TOPOLOGIES ON RIESZ GROUPS AND APPLICATIONS TO MEASURE THEORY

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1. Introduction

In [**16**] the author proved a theorem equivalent to the statement that a finitely additive measure defined on a σ-algebra $\mathscr{S}$ and taking values in a complete metric separable additive group G is countably additive provided it is countably additive for some weaker Hausdorff group topology on G. This generalized to abelian groups the well-known Orlicz–Pettis theorem (Dunford and Schwartz, [**9**], p. 318). It was then natural to investigate whether any similar result was available for so-called exhaustive or strongly bounded measures, which instead of being countably additive simply have the property that whenever (S_n) is a sequence of disjoint sets then $\mu(S_n) \to 0$. In the context of locally convex spaces, such a result is available (Diestel, [**4**]) and is a close relative of a theorem due to Grothendieck ([**13**]) that every bounded linear operator $T: l_\infty \to X$, where X is a complete separable locally convex space, is weakly compact.

The solution of this problem is realized in Theorem 7. However, the course of the proof leads us rather away from the original setting as it requires the definition of a topology, the intrinsic topology (see § 3), which has significance in any lattice-ordered abelian group or more generally in a group with the Riesz interpolation property. Therefore we have prefaced the main theorem with a study of this topology in a very general setting (§§ 3 and 4). In particular, we see that, in certain circumstances, the topological completion of a group in its intrinsic topology is also an order-completion.

The main substance of the paper, however, stems from Theorem 2 in § 5, where it is shown that in a σ-complete group the intrinsic topology has some properties reminiscent of the Baire category theorem; this is in fact a generalization of a theorem of Phillips ([**21**]) and involves no new techniques. These properties lead in § 6 to the closed graph theorem for σ-complete Riesz groups and the applications of this in § 7 solve the problem under consideration.

Finally, in § 8 we show the relevance of the intrinsic topology to attempts to generalize the Riesz representation theorem.

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The intrinsic topology approach to measure theory is very closely related to the approach of Drewnowski ([**6**]) using Fréchet–Nikodym topologies on rings of sets. However, we use a topology on the group of integer-valued measurable functions rather than the algebra of sets itself.

2. Preliminaries

Throughout this paper all groups are abelian and written additively. Let G be a partially ordered group† (Fuchs, [**9**], provides background on such groups); then we say that G is a *Riesz group* if whenever $a, b, c, d \in G$ such that $a \leqslant c$, $b \leqslant c$, $a \leqslant d$, and $b \leqslant d$ (or $(a, b \leqslant c, d)$), then there exists $g \in G$ with $a, b \leqslant g \leqslant c, d$ (the *Riesz interpolation property*). The reader may refer to Fuchs ([**10**]) or Jameson ([**15**]) for the study of the algebraic properties of Riesz groups; particularly important are the equivalent formulations of the property ([**12**], p. 14).

A special case of a Riesz group is a lattice group, also called an l-group ([**11**], p. 66), in which there is a supremum $a \vee b$ and an infimum $a \wedge b$ of any two elements. In this case we use the notation

$$a^+ = a \vee 0, \quad a^- = (-a) \vee 0, \quad |a| = a^+ + a^-.$$

We say that a Riesz group is *σ-complete* if every monotone increasing sequence (a_n) which is bounded above has a supremum, and *order-complete* if every bounded monotone increasing net has a supremum (an order-complete Riesz group is automatically a lattice group).

A *quasi-norm* η on an abelian group G is a map $\eta\colon G \to \mathbf{R}^+$ such that

$$\begin{aligned} \eta(0) &= 0, \\ \eta(a+b) &\leqslant \eta(a) + \eta(b) \quad (a, b \in G), \\ \eta(a) &= \eta(-a) \qquad (a \in G). \end{aligned}$$

Any group topology on G may be induced by a family of quasi-norms. We say that a subset B of G is *bounded* if

$$\sup_{b \in B} \eta(b) < \infty$$

whenever η is a continuous quasi-norm on G. If G has a given quasi-norm then we may also refer to metrically bounded sets on which the given quasi-norm is bounded.

If G is a lattice group then a subset V of G is *solid* if $a \in V$ and $|x| \leqslant |a|$ imply that $x \in V$. If G is a topological lattice group which has a base of solid neighbourhoods of zero then G is called locally solid, and ([**15**], p. 151) the lattice operations are uniformly continuous on G.

† We shall always assume that G is positively generated (see Lemma 3).

A subset V of G is called *order-convex* if a, $b \in V$ and $a \leqslant x \leqslant b$ imply $x \in V$, and an *order-ideal* if it is also directed (that is, given $a, b \in V$, there exists $c \in V$ with $c \geqslant a$, $c \geqslant b$). For details, again see [**11**].

3. A topology for Riesz groups

Suppose G is a Riesz group; we shall say that a *test sequence* is a sequence (a_n) such that

(i) $0 \leqslant a_n$ $(n = 1, 2, \ldots)$,

(ii) there exists $b \in G$ with $\sum_{i=1}^{n} a_i \leqslant b$ $(n = 1, 2, \ldots)$.

We shall also write $a_n \twoheadrightarrow 0$ if $a_n = b_n - c_n$, where both (b_n) and (c_n) are test sequences.

LEMMA 1. *Suppose η is a quasi-norm on G; then the following are equivalent:*

(i) *η is bounded on every order-interval;*

(ii) *η is bounded on every test sequence;*

(iii) *η is bounded on any bounded monotone increasing sequence;*

(iv) *if $a_n \twoheadrightarrow 0$ then $\sup_n \eta(a_n) < \infty$.*

Proof. The implications (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are easy and we omit the details. Clearly also (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) (cf. [**1**], p. 856). It is enough to show that if $a \in G$ with $a \geqslant 0$ then η is bounded on $[0, a]$. For $a \in G$ with $a \geqslant 0$ we define

$$\eta^*(a) = \sup_{0 \leqslant c \leqslant a} \eta(c).$$

Now suppose $a = x + y$ and $c \leqslant a$. By the Riesz decomposition property $c = c_1 + c_2$, where $0 \leqslant c_1 \leqslant x$ and $0 \leqslant c_2 \leqslant y$. Hence

$$\eta(c) \leqslant \eta(c_1) + \eta(c_2) \leqslant \eta^*(x) + \eta^*(y),$$

and therefore

$$\eta^*(a) \leqslant \eta^*(x) + \eta^*(y). \tag{1}$$

Now suppose $\eta^*(a_0) = \infty$; we construct a sequence $(a_n : n = 1, 2, \ldots)$ such that

$$(\alpha) \quad 0 \leqslant a_n \leqslant a_{n-1} \quad (n = 1, 2, \ldots),$$

$$(\beta) \quad \eta^*(a_n) = \infty,$$

$$(\gamma) \quad \eta(a_n) \geqslant n + \eta(a_{n-1}).$$

For given a_{n-1} satisfying (α), (β), and (γ), we may choose x $(0 \leqslant x \leqslant a_{n-1})$ such that

$$\eta(x) \geqslant n + 2\eta(a_{n-1}).$$

Then if $\eta^*(x) = \infty$, let $a_n = x$; otherwise by (1) $\eta^*(a_{n-1} - x) = \infty$ and we let $a_n = a_{n-1} - x$. In either case, (α), (β), and (γ) are satisfied. Now if $c_n = a_{n-1} - a_n$ $(n = 1, 2, \ldots)$ then $c_n \geqslant 0$ and $\sum_{i=1}^{n} c_i = a_0 - a_n \leqslant a_0$, and so

(c_n) is a test sequence. Clearly $\eta(c_n) \geqslant n$ and so we have a contradiction to (ii).

