). This will be trivially true if either \(t < |a_{m,1}|^p\) or \(s \geq \sum_{n=1}^{\infty} |a_{m,n}|^p\). Hence we assume \(t \geq |a_{m,1}|^p\) and \(s < \sum_{n=1}^{\infty} |a_{m,n}|^p\). Then

\[ y_m(t) - y_m(s) = \sum_{n=k+1}^{k} a_{m,n} x_{m,n} \]

where \(0 \leq l < \infty\) and \(1 \leq k \leq \infty\), and

\[ t - s \geq \sum_{n=l+1}^{k} |a_{m,n}|^p \]

\[ \geq \sum_{n=l+1}^{k} |a_{m,n}|^p - \|\mu_m\|_1^p \]

\[ \geq \sum_{n=l+1}^{k} |a_{m,n}|^p - \frac{1}{2} (t - s). \]

Hence

\[ t - s \geq \frac{2}{3} \sum_{n=l+1}^{k} |a_{m,n}|^p \]

and so

\[ (t - s)^{-(1/p)}(y_m(t) - y_m(s)) = \sum_{n=l+1}^{k} a_{m,n} (t - s)^{-(1/p)} x_{m,n} \]

\[ \in \lambda \Delta_p(K) \]
where

$$\lambda^p = (t-s)^{-1} \sum_{i=1}^{k} |a_{m,n}|^p \leq \frac{3}{2}.$$ 

Now considering \((y_m : m \in \mathbb{N})\) as a sequence in the compact space of all \(\Delta_p(K)\)-valued functions on \([0, 1]\) with pointwise convergence we may find a cluster point \(y(t)\). Then since \(\|\mu_m\|_1 \rightarrow 0\) we will have

$$(t-s)^{-(1/p)}(y(t) - y(s)) \in \left(\frac{3}{2}\right)^{(1/p)}\Delta_p(K)$$

whenever \(0 \leq s < t \leq 1\). Hence by the preceding lemma, \(y(t) = y(0) = 0\) for all \(t\). However \(y(1)\) is a cluster point of the sequence \(T\mu_m\) and \(T\mu_m \not\in V\); thus we have arrived at a contradiction.

**Theorem 1.** Let \(K\) be a compact \(p\)-convex subset \((0 < p < 1)\) of a topological vector space \(X\). Then \(K\) can be linearly embedded in a locally \(p\)-convex topological vector space.

**Proof.** Clearly \(\Delta_p(K) = \{ax - by : 0 \leq a, b \leq 1, a^p + b^p \leq 1, x, y \in K\}\) is compact and hence we may construct the continuous operator \(T : \mathcal{M}_p(K), \theta_p \rightarrow X\) as in Lemma 4. Let \(N = T^{-1}(0)\) and consider the quotient space \(\mathcal{M}_p(K)/N\) with quotient \(\theta_p\)-topology, which is locally \(p\)-convex. Then there is an induced injective map \(\tilde{T} : \mathcal{M}_p(K)/N \rightarrow X\). Restricted to \(q(\delta(K))\) (where \(q : \mathcal{M}_p(K) \rightarrow \mathcal{M}_p(K)/N\) is the quotient map), \(\tilde{T}\) is a homeomorphism onto \(K\); \(\tilde{T}^{-1}\) is the required embedding.

In view of Theorem 1, we could appeal to the results of Fuchssteiner ([1], [2]) to demonstrate the existence of \(p\)-extreme points, and an analogue of the Krein-Milman theorem. In fact we may go further and establish a version of Choquet's theorem (improving Theorem 2 of [3]).

**Theorem 2.** Let \(C\) be a compact \(p\)-convex subset \((0 < p < 1)\) of a topological vector space \(X\) and let \(K\) be a closed subset of \(C\) such that \(C\) is the closure of \(\Gamma_p(K)\). Then

1. \(\partial_p C \subset K\).
2. If \(x \in C\) there is a sequence of distinct points \(x_n \in \partial_p C\) and \(a_n \geq 0\) with \(\sum a_n^p = 1\) such that \(x = \sum_{n=1}^{\infty} a_n x_n\).

