NORMALIZATION PROPERTIES OF SCHAUDER BASES

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1. Introduction

A *basis* of a locally convex topological vector space E is a sequence (x_n) such that for every $x \in E$ there is a unique sequence (α_n) of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$; if, further, the linear functionals $f_n(x) = \alpha_n$ are continuous then (x_n) is said to be a *Schauder basis*. The theory of Schauder bases has been developed in considerable detail in Banach spaces, but comparatively little is known about bases of more general types of spaces (see [**1**] and [**2**]).

When E is a normed space, an alternative Schauder basis is given by $y_n = x_n / \|x_n\|$; this basis is normalized so that $\|y_n\| = 1$. In topological terms this means that (y_n) is bounded and 'bounded away from zero' in the sense that there exists a neighbourhood V of 0 such that $y_n \notin V$ for all n. These two properties are extremely useful in the theory of bases in Banach spaces. The purpose of this paper is to consider these properties for bases of general locally convex spaces.

DEFINITION 1.1. A sequence (x_n) is said to be *regular* if there exists a neighbourhood V of 0 such that $x_n \notin V$ for all n.

Even if E is a Fréchet space, it does not necessarily follow that a basis (x_n) can be 'regularized'. If ω denotes the space of all real (or complex) sequences, and φ is the space of all real sequences eventually equal to zero, then ω and φ form a dual pair of sequence spaces and $(\omega, \sigma(\omega, \varphi))$ is an F-space. If e_n denotes the sequence taking 1 in the nth place and zero elsewhere, then (e_n) is a Schauder basis of ω; but for every sequence (α_n) of scalars, $\alpha_n e_n \to 0$. Hence, for every sequence (α_n) with $\alpha_n \neq 0$ for all n, $(\alpha_n e_n)$ is not a regular basis.

DEFINITION 1.2. A bounded regular sequence is called a *normalized sequence*.

DEFINITION 1.3. If (x_n) is a sequence, such that there exist scalars α_n with $(\alpha_n x_n)$ normalized, then (x_n) is said to be *normal*.

In § 2, we shall consider the elementary properties of regular bases, and in § 3 regular bases will be related to bounded bases. An important use of regular bases is shown in § 4, where a regular basis is 'deformed' into an equivalent regular basis. In § 5, we show an important link between normality of basic sequences and Montel spaces; finally in § 6, these results are applied to certain problems about equivalence of basic sequences.

2. Regular bases

If (x_n) is a Schauder basis of E, the dual sequence will always be denoted by (f_n); also P_n will denote the continuous projection given by $P_n x = \sum_{i=1}^{n} f_i(x)x_i$, and P'_n will denote the projection $P'_n f = \sum_{i=1}^{n} f(x)f_i$ in E'. In general, the regularity of (x_n) is very closely related to the equicontinuity of (f_n).

PROPOSITION 2.1. *If (f_n) is equicontinuous, then (x_n) is regular. Conversely, if E is a Mackey space (i.e. the topology on E is the Mackey topology) and $\tau(E', E)$ (the Mackey topology on E') is sequentially complete, then if (x_n) is regular, (f_n) is equicontinuous.*

Proof. If (f_n) is equicontinuous, then let $V = \left\{x : \sup_n |f_n(x)| < \tfrac{1}{2}\right\}$; V is a neighbourhood of 0 with $x_n \notin V$ for all n.

Conversely, if (x_n) is regular, since, for all $x \in E$,

$$x = \sum_{n=1}^{\infty} f_n(x)x_n,$$

$$\lim_{n\to\infty} f_n(x)x_n = 0$$

and so

$$\lim_{n\to\infty} f_n(x) = 0,$$

$$\lim_{n\to\infty} f_n = 0 \quad \text{in the topology } \sigma(E', E).$$

Thus (f_n) is $\tau(E', E)$-bounded, and so, if $\tau(E', E)$ is sequentially complete and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$, then $\sum_{i=1}^{\infty} \alpha_i f_i$ converges in $\tau(E', E)$. We define $T: l^1 \to E'$ by $T(\alpha) = \sum_{i=1}^{\infty} \alpha_i f_i$. Let $x \in E$; then

$$T(\alpha)(x) = \sum_{i=1}^{\infty} \alpha_i f_i(x)$$

and since $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in E$, T is a continuous map from $(l^1, \sigma(l^1, c_0))$ to $(E', \sigma(E', E))$. Let B be the closed unit ball of l^1; then B is

$\sigma(l^1, c_0)$-compact, and so $T(B)$ is $\sigma(E', E)$-compact. However, if $\Delta\{f_n: n = 1, 2, \ldots\}$ denotes the absolutely convex hull of $\{f_n: n = 1, 2, \ldots\}$, then $\Delta\{f_n: n = 1, 2, \ldots\} \subseteq T(B)$; hence $\bar{\Delta}\{f_n: n = 1, 2, \ldots\}$ is $\sigma(E', E)$-compact. Thus, if E is a Mackey space, (f_n) is equicontinuous.