The conditions of Lemma 1 are not 'topological', for any quasi-norm is equivalent to a bounded quasi-norm. However, we can introduce a topological condition thus: a quasi-norm η is called *limiting* if for every test sequence (a_n) we have $\lim_{n\to\infty} \eta(a_n) = 0$. This means that in the topology induced by η every test sequence is null.

LEMMA 2. *Let G be a Riesz group and suppose η is a quasi-norm on G; the following are equivalent:*

(i) *η is limiting;*

(ii) *any bounded monotone increasing sequence has the η-Cauchy property;*

(iii) *any bounded monotone increasing net has the η-Cauchy property;*

(iv) *if $a_n \twoheadrightarrow 0$ then, given $\varepsilon > 0$, there exists an integer m such that for any finite subset F of $\{m+1, m+2, \ldots\}$ we have $\eta(\sum_{i\in F} a_i) \leqslant \varepsilon$.*

Proof. (i) $\Rightarrow$ (iv). If (iv) fails then for some $\varepsilon > 0$ and some sequence $a_n \twoheadrightarrow 0$ we may construct an increasing sequence $(m_n;\ n = 1, 2, \ldots)$ and a sequence (F_n) of finite subsets of integers with $F_n \subseteq \{m_n+1, \ldots, m_{n+1}\}$ such that $\eta(\sum_{i\in F_n} a_i) > \varepsilon$. (The construction is a standard induction argument.) Suppose $a_j = b_j - c_j$, where (b_j) and (c_j) are test sequences; then $\sum_{i\in F_n} a_i = \sum_{i\in F_n} b_i - \sum_{i\in F_n} c_i$ and both $(\sum_{i\in F_n} b_i)$ and $(\sum_{i\in F_n} c_i)$ are test sequences. We quickly obtain a contradiction.

(iv) $\Rightarrow$ (ii). This is immediate, for if a_n is bounded monotone and increasing then $a_{n+1} - a_n \twoheadrightarrow 0$.

(ii) $\Rightarrow$ (iii). Suppose (a_α) is bounded monotone and increasing but does not have the η-Cauchy property. Then, for some $\varepsilon > 0$ and any β, there exists $\alpha \geqslant \beta$ with $\eta(a_\alpha - a_\beta) \geqslant \varepsilon$. Hence we may construct an increasing sequence (α_n) such that $\eta(a_{\alpha_n} - a_{\alpha_{n-1}}) \geqslant \varepsilon$. But (a_{α_n}) must be η-Cauchy by (ii), and we have a contradiction.

(iii) $\Rightarrow$ (i). This is immediate.

Now we are ready to define the *intrinsic* topology on G. This is the topology induced by the family of all limiting quasi-norms; we denote it by λ. We collect together some elementary remarks in

PROPOSITION 1. (i) *λ is the finest topology on G such that every bounded monotone net is a Cauchy net.*

(ii) *A quasi-norm η is λ-continuous if and only if it is limiting.*

Given a quasi-norm η which is bounded on every test sequence we can define a new quasi-norm $|\eta|$ as follows: let

$$\eta^*(a) = \sup\{\eta(x) \colon 0 \leqslant x \leqslant a\} \quad (a \geqslant 0)$$

and define

$$|\eta|(a) = \inf\{\eta^*(b)\colon -b \leqslant a \leqslant b\}.$$

The fact that $|\eta|$ is a quasi-norm follows from (1) without difficulty. We also have $|\eta|(a) = \eta^*(a)$ for $a \geqslant 0$.

LEMMA 3. (i) $\eta(x) \leqslant 2|\eta|(x)$ $(x \in G)$.
(ii) *If* η *is limiting then* $|\eta|$ *is limiting.*

Proof. (i) Suppose $-b \leqslant x \leqslant b$, then by the Riesz property $x = y - z$, where $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant b$. Hence $\eta(x) \leqslant \eta(y)+\eta(z) \leqslant 2\eta^*(b)$.
As this is true for all such b,

$$\eta(x) \leqslant 2|\eta|(x).$$

(ii) If (a_n) is a test sequence then $|\eta|(a_n) = \eta^*(a_n)$, and so we may choose b_n such that $0 \leqslant b_n \leqslant a_n$ and $\eta(b_n) \geqslant \frac{1}{2}\eta^*(a_n)$. Then (b_n) is also a test sequence and so $\eta(b_n) \to 0$; therefore $|\eta|(a_n) \to 0$.

We shall call a quasi-norm *solid* if

$$\eta(x) = \inf\{\eta(a)\colon -a \leqslant x \leqslant a\}.$$

Lemma 3 implies that λ may be determined by a family of solid quasi-norms, and so has a base of neighbourhoods V such that if $b \in V$ and $-b \leqslant a \leqslant b$ then $a \in V$. If G is a lattice, a solid quasi-norm η will satisfy $\eta(x) = \eta(|x|)$ and (G, λ) is locally solid. Thus the lattice operations are continuous, and the positive cone is closed in λ.

We do not know any reasonable conditions to ensure that λ is a Hausdorff topology. In general, the closure of $\{0\}$ in (G, λ), G_0, say, is an order-convex subgroup of G, so that G/G_0 is a partially ordered group under the natural ordering. If, in addition, G is a lattice then G_0 is an order-ideal (that is, G_0 is directed—because if $a \in G_0$ then $|a| \in G_0$); in this case G/G_0 is a lattice. In general, however, G_0 fails to be an order-ideal and so we cannot deduce that G/G_0 is even a Riesz group ([**10**], Proposition 5.3). Even in the lattice case it does not appear obvious that the induced Hausdorff topology on G/G_0 is the intrinsic topology on G/G_0.

We conclude the section with a criterion for continuity of group homomorphisms in the intrinsic topology. Let H be an abelian topological group; we say that a group homomorphism $\alpha\colon G \to H$, where G is a Riesz group, is *exhaustive* (see § 7) if whenever (a_n) is a test sequence then $\alpha a_n \to 0$.

PROPOSITION 2. α *is exhaustive if and only if* α *is continuous for the intrinsic topology on* G.

Proof. Let η be a continuous quasi-norm on H and suppose α is exhaustive. For any test sequence (a_n), $\eta(\alpha a_n) \to 0$ so that $\eta \circ \alpha$ is limiting, that is, α is continuous on (G, λ). Conversely, if α is continuous then, since $a_n \to 0$ in (G, λ), $\alpha a_n \to 0$, that is, α is exhaustive.

COROLLARY. *Let G and H be Riesz groups and let $\alpha: G \to H$ be a positive homomorphism. Then α is continuous for the intrinsic topologies.*

Proof. If (a_n) is a test sequence then

$$\sum_{i=1}^{n} a_i \leqslant b \quad (n = 1, 2, \ldots),$$

so that

$$\sum_{i=1}^{n} \alpha a_i \leqslant \alpha b \quad (n = 1, 2, \ldots)$$

and therefore (αa_n) is a test sequence. Hence α is exhaustive.

