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INTRODUCTION

This present volume is a collection of articles presented during the conference to honor the memory of Wladyslaw Orlicz (1903-1990). This conference was organized by the Department of Mathematics of the University of Mississippi.

Wladyslaw Orlicz, one of founders of modern functional analysis and the last surviving member of the Banach Mathematical School, died on August 9, 1990. With him a chapter of functional analysis passed into history.

Several of his students, some of whom are now at the University of Mississippi, others at various schools in the country, decided to meet to review these chapters of mathematics they studied under the influence of Professor Orlicz. With this in mind a number of colleagues were called upon to meet together. In spite of a short notice, we are pleased that many distinguished mathematicians accepted our invitation and were able to attend.

We were lucky enough to have the support of the University of Mississippi. We owe much thanks for this support to Dr. Michael Dingerson, Dean of the Graduate School and Dr. Dale Abadie, Dean of the College of Liberal Arts. In addition, it is worthwhile to note that owing to the strong Polish participation in the meeting, funds were generously provided by the Kosciuszko Foundation of America. We express our deep appreciation.

We dedicate the present volume to the memory of Professor Wladyslaw Orlicz.

Przemo Kranz   Iwo Labuda

The atomic space problem and related questions for F-spaces

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In this note we consider some open questions concerning linear metric spaces. Although we will prove one or two minor results, our main objective is to discuss the relationship between certain problems related to the fundamental atomic space problem. We will freely use standard notions in the theory of complete metric linear spaces (F-spaces); see [7] and [14].

1. The atomic space problem.

Probably the most fundamental problem concerning F-spaces is:

PROBLEM 1: THE ATOMIC SPACE PROBLEM. *Does every infinite-dimensional F-space contain a proper closed infinite-dimensional subspace?*

This question can be traced to Pelczynski in the early sixties, but perhaps it has been around longer [9] (see also [7] and [14]). It is such an obvious question that it seems quite amazing that the answer is still unknown. We will say that an infinite-dimensional F-space is *atomic* if it contains no proper closed infinite-dimensional subspace. Of course an atomic space must be separable.

In order to discuss this problem further let us define an F-space X to be *minimal* if there is no strictly weaker Hausdorff vector topology on X. In [4] it is shown that the space ω of all sequences is minimal. In fact any minimal space with a basis is isomorphic to ω, by simply noting that the F-space topology must coincide with the topology induced by the biorthogonal functionals. Also note that a closed subspace of a minimal space is minimal. Thus a minimal space which contains a basic sequence contains a subspace isomorphic to ω; in particular a minimal quasi-Banach space contains no basic sequence. The following result is shown in [4] and [8]:

THEOREM 1.1 [8]. *Let X be a non-minimal F-space. Then X contains a basic sequence (x_n). Further (x_n) can be chosen to be regular i.e. bounded away from zero.*

Plainly an atomic space must be minimal. In fact it must also be *quotient-minimal* i.e. every quotient must also be minimal [3].

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THEOREM 1.2. *Let X be an F-space. Then X contains a basic sequence if and only if it contains a decreasing sequence (L_n) of infinite-dimensional closed subspaces so that $\cap L_n = \{0\}$.*

PROOF: One direction is clear. If (x_n) is a basic sequence simply let $L_n = [x_k : k \geq n]$. Conversely if (L_n) is such a sequence of subspaces we can assume that $L_n \neq L_{n+1}$ for all n. Consider the topology τ generated by the F-seminorms $x \to d(x, L_n)$ for $n \in \mathbb{N}$. If τ coincides with the original topology then picking $x_n \in L_n \setminus L_{n+1}$ with $x_n \neq 0$ it is simple to verify that (x_n) is a basic sequence and $[x_n] \sim \omega$. If τ is a strictly weaker topology then X is non-minimal and so contains a basic sequence.∎

PROBLEM 2: THE BASIC SEQUENCE PROBLEM. *Does every infinite-dimensional F-space contain a basic sequence? Does every infinite-dimensional quasi-Banach space contain a basic sequence?*

This in turn relates closely to (cf. [2]):

PROBLEM 3: THE MINIMAL SPACE PROBLEM. *Let X be an infinite-dimensional minimal F-space (i.e. such that there is no strictly weaker Hausdorff vector topology on X). Must X be isomorphic to ω, the space of all sequences? Is there an infinite-dimensional minimal quasi-Banach space?*

Obviously Problems 2 and 3 coincide for quasi-Banach spaces. In this context we note that the results of Bastero [1] on stable p-Banach spaces show that no subspace of $L_p[0,1]$ $(0 < p < 1)$ can be minimal (or atomic). It is apparently unknown, however, whether $L_0[0,1]$ has a minimal subspace.