It may be noted that the conditions of Proposition 2.1 always hold when E is barrelled.

PROPOSITION 2.2. *If E is a Mackey space and $\tau(E', E)$ is sequentially complete, and if E has a basis (x_n), then the following assertions are equivalent.*

(i) *There exist scalars $\alpha_n \neq 0$ such that $(\alpha_n x_n)$ is regular.*

(ii) *There exist scalars $\beta_n \neq 0$ such that $(\beta_n f_n)$ is equicontinuous.*

(iii) *There exists on E a continuous norm.*

Proof. That (i) implies (ii) is immediate from Proposition 2.1 since $(\beta_n f_n)$ is the dual sequence of $(\alpha_n x_n)$ when $\beta_n = 1/\alpha_n$. If (ii) holds then, if $p(x) = \sup_n |\beta_n f_n(x)|$, p is a continuous norm on E; and if p is a continuous norm on E, and $\alpha_n = 1/p(x_n)$, then $(\alpha_n x_n)$ is a regular basis of E.

If E is an F-space, Proposition 2.2 can be strengthened considerably, improving on a result of Bessaga and Pełczyński ([**3**]), who proved the equivalence of conditions (ii) and (iv) of Theorem 2.3. Any basis of an F-space is a Schauder basis.

THEOREM 2.3. *Let E be an F-space with a basis (x_n); then the following assertions are equivalent.*

(i) *There exist $\alpha_n \neq 0$ such that $(\alpha_n x_n)$ is regular.*

(ii) *There exists a continuous norm on E.*

(iii) *There is no complemented subspace isomorphic to ω.*

(iv) *There is no subsequence (x_{n_j}) of (x_n) such that $\sum_{j=1}^{\infty} \alpha_j x_{n_j}$ exists for all $\alpha \in \omega$.*

Proof. Of course ω has the topology $\sigma(\omega, \varphi)$. That (i) and (ii) are equivalent is immediate from Proposition 2.2; certainly (ii) implies (iii) since there is no continuous norm on ω. The proof is completed by showing that (iii) implies (iv), and (iv) implies (ii).

Suppose (x_{n_j}) is a subsequence of (x_n) such that $\sum_{j=1}^{\infty} \alpha_j x_{n_j}$ converges for all $\alpha \in \omega$, and let $W = \left\{\sum_{j=1}^{\infty} \alpha_j x_{n_j}: \alpha \in \omega\right\}$; define the maps $T_m: E \to W$ by

$$T_m(x) = \sum_{j=1}^{m} f_{n_j}(x) x_{n_j}.$$

Then each T_m is continuous and $T(x) = \lim_{m\to\infty} T_m(x)$ exists for each $x \in E$; hence, by the Banach-Steinhaus theorem, T is a continuous projection of E onto W, and since (x_{n_j}) is a basis for W, equivalent to the usual basis of ω, W is isomorphic to ω (see [1]).

As E is barrelled, the set of operators (P_n) is equicontinuous (i.e. (x_n) is an equi-Schauder basis of E as defined in [4]). Thus the topology on E may be defined by an increasing sequence of semi-norms (p_n) such that $p_n(x) = \sup_m p_n(P_m x)$.

Now suppose that there is no continuous norm on E. Let

$$X_n = \{x : p_n(x) = 0\};$$

then X_n is an infinite-dimensional subspace of E, since, if not, there exists a continuous semi-norm q_n on E which restricts to a norm on X_n, and $p_n + q_n$ is a continuous norm on E. Let $Z_n = \{m : \exists\, x \in X_n, f_m(x) \neq 0\}$; then, since X_n is infinite-dimensional, Z_n is infinite. If $m \in Z_n$ and $x \in X_n$ are such that $f_m(x) \neq 0$ then

$$p_n(f_m(x)x_m) \leqslant p_n(P_m x - P_{m-1}x) \leqslant 2p_n(x) = 0.$$

Therefore $p_n(x_m) = 0$.

Thus $Z_n = \{m : p_n(x_m) = 0\}$; thus we can choose an increasing sequence $n_j \in Z_j$, and, if $j \geqslant k$, $p_k(x_{n_j}) \leqslant p_j(x_{n_j}) = 0$. Hence, for all $\alpha \in \omega$, $\sum_{j=1}^{\infty} \alpha_j x_{n_j}$ converges.

To conclude this section, we consider regularity in terms of the sequence spaces determined by the basis (x_n). The sequence space $\left\{\alpha : \alpha \in \omega \text{ and } \sum_{n=1}^{\infty} \alpha_n x_n \text{ converges}\right\}$ will be denoted by λ, and the sequence space $\left\{\alpha : \alpha \in \omega \text{ and } \sum_{n=1}^{\infty} \alpha_n f_n \text{ converges in } \sigma(E', E)\right\}$ will be denoted by μ. Obviously (λ, μ) forms a dual pair of sequence spaces isomorphic to the dual pair (E, E').

