AN ANALOGUE OF THE RADON-NIKODYM PROPERTY FOR NON-LOCALLY CONVEX QUASI-BANACH SPACES

by N. J. KALTON

(Received 1st November 1977)

1. Introduction

In recent years there has been considerable interest in Banach spaces with the Radon-Nikodym Property; see (1) for a summary of the main known results on this class of spaces. We may define this property as follows: a Banach space $X$ has the Radon-Nikodym Property if whenever $T \in \mathcal{L}(L_1, X)$ (where $L_1 = L_1(0, 1)$) then $T$ is differentiable i.e.

$$Tf = \int_0^1 f(x)g(x) \, dx$$

where $g : (0, 1) \to X$ is an essentially bounded strongly measurable function.

In this paper we examine analogues of the Radon-Nikodym Property for quasi-Banach spaces. If $0 < p < 1$, there are several possible ways of defining “differentiable” operators on $L_p$, but they inevitably lead to the conclusion that the only differentiable operator is zero. For example, a differentiable operator on $L_1$ has the Dunford-Pettis property; operators on $L_1$ with the Dunford-Pettis property map the unit ball of $L_1$ to a compact set (cf (12)). However any operator on $L_p$ ($p < 1$) with this property is zero (4).

Thus we define a quasi-Banach space $X$ to be $p$-trivial if $\mathcal{L}(L_p, X) = \{0\}$. The concept of $p$-triviality is then hoped to be an analogue of the Radon-Nikodym property amongst locally $p$-convex quasi-Banach spaces. It turns out that this hope is fulfilled to some extent. Our main results in Sections 4 and 5 demonstrate an analogue of Edgar’s theorem (2) and of the Phelps characterisation of the Radon-Nikodym Property ((1), (9)) to this setting. Precisely we show that a locally $p$-convex quasi-Banach space is $p$-trivial if and only if every closed bounded $p$-convex set is the closed $p$-convex hull of its “strongly $p$-extreme points”. Our analogue of Edgar’s theorem is that if $C$ is a bounded closed $p$-convex subset of a $p$-trivial quasi-Banach space then every $x \in C$ may be represented in the form

$$x = \sum_{n=1}^{\infty} a_n u_n$$

where $a_n \geq 0$, $\sum a_n^p = 1$, and each $u_n$ is a $p$-extreme point of $C$. We observe in this connection that a similar Choquet-type theorem for compact $p$-convex sets was proved in (5).

In our final Section 6 we briefly discuss the associated super-property. Here there
is a slight divergence between the Radon-Nikodym Property for Banach spaces and $p$-triviality for quasi-Banach spaces. A Banach space with the super-Radon-Nikodym property is super-reflexive (11); thus there is a space $X$ such that $\ell_1$ is not finitely representable in $X$ but which fails the Radon-Nikodym Property (3). However if $\ell_p$ ($0 < p < 1$) is not finitely representable in a quasi-Banach space then it is $p$-trivial.

2. Notation

A quasi-norm on a real vector space $X$ is a map $x \mapsto \|x\|$ such that

1. $\|x\| > 0$ if $x \neq 0$.
2. $\|tx\| = |t| \|x\|$, $t \in \mathbb{R}, x \in X$.
3. $\|x + y\| \leq k(\|x\| + \|y\|)$, $x, y \in X$.

where $k$ is the modulus of concavity of the quasi-norm. If $k = 1$, $\| \cdot \|$ is a norm. In general the quasi-norm is $r$-subadditive ($0 < r \leq 1$) if

4. $\|x + y\|^r \leq \|x\|^r + \|y\|^r$, $x, y \in X$.

The sets $\{x : \|x\| < \alpha\}$ define the base of neighbourhoods for a Hausdorff vector topology on $X$. If $X$ is complete, we say that $X$ is a quasi-Banach space; if the quasi-norm is also $r$-subadditive then $X$ is an $r$-Banach space.