THEOREM 1.3. *Let X be an arbitrary infinite-dimensional F-space. Then either X contains an atomic subspace or X contains a subspace F of finite dimension so that X/F contains a basic sequence.*

PROOF: We may suppose that X is separable. Let $\mathcal{L}$ be a maximal collection of infinite-dimensional closed subspaces of X which is closed under finite intersections. Let $F = \cap\{L : L \in \mathcal{L}\}$. If $\dim F = \infty$ then F is atomic. If not then $\dim F < \infty$. Since X is separable, by Lindelof's theorem we can pick a sequence $V_n \in \mathcal{L}$ with $F = \cap V_n$. Let $L_n = \cap_{i=1}^{n} V_i$. Then (L_n) decreases and $\cap L_n = F$. Clearly by Theorem 1.2 X/F contains a basic sequence.∎

COROLLARY 1.4. *Every infinite-dimensional quotient-minimal quasi-Banach space contains an atomic subspace.*

2. Partial results and comments.

We would like first to remark that the problems above seem to have a fundamentally different character if one restricts attention to quasi-Banach spaces. Let us first discuss the problems in the full generality of F-spaces. We start with the recent intriguing example due to Reese [10]:

THEOREM 2.1 (REESE). *There exists an F-space X with a sequence of finite-dimensional subspaces V_n so that $\dim V_n \geq n$ and so that $x_n \in V_n$ for every n and $x_n \neq 0$ for infinitely many n then $[x_n] = X$.*

Following [10] let us call such a space X *almost atomic.* It is clear that an almost atomic space has trivial dual (and, with a little bit more work any compact operator on X is zero). Further a quotient of an almost atomic space remains almost atomic.

It is rather unclear to the author just how unusual almost atomic spaces are. particularly in the general setting of F-spaces. It is natural thus to ask the following:

PROBLEM 4. *Is $L_0[0,1]$ almost atomic?*

It is important to stress that Reese's example is not a quasi-Banach space and it does not appear to be an easy task to make an almost atomic quasi-Banach space.

PROBLEM 5. *Does there exist an almost atomic quasi-Banach space?*

Recently, together with one of my students, S.C. Tam, I have been considering this problem in some special cases. We can show for example that $L_p[0,1]$ $(0 < p < 1)$ has no almost atomic subspace.

Now let's consider another variation. Suppose we try to find a minimal quasi-Banach space without an atomic subspace (e.g. if we believe that no atomic spaces exist). A glance at Theorem 1.3 shows us that we can look for a space X with a finite-dimensional subspace F so that X/F has a basis.

THEOREM 2.2. *Let X be an quasi-Banach space and suppose F is a finite-dimensional subspace so that X/F has a basis. Then X is minimal if and only if the quotient map $Q : X \to X/F$ is strictly singular.*

PROOF: First suppose Q is not strictly singular. Then obviously X contains a basic sequence. Conversely suppose Q is strictly singular. Suppose (x_n) is a basic sequence in X. Then there is a block basic sequence (y_n) of (x_n) with $\|y_n\| = 1$ and such that $\|Qy_n\| < 2^{-n}$. Thus there exist $f_n \in F$ so that $\|y_n - f_n\| < 2^{-n}$. By standard perturbation arguments $(f_n)_{n \geq N}$ is basic for some $N \geq 1$; but this contradicts the fact that F is finite-dimensional. ∎

Now it is simple to show that if we can find a minimal space of this type then we can suppose dim $F = 1$. Finally we can even wonder whether this can be done with X/F a Banach space:

PROBLEM 6. *Does there exist a minimal quasi-Banach space X with a one-dimensional subspace F so that the quotient X/F is a Banach space (e.g. ℓ_1) and the quotient map is strictly singular?*

There are many examples of non-locally convex quasi-Banach spaces X so that for a one-dimensional subspace F we have X/F isomorphic to a Banach space (see e.g. [5],[12], [13]). However, the additional requirement here that the quotient map is strictly singular does not appear to be satisfied in any known example.