4. The Fatou topology for a σ-complete Riesz group

Suppose now that G is a σ-complete Riesz group; that is, if (a_n) is a test sequence then $\sum_{n=1}^{\infty} a_n = \sup_k (\sum_{n=1}^{k} a_n)$ exists in G. Then we say a quasi-norm η on G is a *Fatou* quasi-norm if η is limiting and whenever (a_n) is a test sequence then

$$\eta\left(\sum_{n=1}^{\infty} a_n\right) \leqslant \sum_{n=1}^{\infty} \eta(a_n),$$

(that is, η is countably sub-additive).

We write $a_n \uparrow a$ if (a_n) is a monotone increasing sequence whose supremum is a.

LEMMA 4. *The following are equivalent:*

(i) *η is a Fatou quasi-norm;*

(ii) *η is limiting and whenever $0 \leqslant a_n \uparrow a$ then $\eta(a) \leqslant \sup_n \eta(a_n)$;*

(iii) *if $a_n \uparrow a$ then $\eta(a - a_n) \to 0$.*

Proof. (i) $\Rightarrow$ (iii). If $a_n \uparrow a$, let $c_n = a_{n+1} - a_n$. Then (c_n) is a test sequence and $\sum_{n=1}^{\infty} c_n = a - a_1$. By Lemma 2(iv) for $\varepsilon > 0$, we can select a sequence (m_n) of increasing positive integers such that for any subset F of $\{m_n + 1, \ldots, m_{n+1}\}$

$$\eta\left(\sum_{i \in F} c_i\right) \leqslant \varepsilon / 2^n.$$

Now for $k > m_1$,

$$\sum_{i=k}^{\infty} c_i = \sum_{n=1}^{\infty} \sum_{i \in F_n} c_i,$$

where $F_n = \{j \geqslant k \mid m_n + 1 \leqslant j \leqslant m_n\}$ and $(\sum_{i \in F_n} c_i)$ is a test sequence. Therefore

$$\eta\left(\sum_{i=k}^{\infty} c_i\right) \leqslant \sum_{n=1}^{\infty} \varepsilon/2^n \leqslant \varepsilon$$

that is, $\eta(a - a_k) \leqslant \varepsilon$ for $k \geqslant m_1$. Therefore $\eta(a - a_n) \to 0$.

(iii) $\Rightarrow$ (ii).

$$\begin{aligned}\eta(a) &\leqslant \limsup_{n\to\infty}(\eta(a_n) + \eta(a - a_n))\\ &\leqslant \limsup_{n\to\infty} \eta(a_n) + \limsup_{n\to\infty} \eta(a - a_n)\\ &\leqslant \sup \eta(a_n).\end{aligned}$$

(ii) $\Rightarrow$ (i). If (a_n) is a test sequence,

$$\eta\left(\sum_{n=1}^{\infty} a_n\right) \leqslant \sup_k \eta\left(\sum_{n=1}^{k} a_n\right) \leqslant \sup_k \sum_{n=1}^{k} \eta(a_n) \leqslant \sum_{n=1}^{\infty} \eta(a_n).$$

We define the *Fatou topology* γ on G in the obvious way as the topology induced by all Fatou quasi-norms. We have immediately:

PROPOSITION 3. *γ is the finest topology on G such that every bounded increasing sequence converges to its supremum, and γ is weaker than λ.*

In the remainder of this section we show that every lattice group G in which the intrinsic topology is a Hausdorff topology may be completed to an order-complete lattice group; and then (G, λ) can be embedded densely in a σ-complete lattice group G^σ with its Fatou topology.

If (G, λ) has the Hausdorff property then let $(\bar{G}, \bar{\lambda})$ be its completion; we identify G as a dense subgroup of $\bar{G}$. Then we define the positive cone $\bar{P}$ in $\bar{G}$ as the closure of the positive cone P in G; thus $\bar{G}$ is partially ordered. Since G is a lattice group, P is λ-closed in G and therefore the partial order in $\bar{G}$ extends the partial order on G.

THEOREM 1. *$\bar{G}$ is an order-complete lattice group.*

Proof. By Lemma 3 and [**15**], p. 151, the lattice operations are uniformly continuous on (G, λ) and therefore extend to continuous operations on $\bar{G}$; it is a matter of trivial verification that these operations are the lattice operations induced by the partial ordering in $\bar{G}$. Thus $\bar{G}$ is a lattice group.

Now any limiting solid quasi-norm is also solid when extended to $\bar{G}$ as then $(a\colon -a \leqslant x \leqslant a)$ converges to $(2x)^+ - x = 2x^+ - x = |x|$ ([**11**], p. 75). We now demonstrate that the extended quasi-norm is also limiting. For suppose $a_n \in \bar{G}$ and $a_n \uparrow$ with $a_n \leqslant 0$ for all n. Suppose η is a solid continuous quasi-norm on $\bar{G}$ and that $\varepsilon > 0$. We select $x_n \in G$ with $x_n \leqslant 0$

and $\eta(x_n - a_n) \leqslant \varepsilon/2^{n+2}$. Let $y_n = x_1 \vee \ldots \vee x_n$. Now

$$\begin{aligned} y_n \vee x_{n+1} - a_{n+1} &= (y_n - a_{n+1}) \vee (x_{n+1} - a_{n+1}) \\ &\leqslant (y_n - a_n) \vee (x_{n+1} - a_{n+1}) \\ &\leqslant |y_n - a_n| + |x_{n+1} - a_{n+1}| \end{aligned}$$

and

$$\begin{aligned} y_n \vee x_{n+1} - a_{n+1} &\geqslant x_{n+1} - a_{n+1} \\ &\geqslant -|x_{n+1} - a_{n+1}|. \end{aligned}$$

Therefore

$$|y_n \vee x_{n+1} - a_{n+1}| \leqslant |x_{n+1} - a_{n+1}| + |y_n - a_n|$$

and so

$$\eta(y_{n+1} - a_{n+1}) \leqslant (\varepsilon/2^{n+3}) + \eta(|y_n - a_n|).$$

Hence

$$\eta(y_n - a_n) \leqslant \sum_{i=0}^{n-1} \varepsilon/2^{i+3} \leqslant \tfrac{1}{4}\varepsilon.$$

Now $y_n \in G$ is increasing and $y_n \leqslant 0$. Hence, for large enough n, and $m, p \geqslant n$,

$$\eta(y_m - y_p) \leqslant \tfrac{1}{2}\varepsilon.$$

Then, for $m, p \geqslant n$,

$$\eta(a_m - a_p) \leqslant \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon = \varepsilon,$$

that is, (a_n) has the η-Cauchy property.

Thus every bounded monotone sequence in $(\bar{G}, \bar{\lambda})$ has the Cauchy property, and therefore, by Lemma 2, so has every bounded monotone net. However, $(\bar{G}, \bar{\lambda})$ is complete and therefore if x_α is an increasing bounded set then $x_\alpha \to x$ in $\bar{G}$. Then $x \geqslant x_\alpha$ for all α, and if $y \geqslant x_\alpha$ for all α then $y - x_\alpha \to y - x \geqslant 0$, that is, $y \geqslant x$; thus $x = \sup(x_\alpha)$. As $\bar{G}$ is a lattice, it follows that $\bar{G}$ is order-complete.

COROLLARY. *Suppose G is a lattice group with a Hausdorff intrinsic topology; then (G, λ) may be embedded as a dense subgroup of a σ-complete lattice group G^σ with its Fatou topology.*

Proof. Embed G in its completion $\bar{G}$ and let G^σ be the smallest subset of $\bar{G}$ containing G, which is stable under convergence of monotone sequences. For $x \in G^\sigma$ let $H_x = \{y : y \in G^\sigma, x + y \in G^\sigma\}$, then if $x \in G$, $H_x \supseteq G$ and is also stable under convergence of monotone sequences, so that $H_x = G^\sigma$.

