Positivity **8:** 101–107, 2004.

Invariant Subspaces and the Exponential Map

A. ATZMON[1], G. GODEFROY[2] and N. J. KALTON[3]
[1]*School of Mathematical Analysis, Tel-Aviv University, Tel-Aviv 69978, Israel. (E-mail: aatzmon@post.tau.ac.il);* [2]*Equipe d'Analyse, Université Paris 6, Case 186, 4, Place Jussieu, 75252 Paris Cedex 05, France. (E-mail: gig@ccr.jussieu.fr);* [3]*University of Missouri-Columbia, Department of Mathematics, Columbia, MO 65211, USA. (E-mail: nigel@math.missouri.edu)*

(Received 20 September 2001; accepted 10 February 2003)

Bounded operators with no non-trivial closed invariant subspace have been constructed by P. Enflo [6]. In fact, there exist bounded operators on the space ℓ_1 with no non-trivial closed invariant subset [12]. It is still unknown, however, if such operators exist on reflexive Banach spaces, or on the separable Hilbert space. The main result of this note (Theorem 1) asserts that the existence of an invariant non-trivial closed *subset* for the image of an algebra under the exponential map implies the existence of an invariant non-trivial closed *subspace* for the operators in the algebra. The proof relies on a simple differentiation argument. Several consequences of the main result are gathered.

This work relies in part on the Note [4]. However, Corollary 2 and Corollary 5 are the only statements of this work which go back to [4].

NOTATION 1. We denote by X a separable *real* Banach space, of dimension greater than one. We say that a closed subset F of a Banach space X is non-trivial if it is different from the singleton $\{0\}$ and the whole space X. If S is a subset of the algebra $L(X)$ of bounded operators on the space X and $x \in X$, we denote

$$S(x) = \{T(x); T \in S\}.$$

A subalgebra $\mathcal{A}$ of $L(X)$ is transitive if for every $x \neq 0$ in X, the subspace $\mathcal{A}(x)$ is dense in X. Equivalently, the algebra $\mathcal{A}$ is not transitive if there is a non-trivial closed $\mathcal{A}$-invariant subspace, that is, a non-trivial closed linear subspace M such that $T(M) \subset M$ for every $T \in \mathcal{A}$ (see [11]). We denote by *Exp* the usual exponential map from $L(X)$ to the group $GL(X)$ of invertible elements of $L(X)$. The algebra of real polynomials is denoted $\mathbf{R}[\xi]$.

We now state and prove our main result.

THEOREM 1. *Let $\mathcal{A}$ be a subalgebra of $L(X)$. If there exists a non-trivial closed subset F of X such that $Exp(T)(F) \subset F$ for every $T \in \mathcal{A}$, then the algebra $\mathcal{A}$ is not transitive.*

Proof. Since X is separable, there exists a Lipschitz and Gâteaux smooth bump function on X. Following [4], an application of the smooth variational principle from [5] provides a Gâteaux smooth and Lipschitz function H and a point $w \neq 0$ in F such that

(i) $H(w) = min\{H(x); x \in F\}$
(ii) $H'(w) \neq 0$

Pick any $T \in \mathcal{A}$ and consider the following real function

$$\phi_T(s) = H(\mathrm{Exp}(sT)w).$$

It follows from our assumptions that the function ϕ_T attains its minimum at $s = 0$, and thus its derivative vanishes at $s = 0$. Since H is Gâteaux smooth and Lipschitz, we may apply that chain rule and thus

$$< H'(w), T(w) > = 0. \tag{1}$$

If $T(w) = 0$ for every $T \in \mathcal{A}$, then the one-dimensional space generated by w is a non-trivial closed $\mathcal{A}$-invariant subspace. If not, then by (1) and since $\mathcal{A}$ is a subalgebra, the closure of the space $\mathcal{A}(w)$ is a non-trivial closed $\mathcal{A}$-invariant subspace.

REMARKS 1. (1) Up to the notation, Theorem 1 is in fact a special case of [4, Theorem 2] where only exponential functions are considered. However, if an operator A satisfies the assumptions of [4, Theorem 2], then in the notation used there we have $\mathrm{Exp}(sA)(F) \subset F$ for every $s \in \mathbf{R}$, since for every $x \in X$,

$$\lim_{n \to \infty} [f_A(s/n)]^n(x) = \mathrm{Exp}(sA)(x).$$

Hence there is no loss of generality to only consider the exponential function in Theorem 1 instead of a general differentiable function as in [4, Theorem 2]. (2) The converse to Theorem 1 is clearly true: if $\mathcal{A}$ is not transitive, let M be a non-trivial closed $\mathcal{A}$-invariant subspace. Then $\mathrm{Exp}(T)(M) \subset M$ for every $T \in \mathcal{A}$. (3) The proof of Theorem 1 works as well for any Banach space X on which there is a Lipschitz Gâteaux smooth bump. In particular, it works for every reflexive Banach space and every space $L^1(\nu)$, where ν denotes a probability measure.

