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Uniqueness of the Unconditional Basis of $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$, $0<p<1$

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Abstract. We prove that the quasi-Banach spaces $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0<p<1)$ have a unique unconditional basis up to permutation.

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1. Introduction

If $(X, \|\cdot\|)$ is a quasi-Banach space (in particular a Banach space) with normalized unconditional basis $(e_n)_{n=1}^{\infty}$, we say that X has *unique unconditional basis (up to permutation)* if whenever $(x_n)_{n=1}^{\infty}$ is another normalized unconditional basis of X, then $(x_n)_{n=1}^{\infty}$ is equivalent to (a permutation of) $(e_n)_{n=1}^{\infty}$. We will write it as $(x_n)_{n=1}^{\infty} \sim (e_n)_{n=1}^{\infty}$.

The question of uniqueness of unconditional basis up to permutation is a very natural one from the point of view of (quasi-)Banach lattices, because an unconditional basis induces a lattice ordering. Thus if X is a separable discrete order-continuous quasi-Banach lattice it has a unique unconditional basis up to permutation if and only if whenever X is linearly isomorphic to a discrete order-continuous quasi-Banach lattice Y then X is also lattice-isomorphic to Y.

The only Banach spaces with unique normalized unconditional basis are c_0, ℓ_1 and ℓ_2 [12, 14]. For Banach spaces without symmetric basis, this problem was first treated by Edelstein and Wojtaszczyk [5] who proved that the spaces $c_0 \oplus \ell_1$, $c_0 \oplus \ell_2$, $\ell_1 \oplus \ell_2$ and $c_0 \oplus \ell_1 \oplus \ell_2$ have unique unconditional basis up to permutation. Bourgain et al. studied in [3] the infinite direct sums of the spaces c_0, ℓ_1 and ℓ_2, showing that in $c_0(\ell_1)$, $c_0(\ell_2)$, $\ell_1(c_0)$ and $\ell_1(\ell_2)$ the canonical unit vector basis is unique up to permutation, while the result is not true for $\ell_2(c_0)$ and $\ell_2(\ell_1)$. So, in the context of Banach spaces it is quite exceptional for a space to have a unique unconditional basis. In the wider class of quasi-Banach spaces, however, we find many other spaces with that property, including ℓ_p when $0<p<1$ (see [6, 10, 16]).

Motivated by the results in [3], it was shown in [11] that the spaces $c_0(\ell_p)$ $(0<p<1)$ have unique normalized unconditional basis up to permutation and, later on, the authors proved in [1] the uniqueness of unconditional basis up to permutation of $\ell_p(c_0)$ and $\ell_p(\ell_2)$ for $0<p<1$. In this paper we prove that the same result holds for the quasi-Banach spaces $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0<p<1)$. The techniques we use here are completely different.

2. Terminology

Next we give some remarks on terminology and assumptions. An unconditional basic sequence $(u_n)_{n=1}^{\eta}$ in a quasi-Banach space X is *complemented* if there is a bounded linear projection $P: X \longrightarrow [u_n]$. This implies the existence of $(u_n^*)_{n=1}^{\eta} \subset X^*$ so that $Px = \sum_{n=1}^{\eta} u_n^*(x) u_n$. We sometimes refer to this as the biorthogonal sequence. The symbol η can denote either a positive integer or ∞.

We will use the term *sequence space* to mean a quasi-Banach space of sequences $\xi = (\xi_n)_{n=1}^{\infty}$ so that the canonical basis vectors $(e_n)_{n=1}^{\infty}$ form a 1-unconditional basis. The corresponding coordinate functionals are denoted $(e_n^*)_{n=1}^{\infty}$. Of course, any unconditional basis $(u_n)_{n=1}^{\infty}$ of a quasi-Banach space Y induces an isomorphism $S: X \to Y$ where X is a sequence space and $Se_n = u_n$.

Note that any disjoint sequence (u_n) in a sequence space is an unconditional basic sequence. We refer to such a sequence as complemented if $[u_n]$ is the range of a projection.

If $(u_n)_{n=1}^{\infty}$ is an unconditional basis for X and N is a natural number we denote by $(u_n^{(k)})_{n \in \mathbb{N}, 1 \leqslant k \leqslant N}$ the naturally induced unconditional basis of X^N (the direct sum of N copies of X).

A quasi-Banach lattice is called *sufficiently lattice Euclidean* if there is a constant M so that for any n there are operators $S: X \longrightarrow \ell_2^n$ and $T: \ell_2^n \longrightarrow X$ so that $ST = I_{\ell_2^n}$ and $\|S\|\|T\| \leqslant M$, and S is a lattice homomorphism. This is equivalent to asking that ℓ_2 is finitely representable as a complemented sublattice of X. We will say that X is *lattice anti-Euclidean* if it is not sufficiently lattice Euclidean. We use the same terminology for an unconditional basic sequence, which we regard as inducing a lattice structure on its closed linear span.