In general, therefore, if $x \in G^\sigma$ then $H_x \supseteq G$ and so $H_x = G^\sigma$. Therefore G^σ is stable under addition; but clearly $-G^\sigma \supseteq G$ and so $-G^\sigma \supseteq G^\sigma$, so that G^σ is a subgroup of $\bar{G}$. A similar argument shows that G^σ is a lattice and therefore a σ-complete lattice.

Now let η be a Fatou quasi-norm on G^σ; then η on G is limiting and so has a unique continuous extension to $\bar{G}$, η' say. But $\{x: \eta(x) = \eta'(x)\}$ is stable under monotone convergence of sequences, so $\eta = \eta'$ on G^σ; clearly any $\bar{\lambda}$-continuous quasi-norm must be Fatou on G^σ, and hence the corollary follows.

5. 'Category' theorems in σ-complete lattice groups

The next theorem is a modification of a celebrated result of Phillips ([**21**]); indeed the method of proof also stems from Phillips. We call it a 'category' theorem as it has a resemblance to the Baire category theorem for complete metric spaces.

THEOREM 2. *Let G be a σ-complete Riesz group, and let (C_n) be an increasing sequence of subsets C_n of G containing 0 and such that $\bigcup C_n = G$. Suppose (a_n) is a test sequence; then, for some m, we have $\{a_n\} \subset \overline{C_m - C_m}$ (closure in the intrinsic topology).*

Proof. If $\Gamma = (\Gamma_n)$ is a subsequence of $\mathbf{N}$, we denote by $\sum_{i \in \Gamma} a_i$ the order-sum $\sup_m \sum_{i=1}^m a_{\Gamma_i}$, in G (G is σ-complete).

Now suppose the theorem false; then we may determine for each n a λ-neighbourhood V_n of 0 such that $\{a_n\} \not\subseteq C_n - C_n + V_n - V_n$. We may further suppose that each V_n is symmetric and has the property that if $0 \leqslant a \leqslant b$ and $b \in V_n$ then $a \in V_n$ (cf. Lemma 3). Now we select two increasing sequences $(p(n))$ and $(q(n))$ such that

(i) $a_{p(n)} \notin C_{q(n)} - C_{q(n)} + 2V_{q(n)}$;

(ii) if F is a finite subset of $\{p(n), p(n)+1, \ldots\}$ then $\sum_{i \in F} a_i \in V_{q(n-1)}$;

(iii) if $F \subset \{1, 2, \ldots, p(n-1)\}$, $\sum_{i \in F} a_i \in C_{q(n)}$.

First, we select $p(1)$ and $q(1)$ so that (i) holds; next suppose $p(n-1)$ and $q(n-1)$ have been chosen. Now by Lemma 2, we determine $m \geqslant p(n-1)$ such that $\sum_{i \in F} a_i \in V_{q(n-1)}$ for $F \subseteq \{m+1, m+2, \ldots\}$. As the set $\{1, 2, \ldots, m\}$ is finite, there exists $q(n)$ such that $\sum_{i \in F} a_i \in C_{q(n)}$ for $F \subset \{1, 2, \ldots, m\}$. Next, choose $p(n)$ so that (i) holds; certainly $p(n) > m$ and so (ii) also holds.

Now let $\Gamma_0 = (p(1), p(2), \ldots)$; we shall next determine a sequence (Γ_n) of subsequences of $\mathbf{N}$ so that

(iv) Γ_n is a subsequence of Γ_{n-1} $(n \geqslant 1)$,

(v) $\sum_{i \in \Gamma_n} a_i \in V_{q(n)}$ $(n \geqslant 1)$.

Suppose Γ_{n-1} has been determined; then let $\Gamma_{n-1} = \bigcup_{k=1}^\infty \Delta_k$, where (Δ_k) is a sequence of disjoint subsequences of Γ_{n-1}. Then

$$\sum_{k=1}^{m} \sum_{i \in \Delta_k} a_i \leqslant \sum_{i \in \Gamma_{n-1}} a_i \quad (m = 1, 2, \ldots)$$

and so $\sum_{i\in\Delta_k} a_i$ is a test sequence. Hence for some m we have

$$\sum_{i\in\Delta_m} a_i \in V_{q(n)}.$$

Now let $\Gamma_n = \Delta_m$ for this choice of m.

The next stage is to define $\Pi = (\pi(n))$ as the sequence $(\Gamma_{n,n})$ of the nth term of each Γ_n (clearly Π is an increasing sequence). Suppose $p(m) \in \Pi$; in fact let $p(m) = \pi(r)$, where $r \leqslant m$ (obviously). Then we have, if $\chi = \sum_{i\in\Pi} a_i$,

$$\chi = \sum_{i=1}^{r-1} a_{\pi(i)} + a_{p(m)} + \sum_{i=r+1}^{m} a_{\pi(i)} + \sum_{i=m+1}^{\infty} a_{\pi(i)}.$$

Now by (iii)

$$\sum_{i=1}^{r-1} a_{\pi(i)} \in C_{q(m)},$$

by (ii)

$$\sum_{i=r+1}^{m} a_{\pi(i)} \in V_{q(m)} \quad (\text{for } \pi(r+1) \geqslant p(m+1)),$$

and

$$0 \leqslant \sum_{i=m+1}^{\infty} a_{\pi(i)} \leqslant \sum_{i\in\Gamma_m} a_i;$$

and so

$$\sum_{i=m+1}^{\infty} a_{\pi(i)} \in V_{q(m)} \quad \text{by (v)}.$$

Combining these remarks, we obtain

$$a_{p(m)} \in \chi - C_{q(m)} + 2V_{q(m)}.$$

This result is true for infinitely many m, and as $a_{p(m)} \notin C_{q(m)} - C_{q(m)} + 2V_{q(m)}$ this means that $\chi \notin C_{q(m)}$ infinitely often, which contradicts our assumptions.

COROLLARY. *Let η be a λ-lower-semi-continuous quasi-norm on G; then η is bounded on any order-interval.*

Proof. Let $C_m = \{a\colon \eta(a) \leqslant \frac{1}{2}m\}$ $(m = 1, 2, \ldots)$; then $\bigcup C_m = G$ and so, by Theorem 2, for any test sequence (a_n) there exists m_0 such that $\{a_n\} \subset \overline{C_{m_0} - C_{m_0}} \subset \{a\colon \eta(a) \leqslant m_0\}$ since η is lower-semi-continuous. Therefore $\sup \eta(a_n) < \infty$, and we apply Lemma 1.

The topology induced by all λ-lower-semi-continuous quasi-norms is the analogue of the order-bound topology in locally convex vector lattice theory. It may, therefore, be of some interest in itself.

THEOREM 3. *Suppose G is a σ-complete Riesz group, and H an abelian topological group.*

(i) *Suppose $(\alpha_i: i \in I)$ is a collection of exhaustive homomorphisms $\alpha_i: G \to H$ which are pointwise bounded; then the α_i are uniformly bounded on order-intervals.*

(ii) *Suppose $(\alpha_n: n = 1, 2, \ldots)$ is a sequence of exhaustive homomorphisms which converge pointwise to $\alpha: G \to H$. Then α is exhaustive, and the convergence is uniform on any test sequence.*

Proof. (i) Suppose η is a continuous quasi-norm on H and let $\hat{\eta}(x) = \sup_{i \in I} \eta(a_i x)$ on G. $\hat{\eta}$ is λ-lower-semicontinuous and therefore bounded on order-intervals by the corollary to Theorem 2.

