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The *M*-ideal structure of some algebras of bounded linear operators

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Let $1 < p, q < \infty$. It is shown for complex scalars that there are no nontrivial M-ideals in $\mathscr{L}(L_p[0,1])$ if $p \neq 2$, and $\mathscr{K}(l_p(l_q^n))$ is the only nontrivial M-ideal in $\mathscr{L}(l_p(l_q^n))$.

1. Introduction

A subspace J of a Banach space X is called an M-ideal if there is an l_1-direct decomposition of the dual space X^* of X into the annihilator $J^\perp$ of J and some subspace $V \subset X^*$:

$$X^* = J^\perp \oplus_1 V;$$

it is called nontrivial if $\{0\} \neq J \neq X$. This notion was introduced by Alfsen and Effros [**1**] and has proved useful in Banach space geometry, approximation theory and harmonic analysis; see [**12**] for a detailed account.

A number of authors have studied the M-ideal structure in $\mathscr{L}(X)$, the space of bounded linear operators on a Banach space X, with special emphasis on the question whether $\mathscr{K}(X)$, the subspace of compact operators, is an M-ideal; see for instance [**3, 7–11, 13, 14, 16–18, 21, 22, 24**] or [**12**, Chap. VI]. In particular, we mention the facts that, for a Hilbert space H, the M-ideals of $\mathscr{L}(H)$ coincide with its closed two-sided ideals [**20**] and that, for a subspace X of l_p, $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if X has the metric compact approximation property [**7**].

In this paper we show that in many cases the ideal of compact operators is the only candidate for an M-ideal in $\mathscr{L}(X)$. For $X = l_p$, this was done by Flinn [**9**]. For the function space $L_p = L_p[0,1]$, it has long been known that the compact operators do not form an M-ideal if $p \neq 2$ ([**14**] or, for another approach, [**17**]); in [**12**, p. 252] the problem is posed to determine the M-ideal structure of $\mathscr{L}(L_p)$ completely. This is done in Section 2, where we prove that there are no nontrivial M-ideals in the algebra of operators on the complex space L_p, $p \neq 1, 2, \infty$. In Section 3 we study the M-ideals in $\mathscr{L}(l_p(l_q^n))$, for $1 < p, q < \infty$. We show, for complex spaces again, that here the compact operators form the only M-ideal, thus proving a conjecture of Cho and

Johnson [8]. (Here, as usual, $l_p(E_n)$ denotes the l_p-direct sum of the Banach spaces E_n, i.e. $l_p(E_n)$ consists of all sequences of vectors $x_n \in E_n$ such that $\|(x_n)\| = (\Sigma \|x_n\|^p)^{1/p} < \infty$.)

For our arguments we need the notion of the (spatial) numerical range of an operator T on a complex Banach space E. This is the set

$$V(T) = \{x^*(Tx)\colon \|x^*\| = \|x\| = x^*(x) = 1\}.$$

The operator T is called hermitian if $V(T) \subset \mathbb{R}$. It is a well-known fact that an operator on L_p for $p \neq 2$ is hermitian if and only if it is a multiplication operator, i.e. $Tf = hf$ for some real-valued $h \in L_\infty$. (See [4] and [5] for details.)

2. *M*-ideals of operators on $L_p[0, 1]$

To prove the main result of this section, Theorem 2.4, we need the following lemmas.

LEMMA 2.1. *For all $\varepsilon > 0$, there is some $\delta > 0$ such that whenever A and B are operators on a complex Banach space satisfying* $\|\sin A\| \leqq 1 - \delta$, $\|\sin B\| \leqq 1$, $\arcsin(\sin A) = A$, $\arcsin(\sin B) = B$ *and* $\|\sin A - \sin B\| \leqq \delta$, *then* $\|A - B\| \leqq \varepsilon$. *The condition* $\arcsin(\sin A) = A$ *holds if* $V(A) \subset R := [-0.1, 1.1] \times [-0.1, 0.1]i$.