COROLLARY 1. *The following assertions are equivalent*:

(i) *There is a commutative transitive subalgebra* $\mathcal{A}$ *of* $L(X)$.
(ii) *There is a commutative subgroup* G *of* $GL(X)$ *such that* $G(x)$ *spans a dense linear subspace of* X *for every* $x \in X \setminus \{0\}$.
(iii) *There is a commutative subgroup* G *of* $GL(X)$ *such that* $G(x)$ *is a dense subset of* X *for every* $x \in X \setminus \{0\}$.

Proof. The implication (iii) implies (ii) is obvious. If G satisfies (ii), then the algebra $\mathcal{A}$ generated by G clearly satisfies (i). Finally, if $\mathcal{A}$ satisfies (i), then the set

$$G = \{\mathrm{Exp}(T); T \in \mathcal{A}\}$$

is a group, which satisfies (iii) by Theorem 1 since the algebra $\mathcal{A}$ is transitive.

It is not known whether the Hilbert space $X = l^2(\mathbf{N})$ satisfies the above equivalent conditions. We can reformulate the equivalence between (ii) and (iii) by saying that if there is an irreducible representation of a commutative group on X, then there is a 'hyper-irreducible' representation of a commutative group on X. Note also that if $G = \{\mathrm{Exp}(T); T \in \mathcal{A}\}$, where $\mathcal{A}$ is a commutative algebra, then every nonzero vector is G-cyclic if and only if every nonzero vector is G-hypercyclic.

If $L(X)$ contains an operator with no nontrivial closed invariant subspace, then X satisfies the equivalent conditions of Corollary 1. Indeed we have:

COROLLARY 1. *If $T \in L(X)$ has no non-trivial closed invariant subspace, then for every $x \in X \backslash \{0\}$, the set*

$$E(x) = \{\mathrm{Exp}(P(T))(x); P \in \mathbf{R}[\xi]\}$$

is dense in X.

Proof. We consider the algebra $\mathcal{A} = \{P(T); P \in \mathbf{R}[\xi]\}$. For every $x \in X \backslash \{0\}$, we call $F(x)$ the closure of the set $E(x)$. It is clear that $\mathrm{Exp}(S)(F(x)) \subset F(x)$ for every $S \in \mathcal{A}$. The conclusion follows from Theorem 1 since by assumption the algebra $\mathcal{A}$ is transitive.

Let $C \subset X$ be a non-trivial closed convex cone. We denote by $\leqslant_C$ the corresponding ordering on $L(X)$ defined by $T \leqslant_C S$ if $(S-T)(C) \subset C$. We say that T is C-order bounded if there exists $\lambda \geqslant 0$ such that

$$-\lambda I \leqslant_C T \leqslant_C \lambda I.$$

We denote by $\mathcal{A}_C$ the set of C-order bounded operators. It is easy to check that $\mathcal{A}_C$ is a subalgebra of $L(X)$. With this notation, the following holds.

COROLLARY 2. [4, Cor. 6]. *The algebra $\mathcal{A}_C$ is not transitive.*

Proof. Pick any $T \in \mathcal{A}_C$. There is $\lambda \in \mathbf{R}$ such that $(T + \lambda I)(C) \subset C$, and thus

$$\mathrm{Exp}(T + \lambda I)(C) \subset C$$

since C is a closed convex cone. But since

$$\mathrm{Exp}(T + \lambda I) = e^{\lambda}\mathrm{Exp}(T)$$

it follows that $\mathrm{Exp}(T)(C) \subset C$ and Theorem 1 concludes the proof.

EXAMPLE 1. In this notation of the proof of Corollary 5 in [4], the operators A_g and B_g are F-order bounded [3], and [4, Corollary 5] follows since the algebra they generate is contained in $\mathcal{A}_F$.

