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INTERPOLATION OF COMPACT OPERATORS BY THE METHODS OF CALDERÓN AND GUSTAVSSON–PEETRE

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Let $\mathbf{X}=(X_0,X_1)$ and $\mathbf{Y}=(Y_0,Y_1)$ be Banach couples and suppose $T:\mathbf{X}\to\mathbf{Y}$ is a linear operator such that $T:X_0\to Y_0$ is compact. We consider the question whether the operator $T:[X_0,X_1]_\theta\to[Y_0,Y_1]_\theta$ is compact and show a positive answer under a variety of conditions. For example it suffices that X_0 be a UMD-space or that X_0 is reflexive and there is a Banach space so that $X_0=[W,X_1]_\alpha$ for some $0<\alpha<1$.

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1. Introduction

Let $\mathbf{X}=(X_0,X_1)$ and $\mathbf{Y}=(Y_0,Y_1)$ be Banach couples and let T be a linear opeator such that $T:\mathbf{X}\to\mathbf{Y}$ (meaning, as usual, that $T:X_0+X_1\to Y_0+Y_1$ and $T:X_j\to Y_j$ boundedly for $j=0,1$). Interpolation theory supplies us with a variety of *interpolation functors F* for generating *interpolation spaces*, i.e. functors F which when applied to the couples $\mathbf{X}$ and $\mathbf{Y}$ yield spaces $F(\mathbf{X})$ and $F(\mathbf{Y})$ having the property that each T as above maps $F(\mathbf{X})$ into $F(\mathbf{Y})$ with bound

$$\|T\|_{F(\mathbf{X})\to F(\mathbf{Y})}\leqq C\max(\|T\|_{X_0},\|T\|_{X_1})$$

for some absolute constant C depending only on the functor F. It will be convenient here to use the customary notation $\|T\|_{\mathbf{X}\to\mathbf{Y}}=\max(\|T\|_{X_0},\|T\|_{X_1})$. Further general background about interpolation theory and Banach couples can be found e.g. in [1], [3] or [5].

In this paper we shall be concerned with the following question.

Question 1. *Suppose that the operator $T:\mathbf{X}\to\mathbf{Y}$ also has the property that $T:X_0\to Y_0$ is compact. Let F be some interpolation functor. Does it follow that $T:F(\mathbf{X})\to F(\mathbf{Y})$ is compact?*

The first positive answer to a question of this type was given by Krasnolsel'skiĭ [17]

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in 1960 in the special context of L_p spaces. Since then Question 1 has been answered in the affirmative in many particular cases:

In 1964 Calderón [8, Section 9.6, 10.4] gave partial results for the case where F is the functor $F(\mathbf{X})=[X_0,X_1]_\theta$ of his complex method [8]. In the same year Lions–Peetre [18, pp. 36–37] obtained results which apply under suitable conditions to the complex method, to their real method, and to certain other methods also.

In 1992 one of us gave the complete answer for the real method [12], using results and methods suggested by the work of Hayakawa [15] and Cobos–Peetre [10].

Question 1 is still open in the case where F is the functor of Calderón's complex method [8]. Among the partial solutions which have been given to date, in addition to the work of Calderón referred to above, we mention results of Persson [21], Cwikel [12], Cobos–Kuhn–Schonbek [9] and a forthcoming paper of Mastyło [19]. In this paper we present some further partial results for this functor. We are able to answer Question 1 in the affirmative in each of the following four cases:

(i) if X_0 has the UMD property,

(ii) if X_0 is reflexive and is given by $X_0=[W,X_1]_\alpha$ for some Banach space W and some $\alpha\in(0,1)$,

(iii) if Y_0 is given by $Y_0=[Z,Y_1]_\alpha$ for some Banach space Z and some $\alpha\in(0,1)$,

(iv) if X_0 and X_1 are both complexified Banach lattices of measurable functions on a common measure space.

Our result (iv) strengthens Theorem 3.2 of [9] where both $\mathbf{X}$ and $\mathbf{Y}$ are required to be such couples of complexified lattices with some other mild requirements. We obtain (iv) as a corollary of the result that $T:\langle X_0,X_1,\theta\rangle\to[Y_0,Y_1]_\theta$ is compact for arbitrary Banach couples $\mathbf{X}$ and $\mathbf{Y}$. Here $\langle X_0,X_1,\theta\rangle=G_{3,\theta}(\mathbf{X})$ denotes the interpolation space defined by Gustavsson–Peetre [13] and characterized as an orbit space by Janson [16]. Mastyło [19] has obtained an alternative proof of (iv) as a consequence of other results of his which answer Question 1 in the cases where F is the Gustavsson–Peetre functor, or other related functors introduced by Peetre and by Ovchinnikov.

We now recall the definitions of the main interpolation functors to be used in this paper:

1. Calderón's complex method $[\cdot,\cdot]_\theta$

For each Banach couple $\mathbf{X}=(X_0,X_1)$ we let $\mathscr{H}=\mathscr{H}(\mathbf{X})$ denote the space of all X_0+X_1-valued functions which are analytic on the open annulus $\Omega=\{z:1<|z|<e\}$ and continuous on the closure of Ω. This space is normed by $\|f\|_{\mathscr{H}}=\max_{z\in\Omega}\|f(z)\|_{X_0+X_1}$.

The space $\mathscr{F}=\mathscr{F}(\mathbf{X})$ is defined to be the subspace of $\mathscr{H}$ which consists of those functions f which are X_0-valued and X_0-continuous on the circle $|z|=1$ and X_1-valued and X_1-continuous on the circle $|z|=e$. We define $\|f\|_{\mathscr{F}}=\max_{j=0,1}(\max_{|z|=e^j}\|f(z)\|_{X_j})$. For each $\theta\in[0,1]$ the interpolation space $[X_0,X_1]_\theta$ generated by Calderón's complex method is the set of all elements $x\in X_0+X_1$ of the form $x=f(e^\theta)$ where $f\in\mathscr{F}$. Its norm is given by $\|x\|_{X_\theta}=\inf\{\|f\|_{\mathscr{F}}:f\in\mathscr{F},f(e^\theta)=x\}$. In fact this definition differs slightly from the one given in Calderón's classical paper [8] where the unit strip $\{z: 0<\Re z<1\}$ replaces the annulus Ω but, as shown in [11], the two definitions coincide to within equivalence of norms.

We will sometimes use the notation $X_\theta=[X_0,X_1]_\theta$ when there is no danger of confusion. Actually this could be ambiguous for the (sometimes forgotten) endpoint values $\theta=j=0,1$ since then $[X_0,X_1]_j$ is the closure of $X_0\cap X_1$ in X_j. (see [3, Theorem 4.2.2, p. 91] or [8, Sections 9.3 and 29.3, pp. 116, 133–4].)