If X is a quasi-Banach space, by $\hat{X}$ we will denote its Banach envelope (cf. [9, 10]).

We recall that a quasi-Banach lattice X is said to be *p-convex*, where $0<p<\infty$, if there is a constant $M>0$ such that for any $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$ we have

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leqslant M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}.$$

The procedure to define the element $(\sum_{i=1}^{n} |x_i|^p)^{1/p} \in X$ is described in [13], pp. 40–41. We recall that not every quasi-Banach lattice is p-convex for some $p<2$, but if X is a quasi-Banach lattice which is isomorphic to a subspace of a p-convex quasi-Banach lattice then X is also p-convex. For details we refer the reader to [7].

For $0<p\leqslant 1$ fixed, we denote

$$\ell_1(\ell_p)=\left\{(\overline{x}_l)_{l=1}^{\infty}\,;\,\overline{x}_l\in\ell_p \text{ for each } l \text{ and } (\|\overline{x}_l\|_p)_{l=1}^{\infty}\in\ell_1\right\}.$$

This space endowed with the p-norm

$$\|(\overline{x}_l)_l\|_{\ell_1(\ell_p)}=\sum_{l=1}^{\infty}\|\overline{x}_l\|_p$$

is a p-Banach space. It is also p-convex as a quasi-Banach lattice, and hence every unconditional basic sequence generates a p-convex lattice structure.

For each $l\in\mathbb{N}$, we can write $\overline{x}_l=(x_{l1},x_{l2},\dots,x_{lk},\dots)\in\ell_p$, and then identify $\ell_1(\ell_p)$ with the space of infinite matrices $(x_{lk})_{l,k=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty}|x_{lk}|^p<\infty$ for all l and

$$\|(x_{lk})_{l,k}\|_{\ell_1(\ell_p)}=\sum_{l=1}^{\infty}\left(\sum_{k=1}^{\infty}|x_{lk}|^p\right)^{1/p}<\infty.$$

Analogously, for $0<p\leqslant 1$ fixed,

$$\ell_p(\ell_1)=\left\{(\overline{x}_l)_{l=1}^{\infty}\,;\,\overline{x}_l\in\ell_1 \text{ for each } l \text{ and } (\|\overline{x}_l\|_1)_{l=1}^{\infty}\in\ell_p\right\}.$$

This space endowed with the p-norm

$$\|(\overline{x}_l)_l\|_{\ell_p(\ell_1)}=\left(\sum_{l=1}^{\infty}\|\overline{x}_l\|_1^p\right)^{\frac{1}{p}}$$

is a p-Banach space.

For each $l\in\mathbb{N}$, we can write $\overline{x}_l=(x_{l1},x_{l2},\dots,x_{lk},\dots)\in\ell_1$, and then identify $\ell_p(\ell_1)$ with the space of infinite matrices $(x_{lk})_{l,k=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty}|x_{lk}|<\infty$ for all l and

$$\|(x_{lk})_{l,k}\|_{\ell_p(\ell_1)}=\left(\sum_{l=1}^{\infty}\left(\sum_{k=1}^{\infty}|x_{lk}|\right)^p\right)^{1/p}<\infty.$$

The dual space of both $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0<p<1)$ can be identified with $\ell_\infty(\ell_\infty)=\ell_\infty$, the Banach space of infinite matrices $a=(a_{lk})_{l,k=1}^{\infty}$ such that

$$\|a\|_\infty=\sup_{l,k}|a_{lk}|<\infty.$$

The Banach envelope of both quasi-Banach spaces $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0<p<1)$ is $(\ell_1(\ell_1),\|\cdot\|_1)=\ell_1$.

$\|\cdot\|$ will denote, according to context the quasi-norm in the spaces $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0<p<1)$ and the norm in the dual ℓ_∞.

The canonical basis for the spaces $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ will be denoted by $(e_{lk})_{l,k=1}^{\infty}$.

Our aim is to prove the following result:

THEOREM 2.1 (Main Theorem). *Let* $0<p<1$. *Let* $(x_n)_{n=1}^{\infty}$ *be a normalized unconditional basis of* $\ell_1(\ell_p)$ (*respectively* $\ell_p(\ell_1)$). *Then* $(x_n)_{n=1}^{\infty}$ *is equivalent to a permutation of the canonical basis of* $\ell_1(\ell_p)$ (*respectively* $\ell_p(\ell_1)$).