EXAMPLE 2. [7], *see [8, p. 267].* If C is a non-trivial closed convex cone with non-empty interior, then every operator T such that $T \geqslant_C 0$ has a non-trivial invariant subspace. Indeed, if $V = C \cap (-C) \neq \{0\}$, then V is such a subspace. Assume now that $C \cap (-C) = \{0\}$, and pick $x \in C \backslash \{0\}$. We claim that $\mathrm{Exp}(P(T))(x) \notin (-C)$ for every $P \in \mathbf{R}[\xi]$. Indeed, write $P = P_1 - P_2$, where P_1 and P_2 have positive coefficients. If $\mathrm{Exp}(P(T))(x) \in (-C)$, then since P_2 has positive coefficients, one has

$$\mathrm{Exp}(P_2(T))\mathrm{Exp}(P(T))(x) \in (-C)$$

but this means that

$$\mathrm{Exp}(P_1(T))(x) \in (-C)$$

and since we also have $\mathrm{Exp}(P_1(T))(x) \in C$ because P_1 has positive coefficients, it follows that $\mathrm{Exp}(P_1(T))(x) = 0$, but this contradicts $x \neq 0$ since the operator $\mathrm{Exp}(P_1(T))$ is invertible. We found a vector, namely x, such that $E(x)$ is not dense since C has non empty interior, and Corollary 1 concludes the proof.

It is easily seen that the following Corollary 3 generalizes the above Example 2.

COROLLARY 3. *Let $T \in L(X)$ be such that there exists a scalar $a > 0$ and a vector $x_0 \in X$ such that for all polynomials P and Q with positive coefficients, one has*:

(i) $\|P(T) + Q(T)\| \geqslant a\|P(T)\|$.
(ii) $\|P(T)(x_0)\| \geqslant a\|P(T)\|$.

Then T has a non-trivial closed invariant subspace.

Proof. Pick $\epsilon \in (0, a^2)$. If $T = 0$, there is nothing to prove. If not, then necessarily $x_0 \neq 0$, and by Corollary 1 there is a polynomial $P \in \mathbf{R}[\xi]$ such that

$$\|\mathrm{Exp}(P(T))(x_0) + x_0\| < \epsilon.$$

We write $P = Q - R$, where Q and R have positive coefficients. We have

$$\begin{aligned}\|\mathrm{Exp}(Q(T))(x_0) + \mathrm{Exp}(R(T))(x_0)\| &= \|\mathrm{Exp}(R(T))[\mathrm{Exp}(P(T))(x_0) + x_0]\| \\ &\leqslant \epsilon \|\mathrm{Exp}(R(T))\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\mathrm{Exp}(Q(T))(x_0) + \mathrm{Exp}(R(T))(x_0)\| &\geqslant a\|\mathrm{Exp}(Q(T)) + \mathrm{Exp}(R(T))\| \\ &\geqslant a^2 \|\mathrm{Exp}(R(T))\|\end{aligned}$$

and this contradicts the choice of ϵ since $\mathrm{Exp}(R(T)) \neq 0$.

COROLLARY 4. *If X is a Banach lattice and $T \in L(X)$ is a positive operator, then T has a non-trivial closed invariant subspace provided there exists $x_0 \in X$ such that (ii) holds for every polynomial P with positive coefficients.*

Indeed, in this case (i) is clearly satisfied. It is not known if every positive operator on a Banach lattice has a non-trivial invariant subspace (see [1]).

We say that an operator T has a moment sequence (see [2]) if there exists $x_0 \in X \setminus \{0\}, x_0^* \in X^* \setminus \{0\}$ and a positive measure μ on $\mathbf{R}$ such that for every $n \geqslant 0$, one has

$$< x_0^*, T^n(x_0) > = \int u^n \mathrm{d}\mu(u). \tag{2}$$

With this notation, one has

COROLLARY 5. [4, Cor. 4]. *If $T \in L(X)$ has a moment sequence, then T has a non-trivial closed invariant subspace.*

Proof. It follows from the spectral radius formula that if μ satisfies (2) then its support is bounded. If $Q \in \mathbf{R}[\xi]$, we have therefore

$$< x_0^*, \mathrm{Exp}(Q(T))(x_0) > = \int e^{Q(u)} \mathrm{d}\mu(u).$$

It follows that $E(x_0) \subset \{x; < x_0^*, x > \geqslant 0\}$, and the conclusion follows from Corollary 1.