(ii) Again suppose η is a continuous quasi-norm on H and $\varepsilon > 0$. Let $C_m = \{x \in G: \eta(\alpha_p x - \alpha_q x) \leqslant \frac{1}{2}\varepsilon,\ p \geqslant q \geqslant m\}$. Then $\bigcup C_m = G$, and hence, if (a_n) is a test sequence then for some k, we have $\{a_n\} \subset \overline{C_k - C_k}$. If $x \in C_k - C_k$, then $\eta(\alpha_p x - \alpha_q x) \leqslant \varepsilon\ (p \geqslant q \geqslant k)$; hence this is also true for each a_n. Therefore $\alpha_n a_m \to \alpha a_m$ uniformly in m, and

$$\lim_{m\to\infty} \alpha a_m = \lim_{n\to\infty} \lim_{m\to\infty} \alpha_n a_m = 0,$$

that is, α is exhaustive.

6. The closed graph theorem

THEOREM 4. *Let G be a σ-complete Riesz group and let H be a complete metric separable abelian group. Suppose $\alpha: G \to H$ is a homomorphism with closed graph in $(G, \lambda) \times H$. Then $\alpha: (G, \lambda) \to H$ is continuous (that is, α is exhaustive).*

Proof. We use the closed graph theorem stated in [**17**], p. 213, Example R. According to this we require only to show that α is almost continuous, that is, that V is a neighbourhood of 0 in H then the closure of $\alpha^{-1}(V)$ is a neighbourhood of 0 in (G, λ). Let η be a continuous quasi-norm on H and let $V_\varepsilon = \{x: \eta(x) \leqslant \varepsilon\}$, $U_\varepsilon = \alpha^{-1}(V_\varepsilon)$. We define a quasi-norm $\bar{\eta}$ on G by

$$\bar{\eta}(x) = \inf\{\theta: x \in \overline{U}_\theta\}.$$

As $\alpha(G) \subseteq H$ is separable, we can find, for $\varepsilon > 0$, $x_n \in G$ with

$$\bigcup_{n=1}^{\infty} \{\alpha(x_n) + V_{\frac{1}{2}\varepsilon}\} \supset \alpha(G),$$

that is,

$$\bigcup_{n=1}^{\infty} \{x_n + U_{\frac{1}{2}\varepsilon}\} = G.$$

Let

$$C_m = \bigcup_{n=1}^{m} \{x_n + U_{\frac{1}{2}\varepsilon}\}.$$

By Theorem 2, if (a_n) is a test sequence then, for some p, $\{a_n\} \subset \overline{C_p - C_p}$. However,

$$C_p - C_p = \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \{x_i - x_j + U_{\frac{1}{2}\varepsilon} - U_{\frac{1}{2}\varepsilon}\}$$

$$\subseteq \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \{x_i - x_j + \bar{U}_\varepsilon\}.$$

Therefore $\{a_n\} \subseteq \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \{x_i - x_j + \bar{U}_\varepsilon\}$, that is, $\{a_n\}$ is $\bar{\eta}$-precompact.

Now suppose $\liminf \bar{\eta}(a_n) > 0$, so that we may find a subsequence (c_n) with $\bar{\eta}(c_n) \geqslant 3\delta > 0$; we may suppose this subsequence to have the $\bar{\eta}$-Cauchy property as $\{a_n\}$ is $\bar{\eta}$-precompact. Hence, for some r and all $s \geqslant r$, we have $\bar{\eta}(c_s - c_r) \leqslant \delta$. Therefore $c_s - c_r \in \bar{U}_{2\delta}$ for all $s \geqslant r$ and, as $c_s \to 0$ in λ, we have $c_r \in \bar{U}_{2\delta}$, that is, $\bar{\eta}(c_r) \leqslant 2\delta < 3\delta$, contrary to assumption. Therefore $\bar{\eta}(a_n) \to 0$ and $\bar{\eta}$ is limiting. Hence, for arbitrary $\varepsilon > 0$, we have $\bar{U}_\varepsilon \supseteq \{a;\ \bar{\eta}(a) \leqslant \frac{1}{2}\varepsilon\}$, which is a λ-neighbourhood of zero, and α is almost continuous.

Corollary. *Under the hypotheses of Theorem* 4, *suppose* $\alpha\colon (G,\gamma) \to H$ *has closed graph. Then* $\alpha\colon (G,\gamma) \to H$ *is continuous.*

Proof. By Theorem 4, $\alpha\colon (G,\lambda) \to H$ is continuous, and so if $a_n \uparrow a$ in G then αa_n is Cauchy in H. Then by the closed graph assumption, $\alpha a_n \to \alpha a$ in H and it quickly follows that α is Fatou-continuous.

Theorem 5. *Let G be a σ-complete lattice group in which the intrinsic topology is a Hausdorff topology. Suppose τ is a locally solid separable complete metric group topology on G. Then*

(i) *τ coincides with the Fatou topology and the intrinsic topology,*

(ii) *G is order-complete.*

Proof. (i) Let θ be a solid quasi-norm determining the topology τ. Then suppose $\theta(x_n) \leqslant 2^{-n}$ for a sequence (x_n); let $a_n = |x_n|$ and then $\theta(a_n) \leqslant 2^{-n}$. Now $\sum a_n$ converges in (G,τ), to b say; as the positive cone is τ-closed (G is a lattice group; see **[15]**, p. 150) it follows that $b \geqslant \sum_{i=1}^{n} a_i$ $(n = 1, 2, \ldots)$. Hence (a_n) is a test sequence and so $a_n \to 0$ in (G,λ); but λ is also locally solid and so $x_n \to 0$ in λ. Hence the identity map $i\colon (G,\tau) \to (G,\lambda)$ is continuous; but by the closed graph theorem the inverse is also continuous, that is, $\lambda = \tau$. However, if (a_n) is a test sequence, then, as τ is complete, $\sum a_n$ converges in (G,τ), to b say, and

$b \geqslant \sum_{i=1}^{n} a_i$ $(n = 1, 2, \ldots)$. If $c = \sup_n \sum_{i=1}^{n} a_i$ then $c - \sum_{i=1}^{n} a_i \to c - b$ in τ and therefore $c \geqslant b$; hence $c = b$ and so τ is weaker than the Fatou topology γ, that is, $\lambda = \tau = \gamma$.

Part (ii) follows from Theorem 1.

7. Applications to group-valued measures

Let $\mathscr{S}$ be a σ-algebra of subsets of a set T, and H be an (abelian) topological group. By an *H-valued measure* on $\mathscr{S}$ we mean a finitely additive map $\mu: \mathscr{S} \to H$. We shall say μ is *exhaustive* if whenever (E_n) is a sequence of mutually disjoint sets in $\mathscr{S}$ then $\mu(E_n) \to 0$ (Labuda, [**18**], and Drewnowski, [**6**]); in the case when H is a Banach space such a measure is often called *strongly bounded* (Rickart, [**23**], and Diestel, [**4**]).