3. Preliminary results

The canonical basis of both $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0<p<1)$ is also an unconditional basis of their Banach envelope, ℓ_1, where all normalized unconditional bases are equivalent and, therefore, symmetric:

THEOREM 3.1 ([12]). *Let* $(x_n)_{n=1}^{\infty}$ *be a normalized* K*-unconditional basis of* ℓ_1. *Then, there exists a constant* $c>0$ (*depending only on* K) *so that*

$$c\sum_{n=1}^{N}|a_n| \leqslant \left\|\sum_{n=1}^{N}a_n x_n\right\| \leqslant \sum_{n=1}^{N}|a_n| \tag{2.1}$$

for any $(a_n)_{n=1}^{N}$ *scalars*, $N\in\mathbb{N}$.

Let us notice that Theorem 3.1 does not give any information about the different behavior of the subsets of any unconditional basis of $\ell_p(\ell_1)$ or $\ell_1(\ell_p)$ $(0<p<1)$ seen as unconditional basic sequences of their Banach envelope, in contrast with what happened in $c_0(\ell_p)$, $\ell_p(c_0)$ and $\ell_p(\ell_2)$, $0<p<1$. In these cases, the corresponding theorems of uniqueness of unconditional basis up to permutation in their Banach envelopes ([3]), where the canonical basis is not symmetric, were the starting point of the proofs. Since the canonical basis of ℓ_1 is symmetric we cannot approach the proofs in the same way. Instead, the proof of the Main Theorem relies upon Theorem 3.4 below which will allow us to unravel the form in which any complemented, normalized unconditional basic sequence $(u_n)_{n=1}^{\infty}$ in $\ell_1(\ell_p)$ or $\ell_p(\ell_1)$ $(0<p<1)$ can be written in terms of the canonical basis.

First we recall the Cantor-Bernstein principle for unconditional bases. This was first observed by Mityagin [15]. See also [16] and [17].

PROPOSITION 3.2. *Suppose* $(u_n)_{n=1}^{\infty}$ *and* $(v_n)_{n=1}^{\infty}$ *are two unconditional basic sequences. In order that* (u_n) *and* (v_n) *be equivalent* (*up to permutation*) *it is necessary and sufficient that* (u_n) *is equivalent* (*up to permutation*) *to a subsequence of* (v_n) *and* (v_n) *is equivalent* (*up to permutation*) *to a subsequence of* (u_n).

Using the Cantor-Bernstein principle it is easy to classify the subbases of the canonical basis of $\ell_p(\ell_q)$ when $0<p,q<\infty$.

PROPOSITION 3.3. *Suppose* $0<p\neq q<\infty$. *Let* $\mathcal{N}$ *be an infinite subset of* $\mathbb{N}\times\mathbb{N}$. *Then* $(e_{lk})_{(l,k)\in\mathcal{N}}$ *is equivalent (up to permutation) in* $\ell_p(\ell_q)$ *to the canonical basis of exactly one of the following spaces*

$$\ell_p,\ \ell_q,\ \ell_p\oplus\ell_q,\ \ell_p(\ell_q^n)_{n=1}^{\infty},\ \ell_q\oplus\ell_p(\ell_q^n)\text{ or }\ \ell_p(\ell_q).$$

We omit the easy details.

Theorem 3.4 is an extension to p-convex quasi-Banach lattices of Theorem 3.5 of [4]. It shows that complemented unconditional basic sequences in a p-convex quasi-Banach sequence space whose Banach envelope is lattice anti-Euclidean take a particularly simple form.

THEOREM 3.4. *Let* X *and* Y *be quasi-Banach sequence spaces. Suppose* Y *is* p*-convex for some* $p>0$ *and that* X *is isomorphic to a complemented subspace of* Y. *Suppose* $\hat{X}$ *is lattice anti-Euclidean. Then there exists* $N\in\mathbb{N}$ *such that* X *is isomorphic to a complemented sublattice of* Y^N. *More specifically, there exists a complemented disjoint positive sequence* (u_n) *in* Y^N *equivalent to the unit vector basis* (e_n) *in* X. *Furthermore, the projection* P *of* Y^N *onto* $[u_n]$ *may be given in the form*

$$Px=\sum_{n=1}^{\infty}\langle x,u_n^*\rangle u_n$$

where $u_n^*\geqslant 0$ *and* $\operatorname{supp}u_n^*\subseteq\operatorname{supp}u_n$ *for every* n.

