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Differential games with optional stopping

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1. *Introduction*. Consider a differential game of survival governed by the differential equation

$$\frac{dx}{dt} = f(t, x, y, z),$$
$$x(t_0) = x_0$$

in $\mathscr{R}^m$, with pay-off

$$P = \int_{t_0}^{t_F} h(t, x(t), y(t), z(t))\,dt + g(t_F, x(t_F)),$$

where t_F is the entry time of the trajectory $(t, x(t))$ into a given terminal set F. Under suitable conditions on f, g, h and the terminal set F, it was shown in (3) that the question of existence of value of such a game can be approached by considering a certain pair of partial differential equations called the Isaacs–Bellman equations.

Suppose for all $p \in \mathscr{R}^m$ we have

$$\min_{z\in Z} \max_{y \in Y} (p.f + h) = \max_{y\in Y} \min_{z \in Z} (p.f+h)$$
$$= H(t, x, p)$$

(the Isaacs condition), then the two Isaacs–Bellman equations reduce to one:

$$L\theta = \frac{\partial \theta}{\partial t} + H(t, x, \nabla\theta) = 0$$

in $\mathscr{R}^{m+1} - F$ and $\theta(t, x) = g(t, x)$ for $(t, x) \in \partial F$.

In (3) it was shown that if there exist C^1-functions ϕ and ψ on $\mathscr{R}^{m+1}$ such that

$$L\phi \geqslant 0 \geqslant L\psi$$

in $\mathscr{R}^{m+1} - F$, and

$$\phi(t, x) = \psi(t, x) = g(t, x) \tag{1}$$

for $(t, x) \in \partial F$, then the game has value V and furthermore $\phi(t_0, x_0) \leqslant V \leqslant \psi(t_0, x_0)$. Indeed only a local version of these conditions near ∂F is required. The main result of this paper is that if a one-sided version of (1) is available (for example if only ϕ can be determined to fit the conditions of (1)) then the game has extended value in the sense described in (2) (see also the work of Friedman (9), (10), (11) on generalized values). This extends the result of (2), both in dropping the 'reverse Isaacs condition' and also in the fact that a generalized pursuit-evasion automatically leads to a one-sided version of (1) with $\phi \equiv 0$.

It turns out that in order to obtain results on generalized values it is necessary to study another type of differential game, one in which we assume that the minimizer has the option of stopping the game at any time τ prior to the entry time into the terminal set t_F, and then accepting a pay-off depending on the time τ. Such games we call games of *optional stopping* and we develop for them a theory completely analogous to the theory developed in (3) for games of survival.

2. *Values in evolutionary games.* Suppose Y and Z are two fixed compact metric spaces. For $0 \leqslant t_0 < \infty$ we define $M_Y(t_0)$ as the set of all functions $y\colon [t_0, \infty) \to Y$ which are measurable in the sense that if $\phi \in C(Y)$ then $\phi \circ y$ is Lebesgue measurable; we identify functions equal almost everywhere in $M_Y(t_0)$. Similarly we define $M_Z(t_0)$. We also define, for $s > 0$, the set $M^s_Y(t_0)$ as the set of Lebesgue measurable functions $y\colon [t_0, t_0+s) \to Y$ (again identifying functions equal almost everywhere); we similarly define $M^s_Z(t_0)$.

A map $\alpha\colon M_Z(t_0) \to M_Y(t_0)$ is called a strategy for the player J_Y (who controls the Y-variable) if whenever
$$z_1(t) = z_2(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau$$
then
$$\alpha z_1(t) = \alpha z_2(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau.$$
The set of all such strategies is denoted by Γ_{t_0}. For $s > 0$ we define $\Gamma_{t_0}(s)$ as the set of all $\alpha \in \Gamma_{t_0}$ such that if
$$z_1(t) = z_2(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau$$
then
$$\alpha z_1(t) = \alpha z_2(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau + s.$$
We define $\Gamma_{t_0}(s|y)$for $y \in M^s_Y(t_0)$ as the set of $\alpha \in \Gamma_{t_0}(s)$ such that, in addition,
$$\alpha z(t) = y(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant t_0 + s$$
for any $z \in M_Z(t_0)$.

The corresponding sets for J_Z are denoted by Δ_{t_0}, $\Delta_{t_0}(s)$ and $\Delta_{t_0}(s|z)$.

Now let $P\colon M_Y(t_0) \times M_Z(t_0) \to \mathscr{R}$ be any pay-off function; then P determines an *evolutionary game* in which we assume that J_Y is the maximizer and J_Z the minimizer. For $\alpha \in \Gamma_{t_0}$ we define
$$u(\alpha) = \inf(P(\alpha z, z);\, z \in M_Z(t_0))$$
and for $\beta \in \Delta_{t_0}$
$$v(\beta) = \sup(P(y, \beta y);\, y \in M_Y(t_0)).$$
Then we define the following values:
$$V = \inf(v(\beta)\colon \beta \in \Delta_{t_0}),$$
$$U = \sup(u(\alpha)\colon \alpha \in \Gamma_{t_0}),$$
$$V^+ = \inf(v(\beta)\colon \beta \in \bigcup_{s>0} \Delta_{t_0}(s)),$$
$$V^- = \sup(u(\alpha)\colon \alpha \in \bigcup_{s>0} \Gamma_{t_0}(s)),$$
$$Q^+_s = \sup_{z \in M^s_Z(t_0)} \inf_{\beta \in \Delta_{t_0}(s|z)} v(\beta),$$
$$Q^-_s = \inf_{y \in M^s_Y(t_0)} \sup_{\alpha \in \Gamma_{t_0}(s|y)} u(\alpha),$$
$$Q^+ = \inf_{s>0} Q^+_s,$$
$$Q^- = \sup_{s>0} Q^-_s.$$

It is easy to verify that:

$$Q_s^+ \downarrow Q^+,$$
$$Q_s^- \uparrow Q^-,$$
$$Q^+ \geqslant V^+,$$
$$Q^- \leqslant V^-,$$
$$V^+ \geqslant U \geqslant V^-,$$
$$V^+ \geqslant V \geqslant V^-.$$

We say that the game has *value* if

$$V^+ = V^-$$

and *strong value* if

$$Q^+ = Q^-.$$

Throughout this paper the evolutionary game under consideration will be governed by a differential equation controlled by J_Y, and J_Z. In sections 2–6 we consider a particle whose position in $\mathscr{R}^{m+1}$ is given by co-ordinates (x, ξ) $(x \in \mathscr{R}^m,\ \xi \in \mathscr{R})$ controlled by an equation

$$\left.\begin{aligned} \frac{dx}{dt} &= f(t, x, y, z), \\ \frac{d\xi}{dt} &= h(t, x, y, z), \end{aligned}\right\} \tag{2}$$

where $$f\colon [0, \infty) \times \mathscr{R}^m \times Y \times Z \to \mathscr{R}^m$$

and $$h\colon [0, \infty) \times \mathscr{R}^m \times Y \times Z \to \mathscr{R}.$$

The functions f and h are assumed to be continuous and obey Lipschitz condition in (t, x) thus,

$$\left.\begin{aligned} \|f(t_1, x_1, y, z) - f(t_2, x_2, y, z)\| &\leqslant K_f(|t_1 - t_2| + \|x_1 - x_2\|), \\ \|h(t_1, x_1, y, z) - h(t_2, x_2, y, z)\| &\leqslant K_h(|t_1 - t_2| + \|x_1 - x_2\|). \end{aligned}\right\} \tag{3}$$

(We shall adopt the convention of labelling the Lipschitz constant of a function ϕ by K_ϕ.) For convenience we also make the assumption that both f and h vanish outside some fixed compact set $[0, T] \times \{x\colon \|x\| \leqslant R\} \times Y \times Z$; there is no loss of generality in this (see (1), section 9).

We also introduce the following Hamiltonian functions

$$\left.\begin{aligned} \tilde{H}^+(t, x, p, q) &= \min_{z \in Z} \max_{y \in Y} (p.f + qh), \\ \tilde{H}^-(t, x, p, q) &= \max_{y \in Y} \min_{z \in Z} (p.f + qh), \end{aligned}\right\} \tag{4}$$

for $(t, x) \in [0, \infty) \times \mathscr{R}^m \times \mathscr{R}$, and

$$\left.\begin{aligned} H^+(t, x, p) &= \tilde{H}^+(t, x, p, 1), \\ H^-(t, x, p) &= \tilde{H}^-(t, x, p, 1). \end{aligned}\right\} \tag{5}$$

3. *Fixed time games.* We now assume that the differential game is governed by (2) subject to $x(t_0) = x_0$, $\xi(t_0) = \xi_0$ and the pay-off is given by

$$P = g(x(T), \xi(T)) \tag{6}$$

where $T \geqslant t_0$, and $g: \mathscr{R}^{m+1} \to \mathscr{R}$ is continuous and satisfies a Lipschitz condition

$$|g(x_1, \xi_1) - g(x_2, \xi_2)| \leqslant K_g^x \|x_1 - x_2\| + K_g^\xi |\xi_1 - \xi_2|.$$

By Rademacher's Theorem g is differentiable almost everywhere and satisfies

$$\|\nabla g\| \leqslant K_\xi^x,$$
$$\left|\frac{\partial g}{\partial \xi}\right| \leqslant K_s^\xi.$$

In (3) it is shown that a game of this type satisfies $Q^+ = V^+$ and $Q^- = V^-$. The results of (1) show that the condition

$$\tilde{H}^+(t, x, p, q) \equiv \tilde{H}^-(t, x, p, q) \tag{7}$$

(which we shall call the *extended Issacs condition*) is sufficient to ensure that $V^+ = V^-$ (Theorem 9·2). We shall show in this section that if

$$\frac{\partial g}{\partial \xi} \geqslant 0 \quad \text{a.e.}$$

then (7) may be replaced by the *Isaacs condition*

$$H^+(t, x, p) \equiv H^-(t, x, p). \tag{8}$$

A similar problem was met in (2) where a condition similar to (7) was assumed; later Friedman (10) showed that (8) only was necessary, although his results referred to a slightly different concept of value. However, Friedman's arguments form the basis of this section.