We construct for $\mathscr{S}$ a σ-complete lattice group $\Gamma(\mathscr{S})$; $\Gamma(\mathscr{S})$ consists of all bounded **Z**-valued functions on T which are $\mathscr{S}$-measurable. $\Gamma(\mathscr{S})$ consists only of simple functions and is a σ-complete lattice group. Given an H-valued measure $\mu: \mathscr{S} \to H$, we can determine a homomorphism $\alpha_\mu: \Gamma(\mathscr{S}) \to H$ by

$$\alpha_\mu\left(\sum_{i=1}^{n} \xi_i \chi(S_i)\right) = \sum \xi_i \mu(S_i),$$

where $\chi(S)$ is the characteristic function of $S \in \mathscr{S}$ and $\xi_i \in Z$.

PROPOSITION 5. *μ is exhaustive if and only if α_μ is exhaustive.*

Proof. Suppose μ is exhaustive and $x_n \in \Gamma(\mathscr{S})$ is an increasing sequence with $0 \leqslant x_n \leqslant x$. Let $\max_{t \in T} x(t) = M$ and define $S_n^{(k)} = \{t \in T: x_n(t) \geqslant k\}$. Then $S_n^{(k)} \uparrow$ in $\mathscr{S}$ for each k and so

$$\lim_{n\to\infty} \mu(S_{n+1}^{(k)} - S_n^{(k)}) = 0,$$

that is,

$$\lim_{n\to\infty} \alpha_\mu(\chi(S_{n+1}^{(k)}) - \chi(S_n^{(k)})) = 0.$$

Therefore

$$\lim_{n\to\infty} \alpha_\mu\left(\sum_{k=1}^{M} (\chi(S_{n+1}^{(k)}) - \chi(S_n^{(k)}))\right) = 0,$$

that is,

$$\lim_{n\to\infty} \alpha_\mu(x_{n+1} - x_n) = 0,$$

and hence α_μ is exhaustive.

Now the results of §§ 5 and 6 apply immediately in this setting.

THEOREM 6. (i) *Let* $(\mu_i\colon i \in I)$ *be a collection of exhaustive H-valued measures which are pointwise bounded; then* $(\mu_i\colon i \in I)$ *is uniformly bounded on* $\mathscr{S}$. *If H is quasi-normed, the same statement is valid for metric boundedness.*

(ii) (Vitali–Hahn–Saks.) *Suppose* (μ_n) *is a sequence of exhaustive H-valued measures converging to a measure* μ *on each* $S \in \mathscr{S}$. *Then* μ *is exhaustive and the convergence is uniform on any disjoint sequence in* $\mathscr{S}$.

Proof. See Theorem 3.

Here (i) is due to Drewnowski ([**7**]); (ii) is due to Drewnowski and Labuda ([**18**]), and for vector-valued measures to Brooks and Jewett ([**3**]).

THEOREM 7. *Let* (H, τ) *be a complete metric separable (abelian) group, and suppose* ρ *is a Hausdorff group topology on H with* $\rho \leqslant \tau$. *If* $\mathscr{S}$ *is a* σ*-algebra and* $\mu\colon \mathscr{S} \to H$ *is a* ρ*-exhaustive measure then* μ *is* τ*-exhaustive.*

Proof. $\alpha_\mu\colon \Gamma(\mathscr{S}) \to H$ is $\lambda - \rho$ continuous and therefore, by Theorem 4, $\lambda - \tau$ continuous, and the result follows.

COROLLARY 1. *Let X be a separable complete metric topological vector space with a Hausdorff weak topology. Then a weakly bounded vector measure* $\mu\colon \mathscr{S} \to X$ *is exhaustive.*

Proof. Clearly μ is weakly exhaustive. If X is a separable Banach space, this result is due to Diestel ([**4**]), who obtains it (essentially) from a theorem of Grothendieck that a bounded linear operator from l_∞ to X is weakly compact. If X is a complete metric topological vector space with a Schauder basis then Corollary 1 could be obtained from Theorem 6(ii).

COROLLARY 2. *If, in Theorem* 7, μ *is* ρ*-countably additive then* μ *is also* τ*-countably additive.*

This result is proved in [**16**] by a rather different approach.

THEOREM 8. *Let* (H, ρ) *be a Hausdorff topological group and let* $\mu\colon \mathscr{S} \to H$ *be a H-valued exhaustive measure. Suppose* $\tau \geqslant \rho$ *is a group topology on H with a base of* ρ*-closed neighbourhoods of* 0 *and such that* (H, τ) *is separable. Then* μ *is* τ*-exhaustive.*

We omit the proof as a similar argument is employed in [**16**] to prove a countably additive version of Theorem 8.

We conclude this section by pointing out an application of the σ-completeness of $\Gamma(\mathscr{S})$ to the structure of the Banach space $B(\mathscr{S})$ of all bounded real-valued $\mathscr{S}$-measurable functions on T. Let $\Gamma_{\mathbf{R}}(\mathscr{S})$ be the linear span of $\Gamma(\mathscr{S})$, consisting of the simple functions in $B(\mathscr{S})$. The following result is equivalent to a theorem of Seever ([**24**]) or Dieudonné ([**5**]); see also [**2**].

PROPOSITION 6. *$\Gamma_{\mathbf{R}}(\mathscr{S})$ is a barrelled subspace of $B(\mathscr{S})$.*

Proof. Let η be a lower-semicontinuous (l.s.c.) semi-norm on $\Gamma_{\mathbf{R}}(\mathscr{S})$; then η is l.s.c. with respect to the weak topology. Hence, restricted to $\Gamma(\mathscr{S})$, η is l.s.c. with respect to the intrinsic topology (any test sequence tends to zero weakly). Therefore, by the Corollary to Theorem 2,

$$\sup_{\substack{x \in \Gamma(\mathscr{S}) \\ -1 \leqslant x \leqslant 1}} \eta(x) < \infty.$$

As η is a semi-norm this extends to the convex hull of $\{x \in \Gamma(\mathscr{S}) : \|x\| \leqslant 1\}$, which is the set $\{x \in \Gamma_{\mathbf{R}}(\mathscr{S}) : \|x\| \leqslant 1\}$. Hence η is bounded on the unit ball of $\Gamma_{\mathbf{R}}(\mathscr{S})$ and is norm-continuous.

8. The Riesz representation theorem

Suppose now that T is a compact Hausdorff space and that $\mathscr{S}$ is the σ-algebra of Borel subsets of T. Let $B(T)$ denote the space of all bounded real-valued Borel functions on T; then $B(T)$ is a σ-complete vector lattice. We shall call a linear map $\Psi : B(T) \to X$, where X is a topological vector space, an *X-valued integral* if it is Fatou-continuous; it is easy to show that Ψ is an integral if and only if the measure $\mu(S) = \Psi(\chi(S))$ is countably additive on $\mathscr{S}$, and Ψ is bounded.

Now let L be the subset of l.s.c. functions in $B(T)$, and let $V = -L$ be the subset of u.s.c. functions. For $x \in B(T)$, we denote by $L(x)$ the set of all $l \in L$ such that $l \geqslant x$; then $L(x)$ is directed downwards and so may be considered as a monotone net. Similarly $V(x) = (v : v \in V; v \leqslant x)$ is a monotone net. We shall say that an X-valued integral Ψ is *regular* if $\lim \Psi(L(x)) = \lim \Psi(V(x)) = \Psi(x)$ for all $x \in B(T)$. This is equivalent to the associated measure μ being regular.