The Aoki-Rolewicz theorem (10, p. 57) asserts that every quasi-norm is equivalent to a quasi-norm which is $r$-subadditive for some $r > 0$. Here $\| \cdot \|$ and $\| \cdot \|^*$ are equivalent if there exists $0 < m \leq M < \infty$ such that

$$m\|x\| \leq \|x\|^* \leq M\|x\|, \quad x \in X.$$  

A subset $C$ of $X$ is $p$-convex (where $0 < p \leq 1$) if given $x, y \in C$ and $0 \leq a, b \leq 1$ with $a^p + b^p = 1$, then $ax + by \in C$. Observe that if $0 < p < 1$ and $C$ is a closed $p$-convex set then $C$ contains 0. We say that $X$ is (locally) $p$-convex if there is a bounded $p$-convex neighbourhood of zero; this is equivalent to the existence of an equivalent $p$-subadditive quasi-norm on $X$.

If $C$ is a $p$-convex subset of $X$ then a point $x$ of $C$ is $p$-extreme if $x = a_1x_1 + a_2x_2$ with $x_1, x_2 \in X$ and $0 < a_1, a_2 < 1$, $a_1^p + a_2^p = 1$ implies that $x = x_1 = x_2$.

A point $x \in C$ is strongly $p$-extreme if whenever $y_n, z_n \in C$, $0 \leq a_n, b_n \leq 1$, $a_n^p + b_n^p = 1$ and $a_ny_n + b_nz_n \rightarrow x$ then $\max(a_n, b_n) \rightarrow 1$. According to our definition 0 is never strongly $p$-extreme, although it may well be $p$-extreme. We regard strongly $p$-extreme points as an analogue of denting points.

The set of $p$-extreme points of $C$ is denoted $\partial pC$. If $A$ is any set its $p$-convex hull is denoted by $\text{co}_pA$ and its closed $p$-convex hull by $\text{co}_p\overline{A}$.

3. $p$-trivial spaces

We define a quasi-Banach space $X$ to be $p$-trivial ($0 < p < 1$) if $\mathcal{L}(L_p, X) = \{0\}$, where $L_p = L_p(0, 1)$. As we observed in the introduction, this is the appropriate generalisation, to the case $p < 1$, of the Radon-Nikodym property for Banach spaces. In this section, we observe some examples of $p$-trivial quasi-Banach spaces.
Theorem 3.1. Suppose $X$ satisfies either of the following conditions:
(a) For any closed infinite-dimensional subspace $Y$ of $X$ there exists $q > p$ and a $q$-convex quasi-Banach space $Z$ such that $\mathcal{L}(Y, Z) \neq \{0\}$.
(b) For any closed infinite-dimensional subspace $Y$ of $X$ there exists an $F$-space and a non-zero compact linear operator $T : Y \to Z$.

Then $X$ is $p$-trivial.

Proof. We prove only (b). Suppose $S \in \mathcal{L}(L_p, X)$ and $S \neq 0$. Then $L_p^* = \{0\}$, $Y = \overline{S(L_p)}$ is infinite-dimensional. Let $T : Y \to Z$ be a non-zero compact operator on $Y$. Then $TS$ is a non-zero compact operator on $L_p$, contradicting the results of (4).

A quasi-Banach space $X$ is pseudo-dual if there exists a Hausdorff vector topology $\tau$ on $X$ such that the unit ball of $x$ is relatively compact (cf (8)).

Theorem 3.2. Let $X$ be a $p$-trivial quasi-Banach space and let $Y$ be a closed subspace of $X$ which is either $q$-convex for some $q > p$ or isomorphic to a pseudo-dual space. Then $X/Y$ is $p$-trivial.

Proof. In either case a linear operator $S : L_p \to X/Y$ may be lifted to a linear operator $\tilde{S} : L_p \to X$ (see (8)).

Theorem 3.3. Let $X$ be a quasi-Banach space, and let $Y$ be a closed $p$-trivial subspace of $X$ such that $X/Y$ is $p$-trivial.