Proof. Consider the power series expansion

$$\arcsin z = \sum_{k=1}^{\infty} c_k z^k, \quad |z| \leqq 1,$$

in which all $c_k \geqq 0$. Since $\sin A$ is contractive, the series $\Sigma c_k(\sin A)^k$ converges, and the operator $\arcsin(\sin A)$ is defined (likewise for B).

Write $v = \sin A$, $u = \sin B$, $w = u - v$. By the uniform continuity of the arcsin-function on $[0, 1]$, we conclude for sufficiently small δ that

$$\begin{aligned}
\|B - A\| &= \|\arcsin(v + w) - \arcsin v\| \\
&= \left\| \sum_{k=1}^{\infty} c_k((v + w)^k - v^k) \right\| \\
&\leqq \sum_{k=1}^{\infty} c_k \sum_{l=0}^{k-1} \binom{k}{l} \|v\|^l \|w\|^{k-l} \\
&= \sum_{k=1}^{\infty} c_k((\|v\| + \|w\|)^k - \|v\|^k) \\
&= \arcsin(\|v\| + \|w\|) - \arcsin \|v\| \\
&\leqq \varepsilon,
\end{aligned}$$

if $\|w\| = \|\sin A - \sin B\| \leqq \delta$.

We finally discuss the numerical range condition. If $V(A) \subset R$, then the spectrum $\sigma(A)$ is also contained in R [4, p. 19]. Now $\sin R$ is a subset of the open unit disk. Hence $\arcsin(\sin z) = z$ for $z \in \sigma(A) \subset R$, and we deduce by the functional calculus that $\arcsin(\sin A) = A$. □

LEMMA 2.2. *Let X be a Banach space, $\mathscr{J} \subset \mathscr{L}(X)$ a two-sided ideal and P a projection onto a complemented subspace E of X which is isomorphic to X.*

(a) *If $P \in \mathscr{J}$, then $\mathscr{J} = \mathscr{L}(X)$.*

(b) *If E is C-isomorphic with X and $\mathscr{J}$ contains an operator T with $\|T-P\| < (C\|P\|)^{-1}$, then $\mathscr{J} = \mathscr{L}(X)$.*

Proof. (a) Let $j: E \to X$ denote the canonical embedding and let $\Phi: E \to X$ denote an isomorphism. Then $Id = (\Phi P)P(j\Phi^{-1}) \in \mathscr{J}$.

(b) We retain the above notation and let $S = (\Phi P)T(j\Phi^{-1}) \in \mathscr{J}$. Then, if $\|\Phi\| \, \|\Phi^{-1}\| \leqq C$,

$$\begin{aligned}\|S - Id\| &\leqq \|\Phi\| \, \|P\| \, \|T-P\| \, \|\Phi^{-1}\| \\ &\leqq C\|P\| \, \|T-P\| < 1;\end{aligned}$$

thus S is invertible and $Id = SS^{-1} \in \mathscr{J}$. □

LEMMA 2.3. *Let $1 \leqq p < \infty$ and let φ and φ' be bimeasurable bijective transformations on $[0, 1]$ and λ and λ' be measurable functions on $[0, 1]$. Suppose $I: f \mapsto \lambda \cdot f \circ \varphi$ and $J: f \mapsto \lambda' \cdot f \circ \varphi'$ are two isometric isomorphisms on L_p such that $\{\varphi \neq \varphi'\}$ has positive measure. Then $\|I - J\| \geqq 2^{1/p}$.*

Proof. It is enough to prove this for $J = Id$, that is, for $\lambda' = 1$ and $\varphi'(\omega) = \omega$. By Lusin's theorem, $\{\omega : \varphi(\omega) \neq \omega\}$ contains a compact set C of positive measure such that $\varphi|_C$ is continuous. A compactness argument now yields some compact subset $A \subset C$ of positive measure such that $\varphi(A) \cap A = \varnothing$. Pick α so that $f = \alpha\chi_{\varphi(A)}$ has norm 1. Then $\|If\| = 1$, If and f have disjoint supports and thus $\|If - f\| = 2^{1/p}$. □

We now state and prove the main theorem of this section.