PROPOSITION 2.4. *Let (x_n) be a basis for E, where E is a Mackey space; then (f_n) is equicontinuous if and only if*

(i) $\lambda \subseteq c_0$ *and*

(ii) $l^1 \subseteq \mu$.

Proof. If (f_n) is equicontinuous, then $p(x) = \sup_n |f_n(x)|$ is a continuous norm on E, and so, if $\sum_{i=1}^{\infty} \alpha_i x_i$ converges,

$$\lim_{n\to\infty} p\left(\sum_{i=n+1}^{\infty} \alpha_i x_i\right) = 0$$

i.e.

$$\lim_{n\to\infty} \sup_{m>n} |\alpha_m| = 0.$$

Thus $\alpha \in c_0$ and $\lambda \subseteq c_0$.

If $\alpha \in l^1$, then $\left(\sum_{i=1}^{n} \alpha_i f_i\right)_{n=1}^{\infty}$ is contained in the closed absolutely convex cover of $(f_n)_{n=1}^{\infty}$, which is $\sigma(E', E)$-compact, since (f_n) is equicontinuous. Hence there exists a cluster point f of the sequence, and $f(x_j)$ is a cluster point of $\left(\sum_{i=1}^{n} \alpha_i f_i(x_j)\right)_{n=1}^{\infty}$ for each j; thus $f(x_j) = \alpha_j$, and $f = \sum_{i=1}^{\infty} \alpha_i f_i$. Therefore $l^1 \subseteq \mu$.

Conversely, suppose $\lambda \subseteq c_0$ and $l^1 \subseteq \mu$; define $J: E \to c_0$ by

$$(J(x))_i = f_i(x).$$

If $\alpha \in l^1$, then, since $l^1 \subseteq \mu$, the linear functional $x \to \sum_{i=1}^{\infty} \alpha_i f_i(x) = \sum_{i=1}^{\infty} \alpha_i (J(x))_i$ is continuous; hence $J: (E, \sigma(E, E')) \to (c_0, \sigma(c_0, l^1))$ is continuous. E is a Mackey space, and so $J: E \to (c_0, \tau(c_0, l^1))$ is continuous; but $\tau(c_0, l^1)$ is the norm topology with $\|\alpha\| = \sup_n |\alpha_i|$. Thus $p(x) = \sup_n |f_n(x)|$ is a continuous norm on E, and so (f_n) is equicontinuous.

3. Bounded and normalized bases

Simple Schauder bases were introduced in [4]; a Schauder basis (x_n) is simple if for each $f \in E'$ the sequence $\left(\sum_{i=1}^{n} f(x_i) f_i\right)_{n=1}^{\infty}$ is strongly bounded. It is shown in [4] that in this case $(f_n)_{n=1}^{\infty}$ is an equi-Schauder basis for its closed linear span in $(E', \beta(E', E))$. A sequence (y_n) which is a (Schauder) basis for the closure of its linear span, $\overline{\text{lin}}(y_n)$, is called a (Schauder) *basic sequence*.

PROPOSITION 3.1. *If (x_n) is a simple basis for E, then (x_n) is bounded if and only if (f_n) is a regular Schauder basic sequence in $(E', \beta(E', E))$.*

Proof. If (x_n) is bounded, then $p(f) = \sup_n |f(x_n)|$ is a continuous semi-norm on $(E', \beta(E', E))$, and $p(f_n) = 1$ for all n; thus (f_n) is $\beta(E', E)$-regular.

Conversely, suppose (f_n) is regular; then there exists a bounded set A in E with

$$\sup_{a \in A} |f_n(a)| > 1 \quad \text{for all } n.$$

Hence there exist $a_n \in A$ with $|f_n(a_n)| > 1$. However

$$f_n(a_n) x_n = P_n(a_n) - P_{n-1}(a_n).$$

If $P(A) = \bigcup_{n=1}^{\infty} P_n(A)$, then $f_n(a_n)x_n \in P(A) - P(A)$; since (x_n) is simple, for any $f \in E'$,

$$\sup_{a \in A} \sup_n |P'_n f(a)| < \infty,$$

i.e.

$$\sup_{a \in A} \sup_n |f(P_n a)| < \infty,$$

and so $P(A)$ is bounded. Hence $(f_n(a_n)x_n)_{n=1}^{\infty}$ is bounded and $|f_n(a_n)| > 1$ for all n; therefore $(x_n)_{n=1}^{\infty}$ is bounded.

The analogous result to Proposition 2.4 is the following.