Lemma 3·1. *Suppose* $\delta \geqslant 0$ *and*

(i) $H^+(t, x, p) - H^-(t, x, p) \leqslant A_1 \|p\| + A_2$,

(ii) $|h(t, x, y, z)| \leqslant B_h$,

(iii) $\dfrac{\partial g}{\partial \xi} \geqslant -\delta$ a.e.

for $t_0 \leqslant t \leqslant T$. *Then*

$$V^+ - V^- \leqslant e(T - t_0)\{B_h^\delta + A_2 K_g^\xi + A_1(K_f + K_h) \exp[2(K_f + K_h)(T - t_0)]\}.$$

Proof. Suppose first that g is C^3 and $\delta > 0$; assume without loss of generality that g vanishes outside some compact set (this is permissible since $\delta > 0$). We consider the Cauchy problems:

$$\frac{\epsilon^2}{2}\left(\nabla^2 w + \frac{\partial^2 w}{\partial \xi^2}\right) + \frac{\partial w}{\partial t} + \tilde{H}\left(t, x, \nabla w, \frac{\partial w}{\partial \xi}\right) = 0, \tag{9}$$

$$\frac{\epsilon^2}{2}\left(\nabla^2 w + \frac{\partial^2 w}{\partial \xi^2}\right) + \frac{\partial w}{\partial t} + \tilde{H}\left(t, x, \nabla w, \frac{\partial w}{\partial \xi}\right) = 0, \tag{10}$$

subject to $w(t, x, \xi) = g(x, \xi)$. Then by results of Friedman ((7), (8), p. 205) and Fleming ((5), (6)) the Cauchy problems (9) and (10) have unique solutions W_ϵ^+, W_ϵ^- in the strip $t_0 \leqslant t \leqslant T$, which vanish at infinity. Furthermore

$$V^+ = \lim_{\epsilon \to 0} W_\epsilon^+(t_0, x_0, \xi_0),$$
$$V^- = \lim_{\epsilon \to 0} W_\epsilon^-(t_0, x_0, \xi_0)$$

by the results of (4).

For $\mu > 0$ we define

$$W_{\epsilon,\mu}^{+}(t,x,\xi) = W_{\epsilon}^{+}(t,x,\xi+\mu).$$

Then $W_{\epsilon,\mu}^{+}$ satisfies (9) subject to

$$w(T,x,\xi) = g(x,\xi+\mu).$$

However
$$g(x,\xi+\mu) \geqslant g(x,\xi)-\mu\delta$$

by (iii) and equally
$$g(x,\xi+\mu) \leqslant g(x,\xi)+K_g^{\xi}\mu.$$

Then by a standard comparison argument, using the fact that $\tilde{H}^{+}$ is independent of ξ we obtain

$$W_{\epsilon,\mu}^{+}(t,x,\xi)-\mu\delta \leqslant W_{\epsilon,\mu}^{+}(t,x,\xi) \leqslant W_{\epsilon}^{+}(t,x,\xi)+K_g^{\xi}\mu$$

for $t_0 \leqslant t \leqslant T$. A similar argument is also used later so we will not justify this step (cf. Friedman (8), pp. 201–202). Thus we have

$$-\delta \leqslant \frac{\partial W_{\epsilon}^{+}}{\partial \xi} \leqslant K_g^{\xi}$$

and similarly
$$-\delta \leqslant \frac{\partial W_{\epsilon}^{-}}{\partial \xi} \leqslant K_g^{\xi}.$$

Now consider the function

$$\theta(t,x,\xi) = e^{\lambda t}(W_{\epsilon}^{+}-W_{\epsilon}^{-})$$

for some $\lambda \in \mathscr{R}$. As in Lemma 6·2 of (1) θ is bounded and approaches zero at infinity for $t_0 \leqslant t \leqslant T$. Hence θ must obtain its maximum at a point $(\bar{t},\bar{x},\bar{\xi})$ where $t_0 \leqslant \bar{t} < T$. At this point we have

$$\frac{\partial\theta}{\partial t}+\frac{\epsilon^2}{2}\left(\nabla^2\theta+\frac{\partial^2\theta}{\partial\xi^2}\right) \leqslant 0,$$

$$\frac{\partial\theta}{\partial\xi} = \nabla\theta = 0.$$

Let $\bar{p} = \nabla W_{\epsilon}^{+}$ and $\bar{g} = \dfrac{\partial W_{\epsilon}^{+}}{\partial \xi^2}$ at $(\bar{t},\bar{x},\xi)$. Then $\bar{p} = \nabla W_{\epsilon}^{-}$ and $\bar{q} = \dfrac{\partial W_{\epsilon}^{-}}{\partial \xi}$ and

$$\lambda\theta+e^{\lambda\bar{t}}\left[\frac{\partial W_{\epsilon}^{+}}{\partial t}+\frac{\epsilon^2}{2}\left(\frac{\partial^2 W_{\epsilon}}{\partial\xi^2}+\nabla^2 W_{\epsilon}^{+}\right)-\frac{\partial W_{\epsilon}^{-}}{\partial t}-\frac{\epsilon^2}{2}\left(\frac{\partial^2 W_{\epsilon}^{+}}{\partial t}+\nabla^2 W_{\epsilon}^{+}\right)\right] \leqslant 0,$$

i.e.
$$\lambda\theta \leqslant e^{\lambda\bar{t}}(\tilde{H}^{+}(\bar{t},\bar{x},\bar{p},\bar{q})-\tilde{H}^{-}(\bar{t},\bar{x},\bar{p},\bar{q})).$$

By Lemma 6·1 of (1)

$$\|\bar{p}\| \leqslant (K_f+K_h)\exp(2(K_f+K_h)(T-t_0)) = K^* \quad \text{say}$$

and by the preceding remarks

$$|q| \leqslant K_g^{\xi}.$$

Thus if $\bar{q} \geqslant 0$,
$$\lambda\theta(\bar{t},\bar{x},\bar{\xi}) \leqslant e^{\lambda\bar{t}}(A_1K^*+A_2K_g^{\xi})$$

while if $\bar{q} < 0$, then $-\delta \leqslant \bar{q} < 0$, and

$$\lambda\theta(\bar{t},\bar{x},\bar{\xi}) \leqslant e^{\lambda\bar{t}}(A_1K^*+B_h\delta).$$

In general $$\theta(\bar{t},\bar{x},\bar{\xi}) \leqslant \lambda^{-1}e^{\lambda\bar{t}}(A_1K^*+B_h\delta+A_2K_g^\xi)$$
and $$\theta(t,x,\xi) \leqslant \lambda^{-1}e^{\lambda T}(A_1K^*+B_h\delta+A_2K_g^\xi)$$
for all (t,x,ξ). Thus
$$W_\epsilon^+ - W_\epsilon^- \leqslant \lambda^{-1}e^{\lambda(T-t_0)}(A_1K^*+B_h\delta+A_2K_g^\xi).$$

Taking $\lambda = (T-t_0)^{-1}$ we obtain the result (for g in C^3).

For g Lipschitz but not C^3 and $\delta > 0$, we define, for $\epsilon > 0$, $g_n: \mathscr{R}^m \to \mathscr{R}$ by
$$g_n(x) = g(x, n\epsilon)$$
$n = 0, \pm 1, \pm 2, \ldots$.

For each n choose a C^3-function γ_n with
$$\|\nabla\gamma_n\| \leqslant K_g^x$$
and $$|g_n(x)-\gamma_n(x)| \leqslant \tfrac{1}{2}\epsilon^2 \quad (x\in\mathscr{R}^m).$$

Let π be a function $\pi: \mathscr{R} \to \mathscr{R}$ such that π is C^3 and
$$\pi(\xi) = 0, \quad |\xi| \geqslant 1,$$
$$\pi(0) = 1,$$
$$0 \leqslant \pi(\xi) \leqslant 1, \quad \xi\in\mathscr{R},$$
$$\pi(\xi) = \pi(-\xi) \quad \xi\in\mathscr{R},$$
$$\pi(\xi)+\pi(1-\xi) = 1, \quad 0 \leqslant \xi \leqslant 1,$$
$$0 \leqslant \pi'(\xi) \leqslant 1+\epsilon, \quad \xi \leqslant 0.$$

Then we define
$$\gamma(x,\xi) = \sum_{n=-\infty}^{\infty} \pi(\epsilon^{-1}\xi-n)\,\gamma_n(x)$$
and $\gamma\in C^3$ with
$$|\gamma(x,\xi)-g(x,\xi)| \leqslant K_g\epsilon+\tfrac{1}{2}\epsilon^2,$$
$$\|\nabla\gamma\| \leqslant K_g^x$$
everywhere. Also
$$\begin{aligned}\frac{\partial\gamma}{\partial\xi} &= \sum_{n=-\infty}^{\infty} \epsilon^{-1}\pi'(\epsilon^{-1}\xi-n)\,\gamma_n(x)\\ &= \epsilon^{-1}\{\pi'(\epsilon^{-1}\xi-n)\,\gamma_n(x)+\pi'(\epsilon^{-1}\xi-n-1)\,\gamma_{n+1}(x)\},\end{aligned}$$
where $n\epsilon \leqslant x \leqslant (n+1)\,\epsilon$. Hence
$$\begin{aligned}\frac{\partial\gamma}{\partial\xi} &= \epsilon^{-1}\{\pi'(\epsilon^{-1}\xi-n)\,\gamma_n(x)-\pi'(n+1-\epsilon^{-1}\xi)\,\gamma_{n+1}(x)\}\\ &= \epsilon^{-1}\pi'(\epsilon^{-1}\xi-n)\,(\gamma_n(x)-\gamma_{n+1}(x))\\ &\geqslant -\epsilon^{-1}(1+\epsilon)\,(\epsilon^2+\delta\epsilon)\\ &= -(1+\epsilon)\,(\delta+\epsilon).\end{aligned}$$
At the same time
$$K_\gamma^\xi \leqslant (1+\epsilon)\,(K_g^\xi+\epsilon).$$

Hence applying the first part to γ, we obtain the lemma. Similarly we obtain the result for $\delta = 0$ by approximation from the case $\delta > 0$.