The couple $\mathbf{X}$ is said to be *regular* if $X_0\cap X_1$ is dense in X_0 and also in X_1. If $\mathbf{X}$ is regular then the dual spaces X_0^* and X_1^* also form a Banach couple. Calderón's duality theorem ([8, Sections 12.1, 32.1]) states that for regular couples and $\theta\in(0,1)$ the dual of $[X_0,X_1]_\theta$ coincides with the space $[X_0^*,X_1^*]^\theta$ obtained by applying a variant of Calderón's construction to the couple (X_0^*,X_1^*). We refer to [8] for the exact definition of this second Calderón method $[\cdot,\cdot]^\theta$.

2. Peetre's method $\langle\cdot,\cdot\rangle_\theta$

For each Banach couple $\mathbf{X}$ and each $\theta\in(0,1)$ the space $\langle X_0,X_1\rangle_\theta$ is the set of all elements $x\in X_0+X_1$ which are sums of the form $x=\sum_{k\in\mathbf{Z}}x_k$ where the elements $x_k\in X_0\cap X_1$ are such that $\sum_{k\in\mathbf{Z}}e^{-\theta k}x_k$ is unconditionally convergent in X_0 and $\sum_{k\in\mathbf{Z}}e^{(1-\theta)x}x_k$ is unconditionally convergent in X_1. $\langle X_0,X_1\rangle_\theta$ is normed by

$$\|x\|_{\langle X_0,X_1\rangle_\theta}=\inf\max_{j=0,1}\sup\left\|\sum_{k\in\mathbf{Z}}\lambda_k e^{(j-\theta)k}x_k\right\|_{X_j}$$

where the supremum is taken over all complex valued sequences (λ_k) with $|\lambda_k|\leqq 1$ for all k, and the infimum is taken over all representations as above $x=\sum_{k\in\mathbf{Z}}x_k$. We refer to [20] and [16] for more details.

3. Gustavsson–Peetre's method $\langle\cdot,\cdot,\theta\rangle$

This is a variant of Peetre's method (see [13] and [16]). The space $\langle X_0,X_1,\theta\rangle$ is defined like $\langle X_0,X_1\rangle_\theta$ except that the series $\sum_{k\in\mathbf{Z}}e^{(j-\theta)k}x_k$ need only be *weakly* unconditionally Cauchy in X_j. The norm is accordingly given by

$$\|x\|_{\langle X_0,X_1,\theta\rangle}=\inf\max_{j=0,1}\sup\left\|\sum_{k\in F}\lambda_k e^{(j-\theta)k}x_k\right\|_{X_j}$$

where the supremum is over all λ_k's as before and over all subsets F of $\mathbf{Z}$.

Finally, we discuss the class of UMD-spaces. Let X be a Banach space and let $\mathbf{T}$ denote the unit circle with normalized Haar measure $dt/2\pi$. If $f\in L_2(\mathbf{T},\mathbf{X})$ we denote its Fourier coefficients

$$\hat{f}(k)=\int_0^{2\pi} e^{-ikt}f(e^{it})\frac{dt}{2\pi}.$$

Then the (formal) Fourier series of f is $f\sim\sum_{k\in\mathbf{Z}}\hat{f}(k)z^k$. We recall that X is a UMD-space if the vector-valued Reisz projection $\mathscr{R}: L_2(\mathbf{T},X)\to L_2(\mathbf{T},X)$ is bounded where $\mathscr{R}f\sim\sum_{k\geqq 0}\hat{f}(k)z^k$. In fact UMD-spaces were introduced by Burkholder in [6] with a different definition, but the above characterization follows from results of Burkholder [7] and Bourgain [4].

It is perhaps important to stress that although the condition of being a UMD-space is fairly stringent many of the well-known spaces used in analysis are in fact UMD. The spaces L_p and the Schatten ideals $\mathscr{C}_p$ for $1<p<\infty$ are UMD; further examples are reflexive Orlicz spaces and the Lorentz spaces $L(p,q)$ where $1<p,q<\infty$ (see [14]). The class of UMD-spaces is closed under quotients, duals and subspaces. All UMD-spaces are superreflexive but the converse is false even for lattices [4].

2. Some preliminary results

We will make repeated use of the following simple lemma.

Lemma 1. *Let X and Y be Banach spaces and suppose $T: X\to Y$ is a compact operator. Suppose (f_n) is a bounded sequence in $L_2(\mathbf{T},X)$. Let H be the subspace of all elements $y^*\in Y^*$ which satisfy*

$$\lim_{n\to\infty}\int_0^{2\pi}|\langle Tf_n(e^{it}),y^*\rangle|^2\frac{dt}{2\pi}=0.$$

Suppose H is weak-dense in Y^* (i.e. H separates the points of Y). Then*

$$\lim_{n\to\infty}\int_0^{2\pi}\|Tf_n(e^{it})\|_Y^2\frac{dt}{2\pi}=0.$$

Proof. Let (y_m^*) be a sequence in $H\cap B_{Y^*}$ such that $(T^*y_m^*)_{m=1}^\infty$ is norm dense in $T^*(H\cap B_{Y^*})$. Then any bounded sequence (x_n) in X such that $\lim_{n\to\infty}\langle Tx_n,y_m^*\rangle=0$ for each m must satisfy $\lim_{n\to\infty}\langle Tx_n,y^*\rangle=0$ for all $y^*\in H$. Consequently, by compactness, $\lim\|Tx_n\|_Y=0$. From this it follows easily that for every $\varepsilon>0$ there exists a constant $C=C(\varepsilon)$ such that

$$\|Tx\|_Y^2\leqq\varepsilon\|x\|_X^2+C\sum_{m=1}^\infty 2^{-m}|\langle Tx,y_m^*\rangle|^2$$

for every $x\in X$. Now the lemma follows easily. □

We shall need the following properties of the complex interpolation spaces X_θ, most of which are well known.

Lemma 2. (i) *For each* $0<\theta<1$ *there is a constant* $C=C(\theta)$ *such that, for all* $f\in\mathscr{F}$,

$$\|f(e^{\theta})\|_{X_\theta}\leqq C\left(\int_0^{2\pi}\|f(e^{it})\|_{X_0}\frac{dt}{2\pi}\right)^{1-\theta}\left(\int_0^{2\pi}\|f(e^{1+it})\|_{X_1}\frac{dt}{2\pi}\right)^{\theta}. \tag{1}$$

In particular, for all $x\in X_0\cap X_1$,

$$\|x\|_{X_\theta}\leqq C\|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}. \tag{2}$$

(ii) *For each* $0<\theta<1$, $X_0\cap X_1$ *is a dense subspace of* X_θ.