3. The Hahn-Banach extension problem.

The author's interest in minimal spaces and basic sequences originated in the study of the Hahn-Banach Extension Property for F-spaces [4]. We say that a topological vector space X has the ***Hahn-Banach Extension Property or (HBEP)*** if whenever V is a linear subspace of X and f is a continuous linear functional on V then f has a continuous extension F on X. The main result of [4] is that an F-space with (HBEP) is locally convex. However, (cf. [7]) a topological vector space with (HBEP) need not be locally convex. At the meeting, Przemo Kranz called my attention again to:

PROBLEM 7. *Let X be a metrizable topological vector space with (HBEP); is X locally convex?*

It should be mentioned that Ribe [11] proved some partial results in this direction. He showed that if X is metrizable and isomorphic to its own square $X \oplus X$ then for X, (HBEP) is equivalent to local convexity.

In this section we will discuss this problem and give some observations. For convenience we will discuss only quasi-normed (i.e. locally bounded) spaces. Let us say that a quasi-Banach space X has dense-(HBEP) if there is a dense linear subspace V of X with (HBEP).

LEMMA 3.1. *Let X be a quasi-Banach space with dense-(HBEP). If X has a separating dual then X is locally convex.*

PROOF: This essentially follows from the arguments in [4]. However let us indicate a brief proof when X is separable. In this case, if X is non-locally convex it contains a proper closed weakly dense (PCWD) subspace W, [6]. Then it follows easily from the (HBEP) for V that $W \supset V$ and this is a contradiction. From this it is easy to show the general case since if X is non-locally convex it contains a separable non-locally convex subspace X_0 so that $X_0 \cap V$ is dense in X_0. ■

For an arbitrary F-space X let X^* denote its dual and put $N = N_X = \{x : x^*(x) = 0 \ \forall x^* \in X^*\}$.

LEMMA 3.2. *Let X be a quasi-Banach space with dense-(HBEP). If X contains a basic sequence then X is locally convex.*

PROOF: Clearly $N \cap V = \{0\}$. Pick any $u \in N$ and non-zero $w \in V$. By standard perturbation arguments, X contains a bounded basic sequence (x_n) so that $x_n \in n(u+w) + V$ for each n. Let us write $x_n = n(u+w) + v_n$ where $v_n \in V$. For some $m_0 \geq 1$ and all $m \geq m_0$, we have $u + w \notin Z_m = [x_k : k \geq m]$. Let Y_m be the (closed) linear span of Z_m and $u + w$. Then $v_n \in Y_m \cap V$ for $n \geq m$. Let $x^* \in X^*$ be a linear functional which vanishes on $Y_m \cap V$. Then $x^*(v_n) = 0$ and hence $x^*(u + w - n^{-1}x_n) = 0$. Thus $x^*(u+w) = 0$ and so $x^*(w) = 0$. Hence w is in the weak-closure of $Y_m \cap V$ and by (HBEP) for V this means that w is in Y_m for every $m \geq m_0$. Now the vectors $u + w$ and $\{x_k : k \geq m_0\}$ form a basis for Y_{m_0} and so it follows easily that w is a multiple of $u + w$ and thus $u = 0$. Hence $N_X = 0$ and so by the preceding Lemma 3.1, X is locally convex.■

LEMMA 3.3. *Let X be a quasi-Banach space with dense-(HBEP). Let E be a closed subspace of N_X. Then X/E has dense-(HBEP).*

PROOF: Let V as usual be the dense subspace of X with (HBEP). If $Q : X \to X/E$ is the quotient map then $Q(V)$ is a dense subspace of X/E with (HBEP). In fact if W is a subspace of $Q(V)$ and f is a continuous linear functional on W then $f \circ Q$ can be extended to a continuous linear functional x^* on X. Now since x^* vanishes on E it factors to a continuous linear functional on X/E.■