PROPOSITION 3.2. *Let E be sequentially complete; then a Schauder basis (x_n) of E is bounded if and only if $l^1 \subseteq \lambda$.*

Proof. Certainly if (x_n) is bounded and $\alpha \in l^1$, then $\sum_{n=1}^{\infty} \alpha_n x_n$ converges. Conversely if $l^1 \subseteq \lambda$, then $\mu \subseteq l^\infty$, and so, if $f \in E'$, $\sup_n |f(x_n)| < \infty$, i.e. (x_n) is bounded.

In § 2 it was shown that if, E is an F-space, a basis (x_n) might fail to be regular for all sequences $\alpha_n x_n$ with $\alpha_n \neq 0$; however, for boundedness this is not the case.

PROPOSITION 3.3. *If E is an F-space with a basis (x_n), then there exist $\alpha_n > 0$ with $(\alpha_n x_n)$ bounded.*

It is, in fact, an elementary property of metrizable locally convex spaces that if (y_n) is any sequence there exists a sequence $\alpha_n > 0$ such that $\alpha_n y_n \to 0$.

COROLLARY. *A DF-space with a simple Schauder basis (x_n) admits a continuous norm.*

Proof. Since (x_n) is simple, (f_n) is a basis for its closed linear span H in $(E', \beta(E', E))$. Then H is an F-space, and so there exist $\alpha_n \neq 0$ such that $(\alpha_n f_n)$ is bounded in H; but every strongly bounded sequence in the dual of a DF-space is equicontinuous, so that $(\alpha_n f_n)$ is equicontinuous, and $p(x) = \sup |\alpha_n f_n(x)|$ is a continuous norm on E.

Finally we consider normalized bases.

THEOREM 3.4. *Let E be a locally convex space such that every strongly bounded sequence in E' is equicontinuous (e.g. E is quasi-barrelled); then a simple Schauder basis (x_n) of E is normalized if and only if (f_n) is a normalized basis with respect to $\beta(E', E)$.*

Proof. By Proposition 3.1, (x_n) is bounded if and only if (f_n) is regular. If (f_n) is bounded then (f_n) is equicontinuous and so by Proposition 2.1 (x_n) is regular.

If (x_n) is regular and A is a bounded set in E, then since (x_n) is simple $B = P(A) - P(A)$ is bounded; for each $a \in A$, $f_n(a)x_n \in B$. Thus $\{f_n(a)x_n : a \in A, n \in \mathbf{Z}\}$ is bounded; but (x_n) is regular and so

$$\sup_n \sup_{a \in A} |f_n(a)| < \infty.$$

Therefore (f_n) is $\beta(E', E)$-bounded.

4. Deformations of basic sequences

Two basic sequences (y_n) and (z_n) are said to be *equivalent* if $\sum_{n=1}^{\infty} \alpha_n y_n$ converges if and only if $\sum_{n=1}^{\infty} \alpha_n y_n$ converges. In [**5**], Bessaga and Pełczyński describe a particular way of perturbing a basic sequence of a Banach space to obtain an equivalent basic sequence, and add, without proof, extensions of these results to F-space with continuous norms. In this section it will be shown that the same method may be applied to basic sequences (x_n) whose dual sequence (f_n) is equicontinuous.

Let E be a locally convex space and let (x_n) be a Schauder basis for a closed subspace E_0 of E; suppose the dual sequence (f_n) in E_0' is equicontinuous, and that p_0 is a continuous semi-norm on E with $|f_n(x)| \leqslant p_0(x)$ for all n, and all $x \in E_0$. Suppose (u_n) is any sequence in E such that

(i) $\sum_{i=1}^{\infty} u_i$ converges absolutely (i.e. for all continuous semi-norms p on E, $\sum_{i=1}^{\infty} p(u_i) < \infty$),

(ii) $\sum_{i=1}^{\infty} p_0(u_i) = K_0 < 1$.

Then the sequence $(y_n = x_n + u_n)$ will be called a deformation of (x_n).

Proposition 4.1. *If E is complete, then a deformation (y_n) of (x_n) is a Schauder basic sequence equivalent to (x_n), and whose closed linear span E_1 is isomorphic to E_0.*

Proof. Let p be a continuous semi-norm on E, and let $K_p = \sum_{i=1}^{\infty} p(u_i)$. Let $x \in E_0$; then

$$\sum_{i=1}^{\infty} |f_i(x)| p(u_i) \leqslant K_p p_0(x).$$

E is complete, and so $\sum_{i=1}^{\infty} f_i(x) u_i$ converges. Therefore $\sum_{i=1}^{\infty} f_i(x) y_i$ converges.