The proof of Theorem 3.4 uses the corresponding Banach lattice result from [4]:

LEMMA 3.5 (Theorem 3.6 of [4]). *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach sequence spaces and suppose* $\mathcal{X}$ *is lattice anti-Euclidean. Suppose* $S:\mathcal{X}\to\mathcal{Y}$ *and* $T:\mathcal{Y}\to\mathcal{X}$ *are bounded operators such that* $TS=I_X$. *Let* $f_n=Se_n$ *and* $g_n=T^*e_n^*$, *for* $n\in\mathbb{N}$, *so that* $\|f_ng_n\|_1\geqslant 1$. *Then, there exist* $\delta>0$, $N\in\mathbb{N}$ *and* (F_{kn}) *of* $\mathbb{N}$ *for* $1\leqslant k\leqslant N$ *and* $1\leqslant n\leqslant\infty$, *satisfying*:

(1) For each fixed k *the sets* $(F_{kn})_{n=1}^{\infty}$ *are pairwise disjoint,*

(2) for each fixed n, *the sets* $(F_{kn})_{k=1}^{N}$ *are pairwise disjoint,*

(3) there is a $\delta>0$ *so that the disjoint sequences* $(f_n')_{n=1}^{\infty}$ *and* $(g_n')_{n=1}^{\infty}$ *defined respectively in* Y^N *and* $(Y^*)^N$ *(which we consider as* ℓ_2*-sums) as*

$$f_n'=(|f_n|\chi_{F_{1n}},\ldots,|f_n|\chi_{F_{Nn}})$$

$$g_n'=(|g_n|\chi_{F_{1n}},\ldots,|g_n|\chi_{F_{Nn}})$$

verify

$$\beta_n=\langle f_n',g_n'\rangle\geqslant\delta,\quad n\in\mathbb{N},$$

(4) $(f_n')_{n=1}^{\infty}$ *is complemented and equivalent to* $(e_n)_{n=1}^{\infty}$.

REMARKS. This statement is obtained as a special case of Theorem 3.6 of [4]. In the statement of that theorem, the sets E_n should not be assumed disjoint, and indeed we obtain the above statement taking $E_n = \mathbb{N}$ for all n (as noted in the remark following the Theorem in [4]). Also in the proof the sets A_n should be defined by $A_n = E_n \cap \{k : |\phi_n(k)| \geqslant 1/N\}$.

[Proof of Theorem 3.4]. We apply Lemma 3.5 with $\mathcal{X} = \hat{X}$, $\mathcal{Y} = \hat{Y}$, and with S and T replaced by their canonical extensions $S : \hat{X} \to \hat{Y}$ and $T : \hat{Y} \to \hat{X}$. Let $Se_n = f_n$ and $T^* e_n^* = g_n$. We find N, δ and subsets (F_{kn}) of $\mathbb{N}$ for $1 \leqslant k \leqslant N$ and $1 \leqslant n \leqslant \infty$ verifying (1) and (2) in the previous Lemma so that the sequences of disjointly supported vectors

$$u_n = (|f_n|\chi_{F_{1n}}, \ldots, |f_n|\chi_{F_{Nn}}) \in Y^N$$
$$v_n^* = (|g_n|\chi_{F_{1n}}, \ldots, |g_n|\chi_{F_{Nn}}) \in (Y^*)^N$$

verify

$$\beta_n = \langle u_n, v_n^* \rangle \geqslant \delta, \quad n \in \mathbb{N}. \tag{2.2}$$

We let $u_n^* = v_n^* \chi_{supp\, u_n}$ so that $\langle u_n, u_n^* \rangle = \langle u_n, v_n^* \rangle$.

To finish our proof we just must show that the operators $U : X \longrightarrow Y^N$ and $V : Y^N \longrightarrow X$ defined as

$$U\Big(\sum_{j=1}^{\infty} \xi(j) e_j\Big) = \sum_{j=1}^{\infty} \xi(j) u_j$$
$$V(\mathbf{y}) = \sum_{n=1}^{\infty} \frac{1}{\beta_n} \langle \mathbf{y}, u_n^* \rangle e_n$$

are bounded.

Suppose $(\xi_j)_j \in c_{00}$. Then, using standard quasi-Banach lattice estimates (Theorem 3.3 of [7] and Proposition 2.1 of [10]), we have (with A a constant depending on X and Y)

$$\begin{aligned}
\Big\| U\Big(\sum_{j=1}^{\infty} \xi_j u_j\Big)\Big\|_{Y^N} &\leqslant N^{1/2} \big\| \max_{j \geqslant 1} |\xi_j f_j| \big\|_Y \\
&\leqslant N^{1/2} \Big\| \Big(\sum_{j=1}^{\infty} |\xi_j f_j|^2\Big)^{1/2} \Big\|_Y \\
&\leqslant A\|S\| N^{1/2} \Big\| \Big(\sum_{j=1}^{\infty} |\xi_j e_j|^2\Big)^{1/2} \Big\|_X \\
&= A\|S\| N^{1/2} \Big\| \sum_{j=1}^{\infty} \xi_j e_j \Big\|_X
\end{aligned}$$

So, U is well defined and $\|U\| \leqslant N^{1/2}A\|S\|$.