THEOREM 2.4. *If $1 < p < \infty$ and $p \neq 2$, then there are no nontrivial M-ideals in $\mathscr{L}(L_p([0, 1], \mathbb{C}))$.*

Proof. Let $\{0\} \neq \mathscr{J} \subset \mathscr{L}(L_p)$ be an M-ideal. Since L_p and its dual are uniformly convex, $\mathscr{J}$ is a two-sided ideal [**8**]. We shall show that $Id \in \mathscr{J}$, thus proving our claim that $\mathscr{J} = \mathscr{L}(L_p)$.

In fact, by Lemma 2.2 it is enough to show that some characteristic projection $P_A: f \mapsto \chi_A f$, for some Borel set of positive measure, is in $\mathscr{J}$. This suffices because then $L_p(A)$ is isometricallly isomorphic to $L_p[0, 1]$ [**19**, p. 321].

We now let $0 < \eta < 0.1$ be a small number. (It will become clear in due course how small η should actually be.) By [**12**, Th. V.5.4] $\mathscr{J}$ contains an operator T with $\|T\| = 1$ and $V(T) \subset [-\eta, 1+\eta] \times [-\eta, \eta]i$. We are going to show that this 'almost' hermitian operator T is close to an hermitian operator $f \mapsto hf$; and we eventually prove that $P_A \in \mathscr{J}$ for $A = \{h \geqq \frac{1}{2}\}$.

We first apply [**4**, Theorem 4, p. 28] to obtain that

$$\|e^{itT}\| \leqq 1 + \alpha|t| \quad \forall\, |t| \leqq 1,$$

where $\alpha = \alpha(\eta) \to 0$ as $\eta \to 0$. That is, for small enough η and $|t| \leqq 1$, the operators e^{itT} are small bound isomorphisms:

$$\frac{1}{1+\alpha}\|f\| \leqq \|e^{itT}f\| \leqq (1+\alpha)\|f\|.$$

By a result due to Alspach [**2**], e^{itT} is close to an isometric isomorphism I_t:

$$\|e^{itT} - I_t\| \leqq \beta,$$

where $\beta=\beta(\eta)\to 0$ as $\alpha(\eta)\to 0$. But the isometric isomorphisms on L_p, $p\neq 2$, are known to have the form

$$I_t f=\lambda_t\cdot f\circ\varphi_t$$

for some measurable function λ_t and some bimeasurable bijection φ_t [**19**, p. 333]. Note that by Lemma 2.3 $\|I_t-I_s\|\geqq 2^{1/p}$ if $\{\varphi_s\neq\varphi_t\}$ has positive measure. This and the fact that $(e^{itT})_{t\in\mathbb{R}}$ is a uniformly continuous group of operators show that $\varphi_t=\varphi_0$ for $|t|\leqq 1$ provided η is small enough, and we must have that $\varphi_0(\omega)=\omega$ since $e^{itT}=Id$ for $t=0$. Furthermore, this enforces $|\lambda_t|=1$ a.e. Writing $\lambda_t=e^{ih_t}$, $-\pi\leqq h_t\leqq\pi$, and M_t for the multiplication operator $f\mapsto h_t f$, we finally obtain

$$\|e^{itT}-e^{iM_t}\|\leqq\beta$$

for $|t|\leqq 1$.