(iii) *Let* X_j^{o} *denote the closed subspace of* X_j *generated by* $X_0\cap X_1$. *Then, for all* $\theta\in[0,1]$,

$$[X_0^o,X_1^o]_\theta=[X_0,X_1^o]_\theta=[X_0^o,X_1]_\theta=[X_0,X_1]_\theta$$

(iv) (*reiteration formulae*)

$$[[X_0,X_1]_{\theta_0},[X_0,X_1]_{\theta_1}]_\sigma=[X_0,X_1]_s \tag{3}$$

with equivalence of norms, for each θ_0, θ_1 *and* σ *in* $[0,1]$, *where* $s=(1-\sigma)\theta_0+\sigma\theta_1$. *Also*

$$[[X_0,X_1]_{\theta_0},X_1]_\sigma=[X_0,X_1]_{(1-\sigma)\theta_0+\sigma} \tag{4}$$

and

$$[X_0,[X_0,X_1]_{\theta_1}]_\sigma=[X_0,X_1]_{\sigma\theta_1}. \tag{5}$$

Proof. Part (i) follows easily from the above-mentioned equivalence of complex interpolation in the annulus with complex interpolation in the unit strip, by applying the estimate (ii) of [**8**, Section 9.4, p. 117] to the function $F(z)=f(e^z)e^{z^2}$.

For parts (ii) and (iii) we refer to [3, Theorem 4.2.2, p. 91] or [**8**, Sections 9.3 and 29.3, pp. 116, 133–4]. For part (iv) the formula (3) is proved in [**11**, pp. 1005–1006], and also in [**16**, Theorem 21, pp. 67–68]. Its variant (4) follows from (3) if $X_0\cap X_1$ is dense in X_1. But it can also be shown in general by slightly modifying Janson's proof of the reiteration formula ([**16**, Theorem 21, pp. 67–68]): One of the things to bear in mind for that proof is that simple estimates with the K-functional show that $[X_0,X_1]_{\theta_0}\cap X_1\subset[X_0,X_1]_{(1-\sigma)\theta_0+\sigma}$. (Cf. [**11**]). The proof of (5) is exactly analogous. □

For each $f\in\mathscr{H}$ we write $f(z)=\sum_{k\in\mathbb{Z}}\hat{f}(k)z^k$ and we let $\mathscr{R}f$ denote the analytic function on Ω defined by $\mathscr{R}f(z)=\sum_{k\geq 0}\hat{f}(k)z^k$. We set $\mathscr{R}_-f=f-\mathscr{R}f$. It is easy to see that f extends to an X_0+X_1-valued analytic function on the open disk $|z|<e$ and similarly $\mathscr{R}_-f$ is analytic on the open set $|z|>1$. It thus follows that $\mathscr{R}f$ extends to an element of $\mathscr{H}$ and that $\|\mathscr{R}f\|_{\mathscr{H}}\leqq C\|f\|_{\mathscr{H}}$ for some absolute constant C.

For each positive integer N and $f \in \mathscr{H}$ we define $\mathscr{S}_N f$ by the formula

$$\mathscr{S}_N(f) = \sum_{|k| \leqq N} \hat{f}(k) z^k + \sum_{N < |k| \leqq 2N} \left(2 - \frac{|k|}{N}\right) \hat{f}(k) z^k.$$

By the uniform L_1-boundedness of the de la Vallée Poussin kernels there exists a constant C such that $\|\mathscr{S}_N f\|_{\mathscr{F}} \leqq C\|f\|_{\mathscr{F}}$ for all $f \in \mathscr{F}$ and all $N > 0$.

Now let $\mathbf{Y} = (Y_0, Y_1)$ be another Banach couple and let $T: \mathbf{X} \to \mathbf{Y}$ be a linear operator with the further property that $T: X_0 \to Y_0$ is compact. We may assume that $\|T\|_{\mathbf{X} \to \mathbf{Y}} \leqq 1$. In fact T will be assumed to have these properties throughout the remainder of this paper.

Lemma 3. (*a*) *The set* $\{T\hat{f}(k): f \in B_{\mathscr{F}}, k \in \mathbf{Z}\}$ *is relatively compact in* Y_0.

(*b*) *We have* $\lim_{k \to \infty} \sup_{f \in B_{\mathscr{F}}} \|T\hat{f}(k)\|_{Y_0} = 0$.

(*c*) *For each* $\delta > 0$ *there exists an integer* $L = L(\delta)$ *so that for each* $f \in B_{\mathscr{F}}$ *the set* $\{k: \|T\hat{f}(k)\|_{Y_0} > \delta\}$ *has at most* L *members.*

(*d*) *For each* $0 < \theta < 1$ *we have*

$$\lim_{|k| \to \infty} \sup_{f \in B_{\mathscr{F}}} \|T\hat{f}(k) e^{k\theta}\|_{Y_0} = 0.$$

Proof. (a) We simply observe that

$$T\hat{f}(k) = T\left(\int_0^{2\pi} f(e^{it}) e^{-ikt} \frac{dt}{2\pi}\right).$$

(b) Since $T: X_0 \to Y_0$ is compact, there exists a function $\eta: [0, \infty) \to [0, \infty)$ with $\lim_{\delta \to 0} \eta(\delta) = \eta(0) = 0$ such that if $\|x\|_{X_0} \leqq 1$ then $\|Tx\|_{Y_0} \leqq \eta(\|x\|_{X_1})$ whenever $\|x\|_{X_1} < \infty$. Now for $f \in B_{\mathscr{F}}$ we have $\|\hat{f}(k)\|_{X_0} \leqq 1$ and $\|\hat{f}(k) e^k\|_{X_1} \leqq 1$. Hence $\|T\hat{f}(k)\|_{Y_0} \leqq \eta(e^{-k})$.

(c) Since T is compact we can pick a finite set of functionals $\{y_1^*, \ldots, y_N^*)$ in $B_{Y_0^*}$ such that for $x \in X_0$ we have

$$\|Tx\|_{Y_0} \leqq \max_{1 \leqq j \leqq N} |\langle Tx, y_j^* \rangle| + \frac{1}{2}\delta\|x\|_{X_0}.$$

Now suppose $f \in B_{\mathscr{F}}$ and that $A = \{k: \|T\hat{f}(k)\|_{Y_0} > \delta\}$. Then for each $k \in A$ we have

$$\sum_{j=1}^{N} |\langle T\hat{f}(k), y_j^* \rangle|^2 \geqq \frac{1}{4}\delta^2.$$

Summing over all k and applying Parseval's identity, we obtain

$$\sum_{j=1}^{N} \int_{0}^{2\pi} |\langle Tf(e^{it}), y_j^*\rangle|^2 \frac{dt}{2\pi} \geqq \frac{1}{4}\delta^2|A|.$$

Thus $|A| \leqq 4N\delta^{-2}$ from which the result follows immediately.