On the other hand, given $\mathbf{y}=(y_1,\dots,y_n)\in Y^N$, by (2.2)

$$\begin{aligned}\|V(\mathbf{y})\| &= \Big\|\sum_{n=1}^{\infty}\frac{1}{\beta_n}\sum_{k=1}^{N}\langle y_k, |g_n|\chi_{F_{kn}}\chi_{\mathrm{supp}\ f_n}\rangle e_n\Big\|_X\\ &\leqslant \frac{1}{\delta}\Big\|\sum_{n=1}^{\infty}\sum_{k=1}^{N}\langle y_k, |g_n|\chi_{F_{kn}}\chi_{\mathrm{supp}\ f_n}\rangle e_n\Big\|_X\\ &\leqslant \frac{\Lambda}{\delta}\sum_{k=1}^{N}\Big\|\sum_{n=1}^{\infty}\langle y_k, |g_n|\chi_{F_{kn}}\chi_{\mathrm{supp}\ f_n}\rangle e_n\Big\|_X,\end{aligned}$$

where the constant Λ (that depends only on X and N) appears when we apply the triangle inequality with N terms in the quasi-Banach space X.

Let (sgn g_n) be a sequence of signs so that $|g_n|=g_n\mathrm{sgn}\ g_n$ for every n. Then

$$\begin{aligned}\langle y_k, |g_n|\chi_{F_{kn}}\chi_{\mathrm{supp}\ f_n}\rangle &= \langle y_k\mathrm{sgn}\ g_n\chi_{F_{kn}}\chi_{\mathrm{supp}\ f_n}, g_n\rangle\\ &= \langle T(y_k\mathrm{sgn}\ g_n\chi_{F_{kn}}\chi_{\mathrm{supp}\ f_n}), e_n\rangle.\end{aligned}$$

Hence, using again Theorem 3.3 of [7],

$$\begin{aligned}\|\sum_{n=1}^{\infty}\langle y_k, |g_n|\chi_{F_{kn}\chi_{\mathrm{supp}\ f_n}}\rangle e_n\|_X &\leqslant A\|T\|\|(\sum_{n=1}^{\infty}|y_k\chi_{F_{kn}}\chi_{\mathrm{supp}\ f_n}|^2)^{\frac{1}{2}}\|_Y\\ &\leqslant A\|T\|\|y_k\|_Y.\end{aligned}$$

Thus $\|V\| \leqslant N\Lambda\delta^{-1}$. This shows that X is isomorphic to a complemented sublattice of Y^N. Furthermore the projection on $[u_n]$ is given by $P=UV$ which has the required form. □

COROLLARY 3.6. *Let Y be a p-convex quasi-Banach sequence space such that $\hat{Y}$ is isomorphic to ℓ_1. Then, for any normalized complemented unconditional basic sequence $(f_n)_{n=1}^{\infty}$ in Y we can find $N\in\mathbb{N}$ and a normalized complemented positive disjoint sequence $(u_n)_{n=1}^{\infty}$ in Y^N equivalent to $(f_n)_{n=1}^{\infty}$. Furthermore, it may be assumed that the projection P has the form*

$$Px=\sum_{n=1}^{\infty}\langle x, u_n^*\rangle u_n$$

where $u_n^*\geqslant 0$ and supp $u_n^*\subseteq$ supp u_n for all n.

Proof. We observe that if $[f_n]_{n=1}^{\infty}=X$ then $\hat{X}$ is isomorphic to ℓ_1 and by Theorem 3.1 this implies that (f_n) is equivalent to the canonical basis of ℓ_1 in $\hat{X}$. Thus we can apply Theorem 3.4. □

COROLLARY 3.7. *Let $\mathcal{N}$ be an infinite subset of* $\mathbb{N}\times\mathbb{N}$. *Let* $(f_n)_{n=1}^{\infty}$ *be a normalized complemented unconditional basic sequence in* $Y=[e_{lk}]_{(l,k)\in\mathcal{N}}$ *in* $\ell_p(\ell_q)$ *where* $0<p,q\leqslant 1$ *with* $p\neq q$. *Then,* $(f_n)_{n=1}^{\infty}$ *is equivalent to a normalized complemented disjoint positive sequence* (u_n) *in* Y *and we can assume the projection has the form*

$$Px=\sum_{n=1}^{\infty}\langle x,u_n^*\rangle u_n$$

where $u_n^*\geqslant 0$ *and supp* $u_n^*\subseteq$ *supp* u_n.