Then $X$ is $p$-trivial.

Proof. Immediate.

Theorem 3.4. Let $X$ be a quasi-Banach space which possesses no infinite-dimensional subspace isomorphic to a Hilbert space. Then $X$ is $p$-trivial.

Proof. This is immediate from (4) Theorem 3.4.

The author has recently constructed a non-$p$-trivial space which is $p$-convex, but contains no copy of $L_p$; details will appear elsewhere.

Theorem 3.5. Suppose $X$ is a subspace of $L_p$. Then $X$ is $p$-trivial if and only if $X$ has no subspace isomorphic to $L_p$.

Proof. By the results of (6), if $T \in \mathcal{L}(L_p, L_p)$ and $T \neq 0$, there is a subspace $Y$ of $L_p$, such that $Y \subseteq L_p$ and $T|Y$ is an isomorphism.

4. Edgar's theorem for $p$-trivial spaces

Our first main result generalises Edgar's theorem (2) on Banach spaces with the Radon-Nikodym property.

Theorem 4.1. Suppose $0 < p < 1$ and that $X$ is a $p$-trivial quasi-Banach space.
Suppose $C$ is a closed bounded $p$-convex subset of $X$ and that $x \in C$. Then there exists a sequence $u_n \in \partial pC$ and $a_n \geq 0$ such that $\sum a_n^p \leq 1$ and

$$x = \sum_{n=1}^{\infty} a_n u_n.$$  

Proof. We shall assume the contrary and produce a contradiction. Let $\mathcal{B}$ be the $\sigma$-algebra of Borel subsets of $[0,1]$. For a sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{B}$ let $L_p(\mathcal{A})$ denote the closed subspace of $L_p[0,1] = L_p(\mathcal{B})$ of all $\mathcal{A}$-measurable functions. Let $\Omega$ denote the first uncountable ordinal. We shall construct, by transfinite induction, an increasing transfinite sequence of $\sigma$-algebras $\mathcal{B}_\alpha$ ($1 \leq \alpha < \Omega$) and of linear operators $T_\alpha : L_p(\mathcal{B}_\alpha) \rightarrow X$ such that

1. $\mathcal{B}_1 = \{[0,1], \emptyset\}$ and $T_1(c.1) = cx$ where 1 denotes the characteristic function of $[0,1]$.
2. If $\alpha < \beta$ and $f \in L_p(\mathcal{B}_\alpha)$ then $T_\beta f = T_\alpha f$.
3. If $f \in L_p(\mathcal{B}_\alpha)$, $f \geq 0$ and $\|f\|_p \leq 1$ then $Tf \in C$.
4. If $\epsilon_\alpha = \inf\{\sum_{n=1}^{\infty} \lambda(B_n) T_{\alpha} : B_n \in \mathcal{B}_\alpha; \bigcup_{n=1}^{\infty} B_n = [0,1]\}$ then $\{\epsilon_\alpha : 1 \leq \alpha < \Omega\}$ is strictly decreasing.

Of course if we can satisfy (1), (2), (3), (4) then we have an immediate contradiction since any well-ordered subset of $\mathbb{R}$ is countable.

Define $\mathcal{B}_1, T_1$ as above. Now suppose $1 < \alpha < \Omega$ and that $\mathcal{B}_\beta, T_\beta$ have been defined for $\beta < \alpha$. If $\alpha$ is a limit ordinal, let $\mathcal{B}_\alpha$ be the $\sigma$-algebra generated by $\bigcup (\mathcal{B}_\beta : \beta < \alpha)$. Since

$$\|T_\beta\| \leq 2k \sup_{y \in \mathcal{C}} \|y\| \quad \beta < \alpha$$

we can define $T_\alpha$ to be the unique extension of each $T_\beta$ to $L_p(\mathcal{B}_\alpha)$. Then conditions (2), (3), (4) are immediate.