(d) First we observe, using (2), that

$$\|T\hat{f}(k)\|_{Y_\theta} \leqq C\|T\hat{f}(k)\|_{Y_0}^{1-\theta}\|T\hat{f}(k)\|_{Y_1}^{\theta} \leqq C\|T\hat{f}(k)\|_{Y_0}^{1-\theta}e^{-k\theta}$$

and so we obviously have from (b) that

$$\lim_{k\to\infty} \sup_{f\in B_{\mathscr{F}}} \|T\hat{f}(k)e^{k\theta}\|_{Y_\theta} = 0.$$

It remains to establish a similar result as $k \to -\infty$. Suppose then that this is false. Then we can find $\delta > 0$, a sequence $(f_n) \in B_{\mathscr{F}}$ and a sequence $k_n \to \infty$ such that $k_n > 2k_{n-1}$ and $\|T\hat{f}_n(-k_n)e^{-k_n\theta}\|_{Y_\theta} \geqq \delta$ for all n. Now, given n and any $\varepsilon > 0$, we can use (a) to find integers m and p such that $m > p \geqq n$ and $\|T(\hat{f}_m(-k_m) - \hat{f}_p(-k_p))\|_{Y_0} \leqq \varepsilon$. However $\|T\hat{f}_m(-k_m)e^{-k_m}\|_{Y_1} \leqq 1$ and $\|T\hat{f}_p(-k_p)e^{-k_m}\|_{Y_1} \leqq e^{k_p - k_m} \leqq 1$. Hence, again by (2),

$$\|T(\hat{f}_m(-k_m) - \hat{f}_p(-k_p))e^{-k_m\theta}\|_{Y_\theta} \leqq C\varepsilon^{1-\theta},$$

where C depends only on θ. It follows that

$$\|T\hat{f}_m(-k_m)e^{-k_m\theta}\|_{Y_\theta} \leqq C(\varepsilon^{1-\theta} + e^{(k_p - k_m)\theta}).$$

Hence

$$\delta \leqq C(\varepsilon^{1-\theta} + e^{-k_n\theta})$$

and this is a contradiction since $\varepsilon > 0$ and k_n are arbitrary. □

Lemma 4. *For $0 < \theta < 1$ and each fixed $N \in \mathbf{N}$ the set $\{\mathscr{S}_N Tf(e^\theta) : f \in B_{\mathscr{F}}\}$ is relatively compact in Y_θ.*

Proof. Suppose $f_n \in B_{\mathscr{F}}$; then by Lemma 3(a) we can pass to a subsequence (g_n) such that for $|k| \leqq 2N$ we have

$$\|T\hat{g}_n(k) - T\hat{g}_{n+1}(k)\|_{Y_0} \leqq 2^{-n}.$$

Thus

$$\|\mathscr{S}_N Tg_n(z) - \mathscr{S}_N Tg_{n+1}(z)\|_{Y_0} \leqq (4N+1)2^{-n}$$

for $|z| = 1$. Also, for $|z| = e$, we have

$$\|\mathscr{S}_N Tg_n(z) - \mathscr{S}_N Tg_{n+1}(z)\|_{Y_1} \leqq C_1$$

for some suitable constant C_1. Thus, by (1)

$$\|\mathscr{S}_N Tg_n(e^\theta)-\mathscr{S}_N Tg_{n+1}(e^\theta)\|_{Y_0}\leqq CC_1^\theta(4N+1)^{1-\theta}2^{-n(1-\theta)}$$

and so $\mathscr{S}_N Tg_n(e^\theta)$ is convergent. □

Let $\mathscr{E}$ be a subset of $\mathscr{F}$. We shall say that $\mathscr{E}$ is *effective* if it is bounded in $\mathscr{F}$ and if for some absolute constant λ and every $f\in\mathscr{E}$ and every $n\in\mathrm{N}$ we have $f-\mathscr{S}_n f\in\lambda\mathscr{E}$. For each $\theta\in(0,1)$ let $\mathscr{E}_\theta=\{f(\theta):f\in\mathscr{E}\}$. We shall say that $\mathscr{E}$ is *θ-effective* if it is effective and if $\mathscr{E}_\theta\cap\gamma B_{X_\theta}$ is norm dense in γB_{X_θ} for some positive constant γ (which may depend on θ). Of course $B_{\mathscr{F}}$ is θ-effective, but there are also clearly smaller sets with the same property, for example the set of those f in $B_{\mathscr{F}}$ with finitely many non-zero coefficients $\hat{f}(k)$. (Cf. [8, Section 9.2 and 29.2].)

Lemma 5. *Let $\mathscr{E}$ be an effective subset of $\mathscr{F}$ and let $\theta\in(0,1)$. Then the following conditions are equivalent:*

(*a*) *$T(\mathscr{E}_\theta)$ is a relatively compact subset of Y_θ.*

(*b*) *Every sequence (f_n) in $\mathscr{E}$ satisfies*

$$\lim_{n\to\infty}\|Tf_n(\varepsilon^\theta)-\mathscr{S}_n Tf_n(e^\theta)\|_{Y_\theta}=0.$$

If $\mathscr{E}$ is θ-effective then the preceding two conditions are also equivalent to

(*c*) *$T:X_\theta\to Y_\theta$ is compact.*

Proof. First suppose that (a) holds. If (f_n) is a sequence in $\mathscr{E}$ we observe that for a suitable constant C depending on θ we have

$$\|\mathscr{R}(f_n-\mathscr{S}_n f_n)(e^\theta)\|_{X_0+X_1}\leqq Ce^{-n(1-\theta)}\max_{|z|=e}\|f_n(z)\|_{X_0+X_1}.$$

Combining this with a similar estimate for $\mathscr{R}_-(f_n-\mathscr{S}_n f)(e^\theta)$ we have

$$\|f_n(e^\theta)-\mathscr{S}_n f_n(e^\theta)\|_{X_0+X_1}\leqq C(e^{-n(1-\theta)}+e^{-n\theta}).$$

Hence

$$\lim_{n\to\infty}\|Tf_n(e^\theta)-\delta_n Tf_n(e^\theta)\|_{Y_0+Y_1}=0.$$

Using the fact that $f_n(e^\theta)-\mathscr{S}_n f_n(e^\theta)\in\lambda\mathscr{E}_\theta$ for each n and condition (a) we deduce that we also have convergence in Y_θ, establishing (b).