The Riesz representation theorem states that a bounded linear map $\Phi : C(T) \to \mathbf{R}$ may be extended uniquely to a regular integral $\Psi : B(T) \to \mathbf{R}$. We show here that if $\mathbf{R}$ is replaced by a complete topological vector space X (not necessarily locally convex) then, for such an extension to exist, it is necessary and sufficient that Φ be exhaustive.

A quasi-norm η on $B(T)$ will be called *regular* if

(i) η is limiting;

(ii) for $x \in B(T)$, $L(x) \to x$ and $V(x) \to x$ in η.

The topology induced on $B(T)$ by all regular quasi-norms is called the *regular* topology. In the regular topology we clearly have $L(x) \to x$ and $V(x) \to x$ for $x \in B(T)$.

If $\xi \in \mathbf{R}$ then by ξe we shall mean the constant function ξ on T. If η is limiting on $C(T)$ it is easy to show that $\lim_{\xi \to 0} \eta(\xi e) = 0$.

For convenience we shall extend the class L to include l.s.c. functions taking the value $+\infty$; we say $l \in L^*$ if $l \colon T \to \mathbf{R} \cup \{+\infty\}$ and $l^{-1}(-\infty, \xi]$ is closed for $-\infty < \xi \leqslant \infty$. Similarly, let $V^* = -L^*$. The following lemma is well known.

LEMMA 5. (i) *If $l \in L^*$ and $u \in V^*$ with $l \geqslant u$ then there exists $f \in C(T)$ with $l \geqslant f \geqslant u$.*

(ii) *If l_α is a monotone increasing net in L^* and $u \in V^*$ is such that $u(t) \leqslant \sup_\alpha l_\alpha(t)$ $(t \in T)$ then for $\varepsilon > 0$ there exists α such that*

$$u(t) \leqslant l_\alpha(t) + \varepsilon \quad (t \in T).$$

PROPOSITION 7. *$C(T)$ is dense in $B(T)$ in the regular topology.*

Proof. For $l \in L$, we have $V(l) \to l$. Let $C(l) = \{f \in C(T) \colon f \leqslant l\}$, which is a subset of $V(l)$; the $C(l)$ is cofinal with $V(l)$, for if $v \in V(l)$ there exists $f \in C(l)$ with $f \geqslant v$. Therefore $C(l) \to l$ and so $l \in \overline{C(T)}$, that is, $L \subseteq \overline{C(T)}$; therefore $B(T) \subseteq \overline{C(T)}$.

PROPOSITION 8. *Let η be a solid limiting quasi-norm on $C(T)$; then η has a unique regular extension $\hat{\eta}$ to $B(T)$, and $\hat{\eta}$ is solid and Fatou-continuous.*

Proof. For $l \in L^*$, the net $C(l)$ has the η-Cauchy property (Lemma 2) and so we may define

$$\theta(l) = \lim \eta(C(l)).$$

We observe the following facts concerning θ.

(1) θ is monotone, that is, $0 \leqslant l_1 \leqslant l_2$ implies $\theta(l_1) \leqslant \theta(l_2)$.

(2) If $l_1, l_2 \in L^*$ then $\theta(l_1 + l_2) \leqslant \theta(l_1) + \theta(l_2)$.

(3) If $0 \leqslant l_n$ and $l_n \in L^*$ then $\theta(\sum_{n=1}^\infty l_n) \leqslant \sum_{n=1}^\infty \theta(l_n)$.

(1) is obvious. To prove (2), suppose $\varepsilon > 0$ and choose $f \in C(T)$ with $f \leqslant l_1 + l_2$ such that if $f \leqslant g \leqslant l_1 + l_2$ and $g \in C(T)$, then $\eta(g - f) \leqslant \varepsilon$. Choose $h \in C(T)$ with $f - l_2 \leqslant h \leqslant l_1$ (Lemma 5). Then $h \leqslant l_1$ and $f - h \leqslant l_2$; thus we may choose h_1 and $h_2 \in C(T)$ such that $h \leqslant h_1 \leqslant l_1$, $f - h \leqslant h_2 \leqslant l_2$, and

$$\theta(l_1) \geqslant \eta(h_1) - \varepsilon, \quad \theta(l_2) \geqslant \eta(h_2) - \varepsilon.$$

Then $f \leqslant h_1 + h_2 \leqslant l_1 + l_2$ and so

$$\begin{aligned} \theta(l_1 + l_2) &\leqslant \eta(h_1 + h_2) + \varepsilon \leqslant \eta(h_1) + \eta(h_2) + \varepsilon \\ &\leqslant \theta(l_1) + \theta(l_2) + 3\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, (2) follows.

For (3), suppose $f \in C(T)$ and $f \leqslant \sum_{n=1}^{\infty} l_n$. Then by Lemma 5 for $\varepsilon > 0$, there exists k such that

$$f \leqslant \sum_{n=1}^{k} l_n + \varepsilon e$$

and so

$$\theta\left(\sum_{n=1}^{k} l_n\right) \geqslant \eta(f - \varepsilon e) \geqslant \eta(f) - \eta(\varepsilon e).$$

Hence

$$\theta\left(\sum_{n=1}^{\infty} l_n\right) \leqslant \sup_k \theta\left(\sum_{n=1}^{k} l_n\right) + \eta(\varepsilon e)$$

$$\leqslant \sum_{n=1}^{\infty} \theta(l_n) + \eta(\varepsilon e)$$

by (ii). Now $\eta(\varepsilon e) \to 0$ as $\varepsilon \to 0$, and so (3) follows.

Now let $Y \subseteq B(T)$ be defined as the set of y such that, given $\varepsilon > 0$, there exists $l \in L^*$ and $v \in V^*$ with $l \geqslant y \geqslant v$ and $\theta(l - v) \leqslant \varepsilon$. Then Y has the following properties (4)—(9).

(4) If $\xi \geqslant 0$, $y \in Y$ then $\xi y \in Y$.

(5) If $y \in Y$ then $-y \in Y$.

(6) If $y_1, y_2 \in Y$ then $y_1 + y_2 \in Y$.

(7) If $y, y_2 \in Y$ then $y_1 \vee y_2 \in Y$.

(4) is immediate; (5) and (6) follow quickly (using (2)); and, for (7), we note that if $l_1 \geqslant y_1 \geqslant v_2$ and $l_2 \geqslant y_2 \geqslant v_2$ then $l_1 \vee l_2 \geqslant y_1 \vee y_2 \geqslant v_1 \vee v_2$ and $l_1 \vee l_2 - v_1 \vee v_2 \leqslant (l_1 - v_1) + (l_2 - v_2)$, and again we apply (2).

(4)—(7) imply that Y is a vector sublattice of $B(T)$. We now define $\hat{\eta}$ on Y by

$$\hat{\eta}(y) = \inf\{\theta(l) : l \in L, \, l \geqslant |y|\}.$$

Then $\hat{\eta}$ is a solid quasi-norm on Y (for the triangle inequality, we appeal to (2)) and $\hat{\eta} = \eta$ on $C(T)$.

(8) $\hat{\eta}$ is limiting on Y.