Proof. We need only observe that from Proposition 3.3, Y can be identified with one of the six specific sequence spaces, and in each one the canonical basis is equivalent to its N-fold product. □

Lemma 3.8 is very well-known and simple to prove.

LEMMA 3.8 (cf. Lemma 6.3 of [6] and Theorem 2.3 of [10]). *Suppose that* X *is a quasi-Banach sequence space. Assume that* $(u_n)_{n=1}^{\infty}$ *is a normalized, complemented, disjoint sequence, with* $(u_n^*)_{n=1}^{\infty}$ *the associated biorthogonal sequence. Suppose that* $|v_n|\leqslant|u_n|$ *and that for some* $\nu>0$ *we have* $|\langle v_n,u_n^*\rangle|\geqslant\nu$ *for all* n. *Then* $(v_n)_{n=1}^{\infty}$ *is a complemented disjoint sequence equivalent to* (u_n). *In particular if there exists* $\beta>0$ *and a* $1-1$ *map* $\sigma:\mathbb{N}\to\mathbb{N}$ *so that* $|\langle u_n,e_{\sigma(n)}^*\rangle|$, $|\langle e_{\sigma(n)},u_n^*\rangle|\geqslant\beta$ *then* $(u_n)_{n=1}^{\infty}$ *is equivalent to* $(e_{\sigma(n)})_{n=1}^{\infty}$.

Our last result before we prove the Main Theorem asserts that the canonical basis of ℓ_p $(0<p<1)$ is strongly absolute (see [11]). This fact will be widely used in our proofs.

LEMMA 3.9 (Lemma 2.2 of [11]). *Fix* $0<p<1$. *For any* $\varepsilon>0$ *there exist* $C_\varepsilon>0$ *so that*

$$\sum_{n=1}^{N}|\alpha_n|\leqslant C_\varepsilon\sup_n|\alpha_n|+\varepsilon\left(\sum_{n=1}^{N}|\alpha_n|^p\right)^{1/p}$$

for any choice of scalars $(\alpha_n)_{n=1}^{N}$ *and* $N\in\mathbb{N}$.

4. Uniqueness of unconditional basis of $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$, $0<p<1$

Let us start with a reduction.

LEMMA 4.1. *Suppose* $0<p,q\leqslant 1$ *with* $p\neq q$ *and that* $\mathcal{N}$ *is an infinite subset of* $\mathbb{N}\times\mathbb{N}$. *Let* $(u_n)_{n=1}^{\infty}$ *be a normalized, complemented, positive disjoint sequence in the subspace* $Y=[e_{lk}]_{(l,k)\in\mathcal{N}}$ *of* $\ell_p(\ell_q)$. *Then, there is a normalized,*

complemented, positive disjoint sequence (v_n) *in* Y *and some* $\nu>0$ *such that* (v_n) *is equivalent to* (u_n), *supp* v_n *is finite for each* n, *supp* $v_n^*\subseteq supp v_n$ *and* $v_n^*\geqslant \nu\chi_{supp v_n}$.

Proof. We have a bound that $\|u_n^*\|\leqslant C$ for all n. For each n, put $S_n=\{(l,k): \langle e_{lk},u_n^*\rangle>(2C)^{-1}\}$. Then $\langle u_n-u_n\chi_{S_n},u_n^*\rangle\leqslant 1/2$. Hence there exists a finite set $F_n\subset S_n$ so that $\langle u_n\chi_{F_n},u_n^*\rangle\geqslant 1/4$. Let $w_n=u_n\chi_{F_n}$ and $w_n^*=u_n^*\chi_{F_n}$ so that $\langle w_n,w_n^*\rangle\geqslant 1/4$. It is easy to see using Lemma 3.8 that the sequence (v_n) obtained by normalizing (w_n) with $v_n^*=c_nw_n^*$ for suitable c_n satisfies the required conditions. □

Note that by use of the Cantor-Bernstein principle (Proposition 3.2), the following result implies the Main Theorem in the case $X=\ell_1(\ell_p)$ $(0<p<1)$.

THEOREM 4.2. *Suppose that* $\mathcal{N}$ *is an infinite subset of* $\mathbb{N}\times\mathbb{N}$. *Let* $(u_n)_{n=1}^{\infty}$ *be a normalized, complemented, unconditional basic sequence in* $Y=[e_{lk}]_{(l,k)\in\mathcal{N}}\subset \ell_1(\ell_p)$ $(0<p<1)$. *Then,* $(u_n)_{n=1}^{\infty}$ *is equivalent to a subbasis of* $(e_{lk})_{(l,k)\in\mathcal{N}}$.