Conversely, notice that if (b) holds then $\lim_{n\to\infty}\|Tf(e^\theta)-\mathscr{S}_n Tf(e^\theta)\|_{Y_\theta}=0$ uniformly for $f\in\mathscr{E}$. It then follows from Lemma 4 that the set $\{Tf(e^\theta):f\in\mathscr{E}\}$ is relatively compact in Y_θ and so (a) holds.

Obviously (c) implies (a). The reverse implication is also trivial whenever $\mathscr{E}$ is θ-effective. □

3. The main results

The following theorem will imply the compactness result when the domain space is $\langle X_0, X_1\rangle_\theta$ or $\langle X_0, X_1, \theta\rangle$ or when $\mathbf{X}$ is a couple of lattices.

Theorem 6. *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are Banach couples and that* $T:\mathbf{X}\to\mathbf{Y}$ *is such that* $T:X_0\to Y_0$ *is compact. Let* $\mathscr{E}$ *be the subset of* $B_{\mathscr{F}(\mathbf{X})}$ *consisting of those elements* f *for which the series* $\sum_{k\in\mathbf{Z}} e^{jk}\hat{f}(k)$ *converges unconditionally in* X_j *for* $j=0,1$ *and* $\|\sum_{k\in\mathbf{Z}}\lambda_k e^{jk}\hat{f}(k)\|_{X_j}<1$ *for every sequence of complex scalars* (λ_k) *with* $|\lambda_k|\leqq 1$ *for all* k. *Then* $T(\mathscr{E}_\theta)$ *is relatively compact in* Y_θ *for every* $0<\theta<1$.

Proof. We may suppose that $\|T\|_{X_j\to Y_j}\leqq 1$ for $j=0,1$. Consider an arbitrary sequence (f_n) in $\mathscr{E}$ such that $\hat{f}_n(k)=0$ for $|k|\leqq n$. Fix any $0<\theta<1$. Clearly $\mathscr{E}$ is effective, so by Lemma 5 it will suffice to show that $\lim_{n\to\infty}\|Tf_n(e^\theta)\|_{Y_\theta}=0$.

For any $N\in\mathbf{N}$ let us pick a subset $A_n(N)$ of $\mathbf{Z}$ so that $|A_n(N)|=N$ and $\|T\hat{f}_n(k)\|_{Y_0}\leqq \|T\hat{f}_n(l)\|_{Y_0}$ whenever $k\notin A_n(N)$ and $l\in A_n(N)$. Appealing to Lemma 3(d) we see that for any fixed N we must have

$$\lim_{n\to\infty}\left\|\sum_{k\in A_n(N)} T\hat{f}_n(k)e^{k\theta}\right\|_{Y_\theta}=0.$$

It is therefore possible to pick a non-decreasing sequence of integers N_n with $\lim N_n=\infty$ so that

$$\lim_{n\to\infty}\left\|\sum_{k\in A_n(N_n)} T\hat{f}_n(k)e^{k\theta}\right\|_{Y_\theta}=0.$$

We define $g_n(z)=\sum_{k\notin A_n(N_n)}\hat{f}_n(k)z^k$. Then it is easy to check that $g_n\in B_{\mathscr{F}}$. Further, if $b_n=\sup_{k\in\mathbf{Z}}\|T\hat{g}_n(k)\|_{Y_0}$, then $\lim_{n\to\infty}b_n=0$ by Lemma 3(c). It remains only to show that $\lim_{n\to\infty}\|Tg_n(e^\theta)\|_{Y_\theta}=0$.

To this end suppose $y^*\in B_{Y_\theta^*}$. Then,

$$\begin{aligned}\int_0^{2\pi}|\langle Tg_n(e^{it}),y^*\rangle|^2\frac{dt}{2\pi}&=\sum_{k\in\mathbf{Z}}|\langle T\hat{g}_n(k),y^*\rangle|^2\\&\leqq b_n\sum_{k\in\mathbf{Z}}|\langle T\hat{f}_n(k),y^*\rangle|\\&=b_n\sup_{|\lambda_k|\leqq 1}\left|\left\langle T\left(\sum_{k\in\mathbf{Z}}\lambda_k\hat{f}_n(k)\right),y^*\right\rangle\right|\\&\leqq b_n.\end{aligned}$$

Now by Lemma 1,

$$\lim_{n\to\infty} \int_0^{2\pi} \|Tg_n(e^{it})\|_{Y_0}^2 \frac{dt}{2\pi} = 0.$$

Finally we can appeal to Lemma 2(i) to obtain $\lim_{n\to\infty} \|Tg_n(e^{\theta})\|_{Y_\theta} = 0$. This completes the proof. □

Corollary 7. *For* **X**, **Y** *and* T *as above,*

(*a*) $T: \langle X_0, X_1 \rangle_\theta \to Y_\theta$ *is compact.*

(*b*) $T: \langle X_0, X_1, \theta \rangle \to Y_\theta$ *is compact.*

Furthermore if **X** *is a couple of complexified Banach lattices of measurable functions on some measure space then*
(*c*) $T: X_\theta \to Y_\theta$ *is compact.*

Proof. As pointed out in [**20**] and in [**16**], $\langle X_0, X_1 \rangle_\theta$ is contained in X_θ. More specifically, we observe that for each series $x = \sum_{k\in\mathbf{Z}} x_k$ arising in the definition of an element $x \in \langle X_0, X_1 \rangle_\theta$ we have $\lim_{N\to\infty} \sup \|\sum_{|k|\geq N} \lambda_k e^{(j-\theta)k} x_k\|_{X_j} = 0$ for $j = 0, 1$ where the supremum is over all choices of λ_k with moduli ≤ 1. Thus the function $f(z) = \sum_{k\in\mathbf{Z}} e^{-\theta k} x_k z^k$ is X_j-continuous on $|z| = e^j$ and so it is an element of $\mathscr{F}(\mathbf{X})$. Consequently $\mathscr{E}_\theta$ is the open unit ball of $\langle X_0, X_1 \rangle_\theta$. This immediately implies (a).