THEOREM 3·2. *Suppose g is continuous (but not necessarily Lipschitz) and increasing in ξ and that the Isaacs condition* (8) *holds. Then $V^+ = V^-$.*

Proof. For g Lipschitz this is Lemma 3·1. In general the result follows by uniform approximation of g.

4. *Optional stopping games.* In this section we describe a rather more complicated type of differential game which we call an *optional stopping game.* We first define *stopping time* τ as a map $\tau: M_Y(t_0) \times M_Z(t_0) \to [t_0, \infty)$ such that

$$y_1(t) = y_2(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau(y(\cdot), z(\cdot))$$
$$z_1(t) = z_2(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau(y(\cdot), z(\cdot))$$

together imply $\tau(y_1(\cdot), z_1(\cdot)) = \tau(y_2(\cdot), z_2(\cdot))$.

Suppose now that $\phi: [0,\infty) \times \mathscr{R}^{m+1} \to \mathscr{R}$ and $\psi: [0,\infty) \times \mathscr{R}^{m+1} \to \mathscr{R}$ are any two real functions and that E is a closed subset of $[0, \infty)$; we define a game $G_{\tau, E}(t_0, x_0, \xi_0; \phi, \psi)$ with dynamics (2), initial condition $x_0(t_0) = x_0$, $\xi(t_0) = \xi_0$ and pay-off,

where
$$\left.\begin{aligned} P &= \min(P_1, P_2), \\ P_1 &= \phi(\tau, x(\tau), \xi(\tau)), \\ P_2 &= \inf_{\substack{\sigma \in E \\ t_0 \leqslant \sigma < \tau}} \psi(\sigma, x(\sigma), \xi(\sigma)), \end{aligned}\right\} \tag{11}$$

where $\tau = \tau(y(\cdot), z(\cdot))$.

We shall regard E and ψ as initially fixed, but allow differing choices of τ and ϕ. Thus we shall refer to the various values of $G_{\tau, E}(t_0, x_0, \xi_0; \phi, \psi)$ by $V_\tau^+(t_0, x_0, \xi_0; \phi)$, etc., suppressing mention of E and ψ.

Suppose F is a closed subset of $[0, \infty) \times \mathscr{R}^{m+1}$ such that for some $T > 0$

$$[T, \infty) \times \mathscr{R}^{m+1} \subset F.$$

Then we can define a particular stopping time t_F as the least time such that the trajectory $(t, x(t), \xi(t)) \in F$; F is called the terminal set. Suppose $g: [0, \infty) \times \mathscr{R}^{m+1} \to \mathscr{R}$ is a fixed continuous function. In the special case $\tau \equiv t_F$, $\phi \equiv g$ we shall suppress mention of ϕ and τ and refer to $G(t_0, x_0, \xi_0)$ and $V^+(t_0, x_0, \xi_0)$, etc. Thus the functions V^+, V^-, etc. defined on $[0, \infty) \times \mathscr{R}^{m+1}$ are the value functions of this particular game.

Before proceeding with the formal discussion of optional stopping games, it might be helpful to describe a particular example, to be discussed in detail in section 7. Suppose a differential game has dynamics

$$\dot{x} = f(t, x, y, z)$$

subject to $x(t_0) = x_0$. Suppose that it is the object of J_Z to force (t, x) as close as possible to some terminal set F before time T, but that J_Z is restricted by a fuel limitation

$$\int_{t_0}^{s} h(t, x, y, z) \leqslant \Lambda$$

(where usually $h \geqslant 0$). This game can be described as follows: it has dynamics

$$\frac{dx}{dt} = f(t, x, y, z),$$

$$\frac{d\xi}{dt} = h(t, x, y, z)$$

subject to $x(t_0) = x_0$, $\xi(t_0) = 0$, and pay-off

$$P = \inf_{t_0 \leqslant t \leqslant t_{F_\Lambda}} \rho(t, x(t)),$$

where $\rho(t,x)$ is the distance of (t,x) from F and the terminal set F^*_Λ is given by $\{(t,x,\xi)$, where $t \geqslant T$ or $\xi \geqslant \Lambda\}$.

This is a particular case of an optional stopping game, which besides being of interest in its own right, can be used to study pursuit-evasion games; in such games the fuel consumption between hitting the terminal set is the pay-off.

In this section our objective is simply to generalize the *dynamic programming* theorems of (3) to this new setting. In fact the proofs are identical and so we shall restrict ourselves to the statement and one sample proof.

THEOREM 4·1. *Suppose τ is a stopping time on $M_Y(t_0) \times M_Z(t_0)$ such that $\tau \leqslant t_F$ everywhere. Then*

(i) $U_\tau(t_0, x_0, \xi_0;\ U) = U(t_0, x_0, \xi_0)$,
$V_\tau(t_0, x_0, \xi_0;\ V) = V(t_0, x_0, \xi_0)$.

(ii) $V^+_\tau(t_0, x_0, \xi_0;\ V^+) \leqslant V^+(t_0, x_0, \xi_0)$,
$V^-_\tau(t_0, x_0, \xi_0;\ V^-) \geqslant V^-(t_0, x_0, \xi_0)$.

(iii) *If further* $0 < s \leqslant \tau$ *everywhere*:
$Q^+_{s,\tau}(t_0, x_0, \xi_0;\ Q^+_s) \geqslant Q^+_s(t_0, x_0, \xi_0)$,
$Q^-_{s,\tau}(t_0, x_0, \xi_0;\ Q^-_s) \leqslant Q^-_s(t_0, x_0, \xi_0)$,
$Q^+_{s,\tau}(t_0, x_0, \xi_0;\ U) \leqslant Q^+_s(t_0, x_0, \xi_0)$,
$Q^-_{s,\tau}(t_0, x_0, \xi_0;\ V) \geqslant Q^-_s(t_0, x_0, \xi_0)$.

Proof. We shall prove the first inequality of (iii). For $\epsilon > 0$ and

$$(t, x, \xi) \in [0, \infty) \times \mathscr{R}^{m+1} \quad \text{and} \quad z = z(\cdot) \in M^s_Z(t)$$

there is a strategy $\beta = \beta(t, x, \xi, z)$ for J_Z such that $\beta \in \Delta_t(s|z)$ and

$$v(\beta) \leqslant Q^+_s(t, x, \xi) + \epsilon,$$

where $v(\beta)$ is the value of β in $G(t, x, \xi)$. Now suppose $\beta^* \in \Delta_t(s)$ and $y(\cdot) \in M_Y(t_0)$; let $(x(t), \xi(t))$ be the $\mathscr{R}^{m+1}$-trajectory corresponding to $(y, \beta^* y)$ and let $\sigma = \sigma(y) = \tau(y, \beta^* y)$. We define $\beta' \in \Delta_{t_0}(s)$ by

$$\begin{aligned} \beta' y(t) &= \beta^* y(t) \quad (t_0 \leqslant t \leqslant \sigma + s) \\ &= \beta(\sigma, x(\sigma), \xi(\sigma), \overline{\beta^* y})\, \bar{y}(t) \quad (\sigma + s \leqslant t < \infty), \end{aligned}$$

where $\bar{y} \in M_Y(\sigma)$ is defined by

$$\bar{y}(t) = y(t) \quad (t \geqslant \sigma)$$

and $\overline{\beta^* y} \in M^s_Z(\sigma)$ by

$$\overline{\beta^* y}(t) = \beta^* y(t) \quad (\sigma \leqslant t \leqslant \sigma + s).$$

We must check $\beta' \in \Delta_{t_0}(s)$; suppose

$$y_1(t) = y_2(t) \quad \text{a.e.} \quad (t_0 \leqslant t \leqslant t_1).$$

If $t_1 \leqslant \sigma(y_1)$ then $t_1 \leqslant \sigma(y_2)$ and it follows that

$$\begin{aligned}\beta' y(t) &= \beta^* y_1(t)\\ &= \beta^* y_2(t)\\ &= \beta' y_2(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant t_1 + s.\end{aligned}$$

If $t_1 > \sigma(y_1) = \sigma$ then $\sigma = \sigma(y_2)$ and $\overline{\beta^* y_1} = \overline{\beta^* y_2}$. Hence

$$\beta(\sigma, x(\sigma), \xi(\sigma), \overline{\beta^* y_1}) = \beta(\sigma, x(\sigma), \xi(\sigma), \overline{\beta^* y_2}) = \beta \quad \text{say.}$$

Now $$\bar{y}_1(t) = \bar{y}_2(t) \quad \text{a.e.} \quad \sigma \leqslant t \leqslant t_1$$

and so $$\beta \bar{y}_1(t) = \beta \bar{y}_2(t) \quad \text{a.e.} \quad \sigma \leqslant t \leqslant t_1 + s$$

and so $\beta' \in \Delta_{t_0}(s)$.