THEOREM 3.4. *Let X be a quasi-Banach space and let $N = \{x \in X; x^*(x) = 0 \ \forall x^* \in X^*\}$. In order that X has dense-(HBEP) it is necessary and sufficient that X/N is infinite-dimensional and locally convex and X has the property that whenever L is a closed subspace such that $\dim\, (L+N)/N = \infty$ then $L \supset N$.*

PROOF: First assume X has dense-(HBEP). Then X/N has dense-(HBEP) and so is locally convex by the preceding lemmas. Now suppose L is a closed subspace of X with dim $(L+N)/N = \infty$. We may assume $N \neq \{0\}$ and so by Lemma 8, X is minimal. Then $Y = X/(L \cap N)$ has dense-(HBEP). If we assume that $L \cap N$ is a proper subset of N then Y is minimal. Let $M = L/L \cap N \subset Y$; then $M \cap N_Y = \{0\}$. However M is minimal and thus the quotient map $Q : Y \to Y/N_Y$ is an isomorphism on M. Thus M is locally convex and so dim $M < \infty$. However dim $M =$ dim $(L+N)/N$ and so we have a contradiction and must conclude that $L \cap N = N$.

Conversely suppose X has the properties listed. Let V be an algebraic complement of N. First notice that the closure of V must include N by the hypotheses and so V is dense in X. Let ϕ be a continuous linear functional defined on a relatively closed subspace V_0 of V. We show that ϕ can be extended to V (or X.) If dim $V_0 < \infty$ this is immediate. Otherwise let L be the closure of V_0 in X. Clearly $L \supset N$, and ϕ extends continuously to L. We also must have ker $\phi \supset N$. Now by applying the Hahn-Banach theorem to X/N we obtain the result.∎

Let us give a simple application extending Ribe's result cited above (Corollary E of [11]).

COROLLARY 3.5. *Let X be a quasi-Banach space with dense-(HBEP). Suppose $X = E \oplus F$ where both E and F are infinite-dimensional. Then X is locally convex.*

PROOF: Note first that $N_X = N_E \oplus N_F$. We may may assume without loss of generality that dim $E/N_E = \infty$. In this case dim $(E + N_X)/N_X = \infty$ and so $N_X \subset E$ and $N_F = \{0\}$. Thus dim $F/N_F = \infty$ and the same reasoning yields $N_E = \{0\}$. Hence X has a separating dual and the result follows from Lemma 3.1.∎

THEOREM 3.6. *Let X be a non-locally convex quasi-Banach space with dense-(HBEP). Then X is minimal and $N = N_X$ is quotient-minimal.*

PROOF: If X has dense-(HBEP) and is non-locally convex then clearly by Lemma 8 X is minimal. Thus N is also minimal. Further if L is a closed subspace of N then N/L is a closed subspace of X/L and so is also minimal.∎

We now can make a few observations on this problem. Let us assume X is a non-locally convex quasi-Banach space with dense-(HBEP) and suppose X has no atomic subspaces (e.g. if there are no atomic quasi-Banach spaces). Then N is finite-dimensional. We can then choose L to a subspace of N of codimension one and replace X by X/L. Thus we can assume that N has dimension one. We conclude that we are looking for a minimal quasi-Banach space X with a subspace N of dimension one so that X/N is a Banach space. This is exactly Problem 6. In fact we can go further and argue conversely that a positive solution to Problem 6 does indeed give a negative solution to Problem 7.

THEOREM 3.7. *Let X be an infinite-dimensional minimal quasi-Banach space with a one-dimensional subspace F so that X/F is locally convex. Then X has dense-(HBEP).*

PROOF: Clearly $F = N_X$. Further if L is an infinite-dimensional closed subspace of X and $F \cap L = \{0\}$ then since L is minimal the quotient map $Q : X \to X/F$ is an isomorphism on L. But then L is locally convex and we have a contradiction. Thus $L \supset F$.∎

Thus if we hold that there are no atomic quasi-Banach spaces, Problems 6 and 7 are equivalent.