Fix $t=\frac{1}{2}$. By continuity of inversion we may assume in addition that

$$\|e^{-iT/2}-e^{-iM_{1/2}}\|\leqq\beta,$$

where $\beta=\beta(\eta)\to 0$ as $\eta\to 0$. Consequently

$$\|\sin T/2-\sin M_{1/2}\|\leqq\beta.$$

If B denotes the operator of multiplication with arcsin (sin $h_{1/2}$), then $\sin B=\sin M_{1/2}$, $\arcsin(\sin B)=B$ and $\|\sin B\|\leqq 1$. Also,

$$\|\sin T/2\|\leqq\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left\|\frac{T}{2}\right\|^{2k+1}\leqq\sinh\tfrac{1}{2}<1.$$

According to Lemma 2.1 we have for the multiplication operator $M:=2B$, say $f\mapsto hf$,

$$\|T-M\|\leqq\gamma,$$

where $\gamma=\gamma(\eta)\to 0$ as $\eta\to 0$. Since $\|T\|=1$, it follows that $A=\{h\geqq\frac{1}{2}\}$ has positive measure if η is sufficiently small.

Now consider $P_A\mathscr{J}P_A$, which can isometrically be identified with a subspace of $\mathscr{L}(L_p(A))$. It is straightforward to check that it is an M-ideal and thus a two-sided ideal. Moreover, since $\|P_ATP_A-P_AMP_A\|\leqq\gamma$ and $M|_{L_p(A)}$ is invertible with an inverse of norm $\leqq 2$, it follows that P_ATP_A is invertible in $L_p(A)$ once $\gamma<\frac{1}{2}$. Therefore the ideal $P_A\mathscr{J}P_A$ coincides with $L_p(A)$. This implies $P_A\in\mathscr{J}$, as requested. □

In the case $p=2$, the M-ideals and the closed two-sided ideals of $\mathscr{L}(L_2)$ coincide, as already mentioned. Therefore the ideal of compact operators is the only nontrivial M-ideal in $\mathscr{L}(L_2[0,1])$. For the M-ideals in $\mathscr{L}(L_1)$ and $\mathscr{L}(L_\infty)$, see [**24**].

For future reference, we record the following result established in the preceding proof.

PROPOSITION 2.5. *Let $1\leqq p<\infty$, $p\neq 2$. For all $\gamma>0$, there is some $\eta>0$ such that, whenever X is a complex $L_p(\mu)$-space and $T\in\mathscr{L}(X)$ satisfies $V(T)\subset[-\eta,1+\eta]\times[-\eta,\eta]i$, then there is some hermitian operator M satisfying $\|T-M\|\leqq\gamma$.*

3. M-ideals of operators on $l_p(l_q^n)$

In this section, we shall primarily deal with the space $l_p(l_q^n)$, $1 < p, q < \infty$. It is known (e.g. [**14**]) that the compact operators form an M-ideal in $\mathscr{L}(l_p(l_q^n))$, and in [**8**] Cho and Johnson exhibit an ideal in $\mathscr{L}(l_p(l_q^n))$ that fails to be an M-ideal. Actually, they conjecture that the compact operators are the only M-ideal. Here we offer a proof of this conjecture in the case of complex scalars.

The following lemma will turn out to be useful; for related results see [**6**].

LEMMA 3.1. *There is a constant C such that, whenever (k_n) is a sequence of positive integers with* $\limsup k_n = \infty$, *then $l_p(l_q^n)$ is C-isomorphic to $l_p(l_q^{k_n})$.*

Proof. Let $E_1, E_2, \ldots$ be an enumeration of the l_q^n-spaces in which each l_q^n-space is repeated infinitely often, and let $X = l_p(E_n)$, $Y = l_p(l_q^{k_n})$. It is enough to show that X is C'-isomorphic to Y for some universal constant C'. This follows immediately from an application of Pełczyński's decomposition method [**15**, p. 54], since there are subspaces $U \subset X$, $V \subset Y$ and natural isometric isomorphisms $X \cong l_p(X)$, $X \cong Y \oplus_p U$, $Y \cong X \oplus V$. Note that X and V are in fact complemented by contractive projections, so Y is 4-isomorphic to $X \oplus_p V$. Thus the decomposition scheme yields the desired isomorphism with $C' = 16$. □

Now for a technical lemma.