Now suppose $y = y(\cdot) \in M_Y(t_0)$ and let P be the pay-off $P(y, \beta' y)$. Then

$$P = \min(P_1, P_2),$$

where $$P_1 = g(t_F, x(t_F), \xi(t_F))$$

$$P_2 = \inf_{\substack{\rho \in E\\ t_0 \leqslant \rho < t_F}} \psi(\rho, x(\rho), \xi(\rho)).$$

Now $$P_2 = \min(P_3, P_4),$$

where $$P_3 = \inf_{\substack{\rho \in E\\ t_0 \leqslant \rho < \sigma}} \psi(\rho, x(\rho), \xi(\rho))$$

and $$P_4 = \inf_{\substack{\rho \in E\\ \sigma \leqslant \rho < t_F}} \psi(\rho, x(\rho), \xi(\rho)).$$

Let $P' = \min(P_1, P_4)$; then P' is the pay-off in $G(\sigma, x(\sigma), \xi(\sigma))$ corresponding to $(\bar{y}, \bar{z})$ where $\bar{y}$ and $\bar{z}$ are the restrictions of y and $\beta' y$ to $M_Y(\sigma)$ and $M_Z(\sigma)$. Now

$$\begin{aligned}\bar{z}(t) &= \beta(\sigma, x(\sigma), \xi(\sigma), \overline{\beta^* y})\, \bar{y}(t) \quad && (\sigma + s \leqslant t)\\ &= \overline{\beta^* y}(t) && (\sigma \leqslant t < \sigma + s).\end{aligned}$$

Since $\beta(\sigma, x(\sigma), \xi(\sigma), \overline{\beta^* y}) \in \Delta_\sigma(s|\overline{\beta^* y})$

$$\bar{z} = \beta(\sigma, x(\sigma), \xi(\sigma), \overline{\beta^* y})\, \bar{y}.$$

Hence $$P' \leqslant Q_s^+(\sigma, x(\sigma), \xi(\sigma)) + \epsilon$$

and $$\begin{aligned}P &= \min(P', P_3)\\ &\leqslant \min(P_3, Q_s^+(\sigma, x(\sigma), \xi(\sigma))) + \epsilon.\end{aligned}$$

However $\min(P_3, Q_s^+(\sigma, x(\sigma), \xi(\sigma)))$ is the pay-off of $(y, \beta' y)$ or $(y, \beta^* y)$ in the game with stopping time τ and $\phi \equiv Q_s^+$. Thus the value $v'(\beta^*)$ of β^* in this game satisfies

$$v'(\beta^*) \geqslant v(\beta') - \epsilon,$$

where $v(\beta')$ is the value of β' in $G(t_0, x_0, \xi_0)$.

If $\beta^* \in \Delta_{t_0}(s|z)$ then $\beta' \in \Delta_{t_0}(s|z)$ and so we obtain the desired inequality, since β^* is arbitrary.

5. *Optional stopping game of fixed duration.* Continuing with our study of optional stopping games, we now impose some restrictions; we assume that g and ψ are continuous functions and that the terminal set F takes the form $[T,\infty)\times\mathscr{R}^{m+1}$, where $T > 0$. In this case, the pay (11) takes the form

$$P = \mu(x(\cdot), \xi(\cdot)),$$

where μ is a continuous functional on the Banach space of $\mathscr{R}^{m+1}$-valued continuous functions on $[t_0, T]$. Using this observation, it follows as in Lemma 2·2 of (3) that

LEMMA 5·1. *If $F = [T,\infty)\times\mathscr{R}^{m+1}$, and g and ψ are continuous then*

$$Q^+(t,x,\xi) = V^+(t,x,\xi),$$
$$Q^-(t,x,\xi) = V^-(t,x,\xi)$$

for $0 \leqslant t$ and $(x,\xi)\in\mathscr{R}^{m+1}$.

LEMMA 5·2. (i) *Under the assumption of Lemma 5·1, for each fixed t_0 the functions Q_s^+, V^+, V, U, V^-, Q_s^- are all continuous in (x, ξ).*

(ii) *Suppose in addition ψ and g are Lipschitz in (x,ξ) and*

$$\max(K_\psi^x, K_g^x) = K^x,$$
$$\max(K_\psi^\xi, K_g^\xi) = K^\xi.$$

Then for $\theta = Q_s^+, V^+, \ldots, Q_s^-$, we have

$$K_\theta^x \leqslant (K^x + K^\xi K_h K_f^{-1})\exp(K_f(T-t_0)),$$
$$K_\theta^\xi \leqslant K^\xi.$$

Proof. Each result is proved by a simple comparison of the effects of an identical pair of control functions with respect to ranging initial conditions; we omit the simple calculations. We establish by a similar argument:

LEMMA 5·3. *Under the assumptions of Lemma 5·1, suppose also that g and ψ are each monotonically increasing in ξ for each fixed (t,x). Then the functions V^+ and V^- are monotonically increasing in ξ for fixed (t,x).*

LEMMA 5·4. *Under the assumptions of Lemma 5·1 suppose $t_0 \leqslant \bar{t}$. We define a constant stopping time $\tau \equiv \bar{t}$; then we have*

$$V^+(t_0,x_0,\xi_0) = V_\tau^+(t_0,x_0,\xi_0;\ V^+),$$
$$V^-(t_0,x_0,\xi_0) = V_\tau^-(t_0,x_0,\xi_0;\ V^-).$$

Proof. By Dini's Theorem and Lemma 5·2 we have $Q_s^+(\bar{t},x,\xi)\to Q^+(\bar{t},x,\xi)$ uniformly on compacta in $\mathscr{R}^{m+1}$. Hence by Theorem 4·1

$$\begin{aligned} V^+(t_0,x_0,\xi_0) &= Q^+(t_0,x_0,\xi_0) \\ &\leqslant Q_s^+(t_0,x_0,\xi_0) \\ &\leqslant Q_{s,\tau}^+(t_0,x_0,\xi_0;\ Q_s^+) \\ &\leqslant Q_{s,\tau}^+(t_0,x_0,\xi_0;\ Q^+) + \eta(s), \end{aligned}$$

where $\lim_{s\to 0} \eta(s) = 0$. Hence

$$V^+(t_0, x_0, \xi_0) \leqslant Q^+_\tau(t_0, x_0, \xi_0;\ Q^+)$$
$$= V^+_\tau(t_0, x_0, \xi_0;\ V^+)$$

by applying Lemma 5·1 to the fixed time game with $T = \bar{t}$. Hence by Theorem 4·1 we obtain the result.

THEOREM 5·5. *Suppose*

(i) g, ψ *are Lipschitz functions in* (x, ξ) *and continuous in* t,

(ii) g, ψ *are monotonically increasing in* ξ,

(iii) $H^+(t,x,p) - H^-(t,x,p) \leqslant A_1\|p\| + A_2$.

Then for $0 \leqslant t_0 \leqslant T$

$$V^+(t_0, x_0, \xi_0) - V^-(t_0, x_0, \xi_0) \leqslant e(T - t_0)\,(A_2 K^\xi + A_1(K_f + K_h)\, e^{2(K_h+K_f)\,T}),$$

where $\max(K^\xi_g, K^\xi_\psi) = K^\xi$.

Proof. First we observe that it is sufficient to prove the result for E finite (or $E \cap [0, T]$ finite). For in the general case there is a sequence E_n of finite subsets of E such that

$$\lim_{n\to\infty} \sup_{\substack{e \in E \\ e \leqslant T}} \inf_{e' \in E_n} |e - e'| = 0.$$

If P_n denotes the pay-off in the game with optional stopping on E_n, then we can easily show, using the continuity of ψ, that

$$\lim_{n\to\infty} P_n(y(\cdot), z(\cdot)) = P(y(\cdot), z(\cdot))$$

uniformly on $(y(\cdot), z(\cdot)) \in M_Y(t_0) \times M_2(t_0)$. Using this the Theorem will follow for E from the result for each E_n.

If $E \cap [0, T]$ is finite we proceed by induction. Let $|E|$ be the number of points of $E \cap (t_0, T]$. If $|E| = 0$ and $t_0 \notin E$ then G is a game with no optional stopping and the result follows from Lemma 3·1 directly. If $t_0 \in E$ the pay-off is of the form.

$$P = \min(\psi(t_0, x_0, \xi_0),\ g(T, x(T),\ \xi(T)))$$

and again Lemma 3·1 yields the result.

