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THE SPACE $Z_2$ VIEWED AS A SYMPLECTIC BANACH SPACE

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1. INTRODUCTION

The Banach space $Z_2$ which we consider in this paper was introduced in [4] as an example of a twisted sum of two Hilbert-spaces. It seems however that the space has many potential applications as a counterexample.

Here we shall sketch an approach to the main properties of the space using the ideas of the symplectic forms. This approach arises out of [5] in which, in collaboration with Richard Swanson, we showed that $Z_2$ can be used to resolve a question of Weinstein. However, symplectic forms seem to give a natural way of looking at the space in general.

This is not intended to be a fully detailed account. However, it is our hope that we supply here enough information for the reader who is conversant with standard Banach space techniques (selection of basic sequences, etc.) to fill in the details. There is also little new material included, although we believe Theorem 8 has not been observed before.

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2. SYMPLECTIC BANACH SPACES

Let $X$ be a real Banach space, and let $\Omega$ be a continuous alternating bilinear form on $X$. Thus $\Omega : X \times X \to \mathbb{R}$ is bilinear and satisfies

$$|\Omega(x,y)| \leq K \|x\| \|y\|$$

for $x, y \in X$

$$\Omega(x,y) = -\Omega(y,x)$$

for $x, y \in X$

where $K$ is some constant independent of $x$ and $y$. Then $\Omega$ induces a linear map $L_\Omega : X \to X^*$ given by

$$L_\Omega(x)(y) = \Omega(x,y)$$

for $x, y \in X$

and $\|L_\Omega\| \leq K$. If $L_\Omega$ is an isomorphism of $X$ onto $X^*$, we say that $\Omega$ is a symplectic form on $X$ and $X$ (or more strictly $(X, \Omega)$) is a symplectic Banach space.

In this situation it follows that $L_\Omega^* : X^{**} \to X^*$ is also an isomorphism and if $x \in X \subset X^{**}$ then $L_{\Omega^*}^* x = -L_\Omega^* x$, so that we must have that $X = X^{**}$ in its canonical embedding. We conclude that a symplectic Banach space is necessarily reflexive and isomorphic to its dual space.

Conversely if $X$ is reflexive and $L_\Omega$ is merely an isomorphism into $X^*$ then in fact $(X, \Omega)$ is a symplectic Banach space. For, if $L_\Omega$ fails to be onto $X^*$ there exists $y \in X$ with $y \neq 0$ and

$$L(x)(y) = 0$$

for $x \in X$

i.e.,

$$\Omega(x,y) = 0$$

for $x \in X$
and we have \( L_{\Omega} y = 0 \) contrary to assumption.

A subspace \( E \) of \( X \) is called \textbf{isotropic} if

\[
\Omega(x, y) = 0 \quad x, y \in E.
\]

If \( X \) can be written as a direct sum \( X = E \oplus G \) where \( E \) and \( G \) are both isotropic then \( M \) is called a \textbf{Lagrangian subspace} of \( X \). If this happens then \( G \cong E^* \) and we can identify the symplectic Banach space \( X \) with the standard example \( E \oplus E^* \) with the canonical form

\[
\Omega((e,e^*),(f,f^*)) = f^*(e) - e^*(f).
\]

The symplectic form \( \Omega \) induces an involution on the algebra \( \mathcal{L}(X) \) of bounded linear operators on \( X \). If \( S \in \mathcal{L}(X) \) we may define \( S \in \mathcal{L}(X) \) by

\[
\Omega(x, Sy) = \Omega(s^*x, y) \quad x, y \in X.
\]

It is readily checked that \( \|S^*\| \leq \|L_{\Omega}^{-1}\| \cdot \|L_{\Omega}\| \cdot \|S\| \), and that \( S^{**} = S \).

We call an operator \( T \) \textbf{isotropic} if its range is an isotropic subspace. Then \( T \) is isotropic if and only if \( T^* T = 0 \).