Define the map $A: E_0 \to E_1$ by

$$Ax = \sum_{i=1}^{\infty} f_i(x) y_i;$$

then

$$p(Ax) \leqslant p(x) + p\left(\sum_{i=1}^{\infty} f_i(x) u_i\right) \leqslant p(x) + K_p p_0(x).$$

Therefore A is continuous. Also

$$p_0(Ax) \geqslant p_0(x) - K_0 p_0(x) = (1-K_0) p_0(x),$$

$$p(x) \leqslant p(Ax) + K_p p_0(x) \leqslant p(Ax) + K_p(1-K_0)^{-1} p_0(Ax).$$

Thus A is injective, and an isomorphism onto its image; in particular $A(E_0)$ is complete, and so closed. Hence $A(E_0) = E_1$, and since $Ax_n = y_n$, (y_n) is a Schauder basis for E_1 equivalent to (x_n).

PROPOSITION 4.2. *If E_0 is complemented in E and $T: E \to E_0$ is a continuous projection of E onto E_0 with $K_0 p_0(Tx) \leqslant \delta p_0(x)$, where $\delta < 1$, then E_1 is also complemented in E.*

Proof. We use the same notation as in 4.1. If $x \in E_0$,

$$p(x - Ax) \leqslant K_p p_0(x).$$

Let $S: E \to E$ be defined by $Sx = Tx - ATx$. Then

$$p(Sx) \leqslant K_p p_0(Tx) \leqslant K_p K_0^{-1} \delta p_0(x).$$

Hence

$$p(S^n x) \leqslant K_p K_0^{-1} \delta p_0(S^{n-1} x) \leqslant K_p K_0^{-1} \delta^n p_0(x).$$

Thus, for each $x \in E$, $\sum_{n=1}^{\infty} S^n(x) = R(x)$ converges absolutely and

$$p(Rx) \leqslant \frac{K_p \delta}{K_0(1-\delta)} p_0(x).$$

R is continuous, and $R(I-S) = (I-S)R = I$, where I is the identity map $I: E \to E$. Let $Q = (I-S)TR$; then Q is a continuous projection on E.

$$(I-S)T = (I - T + AT)T = AT.$$

Thus $Q = ATR$ and $Q(E) \subseteq E_1$.

If $x \in E_1$, then $x = Ay$ for some $y \in E_0$. Hence $x = ATy$, and since R is an isomorphism on E, $x = ATRz$ for some $z \in E$. Therefore Q is a continuous projection of E onto E_1.

The next theorem is stated without proof in [**5**]. We recall that, if (x_n) is a Schauder basis of E, a *block basic sequence* (z_n) of (x_n) is a sequence of the form $z_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i$ where (m_n) is a strictly increasing sequence, with $m_0 = 0$, and each z_n is non-zero. It is trivial to check that (z_n) is a Schauder basic sequence in E.

THEOREM 4.3. *Let E be an F-space with a basis (x_n), and let (y_n) be any sequence such that y_n does not tend to zero, but $\lim_{m\to\infty} f_m(y_n) = 0$ for each m. Then (y_n) contains a subsequence (y_{n_j}) which is a regular basic sequence equivalent to a block basic sequence (z_n) of (x_n).*

Proof. By taking a subsequence of (y_n) if necessary, we may assume that there exists a continuous semi-norm p_1 on E such that there exists $\varepsilon > 0$ with $p_1(y_n) > \varepsilon$ and $\sup_n p_1(P_n x) = p_1(x)$, and that the topology on E is given by an increasing sequence of semi-norms (p_n).

We may choose inductively increasing sequences (n_j) and (m_j) such that

$$p_j\left(\sum_{i=1}^{m_{j-1}} f_i(y_{n_j})x_i\right) < \varepsilon(\tfrac{1}{2})^{j+4},$$

and

$$p_j\left(\sum_{i=m_j+1}^{\infty} f_i(y_{n_j})x_i\right) < \varepsilon(\tfrac{1}{2})^{j+4},$$

where $n_1 = 1$ and $m_0 = 0$.

Let $z_j = \sum_{i=m_{j-1}+1}^{m_j} f_i(y_{n_j})x_i$; then (z_j) is a block basic sequence and $\sum_{j=1}^{\infty} (y_{n_j} - z_j)$ converges absolutely. Let (h_j) be the dual sequence of (z_j) defined on the closed linear span G of (z_j). If $z \in G$,

$$z = \sum_{j=1}^{\infty} h_j(z) z_j,$$

$$h_j(z)z_j = \sum_{i=m_{j-1}+1}^{m_j} f_i(z)x_i = P_{m_j}(z) - P_{m_{j-1}}(z),$$

$$p_1(h_j(z)z_j) \leqslant 2p_1(z).$$

However, $p_1(z_j) \geqslant \varepsilon/2$. Therefore $|h_j(z)| \leqslant (4/\varepsilon)p_1(z)$; and taking

$$p_0(z) = (4/\varepsilon)p_1(z)$$

we have

$$\sum_{j=1}^{\infty} p_0(y_{n_j} - z_j) \leqslant (4/\varepsilon)\sum_{j=1}^{\infty} p_j(y_{n_j} - z_j) \leqslant 4\sum_{j=1}^{\infty} (\tfrac{1}{2})^{j+3} = \tfrac{1}{2},$$

and so, by Proposition 4.1, (y_{n_j}) is a basic sequence equivalent to (z_j), and is regular as required.