Suppose now the result is proved for $|E| \leqslant k$, where $k \geqslant 0$. Suppose $|E| = k+1$ and let $\bar{t} = \inf(E \backslash [0, t_0])$; then if $\tau \equiv \bar{t}$

$$V^+(t_0, x_0, \xi_0) = V^+_\tau(t_0, x_0, \xi_0;\ V^+),$$

$$V^-(t_0, x_0, \xi_0) = V^-_\tau(t_0, x_0, \xi_0;\ V^-).$$

Now
$$V^+_\tau(t_0, x_0, \xi_0;\ V^+) - V^-_\tau(t_0, x_0, \xi_0;\ V^+)$$
$$\leqslant e(\bar{t} - t_0)\,(A_2 K^\xi_{V^+} + A_1(K_f + K_h)\exp(2(K_h + K_f)\,(\bar{t} - t_0))$$

by Lemma 3·1 (again we treat the cases $t_0 \in E$ and $t_0 \notin E$ separately). By Lemma 5·2 we deduce

$$V^+_\tau(t_0, x_0, \xi_0;\ V^+) - V^-_\tau(t_0, x_0, \xi_0;\ V^+) \leqslant e(\bar{t} - t_0)\,(A_2 K^\xi + A_1(K_f + K_h)\, e^{2(K_h+K_f)T}).$$

Therefore

$$\begin{aligned} &V_\tau^+(t_0, x_0, \xi_0;\ V^+) - V_\tau^-(t_0, x_0, \xi_0;\ V^-) \\ &\leqslant e(\bar{t}-t_0)\,(A_2 K^\xi + A_1(K_f + K_h)\, e^{2(K_h+K_f)T}) + \sup_{(x,\xi)}\,(V^+(\bar{t}, x, \xi) - (V^-(\bar{t}, x, \xi)) \\ &\leqslant e(T-t_0)\,(A_2 K^\xi + A_1(K_f + K_h)\, e^{2(K_h+K_f)T}) \end{aligned}$$

by the inductive hypothesis since $|E \cap (\bar{t}, T]| \leqslant K$. Then the induction is complete.

For later use we require a modification of Theorem 5·5, in which we relax the Lipschitz assumption on h; we shall assume instead that h is continuous. We shall write $K_h = \infty$ if h is continuous but not Lipschitz.

THEOREM 5·6. *If $K_h \leqslant \infty$, and $A_1 = 0$, then under the assumptions of Theorem* 5·5

$$V^+(t_0, x_0, \xi_0) - V^-(t_0, x_0, \xi_0) \leqslant A_2 e(T-t_0)\, K^\xi.$$

Proof. For each n choose h_n with $K_{h_n} < \infty$ and such that

$$\sup |h(t, x, y, z) - h_n(t, x, y, z)| \leqslant \frac{1}{n}.$$

(Recall that h is zero outside a compact set.)

Then $$\min_z \max_y\,(p\,.f + h_n) - \max_y \min_z\,(p\,.f + h_n) \leqslant A_2 + \frac{2}{n}.$$

for all $p \in \mathscr{R}^m$. Hence by Theorem 5·5

$$V_n^+(t_0, x_0, \xi_0) - V_n^-(t_0, x_0, \xi_0) \leqslant \left(A_2 + \frac{2}{n}\right) e(T-t_0)\, K^\xi,$$

where the subscript n refers to the game G_n with h_n replacing h.

If $(y(\cdot), z(\cdot))$ is a pair of controls inducing trajectories $(x(\cdot), \xi(\cdot))$ in G and $(x(\cdot), \xi_n(\cdot))$ in G_n then

$$|\xi(t) - \xi_n(t)| \leqslant \frac{2}{n}(T-t_0) \quad (t_0 \leqslant t \leqslant T)$$

and $$|P - P_n| \leqslant \frac{2}{n}(T-t_0)\, K^\xi,$$

where P, P_n are the pay-offs in G and G_n. Thus

$$V^+(t_0, x_0, \xi_0) - V^-(t_0, x_0, \xi_0) \leqslant \left(eA_2 + \frac{2e}{n} + \frac{2}{n}\right)(T-t_0)\, K^\xi.$$

Letting $n \to \infty$ we obtain the theorem.

THEOREM 5·7. *Suppose now $K_g \leqslant \infty$, $K_\psi \leqslant \infty$ and $K_h \leqslant \infty$, and that the Isaacs condition holds* (i.e. $A_1 = A_2 = 0$). *Then*

$$V^+(t_0, x_0, \xi_0) = V^-(t_0, x_0, \xi_0).$$

Proof. For $K_g < \infty$, $K_\psi < \infty$ this is a special case of Theorem 5·6; by approximating g and ψ by Lipschitz functions uniformly on compact sets the general result follows easily (note that the approximations must be chosen to satisfy (i) and (ii) of Theorem 5·5; this is, however, quite simple).

Finally, we observe that similar argument to that of Theorem 5·5, using the results of (4) shows that without any assumptions on the Hamiltonians H^+ and H^- we can deduce that $V^+ = U$ and $V^- = V$. In this case we can again assume $K_h \leqslant \infty$, $K_g \leqslant \infty$ and $K_\psi \leqslant \infty$ as in Theorem 5·7.

6. *Games of survival with optional stopping.* Suppose now that the terminal set satisfies only the condition $F \supset [T, \infty) \times \mathscr{R}^{m+1}$, so that G ceases to be a fixed time game. We suppose now that $\psi \equiv g$ and that g is continuous (i.e. $K_g \leqslant \infty$) so that the pay-off takes the form

$$P = \inf(g(\sigma, x(\sigma), \xi(\sigma)) : \sigma < t_F, \sigma \in E \text{ or } \sigma = t_F).$$

We shall now call G a game of survival with optional stopping. Following section 6 of (3) we introduce $\bar{Q}_s^+$ on $\mathscr{R}^{m+1}$ by

$$\bar{Q}_s^+(t, x, \xi) = \lim_{(t', x', \xi') \to (t, x, \xi)} \sup Q_s^+(t, x, \xi)$$

and say $(t_0, x_0, \xi_0) \in F$ is *Q^+-regular* if

$$\left.\begin{array}{ll}\text{(i)} & \lim\limits_{s \to 0} \bar{Q}_s^+(t_0, x_0, \xi_0) = g(t_0, x_0, \xi_0), \\ \text{(ii)} & U \text{ is continuous at } (t_0, x_0, \xi_0).\end{array}\right\} \tag{12}$$

Similarly we define

$$\bar{Q}^-(t, x, \xi) = \lim_{(t', x', \xi') \to (t, x, \xi)} \inf Q_s^-(t, x, \xi)$$

and say that $(t_0, x_0, \xi_0) \in F$ is *Q^--regular* if

$$\left.\begin{array}{ll}\text{(i)}' & \lim\limits_{s \to 0} \bar{Q}_s^-(t_0, x_0, \xi_0) = g(t_0, x_0, \xi_0), \\ \text{(ii)}' & V \text{ is continuous at } (t_0, x_0, \xi_0).\end{array}\right\} \tag{13}$$

We say that F is *Q^+- (or Q^--) regular* if every point of F is *Q^+- (or Q^--) regular.*

Let $\rho(t, x, \xi) = \text{dist}((t, x, \xi); F)$.

THEOREM 6·1. *Suppose G is a game of survival with optional stopping with $K_h \leqslant \infty$ and F is Q^+-regular. Then Q^+ is continuous in (x, ξ) and $\lim\limits_{s \to 0} Q_s^+ = Q^+$ uniformly on compacta for each fixed t_0. Dually if F is Q^--regular then Q^- is continuous in (x, ξ) and $\lim\limits_{s \to 0} Q_s^- = Q^-$ uniformly on compacta for each fixed t_0.*

Proof. This is Theorem 6·3 of (3) generalized to this new setting.

Let $B_r = \{(x, \xi) \in \mathscr{R}^{m+1} : \|x\| + |\xi| \leqslant r\}$, for $r > 0$. Then there is an $R \geqslant r$ such that every trajectory with initial point in B_r is contained in B_R for $t_0 \leqslant t \leqslant T$. By Lemma 6·1 of (3) there is the function $\eta_1 : (0, \infty) \to (0, \infty)$ with $\lim\limits_{\delta \to 0} \eta(\delta) = 0$ such that for

$$(t, x, \xi) \in [t_0, T] \times B_R,$$

$$Q_s^+(t, x, \xi) \leqslant g(t, x, \xi) + \eta_1[s + \rho(t, x, \xi)]. \tag{14}$$

Since g is uniformly continuous on $[t_0, T] \times B_R$ we can find a similar η_2 so that if $\|x_1 - x_2\| \leqslant \delta$ and $|\xi_1 - \xi_2| \leqslant \delta$ and $(x_1, \xi_1), (x_2, \xi_2) \in B_R$ then for $t_0 \leqslant t \leqslant T$

$$|g(t, x_1, \xi_1) - g(t, x_2, \xi_2)| \leqslant \eta_2(\delta). \tag{15}$$

Similarly, using the regularity of F we can find η_3 so that if

$$\rho(t, x, \xi) \leqslant \delta \quad \text{and} \quad t_0 \leqslant t \leqslant T, \ (x, \xi) \in B_R$$

then

$$g(t, x, \xi) \leqslant U(t, x, \xi) + \eta_3(\delta). \tag{16}$$

The functions f and h are bounded

$$\|f\| \leqslant M,$$
$$|h| \leqslant M$$

everywhere.

Finally, we note that h is uniformly continuous so that there is a function

$$\theta\colon (0,\infty) \to (0,\infty)$$

such that $\lim_{\delta\to 0}\theta(\delta) = 0$ and

$$|h(t,x,y,z) - h(t,x,y,z)| \leqslant \theta(\|x_1 - x_2\|).$$

Now suppose $(x_1,\xi_1),(x_2,\xi_2)\in B_r$ with $\|x_1-x_2\| + |\xi_1-\xi_2| \leqslant \delta$.