Next suppose $\alpha = \gamma + 1$. Let $(B_j : j \in J)$ be a maximal family of disjoint atoms of $\mathcal{B}_\gamma$, i.e., $\lambda(B_j) > 0$ and $B \in \mathcal{B}_\gamma$, $B \subset B_j$ implies either $\gamma(B) = \lambda(B)$ or $\lambda(B) = 0$. $J$ is at most countable; let $B^* = [0,1] \setminus \bigcup J B_j$. Since $X$ is $p$-trivial we have $T_\gamma|L_p(B^*, \mathcal{B}_\gamma) = 0$.

Let $a_j = \lambda(B_j)^{1/p}$ and $v_j = a_j^{-1} T_\gamma 1_{B_j} (j \in J)$. Then

$$x = T_\gamma(1) = \sum_{j \in J} a_j v_j.$$  

Hence by assumption there exists $i$ such that $v_i \notin \partial_p C$ i.e.,

$$v_i = su + tw$$

where $u, w \in C$, $s, t > 0$ and $s^p + t^p = 1$.

Choose $A \in \mathcal{B}$ such that $A \subset B^*_j$, and $\lambda(A) = (sa_j)^p$.

Let $\mathcal{B}_\alpha$ be the $\sigma$-algebra generated by adjoining $A$ to $\mathcal{B}_\gamma$. Extend $T_\gamma$ by defining

$$T_\alpha 1_A = sa_\mu.$$  

Then

$$T_\alpha 1_{B^*_j \setminus A} = ta_j w$$

and conditions (2), (3) follow easily. For (4), observe that
\[ \epsilon_r = \sum_{j \in J} a_j \]
while
\[ \epsilon_a = \sum_{j \neq i} a_j + (s + t)a_i < \epsilon_r \]

This completes the proof.

Remark. It is easy, given Theorem 4.1, to modify the representation of \( x \) so that \( \sum a_n^p = 1 \). This follows from the fact that \( 0 \in C \) (see (5) for the details).

5. Geometric characterisations of \( p \)-trivial spaces

Suppose that \( C \) is a bounded \( p \)-convex set with \( 0 \) as an interior point (this implies that \( X \) is \( p \)-convex). Denote by \( C_0 \) the interior of \( C \). Then if \( x \in C \) and \( 0 \leq t < 1 \), \( tx \in C_0 \). Let us define a function \( \varphi : C_0 \rightarrow \mathbb{R} \) by

\[ \varphi(x) = \inf \sum_{n=1}^{\infty} a_n \]

where the infimum is taken over all non-negative series \( \sum a_n \) such that \( \sum a_n^p = 1 \) and there exist \( u_n \in C_0 \) with

\[ x = \sum_{n=1}^{\infty} a_n u_n \]

Let us observe that the infimum may be taken instead over all non-negative series \( \sum a_n \) such that \( \sum a_n^p \leq 1 \) and

\[ x = \sum_{n=1}^{\infty} a_n u_n. \]

For if

\[ x = \sum_{n=1}^{\infty} a_n u_n \]

where \( a_n \geq 0 \) and \( \sum a_n^p \leq 1 \), then for any \( N \), we may write

\[ x = \sum_{n=1}^{\infty} a_n u_n + \alpha(0 + 0 + \cdots + 0) \]

where \( \alpha^p = N^{-1}(1 - \sum a_n^p) \), and there are \( N \) zero terms. Thus

\[ \varphi(x) \leq \sum a_n + N\alpha \]

\[ = \sum a_n + N^{1-1/p} \left( 1 - \sum a_n^p \right)^{1/p}. \]

Letting \( N \to \infty \) we see that

\[ \varphi(x) \leq \sum a_n \]
For $x \in C$, we define

$$\varphi^*_x(x) = \lim \inf_{y \to x} \varphi(y).$$

$$\varphi^*_x(x) = \lim \sup_{y \to x} \varphi(y).$$

Thus $\varphi^*_x$ is lower-semi-continuous and $\varphi^*_x$ is upper-semi-continuous on $C$ and $\varphi^*_x \leq \varphi^*_x$. Let

$$V = \{x \in C : \varphi^*_x(x) = 1\}$$

$$W = \{x \in C : \varphi^*_x(x) = 1\}.$$

Then $V$ is closed and $W$ is a $G_\delta$-set; also $W \subset V$. Clearly any member of $W$ is strongly $p$-extreme for $C$.