Proof. By appealing to Theorem 3.4, Corollary 3.7 and Lemma 4.1 we can suppose that (u_n) is disjointly supported and positive, with finite support S_n, that for each n the biorthogonal functional u_n^* is supported on S_n and has the property that $u_n^*\geqslant\nu\chi_{S_n}$ for some $\nu>0$. Let K be the norm of the projection $x\to\sum_{n=1}^{\infty}\langle x,u_n^*\rangle u_n$.

In the Banach envelope (which is simply $\ell_1(\mathcal{N})$) the sequence (u_n) is equivalent to the ℓ_1-basis and hence satisfies an estimate

$$\left\|\sum_{n=1}^{\infty}\alpha_nu_n\right\|_1\geqslant c\sum_{n=1}^{\infty}|\alpha_n|,\qquad (\alpha_n)\in c_{00}.$$

Let us fix $\delta=\frac{1}{2}\nu c^{1/(1-p)}K^{-1}$. Put $\mathcal{A}=\{n:\ \|u_n\|_\infty\leqslant\delta\}$ and $\mathcal{B}=\mathbb{N}\setminus\mathcal{A}$. We assume both sets are infinite, since the other cases are simpler.

First notice that by Lemma 3.8, the sequence $(u_n)_{n\in\mathcal{B}}$ is permutatively equivalent to a subbasis of $(e_{lk})_{(l,k)\in\mathcal{N}}$.

We turn to $(u_n)_{n\in\mathcal{A}}$, which we show is equivalent to the canonical basis of ℓ_1. Let $\mathcal{F}$ be any finite subset of $\mathcal{A}$. For each $n\in\mathcal{F}$ we set $F_n=\{l:\ \exists k,(l,k)\in S_n\}$. Suppose we can pick a one-one map $\psi:\mathcal{F}\to\mathbb{N}$ with $\psi(n)\in F_n$ for every n. For each such n pick $\varphi(n)\in\mathbb{N}$ so that $(\psi(n),\varphi(n))\in S_n$. Then, given any sequence of scalars $\alpha=(\alpha_n)$ let

$$x_\alpha=\sum_{n\in\mathcal{F}}\alpha_ne_{\psi(n)\varphi(n)}.$$

Obviously, $\|x_\alpha\|=\sum_{n\in\mathcal{F}}|\alpha_n|$, so

$$\left\|\sum_{n\in\mathcal{F}}\langle x_\alpha,u_n^*\rangle u_n\right\|\leqslant K\sum_{n\in\mathcal{F}}|\alpha_n|.$$

Hence,

$$\Big\|\sum_{n\in\mathcal{F}}\alpha_n u_n\Big\|\leqslant K\nu^{-1}\sum_{n\in\mathcal{F}}|\alpha_n|.$$

Thus if for every finite set $\mathcal{F}$ we can make such a selection we reach the desired conclusion.

We now claim by the Hall–König Lemma (see [2]) that it is possible to find an injective map $\psi:\mathcal{F}\to\mathbb{N}$ such that $\psi(n)\in F_n$ for each n. It suffices to show that

$$|\cup_{n\in\mathcal{M}}F_n|>|\mathcal{M}|$$

for every $\mathcal{M}\subset\mathcal{F}$. If this is false there is a minimal set $\mathcal{M}$ for which this fails, and we must have equality in this case. Then we can pick a one-one map $\psi:\mathcal{M}\to\mathbb{N}$ with $\psi(n)\in F_n$ for each $n\in\mathcal{M}$. In particular we have

$$\Big\|\sum_{n\in\mathcal{M}}u_n\Big\|\leqslant K\nu^{-1}|\mathcal{M}|.$$

On the other hand $\sum_{n\in\mathcal{M}}u_n$ is supported on the set $\{(l,k):\ l\in\cup_{n\in\mathcal{M}}F_n\}$. Hence by Hölder's inequality (using $\|\cdot\|_p$ to denote the ℓ_p-norm on $\mathbb{N}\times\mathbb{N}$),

$$\begin{aligned}|\mathcal{M}|&\leqslant c^{-1}\Big\|\sum_{n\in\mathcal{M}}u_n\Big\|_1\\&\leqslant c^{-1}\delta^{1-p}\Big\|\sum_{n\in\mathcal{M}}u_n\Big\|_p^p\\&\leqslant c^{-1}\delta^{1-p}|\cup_{n\in\mathcal{M}}F_n|^{1-p}\Big\|\sum_{n\in\mathcal{M}}u_n\Big\|\\&\leqslant c^{-1}\delta^{1-p}K^{1-p}\nu^{1-p}|\mathcal{M}|.\end{aligned}$$

Thus $\delta^{1-p}\geqslant cK^{p-1}\nu^{1-p}$, which is a contradiction. This contradiction shows that the selection is always possible and hence $(u_n)_{n\in\mathcal{A}}$ is equivalent to the ℓ_1-basis.