Let
$$\sigma = \min(\rho(t_0,x_1,\xi_1),\rho(t_0,x_2,\xi_2)).$$

Suppose $(y(\cdot),z(\cdot))\in M_Y(t_0)\times M_2(t_0)$ and let $(x_1(t),\ \xi(t))$, $(x_2(t),\ \xi_2(t))$ be the corresponding trajectories subject to the initial conditions $x_1(t_0) = x_1$, $\xi_1(t_0) = \xi_1$ and $x_2(t_0) = x_2, \xi_2(t_0) = \xi_2$ respectively. Then

$$\begin{aligned}\|x_1(t)-x_2(t)\| &\leqslant \delta e^{K(t-t_0)} \quad (t_0 \leqslant t \leqslant T)\\ &\leqslant \delta e^{KT} \quad (t_0 \leqslant t \leqslant T)\end{aligned}$$

and
$$\begin{aligned}|\xi_1(t)-\xi_2(t)| &\leqslant \delta + \int_{t_0}^{t}\theta(\|x_1(t)-x_2(t)\|)\,dt\\ &\leqslant \delta + T\theta(\delta e^{KT}).\end{aligned}$$

Thus
$$\begin{aligned}\|x_1(t)-x_2(t)\| &\leqslant \theta'(\delta),\\ |\xi_1(t)-\xi_2(t)| &\leqslant \theta'(\delta),\end{aligned}$$

where $\theta'(\delta)\to 0$ as $\delta\to 0$.

Now define a stopping time $\tau\colon M_Y(t_0)\times M_2(t_0)\to \mathscr{R}$ by $\tau = \tau(y(\cdot),z(\cdot))$, where τ is the least time such that

$$\min(\rho(\tau,x_1(\tau),\xi_1(\tau)),\rho(\tau,x_2(\tau),\xi_2(\tau))) = 0.$$

If $\sigma > 0$ then $\tau \geqslant M^{-1}\sigma$ and so for $s \leqslant M^{-1}\sigma$ we have, by Theorem 4·1,

$$\begin{aligned}Q_s^+(t_0,x_1,\xi_1) &\leqslant Q_{s,\tau}^+(t_0,x_1,\xi_1;\,Q_s^+)\\ &\leqslant Q_{s,\tau}^+(t_0,x_1,\xi_1;\,g) + \eta_1[s+\theta'(\delta)]\end{aligned}$$

by (14), since,

$$\max(\rho(\tau,x_1(\tau),\xi_1(\tau)),\rho(\tau,x_2,\xi_2(\tau))) \leqslant \theta'(\delta).$$

Now
$$Q_s^+(t_0,x_1,\xi_1) \leqslant Q_{s,\tau}^+(t_0,x_2,\xi_2;\,g) + \eta_1[s+\theta'(\delta)] + \eta_2(\theta'(\delta))$$
using (15). Then

$$Q_s^+(t_0,x_2,\xi_2;\,g) \leqslant Q_{s,\tau}^+(t_0,x_2,\xi_2;\,U) + \eta_3(\theta'(\delta))$$

using (16). Finally, for $s < M^{-1}\sigma$ we obtain

$$Q_s^+(t_0,x_1,\xi_1) \leqslant Q_{s,\tau}^+(t_0,x_2,\xi_2;\,U) + \eta_1[s+\theta'(\delta)] + \eta_2[\theta'(\delta)] + \eta_3[\theta'(\delta)].$$

By Theorem 4·1 again

$$Q_s^+(t_0,x_1,\xi_1) - Q_s^+(t_0,x_2,\xi_2) \leqslant \eta_1[s+\theta'(\delta)] + \eta_2[\theta'(\delta)] + \eta_3[\theta'(\delta)] \tag{17}$$

and by symmetry letting $s\to 0$

$$|Q^+(t_0,x_1,\xi_1) - Q^+(t_0,x_2,\xi_2)| \leqslant \eta_1[\theta'(\delta)] + \eta_2[\theta'(\delta)] + \eta_3[\theta'(\delta)]$$

and this equation is clearly also valid if we have $\sigma = 0$. Thus it follows that Q^+ is continuous in B_r.

Now let $$C_s(\epsilon) = \{(x,\xi)\in B_r\colon Q_s^+(t_0,x,\xi) - Q^+(t_0,x,\xi) \geqslant \epsilon\},$$

and suppose $(x_0,\xi_0)\in \cap \overline{C_s(\epsilon)}$ for some $\epsilon > 0$. If $\rho(t_0,x_0,\xi_0) > 0$ then for small enough $\delta > 0$ and $s > 0$ we can find $(x_1,\xi_1)\in C_s(\epsilon)$ with

$$\|x_1 - x_0\| + |\xi_1 - \xi_0| \leqslant \delta \quad \text{and} \quad s \leqslant M^{-1}\sigma.$$

Then by (17)

$$Q_s^+(t_0,x_0,\xi_0) - Q^+(t_0,x_0,\xi_0) \geqslant \epsilon - \eta_1(s+\theta'(\delta)) - \eta_2(\theta'(\delta)) - \eta_3(\theta'(\delta)).$$

Letting $\delta, s \to 0$ we obtain

$$\lim_{s\to 0} Q_s^+(t_0,x_0,\xi_0) \geqslant Q^+(t_0,x_0,\xi_0) + \epsilon$$

contradicting the definition of Q^+. Using directly the definition of regularity (12), we can also similarly exclude the possibility $\rho(t_0,x_0,\xi_0) = 0$. Therefore $\bigcap_{s>0} \overline{C_s(\epsilon)} = \varnothing$, and by compactness $\exists\, s > 0$ such that $C_s(\epsilon) = \varnothing$, and so $Q_s^+ \to Q^+$ uniformly on B_r.

THEOREM 6·2. *Let G be a game of survival with optional stopping. Suppose F is Q^+- and Q^--regular and that the extended Isaacs condition holds. Then for any (t,x,ξ)*

$$Q^+(t,x,\xi) = V^+(t,x,\xi) = V^-(t,x,\xi) = Q^-(t,x,\xi).$$

This generalizes the Theorems 6·4 and 8·1 of (3) combined. We omit the proof as a similar argument is employed later in a more special case.

7. *Restricted fuel games.* A game of survival with optional stopping will be called a *restricted fuel* game if

(i) $F = \{(t,x,\xi)\, t \geqslant T$ or $\xi \geqslant \Lambda\}$ for some (Λ, T),

(ii) g is monotonically increasing in ξ for each fixed (t,x),

(iii) $E = [0,\infty)$ (i.e. optional stopping is allowed at any time). We interpret such games as allowing the minimizer J_Z to stop at any time before this 'fuel' ξ or time t is exhausted.

LEMMA 7·1. *If G is a restricted fuel game and if*

$$\lim_{s\to 0} \bar{Q}_s^-(t_0,x_0,\xi_0) = g(t_0,x_0,\xi_0)$$

then (t_0,x_0,ξ_0) is Q^+- and Q^--regular.

Proof. For $s > 0$

$$Q_s^+(t,x,\xi) \leqslant g(t,x,\xi)$$

and so $$\lim_{s\to 0} \bar{Q}_s^+(t_0,x_0,\xi_0) \leqslant g(t_0,x_0,\xi_0),$$

and this combined with the hypothesis forces (12) and (13).

THEOREM 7·2. *Suppose G is a restricted fuel game with $K_h \leqslant \infty$ in which*

$$\lim_{s\to 0} \bar{Q}_s^-(t,x,\xi) = g(t,x,\xi)$$

for $(t,x,\xi)\in F$. Suppose also that the Isaacs condition is satisfied; then we have

$$Q^+(t_0,x_0,\xi_0) = Q^-(t_0,x_0,\xi_0)$$

for any (t_0,x_0,ξ_0).

Proof. Suppose

$$Q^+(t_0, x_0, \xi_0) = Q^-(t_0, x_0, \xi_0) + \theta,$$

where $\theta > 0$; clearly $(t_0, x_0, \xi_0) \in [0, \infty) \times \mathscr{R}^{m+1} - F$. Let B_h be a bound for $|h|$ and B_f for $\|f\|$. We define inductively a sequence $\{(t_n, x_n, \xi_n)\, n \geqslant 0\}$ with $t_n \leqslant T$, $\xi_n \leqslant \Lambda$. Suppose (t_n, x_n, ξ_n) has been determined; then we let

$$\begin{aligned} t_{n+1} &= t_n + \frac{1}{2(1+B_h)} \min(\Lambda - \xi_n, T - t_n) \\ &\leqslant T. \end{aligned}$$

On $t = t_{n+1}$, Q^+ is continuous and $Q_s^+ \to Q^+$ uniformly on compacta (Theorem 6·1). Arguing as in Lemma 5·4 we may show that

$$Q^+(t_n, x_n, \xi_n) = Q_\tau^+(t_n, x_n, \xi_n;\, Q^+),$$

where $\tau \equiv t_{n+1}$. It is easy to check further that Q^+ is increasing in ξ for fixed $(\bar{t}, \bar{x})$; for identical controls with the initial conditions $(\bar{t}, \bar{x}, \bar{\xi})$ and $(\bar{t}, \bar{x}, \bar{\xi} + \eta)$ induce trajectories $(x(t), \xi(t))$ and $(x(t), \xi(t) + \eta)$. The latter trajectory hits the terminal set earlier than the former provided $\eta > 0$, and

$$g(t, x(t), \xi(t) + \eta) \geqslant g(t, x(t), \xi(t)).$$

Thus we can quote Theorem 5·7, concerning the fixed time game given by

$$\tau = t_{n+1}, \quad \psi = g$$

and terminal pay-off Q^+. We deduce

$$Q_\tau^+(t_n, x_n, \xi_n;\, Q^+) = Q_\tau^-(t_n, x_n, \xi_n;\, Q^+).$$

Similarly

$$Q_\tau^-(t_n, x_n, \xi_n;\, Q^-) = Q^-(t_n, x_n, \xi_n)$$

and so

$$Q_\tau^-(t_n, x_n, \xi_n;\, Q^+) - Q_\tau^-(t_n, x_n, \xi_n;\, Q^-) = Q^+(t_n, x_n, \xi_n) - Q^-(t_n, x_n, \xi_n).$$