If, in addition, G is complemented, then let T be a continuous projection of E onto G. As G is an F-space (z_n) is an equi-Schauder basis for G, and so the projections $Q_n\left(\sum_{i=1}^{\infty} \alpha_i z_i\right) = \sum_{i=n}^{\infty} \alpha_i z_i$ are equicontinuous on E_1. Let $p_0'(x) = \max\left(p_0(x), \sup_n(Q_n Tx)\right)$; then p_0' is a continuous semi-norm on E. Therefore $\sum_{j=1}^{\infty} p_0'(y_{n_j} - z_j) < \infty$; thus there exists k such that

$$\sum_{j=k}^{\infty} p_0'(y_{n_j} - z_j) < 1.$$

Then $G_k = \overline{\mathrm{lin}}(z_n)_{n=k}^{\infty}$ is complemented in E, and $Q_k T$ is a continuous projection on G_k. Proposition 4.2 may then be applied with p_0' replacing p_0, and G_k replacing E_0; the conclusion is that $\overline{\mathrm{lin}}(y_{n_j})_{j=k}^{\infty}$ is complemented in E. It follows that $\overline{\mathrm{lin}}(y_{n_j})_{j=1}^{\infty}$ is complemented; this fact will be used in a future paper.

THEOREM 4.4. *Let (E, τ) be an F-space with a basis (x_n) and let E_0 be a closed subspace of E; if E_0 is not a Montel space, then E_0 contains a normalized basic sequence.*

Proof. Let π be the topology on E given by the semi-norms $x \to |f_n(x)|$ for each n. Suppose that, for every τ-bounded sequence (y_n) in E_0, if $y_n \to 0\ (\pi)$ then $y_n \to 0\ (\tau)$. Then τ and π coincide on τ-bounded sets in E_0, and, as every τ-bounded set is π-precompact, E_0 is a Montel space. Thus there is a sequence (y_n) which is τ-bounded and such that $y_n \to 0\ (\pi)$ but $y_n \not\to 0\ (\tau)$; the result follows by Theorem 4.3.

Theorem 4.4 should be compared with the similar results of § 5.

5. Normal basic sequences

A basic sequence (z_n) was defined to be normal if there exist scalars $\alpha_n \neq 0$ such that $(\alpha_n z_n)$ is normalized (Definition 1.3). We now define:

DEFINITION 5.1. A basis (x_n) of a locally convex space is *completely normal* if every block basic sequence of (x_n) is normal.

DEFINITION 5.2. A basis (x_n) of a locally convex space is *completely normal* if no block basic sequence of (x_n) is normal.

It will be shown in this section that these two cases characterize two extreme types of F-spaces: Banach spaces and Montel spaces.

THEOREM 5.3. *Let E be an F-space with a basis (x_n). Then (x_n) is completely normal if and only if E is a Banach space.*

Proof. Clearly any basis of a Banach space is completely normal. Conversely, suppose E is not a Banach space, and that (p_n) is a strictly increasing sequence of continuous semi-norms defining the topology on E, so that, if $m > n$,

$$\sup_{x \in E} \frac{p_m(x)}{p_n(x)} = \infty.$$

Without loss of generality, it may be assumed that $\sup_n p_m(P_n x) = p_m(x)$ for all m and all $x \in E$, as (x_n) is an equi-Schauder basis; also that each p_n is a norm, for as (x_n) is a normal basis there exists a continuous norm on E (Theorem 2.3).

Let $F_n = \operatorname{lin}(x_1, \ldots, x_n)$ and let $G_n = \overline{\operatorname{lin}}(x_{n+1}, x_{n+2}, \ldots)$; then since F_n is finite-dimensional

$$\sup_{x \in F_n} \frac{p_i(x)}{p_j(x)} < \infty \quad \text{for all } i, j.$$

Suppose $i > j$ and $\sup_{x \in G_n} p_i(x)/p_j(x) < \infty$; then if $x \in E$

$$\begin{aligned} p_i(x) &\leqslant p_i(P_n x) + p_i(x - P_n x) \\ &\leqslant K(p_j(P_n x) + p_j(x - P_n x)) \\ &\leqslant 3K p_j(x), \end{aligned}$$

where $K = \sup_{x \in F_n \cup G_n} p_i(x)/p_j(x)$. This contradicts the fact that (p_i) is strictly increasing; hence

$$\sup_{x \in G_n} \frac{p_i(x)}{p_j(x)} = \infty \quad \text{whenever } i > j, \text{ for all } n.$$

Let $n \leftrightarrow (i_n, j_n, k_n)$ be a bijective correspondence between the positive integers and the set of triplets (i, j, k) of positive integers with $i > j$. Let $m_0 = 0$; then we may define inductively

(i) y_n such that: (a) $p_{j_n}(y_n) = 1$,

(b) $y_n \in G_{m_{n-1}}$,

(c) $p_{i_n}(y_n) > k_n$,

(ii) m_n such that $p_{i_n}(y_n - P_{m_n}(y_n)) < \frac{1}{2}$.

