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THE SPACE Z_2 VIEWED AS A SYMPLECTIC BANACH SPACE

N. J. Kalton*

Department of Mathematics
University of Missouri
Columbia, Missouri 65211

1. INTRODUCTION

The Banach space Z_2 which we consider in this paper was introduced in [4] as an example of a twisted sum of two Hilbert-spaces. It seems however that the space has many potential applications as a counterexample.

Here we shall sketch an approach to the main properties of the space using the ideas of the symplectic forms. This approach arises out of [5] in which, in collaboration with Richard Swanson, we showed that Z_2 can be used to resolve a question of Weinstein. However, symplectic forms seem to give a natural way of looking at the space in general.

This is not intended to be a fully detailed account. However, it is our hope that we supply here enough information for the reader who is conversant with standard Banach space techniques (selection of basic sequences, etc.) to fill in the details. There is also little new material included, although we believe Theorem 8 has not been observed before.

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2. SYMPLECTIC BANACH SPACES

Let X be a real Banach space, and let Ω be a continuous alternating bilinear form on X. Thus $\Omega : X \times X \longrightarrow \mathbb{R}$ is bilinear and satisfies

$$|\Omega(x,y)| \leq K\|x\|\|y\| \qquad x,y \in X$$

$$\Omega(x,y) = -\Omega(y,x) \qquad x,y \in X$$

where K is some constant independent of x and y. Then Ω induces a linear map $L_\Omega : X \longrightarrow X^*$ given by

$$L_\Omega(x)(y) = \Omega(x,y) \qquad x,y \in X$$

and $\|L_\Omega\| \leq K$. If L_Ω is an isomorphism of X <u>onto</u> X^* we say that Ω is a <u>symplectic form</u> on X and X (or more strictly (X,Ω)) is a <u>symplectic Banach space</u>.

In this situation it follows that $L_\Omega^* : X^{**} \longrightarrow X^*$ is also an isomorphism and if $x \in X \subseteq X^{**}$ then $L_\Omega^* x = -L_\Omega x$, so that we must have that $X = X^{**}$ in its canonical embedding. We conclude that a symplectic Banach space is necessarily reflexive and isomorphic to its dual space.

Conversely if X is <u>reflexive</u> and L_Ω is merely an isomorphism into X^* then in fact (X,Ω) is a symplectic Banach space. For, if L_Ω fails to be onto X^* there exists $y \in X$ with $y \neq 0$ and

$$L(x)(y) = 0 \qquad x \in X$$

i.e.,

$$\Omega(x,y) = 0 \qquad x \in X$$

and we have $L_\Omega y = 0$ contrary to assumption.

A subspace E of X is called isotropic if

$$\Omega(x,y) = 0 \qquad x,y \in E.$$

If X can be written as a direct sum $X = E \oplus G$ where E and G are both isotropic then M is called a Lagrangian subspace of X. If this happens then $G \cong E^*$ and we can identify the symplectic Banach space X with the standard example $E \oplus E^*$ with the canonical form

$$\Omega((e,e^*),(f,f^*)) = f^*(e) - e^*(f).$$

The symplectic form Ω induces an involution on the algebra $\mathcal{L}(X)$ of bounded linear operators on X. If $S \in \mathcal{L}(X)$ we may define $S \in \mathcal{L}(X)$ by

$$\Omega(x,Sy) = \Omega(S^*x,y) \qquad x,y \in X.$$

It is readily checked that $\|S^*\| \le \|L_\Omega^{-1}\| \cdot \|L_\Omega\| \|S\|$, and that $S^{**} = S$.

We call an operator T isotropic if its range is an isotropic subspace. Then T is isotropic if and only if $T^*T = 0$.

3. THE SPACE Z_2: CONSTRUCTION

Let $\mathbb{R}^\infty$ be the space of all finitely supported sequences. We introduce on $\mathbb{R}^\infty$ the alternating form

$$\Omega(x,y) = \sum_{n=1}^{\infty} (x_{2n-1}y_{2n} - x_{2n}y_{2n-1}).$$

For $x \in \mathbb{R}^\infty$ we also define

$$\rho(x) = \left(\sum_{n=1}^{\infty} |x_{2n}|^2\right)^{1/2}.$$

We now write down an expression which will, after some justification, serve as a norm on $\mathbb{R}^\infty$. It is blatantly not a norm, however, although it is homogeneous, i.e., $\|\alpha x\| = |\alpha|\,\|x\|$. We define

$$\|x\| = \rho(x) + \left\{\sum_{n=1}^{\infty}\left(x_{2n-1} - x_{2n} \log \frac{\rho(x)}{|x_{2n}|}\right)^2\right\}^{1/2}.$$

Here we have to interpret $0 \log \frac{0}{0} = 0 \log \infty = 0$.