The following lemmas prepare our main theorem. We assume that $X$ is $p$-trivial.

**Lemma 5.1.** If $x \in C_0$, there exist $v_m \in V$ and $a_m \geq 0$ such that $\Sigma a_m^p \leq 1$ and $\Sigma a_n v_m = x$.

**Proof.** (cf. Theorem 4.1). Suppose $x \in C_0$. Let $B_1 = \{(0,1), 0\}$ and define $T_1 : L_p(\mathcal{B}_1) \to X$ by $T_1(c.1) = cx$ By induction we construct an increasing sequence of atomic sub-$\sigma$-algebras $\mathcal{B}_n$ of $\mathcal{B}$ and a sequence of linear operators $T_n : L_p(\mathcal{B}_n) \to X$ such that

1. $T_{n+1}|L_p(\mathcal{B}_n) = T_n$, $n \geq 2$,
2. $T_n\{f : f \in L_p(\mathcal{B}_n); f \geq 0, \|f\|_p \leq 1\} \subset C_0$.

Indeed suppose $\mathcal{B}_n$ has atoms $(B_j^i : j \in J)$ where $J$ is at most countable. Let $b_i = \lambda(B_j^i)^{1/p}, j \in J$ and

$$u_j = b_i^{-1}T_n1_{B_j^i}.$$

Then $u_j \in C_0$. Then write

$$u_j = \sum_{i=1}^\infty \alpha_i w_{ij}$$

where $w_{ij} \in C_0, \Sigma a_i = 1$

$$\sum_{i=1}^\infty a_{ij} \leq \frac{1}{2}(1 + \varphi(u_j))$$

(the sum may, of course, be finitely non-zero). Split each $B_j^i$ into atoms $\{B_{ij}^i : i = 1, 2, \ldots\}$ where $\lambda(B_{ij}^i) = a_{ij} b_i^j$. Let $\mathcal{B}_{n+1} = \sigma\{B_{ij}^i : j \in J, i = 1, 2, \ldots\}$ and define $T_{n+1}$ on $L_p(\mathcal{B}_{n+1})$ so that

$$T_{n+1}1_{B_{ij}^i} = b_j^{-1}a_{ij}^{-1}w_{ij}.$$
\[ x = \sum_{j \in \mathcal{A}} a_j v_j \]

where \( a_j = \lambda(B_j)^{1/p} \) and \( v_j = a_j^{-1} T1_{B_j} \). Clearly \( v_j \in C \) and \( \sum a_j^p \leq 1 \). It remains to show that \( v_j \in V \).

For each \( n \), let \( A_j^n \) be the atom of \( \mathbb{B}_n \) including \( B_i \). Then \( \bigcap_n A_j^n = B_i \); let \( z_j^n = \lambda(A_j^n)^{-1/p} T1_{A_j^n} \).

Then \( z_j^n \in C_0 \) and \( z_j^n \to v_i \). Now for each \( n \),

\[
\frac{1}{n}(1 + \varphi(z_j^n)) \geq (\lambda(B_i)/\lambda(A_j^n))^{1/p}
\]

and hence \( \varphi(z_j^n) \to 1 \). Thus \( v_j \in V \).

Since \( X \) is necessarily \( p \)-convex we can assume that the norm on \( X \) is \( p \)-subadditive. We also choose \( \delta > 0 \) such that \( \{ x : \| x \| \leq \delta \} \) is contained in \( C \).