LEMMA 3.2. *Let $1 < p < \infty$ and let (E_n) be a sequence of finite-dimensional Banach spaces. Let P_k denote the canonical finite-rank projection from $l_p(E_n)$ onto E_k. Suppose $T \in \mathscr{L}(l_p(E_n))$ satisfies* $\sup_k \|P_k T P_k\| \leqq \varepsilon$ *for some $\varepsilon > 0$. Then there is a subsequence (k_n) of the positive integers such that the inequality*

$$\|PTP\| \leqq 3\varepsilon$$

holds for the canonical projection P from $l_p(E_n)$ onto $l_p(E_{k_n})$.

Proof. We first establish two claims.

Claim 1. For all $\beta > 0$ and all $k \in \mathbb{N}$, there exists $m_1 \in \mathbb{N}$ such that, for all $m \geqq m_1$,

$$\left\| \sum_{l \geqq m} P_l T P_k \right\| \leqq \beta.$$

This holds since $(\Sigma_{l \geqq m} P_l)_m$ is uniformly bounded, converges to 0 pointwise and thus converges to 0 uniformly on the compact set $TP_k(B)$, where B denotes the unit ball of $l_p(E_n)$.

Claim 2. For all $\beta > 0$ and all $k \in \mathbb{N}$, there exists $m_2 \in \mathbb{N}$ such that, for all $m \geqq m_2$,

$$\|P_k T P_m\| \leqq \beta.$$

This holds since $P_m^* \to 0$ pointwise and thus $\|P_k T P_m\| = \|P_m^* T^* P_k^*\| \to 0$.

We now construct the desired sequence inductively. We put $k_1 = 1$. Suppose $k_1, \ldots, k_n$ have already been constructed. We apply Claim 1 with $k = k_n$, $\beta = \varepsilon/2^n$, and we pick $k_{n+1} \geqq m_1$ so that

$$\sum_{j=1}^{n} \|P_{k_j} T P_{k_{n+1}}\| \leqq \frac{\varepsilon}{2^{n+1}},$$

which is possible by Claim 2. Then the resulting sequence (k_n) satisfies

$$\left\| \sum_{j>n} P_{k_j} T P_{k_n} \right\| \leqq \left\| \sum_{l \geqq k_{n+1}} P_l T P_{k_n} \right\| \leqq \frac{\varepsilon}{2^n}.$$

Furthermore, by assumption on T,

$$\left\| \sum_{n=1}^{\infty} P_{k_n} T P_{k_n} \right\| = \sup_n \| P_{k_n} T P_{k_n} \| \leqq \varepsilon,$$

since $\Sigma_n P_{k_n} T P_{k_n}$ is a block-diagonal operator. Therefore,

$$\begin{aligned} \| PTP \| &\leqq \sum_{n=1}^{\infty} \sum_{j<n} \| P_{k_j} T P_{k_n} \| + \left\| \sum_{n=1}^{\infty} P_{k_n} T P_{k_n} \right\| + \sum_{n=1}^{\infty} \left\| \sum_{j>n} P_{k_j} T P_{k_n} \right\| \\ &\leqq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} + \varepsilon + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\ &= 3\varepsilon. \quad \square \end{aligned}$$

THEOREM 3.3. *Consider the complex Banach space* $l_p(l_q^n)$, $1<p$, $q<\infty$. *Then* $\mathscr{K}(l_p(l_q^n))$ *is the only M-ideal in* $\mathscr{L}(l_p(l_q^n))$.

Proof. We have already pointed out that $\mathscr{K}(l_p(l_q^n))$ is an M-ideal. Moreover, it follows e.g. from [**11**, Theorem 4.4] that an M-ideal $\mathscr{J} \neq \{0\}$ contains $\mathscr{K}(l_p(l_q^n))$, and since $l_p(l_q^n)$ and its dual are uniformly convex, we know from [**8**] that an M-ideal in $\mathscr{L}(l_p(l_q^n))$ must be a two-sided ideal.