3. \textbf{THE SPACE \( \mathbb{Z}_2 \): CONSTRUCTION}

Let \( \mathbb{R}^\infty \) be the space of all finitely supported sequences. We introduce on \( \mathbb{R}^\infty \) the alternating form

\[
\Omega(x, y) = \sum_{n=1}^{\infty} (x_{2n-1}y_{2n} - x_{2n}y_{2n-1}).
\]

For \( x \in \mathbb{R}^\infty \) we also define
\[ \rho(x) = \left( \sum_{n=1}^{\infty} |x_{2n}|^2 \right)^{1/2}. \]

We now write down an expression which will, after some justification, serve as a norm on \( \mathbb{R}^\infty \). It is blatantly not a norm, however, although it is homogeneous, i.e., \( \|a x\| = |a| \|x\| \). We define

\[ \|x\| = \rho(x) + \left\{ \sum_{n=1}^{\infty} \left( x_{2n-1} - x_{2n} \log \frac{\rho(x)}{|x_{2n}|} \right)^2 \right\}^{1/2}. \]

Here we have to interpret \( 0 \log \frac{0}{0} = 0 \log \infty = 0 \).

If \( \{ e_n : n \in \mathbb{N} \} \) denote the usual basis vectors we let \( M_0 \) be the linear span of \( \{ e_{2n-1} : n \in \mathbb{N} \} \). For \( x \in M_0 \) we have

\[ \|x\| = \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} = \left( \sum_{n=1}^{\infty} |x_{2n-1}|^2 \right)^{1/2}. \]

Any \( x \in \mathbb{R}^\infty \) may be written in the form

\[ x = u^{(1)} + u^{(2)} \]

where \( u^{(1)} \in M_0 \), \( \|u^{(2)}\| = \rho(x) \) and

\[ \|x\| = \|u^{(1)}\| + \|u^{(2)}\|. \]

In fact we simply take

\[ u^{(1)}_{2n-1} = x_{2n-1} - x_{2n} \log \frac{\rho(x)}{|x_{2n}|} \quad n \in \mathbb{N} \]

\[ u^{(1)}_{2n} = 0 \quad n \in \mathbb{N} \]

and let \( u^{(2)} = x - u^{(1)} \).

**Lemma 1:** For \( x, y \in \mathbb{R}^\infty \),
\[ |\Omega(x,y)| \leq ||x|| ||y||. \]

**Proof:** Write \( x = u^{(1)} + u^{(2)} \) and \( y = v^{(1)} + v^{(2)} \) as above. Then \( \Omega(u^{(1)},v^{(1)}) = 0 \), while
\[
\Omega(u^{(2)},v^{(1)}) = -\sum_{2n} u_{2n} v_{2n-1}^{(1)}
\]
so that
\[
|\Omega(u^{(2)},v^{(1)})| \leq (\sum_{2n} |u_{2n}^{(2)}|^2)^{1/2} (\sum_{2n-1} |v_{2n-1}^{(1)}|^2)^{1/2} 
\leq ||u^{(2)}|| ||v^{(1)}||.
\]

Similarly
\[
|\Omega(u^{(1)},v^{(2)})| \leq ||u^{(1)}|| ||v^{(2)}||.
\]

Finally
\[
\Omega(u^{(2)},v^{(2)}) = \sum_{n=1}^{\infty} x_{2n} y_{2n} \left( \log \frac{\rho(x)}{|x_{2n}|} - \log \frac{\rho(y)}{|y_{2n}|} \right).
\]

Let \( E = \{ n : \rho(x)|y_{2n}| \geq \rho(y)|x_{2n}| \} \). Then
\[
\left| \sum_{n \in E} x_{2n} y_{2n} \log \frac{\rho(x)|x_{2n}|}{\rho(y)|x_{2n}|} \right| \leq \sum_{n \in E} |y_{2n}|^2 \left| \frac{x_{2n}}{y_{2n}} \right| \log \frac{\rho(x)|y_{2n}|}{\rho(y)|x_{2n}|} 
\leq \frac{1}{e} \frac{\rho(x)}{\rho(y)} \sum_{n \in E} |y_{2n}|^2 \leq \frac{1}{e} \rho(x) \rho(y)
\]
since \( t \log \frac{1}{t} \leq e^{-1} \) for \( 0 \leq t \leq 1 \).