If $\{e_n : n \in \mathbb{N}\}$ denote the usual basis vectors we let M_0 be the linear span of $\{e_{2n-1} : n \in \mathbb{N}\}$. For $x \in M_0$ we have

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} |x_{2n-1}|^2\right)^{1/2}.$$

Any $x \in \mathbb{R}^\infty$ may be written in the form

$$x = u^{(1)} + u^{(2)}$$

where $u^{(1)} \in M_0$, $\|u^{(2)}\| = \rho(x)$ and

$$\|x\| = \|u^{(1)}\| + \|u^{(2)}\|.$$

In fact we simply take

$$u^{(1)}_{2n-1} = x_{2n-1} - x_{2n} \log \frac{\rho(x)}{|x_{2n}|} \qquad n \in \mathbb{N}$$

$$u^{(1)}_{2n} = 0 \qquad n \in \mathbb{N}$$

and let $u^{(2)} = x - u^{(1)}$.

<u>Lemma 1</u>: For $x, y \in \mathbb{R}^\infty$,

$$|\Omega(x,y)| \le \|x\|\|y\|.$$

Proof: Write $x = u^{(1)} + u^{(2)}$ and $y = v^{(1)} + v^{(2)}$ as above. Then $\Omega(u^{(1)}, v^{(1)}) = 0$, while

$$\Omega(u^{(2)}, v^{(1)}) = -\sum u_{2n}^{(2)} \; v_{2n-1}^{(1)}$$

so that

$$|\Omega(u^{(2)}, v^{(1)})| \le (\sum |u_{2n}^{(2)}|^2)^{1/2} (\sum |v_{2n-1}^{(1)}|^2)^{1/2}$$
$$\le \|u^{(2)}\|\|v^{(1)}\|.$$

Similarly

$$|\Omega(u^{(1)}, v^{(2)})| \le \|u^{(1)}\|\|v^{(2)}\|.$$

Finally

$$\Omega(u^{(2)}, v^{(2)}) = \sum_{n=1}^{\infty} x_{2n} y_{2n} \left(\log \frac{\rho(x)}{|x_{2n}|} - \log \frac{\rho(y)}{|y_{2n}|} \right).$$

Let $E = \{n : \rho(x)|y_{2n}| \ge \rho(y)|x_{2n}|\}$. Then

$$\left| \sum_{n \in E} x_{2n} y_{2n} \log \frac{\rho(x)|x_{2n}|}{\rho(y)|x_{2n}|} \right| \le \sum_{n \in E} |y_{2n}|^2 \frac{|x_{2n}|}{|y_{2n}|} \log \frac{\rho(x)|y_{2n}|}{\rho(y)|x_{2n}|}$$
$$\le \frac{1}{e} \frac{\rho(x)}{\rho(y)} \sum_{n \in E} |y_{2n}|^2 \le \frac{1}{e} \rho(x)\rho(y)$$

since $t \log \frac{1}{t} \le e^{-1}$ for $0 \le t \le 1$.

With a similar sum for $n \notin E$ we obtain

$$|\Omega(u^{(2)}, v^{(2)})| \le \frac{2}{e} \rho(x)\rho(y)$$

and the lemma follows. □

Define now for $x \in \mathbb{R}^\infty$

$$|||x||| = \sup_{\|y\|\le 1} |\Omega(x,y)|.$$

Then $|||\cdot|||$ is a semi-norm and $|||x||| \le \|x\|$. However by considering only $y \in M_0$ we have

$$\rho(x) \le |||x||| \qquad x \in \mathbb{R}^\infty.$$

If we write $x = u^{(1)} + u^{(2)}$ as in the preceding lemma then

$$\begin{aligned} \|x\| &= \|u^{(1)}\| + \rho(x) \\ &\le |||u^{(1)}||| + \rho(x) \\ &\le 2|||x||| + |||u^{(2)}||| \\ &\le 2|||x||| + \|u^{(2)}\| \\ &\le 3|||x|||. \end{aligned}$$

Thus $|||x||| \le \|x\| \le 3|||x|||$. Then $|||\cdot|||$ is a norm on $\mathbb{R}^\infty$ and $\|\cdot\|$ is a quasi-norm equivalent to $|||\cdot|||$, i.e.,

$$\|x+y\| \le 3(\|x\| + \|y\|) \qquad x,y \in \mathbb{R}^\infty.$$

We could now switch to using $|||\cdot|||$; however, it is much more convenient to use the equivalent quasi-norm.