**Proof.** Construct as in Lemma 4 the map \(T : \mathcal{M}_p(K) \rightarrow X\). Then \(T(U^+_p)\) is a compact \(p\)-convex set containing \(K\), and is clearly the smallest such. Hence \(T(U^+_p) = C\). If \(x \in \partial_p C\) then

$$x = \sum a_n x_n$$

where \(x_n \in K\) and \(\sum a_n^p \leq 1\). Since it is \(p\)-extreme, and using the fact that \(0 \in C\) we see that this representation must be trivial, i.e. \(x \in K\).

For (2) consider the map \(T\) as in (1) but in the case \(K = C\). For \(x \in C\), the set \(T^{-1}\{x\} \cap U^+_p\) is \(w^*\)-compact and hence there exists
\( \nu \in T^{-1}\{x\} \cap U_p^+ \) such that \( \nu(C) \leq \mu(C) \) whenever \( \mu \in T^{-1}\{x\} \cap U_p^+ \). Let

\[
\nu = \sum b_n \delta(y_n)
\]

where the \( y_n \) are distinct and each \( b_n \neq 0 \). Then if some \( y_k \notin \partial_p C \) we have \( y_k = c_1 z_1 + c_2 z_2 \) where \( z_1, z_2 \in C, \ 0 < c_1, c_2 < 1 \) and \( c_1^p + c_2^p = 1 \). Consider

\[
\nu' = \sum_{n \neq k} b_n \delta(y_n) + b_k c_1 \delta(y_1) + b_k c_2 \delta(y_2)
\]

Then \( \nu' \in U_p^+ \cap T^{-1}\{x\} \) but

\[
\nu'(C) = \sum_{n \neq k} b_n + b_k (c_1 + c_2) < \nu(C)
\]

and we have a contradiction.

Now we have \( \sum b_n^p \leq 1 \); if \( \sum b_n^p = 1 \) we are home. Suppose \( 0 \in \partial_p C \); then by the minimality of \( \nu(C), \ 0 \notin \{y_n : n \in \mathbb{N}\} \). Hence if \( \sum b_n^p < 1, \ x \) may be represented in the required form:

\[
x = \sum_{n=1}^{\infty} b_n y_n + \left(1 - \sum b_n^p\right)^{(1/p)} (0)
\]

Next suppose \( 0 \notin \partial_p C \); then \( 0 = c_1 z_1 + c_2 z_2 \) where \( z_1 + z_2 \in C \) and \( 0 < c_1, c_2 < 1 \) and \( c_1^p + c_2^p = 1 \). By the preceding argument there exist non-zero measures \( \nu_1, \nu_2 \in U_p^+ \) such that \( \nu_1(C \setminus \partial_p C) = \nu_2(C \setminus \partial_p C) = 0 \) and \( T \nu_1 = z_1, T \nu_2 = z_2 \). Then \( T(c_1 \nu_1 + c_2 \nu_2) = 0 \), and hence for any \( \lambda \geq 0, \)

\[
T(\nu + \lambda (c_1 \nu_1 + c_2 \nu_2)) = x.
\]

Then \( \lambda \to \|\nu + \lambda (c_1 \nu_1 + c_2 \nu_2)\|_p \) is continuous in \( \lambda \) and tends to infinity as \( \lambda \to \infty \). Hence for suitable \( \lambda > 0, \)

\[
\|\nu + \lambda (c_1 \nu_1 + c_2 \nu_2)\|_p = 1.
\]

Letting

\[
\nu + \lambda (c_1 \nu_1 + c_2 \nu_2) = \sum a_n \delta(x_n)
\]

with the \( x_n \) distinct, we are home.

**REFERENCES**


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