Suppose (a_n) is a test sequence and $\sum_{i=1}^{k} a_i \leqslant \xi e$ for all k. For $\varepsilon > 0$, choose $l_n \in L$ and $v_n \in V$ so that $l_n \geqslant a_n \geqslant v_n \geqslant 0$ and $\theta(l_n - v_n) \leqslant \varepsilon 2^{-n}$. Then, by Lemma 5(i),

$$\hat{\eta}(v_n) = \inf\{\eta(g) : v_n \leqslant g, g \in C(T)\}.$$

Thus if $a_n \leqslant l_n' \leqslant l_n$ and $l_n' \in L$,

$$\theta(l_n') - \hat{\eta}(v_n) \leqslant \theta(l_n' - v_n) \leqslant \varepsilon 2^{-n},$$

and therefore

$$\hat{\eta}(v_n) \geqslant \hat{\eta}(a_n) - \varepsilon 2^{-n}.$$

Now choose $f_n \in C(T)$ with $l_n \geqslant f_n \geqslant v_n$ (Lemma 5) and let $g_n = \sum_{i=1}^{n} f_i$. Then the sequence $(g_n \wedge \xi e : n = 1, 2, \ldots)$ is monotone increasing and bounded, and, as η is limiting on $C(T)$,

$$\lim_{n\to\infty} \eta(g_{n+1} \wedge \xi e - g_n \wedge \xi e) = 0.$$

However,

$$g_{n+1} \wedge \xi e - g_n \wedge \xi e \geqslant \sum_{i=1}^{n+1} v_i - \sum_{i=1}^{n} l_i$$

and therefore

$$0 \leqslant v_{n+1} \leqslant (g_{n+1} \wedge \xi e - g_n \wedge \xi e) + \sum_{i=1}^{n} (l_i - v_i).$$

As $\hat{\eta}$ is monotone,

$$\limsup_{n\to\infty} \hat{\eta}(v_{n+1}) \leqslant \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon$$

and therefore $\limsup_{n\to\infty} \hat{\eta}(a_n) \leqslant \varepsilon$, and, as ε is arbitrary, (8) follows.

(9) Y is σ-complete and therefore $Y = B(T)$ (since Y contains L).

Again suppose (a_n) is a test sequence in Y and $\sum a_n = a \in B(T)$; since $\hat{\eta}$ is limiting on Y we may suppose by grouping terms that $\hat{\eta}(a_n) \leqslant 1/2^n$ for $n \geqslant 2$. As before, choose $l_n \in L$ and $v_n \in V$ so that $l_n \geqslant a_n \geqslant v_n$ and $\theta(l_n - v_n) \leqslant \varepsilon 2^{-n}$; choose also N so that $1/2^N \leqslant \varepsilon$. Let $v = \sum_{i=1}^{N} v_i$ and $l^* = \sum_{i=1}^{\infty} l_i$; then $l^* \in L^*$ and $v \in V$ and, by (3),

$$\begin{aligned}\theta(l^* - v) &\leqslant \sum_{i=1}^{N} \theta(l_i - v_i) + \sum_{i=N+1}^{\infty} \theta(l_i) \\ &\leqslant \sum_{i=1}^{N} \varepsilon 2^{-i} + \sum_{i=N+1}^{\infty} 2^{-i}(1+\varepsilon) \\ &\leqslant 2\varepsilon.\end{aligned}$$

Clearly $l^* \geqslant a \geqslant v$. Choosing $l \in L$ with $l^* \geqslant l \geqslant a$, we see that $a \in Y$.

Thus $\hat{\eta}$ is a quasi-norm defined on $B(T)$ extending η; $\hat{\eta}$ is solid, as already remarked.

Since $\hat{\eta}(l) = \theta(l)$ for positive l.s.c. functions l, the construction of Y shows that $\hat{\eta}$ is also regular. Finally the Fatou property of $\hat{\eta}$ follows quickly from (3).

THEOREM 9. (i) *The restriction of the regular topology to $C(T)$ is the intrinsic topology.*

(ii) *The regular topology is locally solid and weaker than the Fatou topology.*

Proof. (i) The regular topology on $C(T)$ is clearly weaker than the intrinsic topology, and Proposition 8 establishes the converse.

(ii) If η is a regular quasi-norm then (Lemma 3) there is a solid limiting quasi-norm η_1 dominating η on $C(T)$. By Proposition 7, the regular extension $\hat{\eta}_1$ of η_1, constructed in Proposition 8, dominates η on $B(T)$. This regular extension is both a Fatou quasi-norm and solid.

THEOREM 10. *Let $\Phi: C(T) \to X$ be a linear map into a complete topological vector space X; in order that there exist a regular integral $\Psi: B(T) \to X$ extending Φ it is necessary and sufficient that Φ be exhaustive.*

Proof. It is trivial that the restriction of a regular integral to $C(T)$ must be exhaustive. Conversely, if Φ is exhaustive then Φ is intrinsically continuous and extends uniquely to a map $\Psi: B(T) \to X$ which is regularly continuous. It is easy to see that consequently Ψ is a regular integral.

COROLLAPY. *If X is locally convex then Φ is exhaustive if and only if Φ is weakly compact.*

Proof. (This result is due to Pełczyński ([**20**]) by a different proof; note that 'exhaustive' is the same as 'unconditionally converging'.)

If Φ is weakly compact it is easy to show, using the Orlicz–Pettis theorem, that Φ is exhaustive. Conversely, let Ψ be the regular integral extension of Φ and let U be the unit ball of $B(T)$. Suppose $f \in X'$ and suppose

$$m = \sup_{u \in U} f(\Psi(u)).$$

For $u \in U$, let $\theta(u) = m - f(\Psi(u))$. Then we have, for $u_1, u_2 \in U$,

$$u_1 \vee u_2 + u_1 \wedge u_2 = u_1 + u_2$$

and so

$$\begin{aligned} \theta(u_1 \vee u_2) &= \theta(u_1) + \theta(u_2) - \theta(u_1 \wedge u_2) \\ &\leqslant \theta(u_1) + \theta(u_2) \end{aligned}$$

and, in general,

$$\theta(u_1 \vee \ldots \vee u_n) \leqslant \sum_{i=1}^{n} \theta(u_i).$$

As Ψ is a regular integral, that is, Fatou-continuous, for any sequence (u_n) with $u_n \in U$

$$\theta\left(\bigvee_n u_n\right) \leqslant \sum_{i=1}^{\infty} \theta(u_i).$$

Now choose u_n with $\theta(u_n) \leqslant 2^{-n}$ and let $w_n = \bigvee_{i=n+1}^{\infty} u_i$. Then $\theta(w_n) \leqslant 2^{-n}$ and $w_n \downarrow w$ in U, and we have, again by Fatou continuity, $\theta(w) = 0$. Thus f assumes its supremum on $\Psi(V)$ and, by the James–Pryce theorem ([**12**] and [**19**]), $\Psi(V)$ is relatively weakly compact.

In this section we have restricted attention quite naturally to a vector-valued measure. If, however, the space T is stonean (that is, the closure of every open set is again open) then the space $C(T, \mathbf{Z})$ of continuous integer-valued functions is rich enough to consider group-valued measures. There is then available an entirely analogous theorem to Theorem 10 for group homomorphisms $\Phi: C(T, \mathbf{Z}) \to G$, where G is an abelian topological group.

Postscript. After the initial preparation of this paper, the author learned that Theorem 7 has been independently obtained by I. Labuda ([**19**]). Labuda's proof is quite different to that presented here and is attained by reducing the theorem to the countably additive case and applying the results of [**16**]. In this connection it should also be mentioned that L. Drewnowski ([8]) has recently given a shorter proof of the main results of [**16**].

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