With a similar sum for \( n \notin E \) we obtain
\[
|\Omega(u^{(2)},v^{(2)})| \leq \frac{2}{e} \rho(x) \rho(y)
\]

and the lemma follows. \( \square \)
Define now for \( x \in \mathbb{R}^\infty \)

\[
\|\|x\|\| = \sup_{\|y\| \leq 1} |\Omega(x, y)|.
\]

Then \( \|\cdot\| \) is a semi-norm and \( \|\|x\||\| \leq \|x\| \). However by considering only \( y \in M_0 \) we have

\[
\rho(x) \leq \|\|x\||\|, \quad x \in \mathbb{R}^\infty.
\]

If we write \( x = u^{(1)} + u^{(2)} \) as in the preceding lemma then

\[
\|\|x\||\| = \|\|u^{(1)}\||\| + \rho(x)
\]

\[
\leq \|\|u^{(1)}\||\| + \rho(x)
\]

\[
\leq 2 \|\|x\||\| + \|\|u^{(2)}\||\|
\]

\[
\leq 2 \|\|x\||\| + \|\|u^{(2)}\||\|
\]

\[
\leq 3 \|\|x\||\|.
\]

Thus \( \|\|x\||\| \leq \|\|x\||\| \leq 3 \|\|x\||\| \). Then \( \|\cdot\| \) is a norm on \( \mathbb{R}^\infty \) and \( \|\cdot\| \) is a quasi-norm equivalent to \( \|\cdot\| \), i.e.,

\[
\|\|x + y\|\| \leq 3(\|\|x\||\| + \|\|y\||\|)
\]

\( x, y \in \mathbb{R}^\infty \).

We could now switch to using \( \|\cdot\| \); however, it is much more convenient to use the equivalent quasi-norm.

Let \( Z_2 \) be the completion of \( \mathbb{R}^\infty \) and denote by \( \Omega \) the natural extension of \( \Omega \) to \( Z_2 \). The equivalence of \( \|\cdot\| \) and \( \|\cdot\| \) essentially shows that \( L_\Omega : Z_2 \rightarrow Z_2^* \) is an isomorphism into.
Let $M$ be the closure of $M_0$ in $Z_2$. Then $M$ is isometrically a Hilbert space. If $x \in \mathbb{R}^\infty$,

$$d(x, M) = \inf_{m \in M} \| x - m \| = \rho(x)$$

so that $Z_2/M$ is also a Hilbert space. Hence $Z_2$ is reflexive and $\Omega$ must be a symplectic form on $Z_2$.

We shall denote by $Q$ the quotient map $Q : Z_2 \to \mathcal{L}_2$. It is given by

$$(Qx)_n = x_{2n} \quad x \in \mathbb{R}^\infty.$$ 

Denote by $J$ the embedding $J : \mathcal{L}_2 \to M$ given by

$$(Jx)_{2n-1} = x_n \quad n \in \mathbb{N}$$

$$(Jx)_{2n} = 0 \quad n \in \mathbb{N}.$$ 

4. **Bases and Block Bases in $Z_2$**

Let $\sigma$ be a subset of $\mathbb{N}$. We define an operator $P_\sigma \in \mathcal{L}(Z_2)$ by

$$
\begin{align*}
(P_\sigma x)_{2n-1} &= x_{2n-1} \\
(P_\sigma x)_{2n} &= x_{2n}
\end{align*}
$$

if $n \in \sigma$

$$
\begin{align*}
(P_\sigma x)_{2n-1} &= 0 \\
(P_\sigma x)_{2n} &= 0
\end{align*}
$$

if $n \notin \sigma$.

Then $P_\sigma \in \mathcal{L}(Z_2)$ since by direct calculation,
\[ \|P_\sigma x\| \leq \|x\| + \rho(P_\sigma x) - \rho(x) + \rho(P_\sigma x) \log \frac{\rho(x)}{\rho(P_\sigma x)} \]
\[ \leq \|x\|. \]

Thus \( \|P_\sigma\| \leq 1 \) and \( P_\sigma^2 = P_\sigma = P_\sigma^* \).