Let $z_n = P_{m_n}(y_n)$; then $p_{i_n}(z_n) > k_n - \frac{1}{2}$, and $p_{j_n}(z_n) < 1$. Hence

$$\frac{p_{i_n}(z_n)}{p_{j_n}(z_n)} > k_n - \tfrac{1}{2}.$$

Thus (z_n) is a block basic sequence with respect to (x_n), and so if (x_n) is completely normal, there exist $\alpha_n \neq 0$, with $(\alpha_n z_n)$ normalized. Given i

and $j > i$,

$$\sup_n p_j(z_n)/p_i(z_n) = \infty.$$

Thus

$$\sup_n p_j(\alpha_n z_n)/p_i(\alpha_n z_n) = \infty$$

and

$$\sup_n p_j(\alpha_n z_n) < \infty;$$

therefore

$$\inf_n p_i(\alpha_n z_n) = 0 \quad \text{for all } i.$$

Therefore $(\alpha_n z_n)$ is not regular; thus we have reached a contradiction.

We now consider completely abnormal bases; if a basis of a locally convex space E is completely abnormal, then it is completely abnormal for the weak topology on E. If (f_n) is a Schauder basis of $(E', \beta(E', E))$, then (x_n) is said to be *shrinking*.

THEOREM 5.4. *Suppose (x_n) is a simple Schauder basis of E; then (x_n) is completely abnormal for the weak topology on E if and only if (x_n) is shrinking.*

Proof. Suppose (y_n) is a weakly normalized block basic sequence of (x_n); then there exist $g_1, g_2, \ldots, g_m$ in E' such that

$$\sum_{i=1}^{m} |g_i(y_n)| > \varepsilon > 0 \quad \text{for all } n,$$

and so there exists k, $1 \leqslant k \leqslant m$, such that

$$|g_k(y_n)| > \varepsilon/m \quad \text{for infinitely many } n,$$

and $\sup_n |g_k(y_n - P_l y_n)| > \varepsilon/m$ for all l since (y_n) is a block basic sequence. Hence, if P_l' is the adjoint map to P_l,

$$\sup_n |g_k(y_n) - P_l' g_k(y_n)| > \varepsilon/m \quad \text{for all } l.$$

As (y_n) is bounded, $P_l' g_k \not\to g_k$ in the strong topology on E'; thus (x_n) is not shrinking.

Conversely suppose (x_n) is not shrinking; then there exists $g \in E'$ such that $P_n' g \not\to g$ in $\beta(E', E)$. Hence $(P_n' g)$ is not a Cauchy sequence, and there exists a $\beta(E', E)$-bounded set A in E, and an increasing sequence (n_j) such that

$$\sup_{a \in A} |P_{n_j}' g(a) - P_{n_{j-1}}' g(a)| > \varepsilon > 0$$

for all j, and some $\varepsilon > 0$.

Choose $a_j \in A$, such that

$$|P_{n_j}' g(a_j) - P_{n_{j-1}}' g(a_j)| > \varepsilon/2,$$
$$|g(P_{n_j} a_j - P_{n_{j-1}} a_j)| > \varepsilon/2.$$

Then $y_j = P_{n_j}a_j - P_{n_{j-1}}a_j$ is a weakly regular block basic sequence; but, since (x_n) is simple, the set $P(A) = \bigcup_{n=1}^{\infty} P_n(A)$ is bounded, and hence (y_n) is bounded. Thus (y_n) is a weakly normalized block basic sequence.

Lemma 5.5. *Let (E,τ) be a locally convex space with a simple Schauder basis (x_n); if τ and the weak topology $\sigma(E,E')$ do not define the same convergent sequences, then there exists a normalized block basic sequence.*

Proof. Let (y_n) be a sequence such that

(i) $y_n \to 0$ weakly,

(ii) there exists an absolutely convex τ-neighbourhood V of 0 with $y_n \notin V$ for all n.

Then $\lim_{n\to\infty} P_m(y_n) = 0$ for all m. Hence, if $m_1 = 1$, we may choose increasing sequences (m_j), (n_j) with the properties:

(a) $y_{m_j} - P_{n_j}y_{m_j} \in \frac{1}{3}V$,

(b) $P_{n_j}y_{m_{j+1}} \in \frac{1}{3}V$.

Hence, if $z_j = P_{n_j}y_{m_j} - P_{n_{j-1}}y_{m_j}$, where $n_0 = 0$, then $z_j \notin \frac{1}{3}V$. Since (x_n) is simple and (y_n) is bounded, then (z_j) is bounded and hence a normalized block basic sequence.