For (b) let x be an arbitrary element in the open unit ball of $\langle X_0, X_1, \theta \rangle$. Then there exists a representation $x = \sum_{k\in\mathbf{Z}} x_k$ for which the elements $u_N = x = \sum_{|k|\leq N} x_k$ are all in $\mathscr{E}_\theta$. So by Theorem 6 there exists a subsequence of (Tu_N) which converges in the norm of Y_θ to some element in the closure of $T(\mathscr{E}_\theta)$. Since $u_N \to x$ in $X_0 + X_1$ this element must be Tx and we deduce that (b) holds.

If **X** is a couple of complexified Banach lattices then $\langle X_0, X_1 \rangle_\theta = X_\theta$, as follows from [**8**, Section 13.6(ii), p. 125] and [**22**, Lemma 8.2.1, p. 453]. This of course establishes (c). □

Before providing the next theorem we will need a preliminary lemma.

Lemma 8. *Let* X *be a UMD-space and let* $V: X \to Y$ *be a compact linear operator for some Banach space* Y. *Then there exists a function* $\eta: [0,\infty) \to [0,\infty)$ *with* $\lim_{\delta\to 0} \eta(\delta) = \eta(0) = 0$, *such that* $\|\mathscr{R}_- V\phi\|_{L_2(\mathbf{T},Y)} \leq \eta(\|V\phi\|_{L_2(\mathbf{T},Y)})$ *for all* $\phi \in L_2(\mathbf{T}, X)$ *with* $\|\phi\|_{L_2(\mathbf{T},X)} \leq 2$.

Proof. If the result is false then there is a sequence (ϕ_n) for which $\|\phi_n\|_{L_2(\mathbf{T},X)} \leq 2$, $\lim \|V\phi_n\|_{L_2(\mathbf{T},Y)} = 0$ but such that for some $\varepsilon > 0$ we have $\|\mathscr{R}_- V\phi_n\|_{L_2(\mathbf{T},Y)} \geq \varepsilon$. However for all $y^* \in Y^*$ we have

$$\int_0^{2\pi} |\langle \mathscr{R}_- V\phi_n, y^* \rangle|^2 \frac{dt}{2\pi} \leq \int_0^{2\pi} |\langle V\phi_n, y^* \rangle|^2 \frac{dt}{2\pi}$$

and $\|\mathscr{R}_-\phi_n\|_{L_2(\mathbf{T},X)}$ is bounded by the UMD-property of X. Hence by Lemma 1 we obtain a contradiction. □

Theorem 9. *Suppose that* $\mathbf{X}=(X_0,X_1)$ *and* $Y=(Y_0,Y_1)$ *are Banach couples and* X_0 *is a UMD-space. Let* $T:\mathbf{X}\to\mathbf{Y}$ *be such that* $T:X_0\to Y_0$ *is compact. Then* $T:X_\theta\to Y_\theta$ *is compact for every* $0<\theta<1$.

Proof. Using Lemma 2(iii) we see that we may assume without loss of generality that both of the couples $\mathbf{X}$ and $\mathbf{Y}$ are regular. This ensures that the dual spaces also form Banach couples. In particular we will make use of the fact that $(Y_0+Y_1)^*=Y_0^*\cap Y_1^*$ (cf. [3, p. 32]) and so this space separates points of Y_0.

As in the proofs of preceding theorems, it will suffice to consider a sequence $f_n\in B_{\mathscr{F}}$, satisfying the conditions $\hat{f}_n(k)=0$ for $|k|\leq n$ and show that $\lim_{n\to\infty}\|Tf_n(e^\theta)\|_{Y_0}=0$. We may of course suppose as before that $\|T\|_{\mathbf{X}\to\mathbf{Y}}\leq 1$.

We first consider $\mathscr{R}f_n$. We note that for $|z|=1$ we have an estimate $\|\mathscr{R}f_n(z)\|_{X_1}\leq Ce^{-n}$ and, by the UMD-property of X_0, the sequence $\mathscr{R}f_n$ is bounded in $L_2(\mathbf{T},X_0)$. For each $y^*\in Y_1^*\cap Y_0^*$ we see that $\langle\mathscr{R}Tf_n,y^*\rangle$ is uniformly convergent to 0. So we can apply Lemma 1 to deduce that

$$\lim_{n\to\infty}\|\mathscr{R}Tf_n\|_{L_2(\mathbf{T},Y_0)}=0. \tag{6}$$

Let us fix $\varepsilon>0$. Since $T:X_0\to Y_0$ is compact and $X_0\cap X_1$ is dense in X_0 we can find a finite set $\{x_1,x_2,\dots,x_N\}$ in $B_{X_0}\cap X_1$ so that if $\|x\|_{X_0}\leq 1$ then there exists $1\leq j\leq N$ with $\|Tx-Tx_j\|_{Y_0}\leq\varepsilon$. Thus for each n we can find a measurable function $H_n:\mathbf{T}\to\{x_1,\dots,x_n\}$ so that $\|Tf_n(e^{it})-TH_n(e^{it})\|_{Y_0}\leq\varepsilon$ for all t. By convolving with a suitable kernel we can obtain a C^∞-function $h_n:\mathbf{T}\to F$ (where F is the linear span of $\{x_1,\dots,x_n\}$) so that $\|Tf_n(e^{it})-Th_n(e^{it})\|_{Y_0}\leq 2\varepsilon$ and $\|h_n(e^{it})\|_{X_0}\leq 1$ for all t. Let us expand h_n in its Fourier series

$$h_n(e^{it})=\sum_{n\in\mathbf{Z}}\hat{h}_n(k)e^{ikt}.$$

We will define

$$g_n(z)=\sum_{k\leq-(n+1)}\hat{h}_n(k)z^k$$

for $|z|\geq 1$. This defines an F-valued function which is analytic for $|z|>1$ and continuous for $|z|\geq 1$ since h_n is C^∞.

Now $\mathscr{R}_-f_n-g_n=z^{-n}\mathscr{R}_-(z^nf_n-z^ng_n)=z^{-n}\mathscr{R}_-(z^nf_n-z^nh_n)$. Also clearly the functions $\phi_n=z^nf_n-z^nh_n$ satisfy $\|\phi_n(e^{it})\|_{X_0}\leq 2$ and so $\|\phi_n\|_{L^2(\mathbf{T},X_0)}\leq 2$. Thus we can apply Lemma 8 to obtain that $\|\mathscr{R}_-Tf_n-Tg_n\|_{L^2(\mathbf{T},Y_0)}=\|\mathscr{R}_-T\phi_n\|_{L^2(\mathbf{T},Y_0)}\leq\eta(2\varepsilon)$ for some function η which depends only on T and satisfies $\lim_{\delta\to 0}\eta(\delta)=0$. Combining this with (6) we obtain that $\limsup_{n\to\infty}\|Tf_n-Tg_n\|_{L^2(\mathbf{T},Y_0)}\leq\eta(2\varepsilon)$.