4. The weak Hahn-Banach extension problem.

In this section we will discuss another question related to the Hahn-Banach Extension Property. This question is not directly linked to the atomic space problem, but nevertheless it seems to the author to involve similar ideas. In [15] Shapiro proved that an F-space with a basis with (HBEP) is locally convex; on further examination he proved a little more:

THEOREM 4.1 (SHAPIRO). *Let X be a non-locally convex F-space with a basis. Then there is a weakly closed subspace (actually, the closed linear span of a block basic sequence) E and a continuous linear functional $f \in E^*$ which cannot be extended to a continuous linear functional on X.*

Motivated by this, let us say that an F-space X with separating dual has the *weak-(HBEP)* if whenever E is a weakly closed subspace of X and $f \in E^*$ then f can be extended to a continuous linear functional on X.

PROBLEM 8. *Let X be an F-space with weak-(HBEP). Is X locally convex?*

As before, we will specialize this problem to quasi-Banach spaces, primarily for convenience of exposition. Let us make some initial remarks. Under this hypothesis if E is a weakly closed subspace of X then the Mackey topology of E (i.e. the finest locally convex topology with the same dual) is the same as the relativized Mackey topology of X. Thus the Banach envelope norm for E is equivalent to the Banach envelope norm for X. Further the Banach envelope of E is isomorphic to the closure of E in the Banach envelope X_c of X.

THEOREM 4.2. *Let X be a quasi-Banach space with weak-(HBEP). Then for every infinite-dimensional weakly closed subspace E, the quotient X/E is locally convex.*

PROOF: We will assume that X is normed by a p-subadditive quasi-norm where $0 < p < 1$. Let $Q : X \to X/E$ be the quotient map. Let $\| \ \|_c$ denote the Banach envelope norm on either X or X/E. Next let $\| \ \|_0$ be the lower semi-continuous regularization of the quotient norm $\| \ \|$ on X/E with respect to $\| \ \|_c$. Thus $\|\xi\|_0 = \liminf_{\|\eta-\xi\|_c \to 0} \|\eta\|$.

We aim to show that there exists a constant K so that $\|\xi\|_0 \le K\|\xi\|_c$ for $\xi \in X/E$. Once this is achieved then the identity map $I : (X/E, \| \ \|) \to (X/E, \| \ \|_c)$ is surjective and a standard form of the Open Mapping theorem (e.g. Theorem 1.4 of [7]) gives that I is open. This will imply that X/E is locally convex.

Let us therefore assume the contrary that $\sup_{\|\xi\|_c \le 1} \|\xi\|_0 = \infty$ and produce a contradiction. We start with the observation that if F is a finite-dimensional subspace of X/E and B is a compact subset of F then since $\| \ \|_0$ is $\| \ \|_c$-continuous on F and $\| \ \|_c$-lower-semi-continuous on X/E there exists a $\delta > 0$ so that $\|\eta\|_c \le \delta$ and $\xi \in B$ then $\|\xi + \eta\|_0 > \|\xi\|_0 - 1$.

We now choose a sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$ and a sequence $(\xi_n)_{n=1}^{\infty}$ of elements of X/E. To start the induction choose $\delta_1 = 1/10$. Now suppose that δ_k has been chosen for $k \leq n$ and ξ_k has been chosen for $k < n$. We then choose ξ_n so that $\|\xi_n\|_c < \delta_n/2$ and $\|\xi\|_0 > n^3$. Once this is done we let B be the set of all $\sum_{k=1}^{n} a_k \xi_k$ where $|a_k| \leq 1$ for $1 \leq k \leq n$ and choose δ_{n+1} so that $\delta_{n+1} < \delta_n/2$ and if $\|\eta\|_c \leq \delta_{n+1}$ and $\xi \in B$ then $\|\xi + \eta\|_0 > \|\xi\|_0 - 1$.