Let Z_2 be the completion of $\mathbb{R}^\infty$ and denote by Ω the natural extension of Ω to Z_2. The equivalence of $|||\cdot|||$ and $\|\cdot\|$ essentially shows that $L_\Omega : Z_2 \to Z_2^*$ is an isomorphism into.

Let M be the closure of M_0 in Z_2. Then M is isometrically a Hilbert space. If $x \in \mathbb{R}^\infty$,

$$d(x,M) = \inf_{m\in M}\|x-m\|$$

$$= \rho(x)$$

so that Z_2/M is also a Hilbert space. Hence Z_2 is <u>reflexive</u> and Ω must be a symplectic form on Z_2.

We shall denote by Q the quotient map $Q : Z_2 \longrightarrow \ell_2$. It is given by

$$(Qx)_n = x_{2n} \qquad x \in \mathbb{R}^\infty.$$

Denote by J the embedding $J : \ell_2 \longrightarrow M$ given by

$$(Jx)_{2n-1} = x_n \qquad n \in \mathbb{N}$$

$$(Jx)_{2n} = 0 \qquad n \in \mathbb{N}.$$

4. BASES AND BLOCK BASES IN Z_2

Let σ be a subset of $\mathbb{N}$. We define an operator $P_\sigma \in \mathcal{L}(Z_2)$ by

$$\left.\begin{array}{l}(P_\sigma x)_{2n-1} = x_{2n-1} \\ (P_\sigma x)_{2n} = x_{2n}\end{array}\right\} \quad \text{if } n \in \sigma$$

$$\left.\begin{array}{l}(P_\sigma x)_{2n-1} = 0 \\ (P_\sigma x)_{2n} = 0\end{array}\right\} \quad \text{if } n \notin \sigma.$$

Then $P_\sigma \in \mathcal{L}(Z_2)$ since by direct calculation,

$$\|P_\sigma x\| \le \|x\| + \rho(P_\sigma x) - \rho(x) + \rho(P_\sigma x)\log\frac{\rho(x)}{\rho(P_\sigma x)}$$

$$\le \|x\|.$$

Thus $\|P_\sigma\| \le 1$ and $P_\sigma^2 = P_\sigma^* = P_\sigma$.

It follows that the sequence of two-dimensional subspaces $[e_{2n-1}, e_{2n}]$ form an unconditional Schauder decomposition of Z_2, and from this it can be seen that $\{e_n\}$ is a basis of Z_2.

The sequence $\{e_{2n-1}\}$ is equivalent to the standard ℓ_2-basis, but direct calculation shows that $\sum t_n e_{2n}$ converges if and only if

$$\sum t_n^2 (1 + (\log \frac{1}{t_n})^2) < \infty.$$

Since $Z_2/M \cong \ell_2$ we see that $\{e_n\}$ cannot be an unconditional basis for Z_2. In fact M cannot be complemented in Z_2, for the above shows that Z_2 is not isomorphic to a Hilbert space.

Let $w \in \mathbb{R}^\infty \subset \ell_2$. We shall define $F(w) \in Z_2$ by

$$F(w)_{2n} = w_n$$

$$F(w)_{2n-1} = w_n \log \frac{\|w\|_2}{|w_n|}.$$

Then $\|F(w)\| = \|w\|_2$.

Let $w^{(n)}$ be a normalized block basic sequence in ℓ_2. Then we associate to $w^{(n)}$ a block basic operator $W \in \mathcal{L}(Z_2)$ given by

$$We_{2n-1} = Jw^{(n)} \qquad n \in \mathbb{N}$$

$$We_{2n} = F(w^{(n)}) \qquad n \in \mathbb{N}.$$

In fact W is an isometry, i.e., $\|Wx\| = \|x\|$ for $x \in \mathbb{R}^\infty$. If

$x \in M_0$ this is trivial. Otherwise we may assume $\rho(x) = 1$ and write

$$w^{(n)} = \sum_{p_{n-1}+1}^{p_n} a_i e_i$$

where

$$\sum_{p_{n-1}+1}^{p_n} a_i^2 = 1.$$

If $p_{n-1}+1 \leq i \leq p_n$ and $Wx = u$, we have

$$u_{2i} = x_{2n}\, a_i$$

$$u_{2i-1} = x_{2n-1}\, a_i + x_{2n}\, a_i \log \frac{1}{|a_i|}.$$

Thus $\rho(u) = 1$ and

$$\sum_{i=1}^{\infty} \left(u_{2i-1} - u_{2i} \log \frac{1}{|u_{2i}|} \right)^2 = \sum_{n=1}^{\infty} \sum_{p_{n-1}+1}^{p_n} a_i^2 \left(x_{2n-1} - x_{2n} \log \frac{1}{|x_{2n}|} \right)^2$$

$$= \sum_{n=1}^{\infty} \left(x_{2n-1} - x_{2n} \log \frac{1}{|x_{2n}|} \right)^2.$$

It quickly follows that $\|Wx\| = \|x\|$.