Now consider (g_n) on the circle $|z|=e$. For a suitable constant C_1 we have

$$\|g_n(z)\|_{X_0} \leqq C_1 e^{-n} \max_{|\zeta|=1} \|h_n(\zeta)\|_{X_0} \leqq C_1 e^{-n}.$$

Since g_n is F-valued there is a constant C_2, depending only on F and thus on ε, such that $\|x\|_{X_1} \leqq C_2 \|x\|_{X_0}$ for all $x \in F$. Thus we have $\lim_{n\to\infty} \max_{|z|=e} \|g_n(z)\|_{X_1}=0$. From this we conclude that $\limsup_{n\to\infty} \max_{|z|=e} \|Tf_n(z)-Tg_n(z)\|_{Y_1} \leqq 1$.

Now we can deduce, using Lemma 2(i), that

$$\limsup_{n\to\infty} \|Tf_n(e^\theta)-Tg_n(e^\theta)\|_{Y_\theta} \leqq C_3(\eta(2\varepsilon))^\theta$$

for a constant C_3 which depends only on θ.

However we can also estimate $\|g_n(e^\theta)\|_{X_0} \leqq C_4 e^{-n\theta}$ and again using the fact that all g_n have range in F we have $\lim \|g_n(e^\theta)\|_{X_\theta}=0$. Thus we are left with the estimate

$$\limsup_{n\to\infty} \|Tf_n(e^\theta)\|_{Y_\theta} \leqq C_3(\eta(2\varepsilon))^\theta$$

Since $\varepsilon>0$ is arbitrary this completes the proof. □

Remark. See the introduction for a discussion of the class of (UMD)-spaces.

Theorem 10. *Let* $\mathbf{X}$ *be a Banach couple such that* X_0 *is reflexive and is given by* $X_0=[W,X_1]_\alpha$ *for some* $0<\alpha<1$ *and some Banach space* W *which forms a Banach couple with* X_1. *Suppose* $T:\mathbf{X}\to\mathbf{Y}$ *is such that* $T:X_0\to Y_0$ *is compact. Then* $T:X_\theta\to Y_\theta$ *is compact for* $0<\theta<1$.

Proof. Letting the notation X^o now mean the closure of $W\cap X_1$ in X, we observe that $[W^o,X_1^o]_\delta=[W,X_1]_\delta$ for all $\delta\in(0,1)$ (cf. Lemma 2(iii)). Consequently we may assume without loss of generality that $W\cap X_1$ is dense in W and also in X_1.

For each $\theta\in(0,1)$ we have

$$X_\theta=[[W,X_1]_\alpha,X_1]_\alpha,X_1]_\theta=[W,X_i]_\delta$$

where $\delta=(1-\theta)\alpha+\theta$. (Cf. Lemma 2(iv).)

Let $\mathscr{E}$ be the set consisting of all functions in $B_{\mathscr{F}(\mathbf{X})}$ which can be extended to functions f on the closed annulus $\{z: e^{-\beta} \leqq |z| \leqq e\}$ where $\beta = \alpha/(1-\alpha)$ in such a way that f is analytic into $W + X_1$ on the open annulus, $W+X_1$-continuous on the closed annulus, W-continuous on $|z| = e^{-\beta}$ and $\max_{|z|=e^{-\beta}} \|f(z)\|_W \leqq 1$. Clearly $\mathscr{E}$ is effective. Furthermore it is also θ-effective for every $\theta \in (0, 1)$. This can be shown readily using the above reiteration formula together with the observation [11] that the complex interpolation method yields the same spaces on annuli of different dimensions, even if they are not conformally equivalent. (The spaces defined using the strips $\{z: 0 \leqq \Re z \leqq 1\}$ and $\{z: -\beta \leqq \Re z \leqq 1\}$ are obviously identical. Now simply "periodize" the functions on both of these strips with period $2\pi i$ as in [11].)

We will apply Lemma 5. Consider a sequence $f_n \in \mathscr{E}$. Let $g_n = f_n - \mathscr{S}_n f_n$. Suppose $x^* \in W^* \cap X_1^*$. Then

$$\int_0^{2\pi} |\langle g_n(e^{it}), x^* \rangle|^2 \frac{dt}{2\pi} = \sum_{k \in \mathbf{Z}} |\langle \hat{g}_n(k), x^* \rangle|^2$$

$$\leqq \sum_{k \leqq -n} |\langle \hat{f}_n(k), x^* \rangle|^2 + \sum_{k \geqq n} |\langle \hat{f}_n(k), x^* \rangle|^2$$

$$\leqq C(e^{-2\beta n} \|x^*\|_{W^*}^2 + e^{-2n} \|x^*\|_{X_1^*}^2).$$

By our density assumption $W^* \cap X_1^* = (W + X_1)^*$ so this space separates points of $X_0 \subset W + X_1$. It follows that the set U of $x^* \in X_0^*$ such that

$$\lim_{n \to \infty} \int_0^{2\pi} |\langle g_n(e^{it}), x^* \rangle|^2 \frac{dt}{2\pi} = 0$$

is a closed weak* dense subspace of X_0^*. Since X_0 is reflexive $U = X_0^*$ and so $T^*(Y_0^*) \subset U$. By Lemma 1 we obtain that $\lim_{n \to \infty} \|Tg_n\|_{L_2(\mathbf{T}, Y_0)} = 0$ and then an application of Lemma 2(i) gives that $\lim_{n \to \infty} \|Tg_n(e^\theta)\|_{Y\theta} = 0$. □

Remark. The reader may care to note that if the preceding theorem can be proved without the requirement that X_0 is reflexive then Question 1 is completely answered for the complex method, by using the reduction of this problem given in [12, p. 339] to the case where $\mathbf{X} = (l_1(FL_1), l_1(FL_1(e^v))$ and $\mathbf{Y} = (l_\infty(FL_\infty), l_\infty(FL_\infty(e^v))$. In this case we can of course take $W = l_1(FL_1(e^{-\beta v})$ for β as above.

Here is a sort of "dual" result to Theorem 10. Note that it does not require any reflexivity conditions. But unfortunately it is still not sufficient to give a complete answer to Question 1 (cf. the preceding remark) since $[l_\infty(FL_\infty(e^{-\beta v}), l_\infty(FL_\infty(e^v)]_\alpha$ is strictly contained in $l_\infty(FL_\infty)$.