**Lemma 5.2.** Suppose \( x \in C_0 \) and \( 0 \leq t < 1 \). Then

\[ \varphi(tx) \leq t\varphi_*(x). \]

**Proof.** Suppose \( \epsilon > 0 \), and

\[ \epsilon \leq \delta t^{-1}(1 - t^p)^{1/p}. \]

Then there exists \( u, \| u \| < \epsilon \) such that \( x - u \in C_0 \) and \( \varphi(x - u) < \varphi_*(x) + \epsilon \).

Hence

\[ x - u = \sum_{n=1}^{\infty} a_n x_n \]

where \( x_n \in C_0, a_n \geq 0, \sum a_n^p = 1 \) and

\[ \sum a_n \leq \varphi_*(x) + \epsilon. \]

Thus

\[ tx = \sum_{n=1}^{\infty} ta_n x_n + \frac{t\epsilon}{\delta \left( \frac{\delta u}{\epsilon} \right)} \]

and

\[ \sum t^p a_n^p + t^p \epsilon^p \delta^{-p} \leq 1. \]

By the remark at the beginning of the section,

\[ \varphi(tx) \leq t \left( \sum a_n \right) + t\epsilon \delta^{-1} \]

\[ \leq t(\varphi_*(x) + \epsilon) + t\epsilon \delta^{-1}. \]

Now let \( \epsilon \to 0 \),

\[ \varphi(tx) \leq t\varphi_*(x). \]
Lemma 5.3. Suppose $x_n \in C$, $a_n \geq 0$ and $\sum a_n^p \leq 1$. Then

$$\varphi(x) \left( \sum_{n=1}^{\infty} a_n x_n \right) \leq \sum_{n=1}^{\infty} a_n \varphi(x_n).$$

Proof. For $\epsilon > 0$, there exist $u_n \in C_0$ such that

$$\|x_n - u_n\| \leq \epsilon.$$ and

$$u_n = \sum_{k=1}^{\infty} c_{nk} v_{nk}$$

where $v_{nk} \in C_0$, $c_{nk} \geq 0$, $\sum c_{nk}^p \leq 1$ and

$$\sum_{k} c_{nk} \leq \varphi(x_n) + \epsilon.$$ Thus

$$\left\| \sum_{n=1}^{\infty} a_n x_n - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_n c_{nk} v_{nk} \right\| \leq \epsilon.$$ and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_n c_{nk} \leq \sum_{n=1}^{\infty} a_n (\varphi(x_n) + \epsilon) \leq \sum_{n=1}^{\infty} a_n \varphi(x_n) + \epsilon.$$ Letting $\epsilon \to 0$ we obtain the result.

Lemma 5.4. $W$ is dense in $V$.

Proof. Let

$$M = \sup_{x \in C} \|x\|.$$ Fix $n \in \mathbb{N}$ and let

$$W_n = \{ x \in C : \varphi(x) > 1 - 1/n \}.$$ Then $W_n$ is relatively open in $C$. We shall show that $W_n \cap V$ is dense in $V$. Fix $x \in V$ and $\epsilon > 0$.

Choose $\nu > 0$ such that

$$(1 - \nu)^{\frac{1}{p}} - \nu > 1 - 1/n.$$ and

$$\nu^p + M^p [\nu^p + (1 - (1 - \nu)^{1/p})^p + (1 - (1 - \nu)^{1/p^p})] < \epsilon^p.$$ Since $x \in V$ there exists $y \in C_0$ with
\[ \varphi(y) > 1 - \nu \]

and

\[ \|x - y\| < \nu. \]

Since \( y \in C_0 \), there exists \( \tau, 1 < \tau < 1 + \nu \) such that \( \tau y \in C_0 \) and then we have

\[ \varphi_\tau(y) > 1 - \nu \]

by Lemma 5.2.