It follows that the sequence of two-dimensional subspaces \( \{e_{2n-1}, e_{2n}\} \) form an unconditional Schauder decomposition of \( Z_2 \), and from this it can be seen that \( \{e_n\} \) is a basis of \( Z_2 \).

The sequence \( \{e_{2n-1}\} \) is equivalent to the standard \( \ell_2 \)-basis, but direct calculation shows that \( \sum e_{2n} \) converges if and only if
\[ \sum t_n^2 (1 + (\log \frac{1}{t_n})^2) < \infty. \]

Since \( Z_2/M = \ell_2 \) we see that \( \{e_n\} \) cannot be an unconditional basis for \( Z_2 \). In fact \( M \) cannot be complemented in \( Z_2 \), for the above shows that \( Z_2 \) is not isomorphic to a Hilbert space.

Let \( w \in \mathbb{R}^\infty \subset \ell_2 \). We shall define \( F(w) \in Z_2 \) by
\[ F(w)_{2n} = w_n \]
\[ F(w)_{2n-1} = w_n \log \frac{\|w\|_2}{|w_n|}. \]

Then \( \|F(w)\| = \|w\|_2 \).

Let \( w^{(n)} \) be a normalized block basic sequence in \( \ell_2 \). Then we associate to \( w^{(n)} \) a block basic operator \( W \in \mathcal{L}(Z_2) \) given by
\[ W_{2n-1} = Jw^{(n)} \quad n \in \mathbb{N} \]
\[ W_{2n} = F(w^{(n)}) \quad n \in \mathbb{N}. \]

In fact \( W \) is an isometry, i.e., \( \|Wx\| = \|x\| \) for \( x \in \mathbb{R}^\infty \). If
x \in M_0; this is trivial. Otherwise we may assume \( \rho(x) = 1 \) and write

\[ w(n) = \sum_{p_{n-1}+1}^{p_n} a_i e_i \]

where

\[ \sum_{p+1}^{p_n} a_i^2 = 1. \]

If \( p_{n-1} + 1 \leq i \leq p_n \) and \( Wx = u \), we have

\[ u_{2i} = x_{2n} a_i \]
\[ u_{2i-1} = x_{2n-1} a_i + x_{2n} a_i \log \frac{1}{|a_i|}. \]

Thus \( \rho(u) = 1 \) and

\[ \sum_{i=1}^{\infty} \left( u_{2i-1} - u_{2i} \log \frac{1}{|u_{2i}|} \right)^2 = \sum_{n=1}^{\infty} \sum_{p_{n-1}+1}^{p_n} a_i^2 \left( x_{2n-1} - x_{2n} \log \frac{1}{|x_{2n}|} \right)^2 \]
\[ = \sum_{n=1}^{\infty} \left( x_{2n-1} - x_{2n} \log \frac{1}{|x_{2n}|} \right)^2. \]

It quickly follows that \( \|Wx\| = \|x\| \).

We also note that \( W \) preserves \( \Omega \), since by checking on basis elements we have

\[ \Omega(x,y) = \Omega(Wx,Wy) \quad x,y \in \mathbb{R}^\infty. \]

Thus \( W^*W = I \) in \( \mathcal{L}(\mathbb{Z}_2) \). The range \( \mathcal{A}(W) \) of \( W \) is isomorphic to \( \mathbb{Z}_2 \) and complemented in \( \mathbb{Z}_2 \) by the projection \( WW^* \). We shall refer to \( \mathcal{A}(W) \) as a block subspace.

An immediate deduction from the above argument is that \( \{F(w(n))\}_{n=1}^{\infty} \)
is equivalent to \( \{e_{2n}\} \).

**Lemma 2.** Let \((x^{(n)})\) be a block basic sequence in \( \mathbb{L}_2 \) equivalent to the standard \( \mathbb{L}_2 \)-basis. Then \( p(x^{(n)}) \to 0 \).