Therefore there exists a trajectory $(x(t), \xi(t))$ with

$$x(t_n) = x_n\ \xi(t_n) = \xi_n, \quad x(t_{n+1}) = x_{n+1}, \quad \xi(t_{n+1}) = \xi_{n+1}$$

and

$$Q^+(t_{n+1}, x_{n+1}, \xi_{n+1}) - Q^-(t_{n+1}, x_{n+1}, \xi_{n+1}) \geqslant Q^+(t_n, x_n, \xi_n) - Q^-(t_n, x_n, \xi_n) - \frac{\theta}{2^{n+1}}.$$

Clearly

$$|\xi_{n+1} - \xi_n| \leqslant B_h(t_{n+1} - t_n) \leqslant \tfrac{1}{2}(\Lambda - \xi_n)$$

so that

$$\xi_{n+1} \leqslant \Lambda.$$

Hence we deduce

$$Q^+(t_n, x_n, \xi_n) - Q^-(t_n, x_n, \xi_n) \geqslant \tfrac{1}{2}\theta$$

for all n, and since $t_n \leqslant T\ \forall\ n$, $t_n \to \bar{t}$ for some $\bar{t}$. Since also

$$\begin{aligned} \|x_{n+k} - x_n\| &\leqslant B_f(t_{n+k} - t_n), \\ |\xi_{n+k} - \xi_n| &\leqslant B_h(t_{n+k} - t_n) \end{aligned}$$

$x_n \to \bar{x}$ and $\xi_n \to \bar{\xi}$. Now

$$t_{n+1} - t_n = \frac{1}{2(1+B_h)} \min(\Lambda - \xi_n, T - t_n) \to 0.$$

Hence either $\bar{x} = T$ or $\bar{\xi} = \Lambda$.

Now by the regularity of the boundary, condition (12),

$$g(\bar{t},\bar{x},\bar{\xi}) = \lim_{s\to 0} \bar{Q}_s^+(\bar{t},\bar{x},\bar{\xi})$$
$$\geqslant \lim_{s\to 0}\lim_{n\to\infty} Q_s^+(t_n,x_n,\xi_n)$$
$$\geqslant \lim_{n\to\infty} Q^+(t_n,x_n,\xi_n)$$

while

$$g(\bar{t},\bar{x},\bar{\xi}) = \lim_{s\to 0} \bar{Q}_s^-(\bar{t},\bar{x},\bar{\xi})$$
$$\leqslant \lim_{s\to 0}\lim_{n\to\infty} \bar{Q}_s^-(t_n,x_n,\xi_n)$$
$$\leqslant \lim_{n\to\infty} Q^-(t_n,x_n,\xi_n)$$

and we deduce $\theta \leqslant 0$, contradicting our initial assumption.

8. *Quasi-pursuit-evasion games.* We now give some applications of the theory, which we have developed in the preceding sections. We shall consider a differential game of survival $\bar{G}$ with no optional stopping. The game has dynamics

$$\left.\begin{aligned} \frac{dx}{dt} &= f(t,x,y,z),\\ x(t_0) &= x_0 \end{aligned}\right\} \tag{18}$$

and pay-off

$$P = \int_{t_0}^{t_F} h(t,x(t),y(t),z(t))\,dt + \gamma(t_F,x(t_F)), \tag{19}$$

where F is the terminal set, and $F \supset [T,\infty)\times \mathscr{R}^m$ for some T. We introduce an extra co-ordinate ξ given by

$$\left.\begin{aligned} \frac{d\xi}{dt} &= h(t,x,y,z),\\ \xi(t_0) &= 0 \end{aligned}\right\} \tag{20}$$

and then the pay-off may be given as

$$P = \xi(t_F) + \gamma(t_F,x(t_F)) = g(t_F,x(t_F,\xi(t_F))) \tag{21}$$

say.

An approximate strategy, A for J_Y (see (2)) is a sequence (α_n) of delay strategies (i.e. $\alpha_n \in \bigcup_{s>0} \Gamma_s(t_0)$); a similar sequence $B = (\beta_n)$ is an approximate strategy for J_Z. Then (A,B) induce a *unique* sequence $(y_n(\cdot), z_n(\cdot))$ in $M_Y(t_0)\times M_Z(t_0)$ such that

$$\alpha_n z_n = y_n, \quad \beta_n y_n = z_n.$$

The pairs (y_n,z_n) induce trajectories $(x_n(t),\ \xi_n(t))$ and the sequence of trajectories $(x_n(\cdot),\ \xi_n(\cdot))$ can be shown to be relatively compact in the Banach space of all $\mathscr{R}^{m+1}$-valued trajectories on $[t_0,T]$. We define $P[A,B]$ to be the *set* of all pay-offs $P(\bar{x}(\cdot),\bar{\xi}(\cdot))$ corresponding to cluster points $(\bar{x}(\cdot),\bar{\xi}(\cdot))$ of this sequence. Here

$$P(\bar{x}(\cdot),\bar{\xi}(\cdot)) = g(\bar{t}_F,\bar{x}(\bar{t}_F),\bar{\xi}(\bar{t}_F)).$$

We then define the *extended values*

$$V_e^+ = \inf_B \sup_A \sup P[A,B],$$

$$V_e^- = \sup_A \inf_B \inf P[A,B]$$

and we say that $\bar{G}$ has extended value if $V_e^+ = V_e^-$.

$\bar{G}$ is called a *generalized pursuit-evasion* game if $\gamma = 0$ and $h \geqslant 0$. It was shown in (2) that under the extended Isaacs condition (7), that a generalized pursuit-evasion game has extended value. Later Friedman (11) improved this by requiring only the Isaacs condition (8), although his formulation of extended value is different in general.

We shall say that $\bar{G}$ is a *quasi-pursuit-evasion* game if

$$h^*(t,x) = \max_y \min_z h(t,x,y,z) \geqslant 0$$

and $\gamma \equiv 0$.

LEMMA 8·1. *Let $\bar{G}_1$ and $\bar{G}_2$ be differential games of survival with dynamics* (18) *and pay-offs*

$$P_1 = \int_{t_0}^{t_F} h_1(t, x(t), y(t), z(t))\, dt,$$

$$P_2 = \int_{t_0}^{t_F} h_2(t, x(t), y(t), z(t))\, dt,$$

where

$$|h_1(t,x,y,z) - h_2(t,x,y,z)| \leqslant \epsilon$$

for all (t,x,y,z). Then

$$|V_{e,1}^+ - V_{e,2}^+| \leqslant \epsilon(T-t_0),$$

$$|V_{e,1}^- - V_{e,2}^-| \leqslant \epsilon(T-t_0).$$

Proof. Let A and B be any pair of approximate strategies inducing sequences of trajectories $(x_n(\cdot), \xi_n^1(\cdot))$ and $(x_n(\cdot), \xi_n^2(\cdot))$ respectively. Let $(\bar{x}(\cdot), \bar{\xi}^1(\cdot))$ be a cluster point of the former sequence; then there is a sequence $(x_{n_k}(\cdot), \xi_{n_k}^1(\cdot)) \to (\bar{x}(\cdot), \bar{\xi}^1(\cdot))$ uniformly. By selection of subsequence we may suppose $\xi_{n_k}^2(\cdot) \to \bar{\xi}^2(\cdot)$ uniformly. Then

$$|\xi_n^2(t) - \xi_n^1(t)| \leqslant \epsilon(t-t_0) \quad (t_0 \leqslant t \leqslant T)$$

and so

$$|\bar{\xi}^2(t) - \bar{\xi}^1(t)| \leqslant \epsilon(t-t_0) \quad (t_0 \leqslant t \leqslant T).$$

Therefore

$$|P_2(\bar{x}(\cdot), \bar{\xi}^2(\cdot)) - P_1(\bar{x}(\cdot), \bar{\xi}^2(\cdot))| \leqslant \epsilon(T-t_0).$$

It follows that

$$\sup P_1[A,B] \leqslant \sup P_2[A,B] + \epsilon(T-t_0)$$

and by symmetry

$$|\sup P_1[A,B] - \sup P_2[A,B]| \leqslant \epsilon(T-t_0)$$

and the result follows quickly.

We now define the *associated restricted fuel game* of $\bar{G}$, which for $\Lambda \geqslant 0$ we call G_Λ^*. Let $F_\Delta^* = \{(t,x,\xi), \xi \geqslant \Lambda \text{ or } t \geqslant T\}$. Then G_Λ^* has dynamics

$$\left.\begin{aligned} \frac{dx}{dt} &= f(t,x,y,z), \\ \frac{d\xi}{dt} &= h(t,x,y,z), \end{aligned}\right\} \tag{22}$$

$$x(t_0) = x_0, \quad \xi(t_0) = 0,$$

and pay-off
$$P = \min_{t_0 \leqslant t \leqslant t_{F_\Lambda^*}} \rho(t, x(t)), \tag{23}$$

where $\rho(t,x) = \text{dist}\,[(t,x), F]$. This is clearly a restricted fuel game as in section 7. We first apply Lemma 7·1.

LEMMA 8·2. *If $\bar{G}$ is a quasi-pursuit-evasion game in which*
$$\inf_{(t,x)} h^*(t,x) = \lambda > 0,$$
then in the associated restricted fuel game G^*_Λ, F^*_Λ is both Q^+- and Q^--regular.

