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To John Wermer

GAP-INTERPOLATION THEOREMS FOR ENTIRE FUNCTIONS

by

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We give a preliminary report here on our work on the problem of interpolation on a sequence $Z = (z_n)$ of non-zero complex numbers by entire functions f of the form

$$f(z) = \sum_{\lambda \in \Lambda} a_\lambda z^\lambda , \tag{1}$$

where Λ is a given sequence of positive intergers. This problem seems first to have been raised in [2]. In this summary, we limit ourselves to stating the main theorem (and two of its immediate corollaries), which gives a necessary and sufficient condition that such interpolation be possible. This condition is not easy to work with, and we are in the process of finding some more tractable conditions that either imply it or are implied by it. They involve, somewhat to our surprise, questions of diophantine approximation and p-adic analysis. The proof of our theorem, which we omit, involves functional analysis, specifically the main result of [1]. We expect to publish the details elsewhere.

In what follows, Z is a sequence (z_n), (perhaps a terminating one) of non-zero complex numbers, and Λ is a sequence of positive integers. We suppose that $|z_{n+1}| \geq |z_n|$ for all n.

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DEFINITION. Λ is an interpolating sequence of exponents for Z if for every sequence (w_n) of complex numbers, there exists an entire function f of the from (1) such that $f(z_n) = w_n$ for every n.

DEFINITION. Λ is Z linearly independent if

$$\sum_{n=1}^{N} a_n z_n^{\lambda} = 0 \quad \text{for all } \lambda \in \Lambda$$

implies that each a_n vanishes.

DEFINITION. Λ is asymptotically Z linearly independent if for every $\rho>0$, there exists an $N(\rho)$ so that

$$\left|\sum_{n=1}^{K} a_n z_n^{\lambda}\right| \le \rho^{\lambda} \quad \text{for all } \lambda \in \Lambda$$

implies that $a_n = 0$ for all $n \ge N(\rho)$.

DEFINITION. Λ is totally Z linearly independent if it is both Z linearly independent and asymptotically Z linearly independent.

THEOREM. *Λ is an interpolating sequence of exponents for Z if and only if Λ is totally Z linearly independent.*

We state two simple consequences of this theorem.

COROLLARY 1. *Λ is an interpolating sequence of exponents for every sequence Z that has no finite limit point and has $|z_i| \ne |z_j|$ for $i \ne j$, if and only if Λ is infinite.*

COROLLARY 2. *Λ is an interpolating sequence of exponents for every terminating (i.e. finite) sequence Z if and only if every arithmetic progression $\{an+b\}$, $a \ne 0$, contains at least one element of Λ.*

Corollary 1 was proved by other means in [2]. Corollary 2 is an immediate consequence of our main theorem and the Skolem-Lech theorem [3]. We are grateful to D.J. Newman for pointing out to us the relevance of the Skolem-Lech theorem, an interesting feature of which is that the only known proof seems to be dependent on p-adic analysis.

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