We also note that W preserves Ω, since by checking on basis elements we have

$$\Omega(x,y) = \Omega(Wx,Wy) \qquad\qquad x,y \in \mathbb{R}^{\infty}.$$

Thus $W^*W = I$ in $\mathcal{L}(Z_2)$. The range $\mathcal{R}(W)$ of W is isomorphic to Z_2 and complemented in Z_2 by the projection WW^*. We shall refer to $\mathcal{R}(W)$ as a block subspace.

An immediate deduction from the above argument is that $\{F(w^{(n)})\}_{n=1}^{\infty}$

is equivalent to $\{e_{2n}\}$.

Lemma 2: Let $(x^{(n)})$ be a block basic sequence in Z_2 equivalent to the standard ℓ_2-basis. Then $\rho(x^{(n)}) \longrightarrow 0$.

Proof: Suppose the lemma is false. We can suppose $\rho(x^{(n)}) = 1$. Consider the sequence $u^{(n)} = F(Qx^{(n)})$. Then $x^{(n)} - u^{(n)} \in M_0$ and hence $\Sigma t_n(x^{(n)}-u^{(n)})$ converges provided $\Sigma t_n^2 < \infty$. Thus $u^{(n)}$ is equivalent to the ℓ_2-basis; but $u^{(n)}$ is equivalent to (e_{2n}) and we have a contradiction. $\square$

5. OPERATORS ON Z_2

Our first theorem is immediate from Lemma 2.

Theorem 3: The quotient map $Q : Z_2 \longrightarrow \ell_2$ is strictly singular.

Lemma 4: Let $T \in \mathcal{L}(Z_2)$ and let $w^{(n)}$ be a normalized block basic sequence in M. Then $\|QTw^{(n)}\| \longrightarrow 0$.

Proof: $QT : M \longrightarrow \ell_2$ is strictly singular and hence compact. $\square$

Lemma 5: Suppose $T \in \mathcal{L}(Z_2)$ and $T|M$ is strictly singular. Then T is strictly singular.

Proof: Suppose T is an isomorphism on a closed subspace E of infinite dimension. From Theorem 3 we obtain a basic sequence $u^{(n)} \in E$ such that for some block basic sequence $v^{(n)} \in M$, $\|u^{(n)} - v^{(n)}\| \longrightarrow 0$. Clearly T is an isomorphism on the closed linear span of some subsequence of $\{v^{(n)}\}$. $\square$

Lemma 6: Suppose $T \in \mathcal{L}(Z_2)$ and T is not strictly singular. Then there are block basic operators W and V and $\alpha \neq 0$ so that $\alpha V - TW$ is strictly singular.

Proof: Using Lemma 5 we can find a normalized block basic sequence $(w^{(n)})$ in ℓ_2, a normalized block basic sequence $(v^{(n)})$ in ℓ_2 and $\alpha \neq 0$ so that

$$TJw^{(n)} = \alpha Jv^{(n)} + u^{(n)}$$

where $\sum \|u^{(n)}\| < \infty$. Let W and V be the associated block basic operators. Let K be the compact operator on Z_2 so that

$$Ke_{2n-1} = u^{(n)} \qquad n \in \mathbb{N}$$

$$Ke_{2n} = 0 \qquad n \in \mathbb{N}.$$

Then $(TW - \alpha V - K)(M) = 0$ so that by Theorem 3, $TW - \alpha V - K$ is strictly singular. Hence $TW - \alpha V$ is also strictly singular. □

We now derive a theorem essentially due to Johnson, Lindenstrauss and Schechtman [2] (cf. also [4]).

Theorem 7: Let $T \in \mathcal{L}(Z_2)$ and suppose T is not strictly singular. Then there is a closed subspace E of Z_2 with $E \cong Z_2$, $T|_E$ an isomorphism and both E and $T(E)$ complemented in Z_2.