Proof. By Lemma 7·1, it is only necessary to show that
$$\lim_{s\to 0} \bar{Q}^-_s(\bar{t}, \bar{x}, \bar{\xi}) = g(\bar{t}, \bar{x}, \bar{\xi}) = \rho(\bar{t}, \bar{x})$$
for $(\bar{t}, \bar{x}, \bar{\xi}) \in F^*_\Lambda$.

Suppose B_f and B_h are bounds for $\|f\|$ and $\|h\|$, and let C be a bounded subset of $[0,T] \times \mathscr{R}^{m+1}$. By Theorem 5·1 of (3) there is a function $\eta\colon (0,\infty) \to (0,\infty)$ with $\lim_{s\to 0} \eta(s) = 0$ such that whenever $(t_1, x_1, \xi_1) \in C$ and $y(\cdot) \in M^s_Y(t_1)$ then there exists $\alpha \in \Gamma_{t_1}(s|y)$ such that for any $(z(\cdot), \xi(\cdot))$ with
$$h(t, x(t), \alpha z(t), z(t)) \geqslant h^*(t, x(t)) - \eta(s) \quad \text{a.e.}$$
for $t_1 + s \leqslant t \leqslant T$. Then
$$\int_{t_1}^{t_2} h(t, x(t), \alpha z(t), z(t))\, dt \geqslant \lambda(t_2 - t_1) - \eta(s) - B_h s.$$
If $\tau = t_{F^*_\Lambda}$, then we deduce
$$\Lambda - \xi_1 \geqslant \lambda(\tau - t_1) - \eta(s) - B_h s$$
and therefore
$$\tau - t_1 \leqslant \lambda^{-1}(\Lambda - \xi_1 + \eta(s) + B_h s).$$
Therefore for $t_1 \leqslant t \leqslant \tau$
$$\begin{aligned}|\rho(t, x(t)) - \rho(t, x(t))| &\leqslant (\tau - t_1)(1 + B_f)\\ &\leqslant \lambda^{-1}(1 + B_f)(\Lambda - \xi_1 + \eta(s) + B_h s).\end{aligned}$$
Clearly the value of $\alpha, u(\alpha)$, in G^*_Λ satisfies
$$u(\alpha) \geqslant \rho(t, x_1) - \lambda^{-1}(1 + B_f)(\Lambda - \xi_1 + \eta(s) + B_h s)$$
and equally
$$u(\alpha) \geqslant \rho(t_1, x_1) - (1 + B_f)(T - t_1).$$
Therefore
$$Q^-_s(t_1, x_1, \xi_1) \geqslant \rho(t_1, x_1) - (1 + B_f) \min(T - t_1, \lambda^{-1}(\Lambda - \xi_1 + \eta(s) + B_h s)).$$
Hence for $(\bar{t}, \bar{x}, \bar{\xi}) \in \text{int}\, C \cap F^*_\Lambda$
$$\bar{Q}^-_s(\bar{t}, \bar{x}, \bar{\xi}) \geqslant \rho(\bar{t}, \bar{x}) - (1 + B_f)\lambda^{-1}(\eta(s) + B_h s)$$
and therefore
$$\lim_{s\to 0} \bar{Q}^-_s(\bar{t}, \bar{x}, \bar{\xi}) \geqslant \rho(\bar{t}, \bar{x})$$
and this implies $\lim_{s\to 0} \bar{Q}^-_s(\bar{t}, \bar{x}, \bar{\xi}) = \rho(\bar{t}, \bar{x})$, as required.

THEOREM 8·3. *Let $\bar{G}$ be a quasi-pursuit-evasion game satisfying the Isaacs condition. Then $\bar{G}$ has extended value.*

Proof. Suppose first

$$\inf h^*(t,x) = \lambda > 0.$$

For $\Lambda \geqslant 0$, by Lemma 8·2, F_Λ^* to Q^+- and Q^--regular. Hence by Theorem 7·2 G_Λ^* has value V_Λ^* say (i.e. $Q^+(G_\Lambda^*) = V^+(G_\Lambda^*) = V^-(G_\Lambda^*) = Q^-(G_\Lambda^*)$). The map $\Lambda \to V_\Lambda^*$ is clearly monotonically decreasing and so if

$$\Lambda_0 = \inf(\Lambda_0 V_\Lambda^* = 0)$$

then for $\Lambda > \Lambda_0$, $V_\Lambda^* = 0$. Hence for $\epsilon > 0$, there is a delay strategy β_n for J_Z whose value in $G^*_{\Lambda_0+\epsilon}$ is less than $1/n$. Let $B = (\beta_n)$ be the induced approximate strategy for J_Z in $\bar{G}$, and let $A = (\alpha_n)$ be any approximate strategy for J_Y. Then (α_n, β_n) induce a sequence $(x_n(\cdot), \xi_n(\cdot))$ of trajectories, such that for each n there exists t_n with $t_n \leqslant T$, and

$$\xi_n(t) \leqslant \Lambda_0 + s \quad (t \leqslant t_n)$$

$$\rho(t_n, x_n(t_n)) \leqslant \frac{1}{n}.$$

If $(\bar{x}(\cdot), \bar{\xi}(\cdot))$ is any accumulation point of the sequence then for some $\bar{t} \leqslant T$

$$\bar{\xi}(t) \leqslant \Lambda_0 + \epsilon \quad (t \leqslant \bar{t}),$$

$$\rho(\bar{x}(\bar{t}), \bar{\xi}(\bar{t})) = 0.$$

The hitting time $\bar{t}_F$ of $(\bar{x}(\cdot), \bar{\xi}(\cdot))$ satisfies

$$\bar{t}_F \leqslant \bar{t},$$

$$\bar{\xi}(\bar{t}_F) \leqslant \Lambda_0 + \epsilon.$$

Hence
$$\sup P[A,B] \leqslant \Lambda_0 + \epsilon$$

and so
$$V_e^+ \leqslant \Lambda_0 + \epsilon.$$

As $\epsilon > 0$ is arbitrary we conclude $V_e^+ \leqslant \Lambda_0$.

Conversely $V^*_{\Lambda_0-\epsilon} = \delta > 0$, so that for some delay strategy α for J_Y, the value of α in $G^*_{\Lambda_0-\epsilon}$ is at least $\frac{1}{2}\delta$. If $A = (\alpha)$ (i.e. the constant sequence α) then we may show easily (cf. (2)) that for any B

$$\inf P[A,B] \geqslant \Lambda_0 + \epsilon.$$

Hence $V_e^- \geqslant \Lambda_0$ and so $V_e^- = V_e^+ = \Lambda_0$.

Now suppose $\lambda = 0$; for each $\epsilon > 0$ we may replace h by $h_\epsilon = h + \epsilon$ to obtain a game $\bar{G}_\epsilon$ in which the Isaacs condition still holds, and $V^-_{e,\epsilon} = V^+_{e,\epsilon}$. By Lemma 8·2 we obtain that

$$V_e^+ - V_e^- \leqslant 2\epsilon$$

for any $\epsilon > 0$.

9. *General games of survival.* We conclude with our main theorem on games of survival, of which Theorem 8·3 is a special case.

THEOREM 9·1. *Suppose $\bar{G}$ is a differential game of survival given by* (18) *and* (19) *satisfying the Isaacs condition and such that there is a C^1-function $\psi(t,x)$ such that*

$$L\psi = \frac{\partial\psi}{\partial t} + H(t,x,\nabla\psi) \geqslant 0 \quad (t,x) \in \mathscr{R}^{m+1} - F$$

and $\psi = \gamma$ on ∂F. Then $\bar{G}$ has extended value. (*Equally the same result holds if*

$$L\psi \leqslant 0 \quad (t,x) \in \mathscr{R}^{m+1} - F,$$

by symmetry.)

Proof. Let

$$h_1 = \frac{\partial\psi}{\partial t} + \nabla\psi . f + h.$$

Then h_1 is continuous and

$$h_1^*(t,x) = \min_z \max_y h_1 = L\psi \geqslant 0$$

everywhere. However along any trajectory $(x(t): t_0 \leqslant t \leqslant T)$

$$\int_{t_0}^{t_F} h_1(t,x(t),y(t),z(t))\,dt = \int_{t_0}^{t_F} h\,dt + \psi(t_F, x(t_F)) - \psi(t_0,x_0)$$
$$= P - \psi(t_0,x_0)$$

by (19). Therefore except for the constant $\psi(t_0,x_0)$, $\bar{G}$ is equivalent to a quasi-pursuit evasion game given by (18) with pay-off $P = \int h_1(t,x,y,z)\,dt$.

Now

$$\min_z \max_y (p.f + h_1) = \min_z \max_y \left(\frac{\partial\psi}{\partial t} + h + (p + \nabla\psi).f\right)$$
$$= \frac{\partial\psi}{\partial t} + H^+(t,x,p+\nabla\psi)$$
$$= \frac{\partial\psi}{\partial t} + H^-(t,x,p+\nabla\psi)$$
$$= \max_y \min_z (p.f + h_1).$$

Hence it follows that $\bar{G}$ has extended value by Theorem 8·3.

As pointed out in the introduction, the curious feature of this result is that the existence of *two* functions ϕ and ψ with $L\phi \geqslant 0 \geqslant L\psi$ on $\mathscr{R}^{m+1} - F$ and $\phi = \psi = \gamma$ on ∂F implies the existence of value, not just extended value. This result therefore fits the idea of extended value into context as a first step towards the existence of value. In a quasi-pursuit-evasion game the function $\psi \equiv 0$ satisfies the hypotheses.

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