**Proof:** Suppose the lemma is false. We can suppose \( p(x^{(n)}) = 1 \). Consider the sequence \( u^{(n)} = P(Qx^{(n)}) \). Then \( x^{(n)} - u^{(n)} \in M_0 \) and hence \( \sum_n (x^{(n)} - u^{(n)}) \) converges provided \( \sum_n 2^2 < \infty \). Thus \( u^{(n)} \) is equivalent to the \( \mathbb{L}_2 \)-basis; but \( u^{(n)} \) is equivalent to \( \{e_{2n}\} \) and we have a contradiction. \( \square \)

5. **Operators on \( \mathbb{L}_2 \)**

Our first theorem is immediate from Lemma 2.

**Theorem 3.** The quotient map \( Q: \mathbb{L}_2 \to \mathbb{L}_2 \) is strictly singular.

**Lemma 4.** Let \( T \in \mathcal{L}(\mathbb{L}_2) \) and let \( w^{(n)} \) be a normalized block basic sequence in \( M \). Then \( \|QTw^{(n)}\| \to 0 \).

**Proof:** \( QT:M \to \mathbb{L}_2 \) is strictly singular and hence compact. \( \square \)

**Lemma 5.** Suppose \( T \in \mathcal{L}(\mathbb{L}_2) \) and \( T|_M \) is strictly singular. Then \( T \) is strictly singular.

**Proof:** Suppose \( T \) is an isomorphism on a closed subspace \( E \) of infinite dimension. From Theorem 3 we obtain a basic sequence \( u^{(n)} \in E \) such that for some block basic sequence \( v^{(n)} \in M \), \( \|u^{(n)} - v^{(n)}\| \to 0 \). Clearly \( T \) is an isomorphism on the closed linear span of some subsequence of \( \{v^{(n)}\} \). \( \square \)
Lemma 6: Suppose $T \in \mathcal{L}(Z_2)$ and $T$ is not strictly singular. Then there are block basic operators $W$ and $V$ and $\alpha \neq 0$ so that $\alpha W - TW$ is strictly singular.

Proof: Using Lemma 5 we can find a normalized block basic sequence $(w(n))$ in $Z_2$, a normalized block basic sequence $(v(n))$ in $Z_2$ and $\alpha \neq 0$ so that

$$TJw(n) = \alpha Jv(n) + u(n)$$

where $\sum \|u(n)\| < \infty$. Let $W$ and $V$ be the associated block basic operators. Let $K$ be the compact operator on $Z_2$ so that

$$Ke_{2n-1} = u(n) \quad n \in \mathbb{N}$$

$$Ke_{2n} = 0 \quad n \in \mathbb{N}.$$

Then $(TW - \alpha V - K)(M) = 0$ so that by Theorem 3, $TW - \alpha V - K$ is strictly singular. Hence $TW - \alpha V$ is also strictly singular. □

We now derive a theorem essentially due to Johnson, Lindenstrauss and Schechtman [2] (cf. also [4]).

Theorem 7: Let $T \in \mathcal{L}(Z_2)$ and suppose $T$ is not strictly singular. Then there is a closed subspace $E$ of $Z_2$ with $E \cong Z_2$, $T|_E$ an isomorphism and both $E$ and $T(E)$ complemented in $Z_2$.

Proof: By Lemma 6 we can find block basic operators $V,W$ so that $TW = \alpha V + S$, where $S$ is strictly singular. Hence $\alpha^{-1} V^*TW - I$ is strictly singular, so that for some invertible operator $A$
rank\((AV^*TW - I) < \infty\).

For large enough \(n\), \(TW|_F^n\) is an isomorphism where \(F_n = [e_k : k \geq 2n-1]\). By relabelling we may suppose the original \(TW\) is an isomorphism. Now there is a finite rank operator \(G\) so that

\[
-GTW = AV^*TW - I.
\]

Hence

\[
(AV^* + G)TW = I.
\]

Let \(E = W(Z_2)\). The above equation quickly implies that both \(E\) and \(T(E)\) are complemented in \(Z_2^\ast\). In fact, \(TW(AV^* + G)\) is a projection on \(T(E)\). \(\square\)

**Theorem 8:** Let \(X\) be any Banach space and suppose \(T : Z_2 \to X\) is a non-strictly singular operator. Then there is a complemented subspace \(E\) of \(Z_2\) with \(E \cong Z_2\) so that \(T|_E\) is an isomorphism.