Proof: By Lemma 6 we can find block basic operators V, W so that $TW = \alpha V + S$, where S is strictly singular. Hence $\alpha^{-1} V^* TW - I$ is strictly singular, so that for some invertible operator A

$$\operatorname{rank}(AV^*TW - I) < \infty.$$

For large enough n, $TW|F_n$ is an isomorphism where $F_n = [e_k : k \geq 2n-1]$. By relabelling we may suppose the original TW is an isomorphism. Now there is a finite rank operator G so that

$$-GTW = AV^*TW - I.$$

Hence

$$(AV^* + G)TW = I.$$

Let $E = W(Z_2)$. The above equation quickly implies that both E and $T(E)$ are complemented in Z_2. In fact, $TW(AV^* + G)$ is a projection on $T(E)$. □

<u>Theorem 8</u>: Let X be any Banach space and suppose $T : Z_2 \longrightarrow X$ is a non-strictly singular operator. Then there is a complemented subspace E of Z_2 with $E \cong Z_2$ so that $T|_E$ is an isomorphism.

<u>Proof</u>: The argument of Lemma 5 shows that T cannot be strictly singular on M and indeed we can find a block basic operator W so that $TW|M$ is an isomorphism. Since $\mathcal{R}(W)$ is complemented it will suffice to consider the case when $T|M$ is an isomorphism.

Consider the maps $S_1 : Z_2 \longrightarrow X \oplus Z_2$ and $S_2 : Z_2 \longrightarrow X \oplus Z_2$ given by

$$S_1x = (Tx, 0)$$

$$S_2x = (0, x).$$

Let $Y \subset X \oplus Z_2$ be the subspace of all $(Tx, -x)$ for $x \in M$. Then $S_1 - S_2 = 0 \pmod{Y}$ for $x \in M$ and hence if $q : X \oplus Z_2 \longrightarrow X \oplus Z_2/Y$ is the quotient map, $qS_1 - qS_2$ is strictly singular. Now qS_2 is an isomorphism on Z_2 and hence qS_1 is an isomorphism on some $[e_{2n-1}, e_{2n}, \cdots]$. Hence T is an isomorphism on this subspace. □

Theorem 9: If T^*T is strictly singular then T is also strictly singular.

Proof: If T is not strictly singular there are block basic operators W and V so that

$$TW = \alpha V + S$$

where $\alpha \neq 0$ and S is strictly singular. Hence

$$W^*T^*TW = \alpha^2 V^*V + S_1$$

where S_1 is strictly singular. Since $V^*V = I$, this is impossible. □

Corollary 10: Any isotropic operator is strictly singular.

Corollary 11: Any complemented isotropic subspace of Z_2 is finite-dimensional. Z_2 has no Lagrangian subspace.

Corollary 11 answers a question of Weinstein [7].

6. REMARKS AND OPEN PROBLEMS

As noted in the introduction, Z_2 gives an example of non-Hilbertian space which possesses a closed subspace $M \cong \ell_2$ with $Z_2/M \cong \ell_2$. An earlier example was given by Enflo, Lindenstrauss and Pisier [1], but there is some reason to believe Z_2 is an extreme example. See [3] and [4] for calculations of type inequalities in this vein.

It was shown in [4] that Z_2 has no unconditional basis. Recently

Ketonen [6] has shown that there are subspaces of L_p for $1 < p < 2$ which have like Z_2 an unconditional Schauder decomposition into two-dimensional subspaces, but have no unconditional basis. However Z_2 cannot be embedded in an L_p-space ([4]). Johnson, Lindenstrauss and Schechtman show that Z_2 does not have local unconditional structure [2].

The operator properties of Z_2 as in Theorems 7 and 8 are reminiscent of the sequence spaces ℓ_p $(1 \leq p \leq \infty)$ and c_0. Indeed these are the only other known spaces with this property (apart from the analogous Z_p of Z_2 for $p \neq 2$). The spaces ℓ_p and c_0 are the only known prime Banach spaces. We thus ask whether Z_2 can be prime. It is however very possible (cf. [2]) that Z_2 fails to be prime quite spectacularly. We do not know whether $Z_2 \not\cong Z_2 \oplus \mathbb{R}$; thus Z_2 could be a counterexample to the hyperplane problem for Banach spaces, and this would cause it to fail to be prime.

Theorem 7 shows that any complemented subspace of Z_2 contains a complemented copy of Z_2. Thus since $Z_2 \cong Z_2 \oplus Z_2$ a Pelczynski decomposition argument shows that if E is a complemented subspace of Z_2 and $E \oplus E \cong E$ then $E \cong Z_2$.

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