Theorem 11. *Suppose $T: \mathbf{X} \to \mathbf{Y}$ where $T: X_0 \to Y_0$ is compact. Suppose that for some*

Banach space Z, (Z, Y_1) *forms a Banach couple and* $Y_0=[Z, Y_1]_\alpha$ *for some* $\alpha\in(0,1)$. *Then* $T: X_\theta\to Y_\theta$ *is compact for each* $\theta\in(0,1)$.

Proof. We begin by showing that we can reduce the proof to the case where a number of density conditions are satisfied. First, using Lemma 2(iii) and rather similar reasoning to before, we can suppose without loss of generality that **X** is regular, and similarly, that $Y_0\cap Y_1$ is dense in Y_1. (The hypotheses already ensure that $Y_0\cap Y_1$ is dense in Y_0.) In fact we can furthermore suppose that $Z\cap Y_1$ is dense in Y_1, since if that were not so we could replace the couples $\mathbf{X}=(X_0, X_1)$ and $\mathbf{Y}=(Y_0, Y_1)$ by (X_0, Y_σ) and (Y_0, Y_σ) for some number $\sigma\in(\theta,1)$. By several applications of Lemma 2(iv) these latter couples also satisfy all the other required hypotheses of the theorem and we will be able to deduce the original desired conclusion for $T: X_\theta\to Y_\theta$ since $X_\theta=[X_0, X_\sigma]_{\theta/\sigma}$ and $Y_\theta=[Y_0, Y_\sigma]_{\theta/\sigma}$. Finally, given that all the above density conditions hold, we can now, if necessary, replace Z by Z^o, the closure of $Z\cap Y_1$ in Z without changing any of the other spaces. Also of course $Z^o\cap Y_1$ is dense in Y_1. In other words, we can also assume that $Z\cap Y_1$ is dense in Z.

Let $T^*:(Y_0^*+Y_1^*)\to(X_0^*+X_1^*)$ be the adjoint of $T: X_0\cap X_1\to Y_0\cap Y_1$. Clearly T^* maps Y_1^* to X_0^* boundedly to Y_0^* to X_0^* compactly. This means that $T^*: [Z^*, Y_1^*]_\alpha\to X_0^*$ is compact, since by Calderón's duality theorem $Y_0^*=[Z^*, Y_1^*]^\alpha$ and $[Z^*, Y_1^*]_\alpha$ is a closed subspace of $[Z^*, Y_1^*]^\alpha$. (See [2].) Thus the operator T^* satisfies all the hypotheses of Theorem 10, (T^* replaces T, Z^* plays the role of W, and instead of the original couples **X** and **Y** we have $\mathbf{Z}^*=([Z^*, Y_1^*]_\alpha, Y_1^*)$ and $\mathbf{X}^*=(X_0^*, X_1^*)$ respectively) except that $[Z^*, Y_1^*]_\alpha$ is not necessarily reflexive.

We now define $\mathscr{E}$ exactly analogously to the definition in the proof of Theorem 10, i.e. it is the subset of $B_{\mathscr{F}(\mathbf{Z}^*)}$ of functions which are extendable to $Z^*+Y_1^*$-valued continuous functions on the annulus $\{z: e^{-\beta}\leqq|z|\leqq e\}$ which are analytic in the interior of the annulus and are continuous into Z^*, respectively Y_1^* on the inner, respectively outer components of the boundary. Again we consider the sequence $g_n=f_n-\mathscr{S}_nf_n$, where f_n is an arbitrary sequence in $\mathscr{E}$.

This time we let U be the set of all $y\in Y_0$ such that

$$\lim_{n\to\infty}\int_0^{2\pi}|\langle y, g_n(e^{it})\rangle|^2\frac{dt}{2\pi}=0.$$

Using estimates similar to those in the proof of Theorem 10 we obtain that $Z\cap Y_1\subset U$. Since U must be closed in Y_0 it follows that $U=Y_0$. Consequently,

$$\lim_{n\to\infty}\int_0^{2\pi}|\langle x, T^*g_n(e^{it})\rangle|^2\frac{dt}{2\pi}=0$$

for all $x\in X_0$. We can now apply Lemma 1 to $T^*: [Z^*, Y_1^*]_\alpha\to X_0^*$ to obtain that $\lim_{n\to\infty}\|T^*g_n\|_{L_2(T, X_0^*)}=0$. Then Lemma 2 gives that $\lim_{n\to\infty}\|T^*g_n(e^\theta)\|_{[X_0^*, X_1^*]_\theta}=0$. By Lemma 5 we deduce that $T^*: [[Z^*, Y_1^*]_\alpha, Y_1^*]_\theta\to[X_0^*, X_1^*]_\theta$ is compact.

As already remarked above $[Z^*, Y_1^*]_\alpha$ is a closed subspace of Y_0^*. Furthermore, $[Z^*, Y_1^*]_\alpha$ contains $Z^* \cap Y_1^*]_\alpha$ densely and so obviously it is also the closure in Y_0^* of the larger space $[Z^*, Y_1^*]_\alpha \cap Y_1^*$. So Lemma 2(iii) yields that $[[Z^*, Y_1^*]_\alpha, Y_1^*]_\theta = [Y_0^*, Y_1^*]_\theta$.

Let z^* be an arbitrary element of the open unit ball of $[Y_0^*, Y_1^*]^\theta$. Thus $z^* = h'(\theta)$ where h is an element of the unit ball of the space $\bar{\mathscr{F}}(Y_0^*, Y_1^*)$ (of analytic functions on the *unit strip* as defined in [8]). If we set $h_n(z) = n\, e^{(z^2-1)/n}(h(z+1/n) - h(z))$, and $y_n^* = h_n(\theta)$ then it is easy to see that (y_n^*) is a sequence in the unit ball of $[Y_0^*, Y_1^*]_\theta$ which converges to z^* in $Y_0^* + Y_1^*$. (Cf. [11, p. 1006].) In view of the compactness of $T^*: [Y_0^*, Y_1^*]_\theta \to [X_0^*, X_1^*]_\theta$ we can suppose that (some subsequence of) the sequence $(T^* y_n^*)$ is Cauchy in $[X_0^*, X_1^*]_\theta$. Thus its limit in $[X_0^*, X_1^*]_\theta$ is also its limit in $X_0^* + X_1^*$, namely $T^* z^*$. This shows that T^* maps the unit ball of $[Y_0^*, Y_1^*]^\theta$ into a relatively compact subset of $[X_0^*, X_1^*]_\theta \subset [X_0^*, X_1^*]^\theta$. Consequently $T^*: [Y_0^*, Y_1^*]^\theta \to [X_0^*, X_1^*]^\theta$ is compact. This, together with Calderón's duality theorem and the classical Schauder theorem, completes the proof. □

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