**Proof:** The argument of Lemma 5 shows that \(T\) cannot be strictly singular on \(M\) and indeed we can find a block basic operator \(W\) so that \(TW|_M\) is an isomorphism. Since \(K(W)\) is complemented it will suffice to consider the case when \(T|_M\) is an isomorphism.

Consider the maps \(S_1 : Z_2 \to X \oplus Z_2\) and \(S_2 : Z_2 \to X \oplus Z_2\) given by

\[
S_1x = (Tx, 0)
\]

\[
S_2x = (0, x).
\]

Let \(Y \subset X \oplus Z_2\) be the subspace of all \((Tx, -x)\) for \(x \in M\). Then \(S_1 - S_2 = 0 \mod Y\) for \(x \in M\) and hence if \(q : X \oplus Z_2 \to X \oplus Z_2/Y\) is the quotient map, \(qS_1 - qS_2\) is strictly singular. Now \(qS_2\) is an
isomorphism on $Z_2$ and hence $qS_1$ is an isomorphism on some
$[e_2, e_2, \ldots]$. Hence $T$ is an isomorphism on this subspace. □

**Theorem 9:** If $T^*T$ is strictly singular then $T$ is also strictly
singular.

**Proof:** If $T$ is not strictly singular there are block basic operators
$W$ and $V$ so that

$$TW = \alpha V + S$$

where $\alpha \neq 0$ and $S$ is strictly singular. Hence

$$W^*T^*TW = \alpha^2 V^*V + S_1$$

where $S_1$ is strictly singular. Since $V^*V = I$, this is impossible. □

**Corollary 10:** Any isotropic operator is strictly singular.

**Corollary 11:** Any complemented isotropic subspace of $Z_2$ is finite-
dimensional. $Z_2$ has no Lagrangian subspace.

Corollary 11 answers a question of Weinstein [7].

6. **REMARKS AND OPEN PROBLEMS**

As noted in the introduction, $Z_2$ gives an example of non-Hilbertian
space which possesses a closed subspace $M \cong l_2$ with $Z_2/M \cong l_2$. An
earlier example was given by Enflo, Lindenstrauss and Pisier [1], but
there is some reason to believe $Z_2$ is an extreme example. See [3]
and [4] for calculations of type inequalities in this vein.

It was shown in [4] that $Z_2$ has no unconditional basis. Recently
Ketonen [6] has shown that there are subspaces of \( L_p \) for \( 1 < p < 2 \) which have like \( Z_2 \) an unconditional Schauder decomposition into two-dimensional subspaces, but have no unconditional basis. However \( Z_2 \) cannot be embedded in an \( L_p \)-space ([4]). Johnson, Lindenstrauss and Schechtman show that \( Z_2 \) does not have local unconditional structure [2].

The operator properties of \( Z_2 \) as in Theorems 7 and 8 are reminiscent of the sequence spaces \( l_p \) (1 \( \leq \) p \( \leq \) \( \infty \)) and \( c_0 \). Indeed these are the only other known spaces with this property (apart from the analogous \( Z_p \) of \( Z_2 \) for \( p \neq 2 \)). The spaces \( l_p \) and \( c_0 \) are the only known prime Banach spaces. We thus ask whether \( Z_2 \) can be prime. It is however very possible (cf. [2]) that \( Z_2 \) fails to be prime quite spectacularly. We do not know whether \( Z_2 \neq Z_2 \oplus \mathbb{R} \); thus \( Z_2 \) could be a counterexample to the hyperplane problem for Banach spaces, and this would cause it to fail to be prime.

Theorem 7 shows that any complemented subspace of \( Z_2 \) contains a complemented copy of \( Z_2 \). Thus since \( Z_2 \cong Z_2 \oplus Z_2 \) a Pelczynski decomposition argument shows that if \( E \) is a complemented subspace of \( Z_2 \) and \( E \oplus E \cong E \) then \( E \cong Z_2 \).

REFERENCES