Our next application relies on a recent result of Lomonosov [10]. We denote by $K(X)$ the space of compact operators on X, and $|\|T|\|$ is the essential norm of the operator T, that is, the quotient norm of T in $L(X)/K(X)$. We denote $S(X)$ the set of operators T such that there is a compact subset L of $\mathbf{R}$ with

$$|\|P(T)|\| \leqslant \sup\{|P(x)|; x \in L\} \tag{3}$$

for every $P \in \mathbf{R}[\xi]$. For instance, it is easily seen that multiplication operators by bounded functions on L^p spaces belong to $S(L^p)$. Every self-adjoint operator S on the Hilbert space belong to $S(L^2)$, and we can take L to be the spectrum of S. We also observe that if T satisfies (3), then $T+K$ satisfies it too for every $K \in K(X)$. Therefore one has $S(X)+K(X)=S(X)$. We now prove:

PROPOSITION 1. *If $T \in S(X)$, then the conjugate operator $T^* \in L(X^*)$ has a non-trivial closed invariant subspace.*

Proof. We apply Lomonosov's theorem ([10]; see [8] fo a short proof of the related result [9]), which easily works in the real case for commutative algebras, and reads as follows: if $\mathcal{R}$ is a proper weakly closed subalgebra of $L(X)$, there exist nonzero vectors $x_0^* \in X^*$ and $x_0^{**} \in X^{**}$ such that $< x_0^{**}, x_0^* > \geqslant 0$ and

$$| < x_0^{**}, B^* x_0^* > | \leqslant |||B||| < x_0^{**}, x_0^* >$$

for every $B \in R$. We apply this theorem to the weakly closed subalgebra generated by T. It then follows form (3) that for every polynomial $P \in \mathbb{R}[\xi]$, we have

$$| < x_0^{**}, P(T^*) x_0^* > | \leqslant \sup\{|P(x)|; x \in L\} < x_0^{**}, x_0^* > \tag{4}$$

It easily follows from (4) that the operator T^* has a moment sequence, with a positive measure μ supported by L. The conclusion follows by Corollary 10.

We recall that an operator A on a Hilbert space is called essentially self-adjoint if $A = S + K$, where S is self-adjoint and K is a compact operator. We observe that such an operator has the form $A = T^*$, where T satisfies the assumptions of Proposition 11. Therefore we have:

COROLLARY 6. [14]. *An essentially self-adjoint operator A on a real Hilbert space has a non-trivial invariant subspace.*

1. Acknowledgements

The work of the first- and second-named authors is part of the research program of the European network "Analysis and operators", contract HPRN-CT-00116-2000, funded by the European Community. The third author was supported by NSF Grant DMS-9870027.

References

1. Abramovich, Y., Aliprantis, C. and Burkinshaw, O.: The invariant subspace problem: some recent advances, *Rend. Istit. Mat. Univ. Trieste*, *Suppl.* **XXIX** (1998), 3–79.
2. Atzmon, A.: The existence of translation invariant subspaces of symmetric self-adjoint sequence spaces on Z, *J. Funct. Anal.* **178** (2000), 372–380.
3. Atzmon, A.: Reducible representations of Abelian groups, *Ann. Inst. Fourier* **51** (2001).
4. Atzmon, A., Godefroy, G.: An application of the smooth variational principle to the existence of nontrivial invariant subspaces, *Note aux C. R. A. S. Paris* **332**(1), (2001), 151–156.
5. Deville, R., Godefroy, G. and Zizler, V.: A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions, *J. Funct. Anal.* **111**(1) (1993), 197–212.
6. Enflo, P.: On the invariant subspace problem for Banach spaces, *Acta Math.* **158** (!987), 213–313.
7. Krein, M. G. and Rutman, M. A.: Linear operators leaving invariant a cone in a Banach space, *Uspehi Mat. Nauk.* (N.S.) 3, 1(23) (1948), 3–95 (Also: *Amer. Math, Soc. Transl.* **26** (1950).
8. Lindström, M. and Schlüchtermann, G.: Lomonosov's techniques and Burnside's theorem, *Canadian Math. Bulletin* **43** (2000), 87–89.
9. Lomonosov, V.: An extension of Burnside's theorem to infinite dimensional spaces, *Israel J. Math.* **75** (1991), 329–339.

10. Lomonosov, V.: Positive functionals on general operator algebras, *J. Math. Anal. Appl.* **245** (2000), 221–224.
11. Radjavi, H. and Rosenthal, P.: Invariant subspaces, *Ergebnisse des Mathematik*, Band 77, Springer, Berlin 1977.
12. Read, C.: The invariant subspace problem for a class of Banach spaces. 2: hypercyclic operators, *Israel J. Math.* **63** (1988), 1–40.
13. Schaefer, H. H.: Topological vector spaces, *Springer-Verlag Graduate Texts in Mathematics* Springer Verlag, Vol. 3 1966.
14. Simonic, A. An extension of Lomonosov's techniques to non-compact operators, *Trans. Amer. Math. Soc.* **348** (1996), 975–995.