If (x_n) is an equi-Schauder basis for E, and τ and $\sigma\gamma(E,E')$ (the topology of uniform convergence on the sets $\{P'_nf\colon n = 1,2,\ldots\}$ for $f \in E'$; see [**4**]), define the same normalized block basic sequences, then they define the same convergent sequences. For if not, the sequence (y_n) of the lemma may be chosen such that $y_n \to 0$ in $\sigma\gamma(E,E')$, and since $(P_n)_{n=1}^{\infty}$ is $\sigma\gamma(E,E')$-equicontinuous, it follows that $z_n \to 0$ in $\sigma\gamma(E,E')$, and so (z_n) is not $\sigma\gamma(E,E')$-normalized.

Theorem 5.6. *Let E be sequentially complete and let (x_n) be a Schauder basis of E; then the following are equivalent.*

(i) *(x_n) is completely abnormal.*

(ii) *Every Schauder basic sequence in E is abnormal.*

(iii) *E is a semi-Montel space (i.e. every bounded set is relatively compact).*

Proof. Suppose E is a semi-Montel space and (y_n) is a bounded Schauder basic sequence in E. Then (y_n) has a cluster point y and $y \in \overline{\mathrm{lin}}(y_n)$; thus $y = \sum_{i=1}^{\infty} g_i(y)y_i$ where (g_i) is the dual sequence of (y_i), and $g_i(y)$ is a cluster point of $g_i(y_n)$. Hence $g_i(y) = 0$ for all i, and $y = 0$; so (y_n) is not normalized. Therefore (iii) implies (ii), and (ii) obviously implies (i).

Suppose (x_n) is completely abnormal; since E is sequentially complete, (x_n) is simple and, by Lemma 5.5, the original topology and the weak topology define the same convergent sequences. Using Theorem 6 of [**6**], we see that τ and $\sigma(E,E')$ define the same compact sets.

Suppose that (α_n) is a sequence of scalars such that the sequence $\left(\sum_{i=1}^{n} \alpha_i x_i\right)_{n=1}^{\infty}$ is bounded, but does not converge. Since E is sequentially complete, there exists a neighbourhood V of the origin and an increasing sequence (n_j) of integers with $n_0 = 0$ such that

$$y_j = \sum_{n_{j-1}+1}^{n_j} \alpha_i x_i \notin V.$$

Then (y_j) is a normalized block basic sequence, contradicting the complete abnormality of (x_n).

Thus (x_n) is γ-complete or boundedly complete, and, by Theorem 5.4, is also shrinking; hence, by a theorem of [**8**], E is semi-reflexive, i.e. the bounded sets of E are weakly relatively compact. However, τ and $\sigma(E, E')$ define the same compact sets, and so E is a semi-Montel space.

6. Affinely equivalent basic sequences

If (x_n) is a basis for E_0, (y_n) is an equivalent basis for E_1, and E_0 and E_1 are F-spaces, then there exists an isomorphism $T\colon E_0 \to E_1$ with $Tx_i = y_i$ for each i (see [**1**]); thus (x_n) is normalized if and only if (y_n) is normalized.

In [**7**] Pełczyński and Singer introduce the notion of affine equivalence: (x_n) is *affinely equivalent* to (y_n) if there exists a sequence of scalars $\alpha_n \neq 0$ such that (x_n) and $(\alpha_n y_n)$ are equivalent. It follows at once that, if (x_n) is affinely equivalent to (y_n) and (x_n) is normal, then so is (y_n).

The two main theorems on equivalence and affine equivalence of bases in Banach spaces are Theorems 6.1 and 6.2 due to Pełczyński and Singer ([**7**]).

THEOREM 6.1. *A Banach space with a basis possesses two non-equivalent normalized bases, one of which is conditional.*

THEOREM 6.2. *Let E be a Banach space with an unconditional basis such that all unconditional basic sequences are affinely equivalent. Then $E \cong l^2$.*

Combining these with the results of § 5, we obtain similar results for F-spaces.

THEOREM 6.3. *Let E be an F-space with a basis such that all basic sequences in E are affinely equivalent. Then either E is a Montel space or E is a Banach space with no unconditional basis.*

Proof. It is in fact only necessary to assume that all block basic sequences with respect to the given basis are affinely equivalent; for the basis is then either completely normal or completely abnormal. In the first case E is a Banach space by Theorem 5.3 and by Theorem 6.1 has no unconditional basis; in the second case by Theorem 5.6 E is a Montel space.

THEOREM 6.4. *Let E be an F-space with an unconditional basis such that all unconditional basic sequences are affinely equivalent. Then either E is a Montel space or $E \cong l^2$.*

Proof. This follows in exactly the same way by Theorems 5.3, 5.6, and 6.2.

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