Note that if $\mathcal{A}$ is infinite then ℓ_1 must embed into Y and so, using Proposition 3.3, the canonical basis of ℓ_1 is also a subbasis of $(e_{lk})_{(l,k)\in\mathcal{N}}$. From this the conclusion follows easily. □

As a straightforward consequence of Theorem 4.2 we obtain:

THEOREM 4.3. *Let* $0<p<1$. *Every normalized unconditional basis of an infinite dimensional complemented subspace of* $\ell_1(\ell_p)$ *is equivalent to a permutation of the unit vector basis of one of the following spaces*: ℓ_1, ℓ_p, $\ell_1\oplus\ell_p$, $\ell_1(\ell_p^n)_{n=1}^\infty$, $\ell_p\oplus\ell_1(\ell_p^n)_{n=1}^\infty$, $\ell_1(\ell_p)$.

THEOREM 4.4. *Let* $0<p<1$. *The following quasi-Banach spaces have unique unconditional basis up to permutation*:

$$\ell_1\oplus\ell_p,\ \ell_1(\ell_p^n)_{n=1}^\infty,\ \ell_p\oplus\ell_1(\ell_p^n)_{n=1}^\infty,\ \ell_1(\ell_p).$$

Next we will prove the Main Theorem for the spaces $\ell_p(\ell_1)$ $(0<p<1)$.

THEOREM 4.5. *Suppose $\mathcal{N}$ is an infinite subset of $\mathbb{N}\times\mathbb{N}$. Let $(u_n)_{n=1}^{\infty}$ be a normalized, complemented, unconditional basic sequence in $Y=[e_{lk}]_{(l,k)\in\mathcal{N}}\subset\ell_p(\ell_1)$ $(0<p<1)$. Then, $(u_n)_{n=1}^{\infty}$ is equivalent to a subbasis of $(e_{lk})_{(l,k)\in\mathcal{N}}$.*

Proof. We can suppose that (u_n) is positive, disjointly supported and that each u_n has finite support S_n. As for the biorthogonal sequence (u_n^*), we can assume that each u_n^* is supported on S_n and that there exists some $\nu>0$ for which $u_n^*\geqslant\nu\chi_{S_n}$ for all n. Let K be the norm of the projection $x\to\sum_{n=1}^{\infty}\langle x,u_n^*\rangle u_n$.

For each $(n,l)\in\mathbb{N}$ we write

$$x_{nl}=\sum_{k=1}^{\infty}\langle u_n,e_{lk}^*\rangle e_{lk}.$$

Then

$$1=\langle u_n,u_n^*\rangle=\sum_{l=1}^{\infty}\langle x_{nl},u_n^*\rangle.$$

But

$$\left(\sum_{l=1}^{\infty}|\langle x_{nl},u_n^*\rangle|^p)^{\frac{1}{p}}\right)\leqslant\left(\sum_{l=1}^{\infty}\|x_{nl}\|^p\right)^{1/p}=K.$$

Hence, by Lemma 3.9, for a suitable constant C we have

$$\frac{1}{2}\leqslant C\sup_l\langle x_{nl},u_n^*\rangle.$$

For each n pick l_n so that

$$\langle x_{nl_n},u_n^*\rangle\geqslant(4C)^{-1}.$$

By Lemma 3.8, the sequence (x_{nl_n}) is equivalent to (u_n). It is then easy to see that this sequence is equivalent to a subbasis of the original basis. □

As a straightforward consequence of Theorem 4.5 we obtain:

THEOREM 4.6. *Let $0<p<1$. Every normalized unconditional basis of an infinite dimensional complemented subspace of $\ell_p(\ell_1)$ is equivalent to a permutation of the unit vector basis of one of the following spaces*: ℓ_p, ℓ_1, $\ell_p\oplus\ell_1$, $\ell_p(\ell_1^n)_{n=1}^{\infty}$, $\ell_1\oplus\ell_p(\ell_1^n)_{n=1}^{\infty}$, $\ell_p(\ell_1)$.

THEOREM 4.7. *Let $0<p<1$. The following quasi-Banach spaces have unique unconditional basis up to permutation*:

$$\ell_1\oplus\ell_p,\ \ell_p(\ell_1^n)_{n=1}^{\infty},\ \ell_1\oplus\ell_p(\ell_1^n)_{n=1}^{\infty},\ \ell_p(\ell_1).$$

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