Now by Lemma 5.1

\[ \tau y = \sum_{m=1}^{\infty} a_m v_m \]

where \( v_m \in V, a_m \geq 0 \) and \( \sum a_m = 1 \). Then by Lemma 5.3

\[ \sum_{m=1}^{\infty} a_m \varphi_\tau(v_m) > 1 - \nu \]

and in particular

\[ \sum a_m > 1 - \nu. \]

Suppose \( a_1 \geq a_2 \geq \ldots \); then

\[ a_1 > (1 - \nu)^{1/(1-p)} \]

and hence

\[ \sum_{m=2}^{\infty} a_m^p < 1 - (1 - \nu)^{p/(1-p)}. \]

Thus

\[ a_1 \varphi_\tau(v_1) > 1 - \nu - \sum_{m=2}^{\infty} a_m \]

\[ > (1 - \nu)^{p/(1-p)} - \nu. \]

In particular

\[ \varphi_\tau(v_1) > (1 - \nu)^{p/(1-p)} - \nu \]

so that \( v_1 \in W_n \); also

\[ \|x - v_1\|^p \leq \|x - y\|^p + \|\tau y - y\|^p + \|\tau y - v_1\|^p \]

\[ \leq \nu^p + \nu^p M^p + (1 - a_1)^p + \sum_{m=2}^{\infty} a_m^p M^p. \]

\[ = \nu^p + M^p \left[ \frac{\nu^p}{p} + \left(1 - (1 - \nu)^{1/(1-p)} \right)^p + 1 - (1 - \nu)^{p/(1-p)} \right] \]

\[ < \varepsilon^p. \]

Thus it follows that \( W_n \cap V \) is dense in \( V \). Since \( V \) is closed in \( X \) and \( W_n \cap V \) is relatively open in \( V \), we may deduce from the Baire Category Theorem that \( (\cap_{n=1}^{\infty} W_n) \cap V \) is dense in \( V \) i.e., \( W \) is dense in \( V \).
Lemma 5.5. \( C = \overline{\co_p W} \).

Proof. \( \overline{\co_p W} = \overline{\co_p V} \supseteq C_0 \) by Lemma 5.1. Since \( \overline{C_0} = C \), we have the result.

The next theorem is our main result of the section, and may be regarded as a \( p \)-convex analogue of the characterisation of the Radon-Nikodym property for Banach spaces given by Phelps (9 Theorem 5).

**Theorem 5.6.** Let \( X \) be a \( p \)-convex quasi-Banach space. Then \( X \) is \( p \)-trivial if and only if every closed bounded \( p \)-convex subset of \( X \) is the closed \( p \)-convex cover of its strongly \( p \)-extreme points.

Proof. Suppose \( X \) is not \( p \)-trivial and that \( T : L_p \to X \) is a bounded linear operator. Let \( U \) be the unit ball of \( L_p \) and consider \( 
ach T(U) \). Suppose \( x \) is strongly \( p \)-extreme for \( T(U) \). Then there exists \( \{ f_n \} \subseteq U \) with \( T(f_n) \to x \). However for each \( f_n \) we may write (by splitting the interval)

\[
\frac{1}{2} 2^{-\frac{1}{p}} T g_n + \frac{1}{2} 2^{-\frac{1}{p}} T h_n \to x
\]

and so we have a contradiction. Hence \( \overline{T(U)} \) has no strongly \( p \)-extreme points.

Conversely suppose \( X \) is \( p \)-trivial and \( D \) is a closed bounded \( p \)-convex subset of \( X \). Let \( S \) be the set of strongly \( p \)-extreme points for \( D \).

Let \( B \) be the closed unit ball of \( X \) and for \( \delta > 0 \) let \( C = C_\delta = \overline{\co_p} (D \cup \delta B) \). Using the notation of the preceding lemmas, \( C = \overline{\co_p W} \). However \( W \) is contained in the set \( T_\delta \) of strongly \( p \)-extreme points for \( C \).