Now pick $x_n \in X$ so that $Qx_n = \xi_n$ and $\|x_n\|_c < \delta_n/2$. (This is possible since $\| \ \|_c$ on X/E is the quotient norm of $\| \ \|_c$.) Since E is infinite-dimensional we may pick a sequence (y_n) in E with $\|y_n\|_c = 1$ so that (y_n) is a basic sequence in the Banach envelope X_c of X with constant at most two, i.e. so that for any $(a_k)_{k=1}^{n}$ and any $m < n$

$$\|\sum_{k=1}^{m} a_k y_k\|_c \leq 2\|\sum_{k=1}^{n} a_k y_k\|_c.$$

Now notice that from the construction $\sum \|x_n\|_c < 1/5$ and so by standard perturbation results if $z_n = x_n + y_n$ then (z_n) is also basic in the envelope X_c. We let Z be the weak closure of $\{z_n\}$ i.e. the intersection of the closed linear span in X_c with X. Let z_n^* be the biorthogonal functionals in Z^*. Since $\|z_n\|_c \geq 9/10$ for all n we must have that (z_n^*) is bounded in $(Z, \| \ \|_c)^*$. Let C be a constant such that $|z_n^*(z)| \leq C\|z\|_c$ for all $z \in Z$.

Suppose $z \in Z$ and $\|z\|_c \leq C^{-1}$. Thus $|z_n^*(z)| \leq 1$ for all n. Now, for any n,

$$\|\sum_{k=n+1}^{\infty} z_k^*(z)\xi_k\|_c < \frac{1}{2}\sum_{k=n+1}^{\infty} \delta_k < \delta_{n+1}.$$

Thus

$$\|\sum_{k=1}^{n} z_k^*(z)\xi_k\|_0 < \|Qz\|_0 + 1.$$

However this implies by the p-subadditivity of $\| \ \|_0$,

$$|z_n^*(z)|\|\xi_n\|_0 < 2^{1/p}(\|Qz\|_0 + 1).$$

By construction we deduce that

$$|z_n^*(z)| \leq 2^{1/p}n^{-3}(\|Qz\|_0 + 1).$$

By homogeneity we deduce that $\sup_{n\in\mathbb{N}} n^3|z_n^*(z)| < \infty$ for every $z \in Z$. It follows that we can define a linear functional f on Z by $f(z) = \sum_{n=1}^{\infty} nz_n^*(z)$ and from the Banach-Steinhaus theorem f is continuous on Z for the original quasi-norm topology. Hence f can be extended continuously to X and is also $\| \ \|_c-$ continuous. Since $f(z_n) = n$ we deduce that $\|z_n\|_c \to \infty$ which is a contradiction, as promised.∎

This theorem allows us to solve the problem under certain circumstances.

COROLLARY 4.3. *Let X be a quasi-Banach space with weak-(HBEP). The X is locally convex if one of the following hypotheses holds:*

(1) X contains an infinite-dimensional weakly closed locally convex subspace.

(2) X contains a weakly closed subspace with a basis.

(3) X contains two infinite-dimensional subpaces E and F so that $X = E \oplus F$.

(4) The Banach envelope X_c of X contains an infinite-dimensional B-convex subspace.

PROOF: (1) In this case suppose E is an infinite-dimensional weakly closed subspace which is locally convex. Suppose Q is the quotient map. If $\|x_n\|_c \to 0$ then $\|Qx_n\|_c \to 0$ and so by Theorem 4.2, $\|Qx_n\| \to 0$. Thus there exist $e_n \in E$ so that $\|x_n - e_n\| \to 0$. Hence $\|e_n\|_c \to 0$. However the relativized $\| \ \|_c$ topology on E coincides with the Mackey topology of E by weak-(HBEP) and since E is locally convex this means that $\|e_n\| \to 0$. Thus $\|x_n\| \to 0$.

(2) Clearly the subspace with a basis also has weak-(HBEP) and so by Shapiro's theorem is locally convex. This reduces us to (1).

(3) E and F are both also weakly closed and $E \sim X/F$ and $F \sim X/E$ are therefore both locally convex.

(4) In this case we may pick a basic sequence (x_n) for X_c with $x_n \in X$ and so that the closed linear span in X_c is B-convex. Let E be the weakly closed linear span in X. By the weak-(HBEP) E_c is isomorphic to $[x_n]$ in X_c. Thus E_c is B-convex and so E is locally convex [5]. Hence by (1) X is locally convex.∎

Let us mention a variant of Problem 8 which may be of interest to specialists in the theory of locally convex spaces.

PROBLEM 9. *Let X be a normed space which is fully barreled (i.e. every closed subspace is barreled). Is X necessarily ultrabarreled?*

An examination of Theorem 4.2 shows that this is what is required to improve the argument to solve Problem 8.

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