Let us now assume that $\mathscr{J}$ is an M-ideal in $\mathscr{L}(l_p(l_q^n))$ strictly containing $\mathscr{K}(l_p(l_q^n))$. We have to prove that $\mathscr{J} = \mathscr{L}(l_p(l_q^n))$. We first claim that an operator $T \in \mathscr{L}(l_p(l_q^n))$ factoring through l_p belongs to $\mathscr{J}$. In fact, let us write $T = T_2 T_1$ with $T_1 \in \mathscr{L}(l_p(l_q^n), l_p)$ and $T_2 \in \mathscr{L}(l_p, l_p(l_q^n))$. Now $\mathscr{J}$ contains a noncompact operator S. The proof of [**15**, Proposition 2.c.3] shows that S acts as an isomorphism on a complemented copy E of l_p, and $F = S(E)$ is complemented, too. Let $\pi: l_p(l_q^n) \to E$ and $\sigma: l_p(l_q^n) \to F$ denote projections. Then $\pi = (S^{-1}\sigma) S \pi \in \mathscr{J}$, and letting $\Phi: l_p \to E$ denote an isomorphism, we see that $T = T_2 T_1 = T_2 \Phi^{-1} \pi \Phi T_1 \in \mathscr{J}$.

Next we observe that it is enough to prove the theorem under the assumption that $q \leqq p$; the remaining case $q > p$ then follows by duality. We may also assume that $q \neq 2$. Indeed, $l_p(l_2^n)$ is known to be isomorphic to a complemented subspace of l_p and thus isomorphic to l_p itself. But the compact operators are the only closed two-sided ideal in $\mathscr{L}(l_p)$ (see the above claim); hence $\mathscr{K}(l_p(l_2^n))$ is the only M-ideal in $\mathscr{L}(l_p(l_2^n))$, since every M-ideal is an ideal. So we suppose that $q \leqq p$ and $q \neq 2$ in the remainder of the proof.

We need some notation. We denote the unit vectors in $l_p(l_q^n)$ by e_{kl}, $k \in \mathbb{N}$, $1 \leqq l \leqq n$. It is clear that $E_1 = \overline{\mathrm{lin}}\,\{e_{k1} : k \in \mathbb{N}\}$ is isometric to l_p. If $q \leqq p$, then the identity map from l_q^n to l_p^n is contractive; thus there is a natural operator A mapping $l_p(l_q^n)$ to $l_p(l_p^n) \cong l_p \cong E_1$. We can think of this operator A as an element of $\mathscr{L}(l_p(l_q^n))$, and by construction A factors through l_p. Consequently $A \in \mathscr{J}$, $\|A\| = 1$, and also $\|Ae_{kl}\| = 1$ for all k and l.

At this stage we invoke a result from [**23**] saying that there is a net $(H_\alpha) \subset \mathscr{J}$