Suppose \( x \in T_\delta \) and \( \|x\| > \delta \). Then there exist \( y_n \in D \) and, \( w_n \in \delta B, 0 \leq a_n \leq 1 \), such that

\[
a_n y_n + (1 - a_n)\|w_n\| \to x.
\]

Hence \( \max(a_n, (1 - a_n)^{1/p}) \to 1 \). It is easy to see that since \( \|x\| > \delta \) we have \( a_n \to 1 \) and hence \( x \in D \). This implies that \( x \in S \).

Now suppose \( z \in D \) and \( z \notin \overline{\co_p} S \). Let

\[
\delta = \frac{1}{2} d(z, \overline{\co_p} S) = \frac{1}{2} \inf\|z - v\|: v \in \overline{\co_p} S \).
\]

Then since \( \lambda \overline{\co_p} S \subseteq \overline{\co_p} S \) for \( 0 \leq \lambda \leq 1 \), we have \( z \notin \overline{\co_p} (S \cup \delta B) \).

However \( S \cup \delta B \supseteq T_\delta \) and hence \( z \notin \overline{\co_p} T_\delta \), and we have a contradiction.

**Corollary 5.7.** \( X \) is \( p \)-trivial if and only if every closed bounded \( p \)-convex set has a strongly \( p \)-extreme point.

6. Remarks on super-properties

For the purposes of this section we shall restrict our comments to quasi-Banach spaces \( X \) which have a quasi-norm which is \( r \)-subadditive for some \( r > 0 \). We say that a quasi-Banach \( Y \) is finitely representable in a quasi-Banach space \( X \) if given any
$\epsilon > 0$ and any finite-dimensional subspace $L$ of $Y$ there is a subspace $M$ of $X$ with
\[ \dim M = \dim L \] such that there is an isomorphism $T : L \to M$ with $\|T\|\|T^{-1}\| < 1 + \epsilon$.

If $(P)$ is a property of quasi-Banach spaces, then we say that $X$ has the property super-$\mathbb{P}$ if any space finitely representable in $X$ has property $(P)$.

Theorem 6.1. If $0 < p < 1$, the following conditions on $X$ are equivalent:

1. $X$ is super-$p$-trivial.
2. $\ell_p$ is not finitely representable in $X$.
3. $X$ is $q$-convex for some $q > p$.

Proof. $(2) \Leftrightarrow (3)$ is proved in (7). $(3) \Rightarrow (1)$ is obvious. For $(1) \Rightarrow (2)$ observe that if $\ell_p$ is finitely representable in $X$ then so is $L_p$.

The interest in the above theorem is that the analogy with the Radon-Nikodym Property breaks down at this point. Pisier (11) has shown that $X$ has the super-Radon-Nikodym property if and only if $X$ is super-reflexive. An example of James (3) shows that this is not the same as "$\ell_1$ is not finitely representable in $X$" (i.e., $X$ is $B$-convex).

The author is grateful to the referee for the following comments.

From our remarks in the introduction, the class of $p$-trivial spaces may also be regarded as a generalisation to quasi-Banach spaces of the class of Banach spaces $X$ such that every $T \in \mathcal{L}(L_p, X)$ has the Dunford-Pettis property. This class is strictly larger than the class of spaces with the Radon-Nikodym Property.

The referee also calls our attention to a paper of W. Fischer and U. Scholer (13) who study a (different) generalisation of the Radon-Nikodym Property in quasi-Banach spaces. It is not clear at present how their work relates to the content of this paper.

REFERENCES


(3) R. C. JAMES, A nonreflexive space which is uniformly nonoctrahedral, Israel J. Math. 18 (1974), 145–155.


(6) N. J. KALTON, The endomorphisms of $L_p$, $0 \leq p \leq 1$, Indiana Univ. Math J. 27 (1978), 353–381.


(8) N. J. KALTON and N. T. PECK, Quotients of $L_p(0, 1)$, $0 \leq p < 1$, Studia Math. to appear.


DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY COLLEGE OF SWANSEA
Singleton Park
Swansea SA2 8PP

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING
MICHIGAN 48824
U.S.A.