converging to Id in the $\sigma(\mathscr{L}, \mathscr{J}^*)$-topology such that

$$\limsup \| \pm A + (Id - H_\alpha) \| = 1. \tag{3.1}$$

An application of [**12**, Theorem V.5.4] and a convex combinations technique described there allows us to assume in addition that the numerical ranges $V(H_\alpha)$ are contained in small rectangles $R_\alpha = [-\eta_\alpha, 1+\eta_\alpha] \times [-\eta_\alpha, \eta_\alpha]i$, with $\eta_\alpha \to 0$. We denote the canonical injection of l_q^k into $l_p(l_q^n)$ by j_k and the projection from $l_p(l_q^n)$ onto l_q^k by P_k. For each k, $V(P_k(Id - H_\alpha)j_k) \subset R_\alpha$; and therefore, by Proposition 2.5, we conclude that, given $\varepsilon > 0$, for large enough α all the operators $P_k(Id - H_\alpha)j_k$, $k = 1, 2, \ldots$, differ from hermitian operators $M_{\alpha k}: l_q^k \to l_q^k$ by less than $\varepsilon/3$. Also, from (3.1) and the uniform convexity of $l_p(l_q^n)$ we infer that $(Id - H_\alpha)e_{kl} \to 0$ (since $\| Ae_{kl} \| = 1$), uniformly in k and l. From this we deduce for large enough α that $\|M_{\alpha k}(e_{kL})\| \leqq \frac{2}{3}\varepsilon$ for all k and l and thus, since the $M_{\alpha k}$ are hermitian (i.e. multiplication operators), $\| M_{\alpha k} \| \leqq \frac{2}{3}\varepsilon$ for all k. This means, for large enough α and $H = H_\alpha$, that

$$\| P_k(Id - H)P_k \| \leqq \| P_k(Id - H)j_k \| \leqq \varepsilon, \quad \forall k \in \mathbb{N}.$$

Hence $Id - H$ meets the assumption of Lemma 3.2. That lemma provides us with a sequence (k_n) such that, for the canonical projection P from $l_p(l_q^n)$ onto $l_p(l_q^{k_n})$,

$$\| P - PHP \| = \| P(Id - H)P \| \leqq 3\varepsilon.$$

Since $PHP \in \mathscr{J}$, an appeal to Lemmas 2.2(b) and 3.1 finishes the proof of Theorem 3.3 provided we have chosen $\varepsilon < 1/(3C)$, where C is the constant appearing in Lemma 3.1. □

A similar, but technically simpler argument yields the following result.

PROPOSITION 3.4. *Let $1 < p < \infty$, and let X be a uniformly convex Banach space isomorphic to a subspace of l_p. Suppose $\mathscr{J} \subset \mathscr{L}(X)$ is a closed ideal strictly containing $\mathscr{K}(X)$. If $\mathscr{J}$ is an M-ideal, then $\mathscr{J} = \mathscr{L}(X)$.*

Proof. As in the proof of Theorem 3.3, we argue that $\mathscr{J}$ contains all l_p-factorable operators. Since X embeds into l_p, X contains a subspace isomorphic to l_p. Let $\Psi: X \to l_p$ and $\Phi: l_p \to X$ denote (into-)isomorphisms, and let $A = \Phi\Psi$. Then A factors through l_p, and hence $A \in \mathscr{J}$.

We may assume that $\| A \| = 1$. There is some $\varepsilon > 0$ such that $\varepsilon \| x \| \leqq \| Ax \|$ for all $x \in X$. Again, there is a net $(H_\alpha) \subset \mathscr{J}$ such that

$$\limsup \| \pm A + (Id - H_\alpha) \| = 1.$$

If α is large enough and $H = H_\alpha$, we have by uniform convexity for some $\delta > 0$ that $\| (Id - H)x \| \leqq 1 - \delta$ whenever $\| x \| = 1$. Hence $\| Id - H \| < 1$, and $\mathscr{J}$ contains an invertible element. This proves that $\mathscr{J} = \mathscr{L}(X)$. □

We recall that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if X is isometric to a subspace of l_p and has the metric compact approximation property.

We finally mention some problems suggested by our work.

(1) The results in Sections 2 and 3 support the conjecture that $\mathscr{K}(X)$ is the only candidate for an M-ideal in $\mathscr{L}(X)$ if X is isometric to a subspace of L_p. Does this conjecture hold?

(2) In the other direction, it would be interesting to find an example of a uniformly

convex Banach space X and an ideal $\mathscr{K}(X) \neq \mathscr{J} \subset \mathscr{L}(X)$ which is a nontrivial M-ideal. It is also still open whether, for a uniformly convex Banach space X, every M-ideal in $\mathscr{L}(X)$ is a two-sided ideal. (An M-ideal is known to be a left ideal if X is uniformly convex [**8**].)

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